On Existence of Equilibria in Bayesian Allocation Mechanisms

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Bayesian Allocation Mechanisms

- In allocation mechanisms, agents choose messages.
- The messages determine an outcome: a (randomized) allocation from a (discrete) set and transfers from agents.
- Examples include auctions, provision of public goods, ...
- Question: under what conditions does such a model have an equilibrium?

The Issue of Equilibrium Existence

- When the mechanism is indirect, the outcome is deterministic for most message profiles.
- The outcome function is therefore a discontinuous function of the messages.
- Therefore, fixed-point theorems cannot be applied directly.

Approaches To The Problem

- Assuming "monotonicity," the classical approach in auctions analyzed differential equations arising from first-order conditions.
- Later literature (Athey (2001), McAdams (2003), Reny-Zamir (2004), Reny (2011)) gave conditions for best replies to monotone strategies to be monotone.
- They used these conditions to provide existence theorems.
- These conditions (typically some kind of affiliation) are very non-generic (de Castro (2009)).

Discontinuous Games

- Recent results on existence of equilibria in discontinuous games (Reny (1999), McLennan et al. (2012), Barelli-Meneghel (2013), Bich-Laraki (2013)) have not proved useful for this problem.
- The conditions in this literature are typically difficult to verify.

A Notable Exception

- Jackson, Simon, Swinkels, and Zame (2002) gave an existence theorem using endogenous tie-breaking: the allocation at points of discontinuity depends on the types of the agents.
- The ideas of this paper are related to earlier work of Simon and Zame (1990) for discontinuous games with complete information.
- Jackson and Swinkels used these results to establish existence of equilibria in mixed strategies for private-value auctions.

Our Approach in This Work

- We investigate the issue of the existence of equilibria in behavioral strategies.
- In particular, we examine whether there exist equilibria for which the probability of "ties" is zero.
- We exploit the fact that discontinuities in Bayesian Allocation-Mechanisms are 'mild," in that the set of discontinuities is a lower-dimensional set.
- We study the related question of existence of ε-Nash equilibria.

Notation

- The set of agents is $N = \{1, \ldots, N\}$.
- The set of types of each agent n is Ω_n. $\Pi_n \Omega_n = \Omega$.
- The prior on Ω is p, assumed to be absolutely continuous w.r.t. to the product $\Pi_n p_n$ of the marginals.
- The set of outcomes is a finite set Z and $\Delta(Z)$ is the set of lotteries over Z.

Notation—continued.

- The set of messages of player n is a compact metric space B_n. $\Pi_n B_n = B$.
- $\lambda \times t : B \to \Delta(Z) \times \mathbb{R}^{NZ}$ is the outcome function: $\lambda(b) \in \Delta(Z)$ is the allocation and $t_n(b, z)$ is the transfer from n if z is the allocation.
- Each n is an expected-utility maximizer and his payoff depends on (ω, z) and on his monetary transfers: $u_n : \Omega \times Z \times \mathbb{R} \to \mathbb{R}$.
- $u_{n,z}(\omega, t)$ is the payoff conditional on z. Assume:
  1. $u_{n,z}$ is a Caratheodory function (measurable w.r.t. ω and continuous w.r.t. t_n) and maps $\Omega \times T_n$ to a bounded set for each compact set T_n.
  2. $u_{n,z}$ is strictly decreasing in the transfer t_n.

Assumptions

- $t : B \to \mathbb{R}^{NZ}$ is continuous.
- Let D be the set of b such that λ is discontinuous.
- $D_{-n}(b_n)$ is the cross-section of D over b_n.
- For $\mu_{-n} \in \Pi_{m \neq n}(\Delta(B_m))$, b_n is a point of continuity if $\mu_{-n}(D_{-n}(b_n)) = 0$.
- Mild discontinuity: For each μ_{-n}, the set of b_n that are points of continuity is dense and measurable in B_n.

Behavioral Strategies

- A pure strategy of n is a measurable map $\beta_n : \Omega_n \to B_n$.
- n's interim payoff from a profile β is:

$$\pi_n(\omega_n, \beta) = \sum_z \int_{\Omega_{-n}} u_{n,z}(\omega, t_{n,z}(\beta(\omega)))\lambda_z(\beta(\omega))dp_{-n}(\omega_{-n}|\omega_n).$$

- A behavioral strategy for player n is a transition-probability function $\sigma_n : \mathcal{B}_n \times \Omega_n \to [0,1]$, where $\mathcal{B}_n$ is the Borel sigma-algebra on B_n.
- $\sigma_n(\cdot|\omega_n)$ is the mixture over B_n of player n if his signal is ω_n.
- n's set of behavioral strategies is Σ_n. $\Sigma = \Pi_n \Sigma_n$.
- The payoff $\pi_n(\omega_n, b, \sigma_{-n})$ is defined by the usual expectation formula.

Assumptions—continued.

- b_n is a point of continuity against σ if σ_{-n} assigns zero probability to $D_{-n}(b_n)$.
- Assumption: Suppose σ is a behavioral-strategy profile such that $\sigma(D|\omega) = 0$ for a.e. ω. There exists another behavioral strategy profile $\tilde{\sigma}$ such that for each n:
  1. each b_n is a point of continuity against $\tilde{\sigma}$;
  2. for each b_n that is a point of continuity against σ, $\pi_n(\omega_n, \sigma_{-n}, b_n) = \pi_n(\omega_n, \tilde{\sigma}_{-n}, b_n)$ for a.e. ω_n.
- In an auction, suppose a subset of players choose a bid with positive probability, even though the bid has zero probability of winning. The assumption allows us to change this bid so that ties could not occur even if somebody else deviates to this bid—e.g., allowing bidders not to enter any bid suffices.

A Class of Tie-Breaking Rules

- Alternate tie-breaking rules: the set of $\tilde{\lambda} : \Omega \times B \to \Delta(Z)$ that agree with λ when $b \notin D$.
- Λ is the set of all such rules.
- The payoff function with $\tilde{\lambda}$ is denoted $\pi_n(\omega_n, \sigma; \tilde{\lambda})$.

Lemma

Let σ^k be a sequence in Σ converging to σ^*. There exists $\lambda^* \in \Lambda$ such that $\pi_n(\cdot, \sigma^k)$ converges weakly to $\pi_n(\cdot, \sigma; \lambda^*)$ in $L_\infty(\Omega_n; p_n)$.

- λ^* is an "endogenous" tie-breaking rule, as in JSSZ.
- Ξ is the set of (σ^*, λ^*) obtained as limits of sequences σ^k as above.

Assumptions—continued.

Let σ be a behavioral strategy profile such that $\sigma(D|\omega) > 0$ for a set of ω with positive p-probability. For each n, let X_n^* be the set of (ω_n, b_n) such that b_n is not a point of continuity against σ_{-n}.

• There does not exist λ^* such that $(\sigma, \lambda^*) \in \Xi$ and for each n and ς_n-a.e. pair (ω_n, b_n) and (ω'_n, b'_n) in a subset X_n of X_n^* with $\varsigma_n^*(X_n) = \varsigma_n^*(X_n^*)$:

1. (Individual Rationality) $\pi_n(\omega_n, b_n, \sigma_{-n}; \lambda^*) \geqslant \pi_n(\omega_n, b'_n, \sigma_{-n})$ for every b'_n that is a point of continuity against σ_{-n};
2. (Incentive Compatibility) $\pi_n(\omega_n, b_n, \sigma_{-n}; \lambda^*) \geqslant \psi_n^*(\omega_n, \omega'_n, b'_n)$, where $\psi_n^*(\omega_n, \omega'_n, b'_n)$ is ω_n's payoff at bid b'_n using ω'_n instead of ω_n to determine the allocation.

Interpretation of Key Assumption •

- There is no endogenous tie-breaking rule that:
  1. (IR): gives agents at least as much as from choosing points of continuity.
  2. (IC): does not benefit an agent with signal ω_n who mimicks a different signal ω'_n.
- This rule has the flavor of better-reply security, in the sense that it is a special case of a Bayesian formulation of the idea.
- A special case arises when the tie-breaking rule λ at a tie produces a Pareto-optimal allocation that is not individually rational.

A Basic Existence Theorem

Theorem

Under our assumptions, the game Γ has a Nash equilibrium $\sigma^* \in \Sigma^*$ such that for each n, every b_n is a point of continuity against σ^*. In particular, σ^* is an equilibrium for all tie-breaking rules in Λ.

Sketch of the Proof:

- Consider a sequence Γ^k of approximations of the game Γ (e.g., discrete approximations).
- For each k, let σ^k be an equilibrium with associated equilibrium payoffs $v_n^k(\omega_n)$.

Sketch of Proof—continued.

- Let (σ^*, λ^*) be the limit of the equilibria and let $v_n^*(\omega_n)$ be the limiting payoff.
- We argue by contradiction that there is no point of discontinuity.
- $v_n^*(\cdot)$ is at least the payoff from choosing a point of continuity against σ^*. Therefore, λ^* satisfies IR.
- IC is satisfied since each type could mimic the strategy of any other type along the sequence σ^k and hence at the limit.
- The theorem goes through if the set of feasible messages is type-dependent, given by a correspondence $\varphi_n : \Omega_n \to B_n$.

ε-Nash Equilibria

- Say that σ is a 0-equilibrium if it is the limit of a sequence of ε-equilibria in which ties occur with zero probability.
- We give conditions for existence of 0-equilibria.
- Assume that λ is deterministic outside D.
- If, as in the proof of the theorem, we obtain the limit (σ^*, λ^*) of equilibria of perturbed games, but λ^* is deterministic, then σ^* is obtainable as a 0-equilibrium (though not necessarily through the sequence σ^k).

The Set-Up

- $\Omega_n = [0, 1]$.
- Z is the set of unit vectors in $\mathbb{R}^n$ along with the origin: it is a specification of who, if anybody, gets the object.
- B_n is the union of an interval $[0, \overline{b}]$ and a distinguished point b^* (the outside option of not participating in the auction, which is different from bidding 0).
- The discontinuity set D is where there is a tie for the highest bid.
- When a bidder chooses the outside option b^*, he cannot cause a tie.

Characterization of Tie-Breaking Rules

- In this set up, given any profile σ of behavioral strategies, there can only be a countable number of points of discontinuities (atoms of the bid distribution).
- For each atomic bid b, a tie-breaking rule λ^* specifies an allocation for the signal profile for which b is the highest bid.
- Individual rationality simplifies to checking whether a bidder could do better than the tie-breaking rule by bidding up or down a little, avoiding the atom.
- Incentive compatibility applies to all signals that choose the bid b.

All-Pay Auctions

- In this case the transfer rule is $t_n(z, b) = b_n$, independent of the outcome z.
- We assume the following for each n:
  1. $u_n(\omega, e_n, b_n) > u_n(\omega, z, b_n)$ for $z \neq e_n$. (Player n strictly prefers to win the object than lose.)
  2. $u_n(\omega, z, \overline{b}) < u_n(\omega, z, b_n^*)$. (The upper bound on bids is dominated by the outside option.)

Theorem: Under these conditions, the all-pay auction has an equilibrium in behavioral strategies.

All-Pay Auctions—continued.

- It is a simple matter to verify that IR is violated by any mechanism λ^*.
- At a tie at a bid less than $\overline{b}$, it is impossible to give every player the good, yet each can get it for himself by bidding a little higher.
- By assumption, at $\overline{b}$ each player would rather opt out.
- There is no assumption that u_n depends on ω at all, which implies that the result applies to a game with complete information as well!

Private-Value First- and Second-Price Auctions

- $u_{n,z}$ depends on ω only through ω_n.
- First-Price Auction: $t_n(z, b_n) = b_n$ if $z = e_n$.
- Second-Price Auction: $t_n(z, b_n) = b_m$ if $z = e_n$ where b_m is the second-highest bid.
- Assume that the action correspondence $\varphi_n : \Omega_n \to B_n$ restricts ω_n to undominated bids, i.e., $u_n(\omega_n, b_n) \geqslant 0$. We make the following further assumptions:
  1. For each b_n, $u_n(\omega_n, b_n) \neq 0$ for a.e. ω_n. (Rules out discrete type space.)
  2. $u_n(\omega_n, \overline{b}) < u_n(\omega_n, b_n^*)$.

Existence

- We now have the theorem proved by Jackson and Swinkels (2002): The game has an equilibrium without ties.
- Verification of the IR condition is straightforward.
- At an atomic bid, all signals want either to win or lose.
- Because of the restriction on the bids, all signals want to win, and can by bidding higher.
- Observe that the argument is the same for both first- and second-price auctions: at a tie, both rules have the same payment.
- Without the first assumption (i.e., even with discrete type spaces) there are 0-equilibria.

Two-Bidder Common-Value Auctions

- Assume that $u_n(\omega, e_n, b_n) = u_m(\omega, e_m, b_n)$.
- The assumptions for the Common-Value case are identical to that for the private-values case, except that we do not need a restriction on the bid correspondence.
- Assume that for each b_n, $u_n(\omega, e_n, b_n) \neq 0$ for a.e. ω.
- Assume that $u_n(\omega, e_n, \overline{b}) < u_n(\omega, z, b_n^*)$. (Better to opt out than pay the highest bid.)

Two-Bidder Common-Value Auctions—continued.

- Under our assumptions there exists an equilibrium without ties.
- The verification of IR is simple. The tie-breaking rule of the game (equal split) is Pareto optimal.
- Thus at least one of the two players is better off under the standard tie-breaking rule rather than the endogenous one λ^*.
- This player can bid up or down depending on whether the joint pie (net of the bid) is positive or not.
- The idea has the flavor of using reciprocal u.s.c. and payoff security to get existence (Dasgupta-Maskin (1986), Reny (1999)).

The Two-Bidder Interdependent-Values Case

- Up to now, we only used the IR condition of the assumption to rule out atoms. This case uses the IC constraint as well.
- In the interdependent-values case, u_n depends on the entire profile ω and not just ω_n.
- For simplicity we assume that $u_n(\omega, z, 0) = 0$ if $z \neq e_n$ (n gets no value if he does not get the object, zero being the value of the outside option).
- We assume that $u_n(\omega, e_n, b_n)$ is strictly increasing in ω_n.

The Crucial Assumption

- Fix a subset $\tilde{\Omega}_m$ of m's signals and fix a bid b_n.
- For each ω_n, we can now look at the conditional expectation $E(u_n(\omega, e_n, b_n)|\omega_n, \tilde{\Omega}_m)$.
- Our main assumption is that this conditional expectation is (a.e.) strictly increasing in ω_n.
- The condition is trivially satisfied in the private-values case.
- In the general case, it is weaker than assuming that u_n is weakly increasing in ω_m and that the distribution p is affiliated.

An Existence Theorem

- Under the above conditions, there exists an equilibrium without ties.
- The proof uses both the IC and IR conditions.
- Assume that there is an atomic bid with a tie-breaking rule λ^*.
- Increasing conditional expectations implies that the probability of winning is monotonically increasing in one's own signals if the rule is to satisfy IC.
- This implies that the probability of beating player m is monotonically decreasing in ω_m.
- The marginal type that ω_n beats with positive probability must be worth beating to get the object.
- This implies that ω_n would prefer to beat all types of m, which he can do by bidding a little above b.

The N-bidder Case

- It is easy to generalize the above assumption to the N-bidder case.
- However, such an assumption does not nest affiliation.
- That is a big problem, since a satisfactory theory would use weaker conditions for (0-)equilibria than is required for the existence of pure strategy equilibria, much less monotone equilibria.
- What is the main issue here?

An Example

- At a tie, a player need not know how many other bidders are involved (e.g. it could be a three-way, or a two-way, or a k-way tie for the object).
- He might want to behave differently depending on what kind of a tie it is.
- Consider a simple three-player common-value auction where there is a tie at some bid. A two-way tie results in a total pie (net of the bid) of 12. A three-way tie results in a pie of -30.
- Assuming symmetry, each player gets half of each of the two-way ties and a third of the three-way tie, to yield a net payoff of 2.
- Bidding up wins against all ties, and thus a payoff of -6. Bidding down gives him 0.
- Thus each bidder prefers to stay at the tie!

A Possible Resolution

- For simplicity, consider a symmetric 3-bidder case.
- In the case of a tie, give players information about how many players are involved in the tie.
- Charge a small fee and then let them bid again in a next round.
- We have an "almost-proof" that this results in a 0-equilibrium.

Conclusion

- Previous existence results rely on strong assumptions
  — affiliated distributions are nongeneric
  — private- and common-value models are restrictive
  — endogenous tie-breaking is impractical
- Structural features of allocation mechanisms can be exploited
  — Payoff discontinuities are "mild", occurring at tied bids
  — "Increasing conditional expectations" is more robust
  — ε- or 0-equilibrium is better predictor in practice
- For simple auctions designs, these assumptions largely suffice for existence
  — Even, apparently, for models with interdependent values

- What further assumptions will yield pure-strategy equilibria?
- Do they suffice for a wider class?
  — Multi-unit auctions, double auctions, multistage dynamic allocation processes,...