Meritocracy, Egalitarianism and the Stability of Majoritarian Organizations*

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Abstract

Egalitarianism and meritocracy are two competing principles to distribute the joint benefits of cooperation. We examine the consequences of letting members of society decide by vote among those two principles, in a context where groups of a certain size must be formed, in order for individuals to become productive. Our setup induces a hedonic game of coalition formation. We study the existence of core stable partitions (organizational structures) of this game. For societies with three types of agents we provide necessary and sufficient conditions under which core stable partitions will exist, and we identify the types of stable organizational structures that will arise. We conclude that the inability of voters to commit to one distributional rule or another is a potential source of instability. But we also prove that, when stable organizational structures exist, they may be rich in form, and quite different than those predicted by alternative models of group formation. In particular, non-segregated groups may arise within core stable structures. Stability is also compatible with the coexistence of meritocratic and egalitarian groups. We also remark that changes in society can alter their distributional regimes and influence their ability to compete.

Key words: Egalitarianism, Meritocracy, Coalition Formation, Hedonic Games, Core Stability, Assortative Mating.

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1. Introduction

Egalitarianism and meritocracy are two competing principles to distribute the joint benefits from cooperation. Though one could debate their relative merits and side for one or the other, we take the opposite route. We analyze the consequences of not taking sides between these two principles, and letting different organizations choose by vote among the two, in a context where this choice is part of group formation decisions. The lack of ability to commit "a priori" on one specific distributional criterion may lead to organizational structures and consequences that would not arise in more classical frameworks. Stability need not imply positive assortative mating, thus allowing for non-segregated groups. Moreover, meritocratic and egalitarian groups may coexist within a stable society.

Specifically, we consider societies within which groups must form to perform a certain task. Each individual is endowed with a productivity level. Groups need a minimal size to be productive. Beyond that size, group production is the sum of the productivities of its members. If a group is formed, its members decide by majority vote whether to distribute their product according to meritocracy or to egalitarianism. Agents prefer to get a higher than a lower pay. If they must choose among organizations that will pay them the same, they prefer those whose average productivity is higher.

We examine the consequences of this form of group governance on the size, stability and composition of organizations, on their endogenous choice of rewards, on their ability to compete for talent, and their ability to keep a competitive edge under changes in their definitional parameters.

Under the assumption that agents know the productivities of all others, they can anticipate what rewards they will get from joining any given group. They will thus play a hedonic game (Drèze and Greenberg (1980)), where agents have preferences over the groups they may belong to and outcomes are partitions of agents into groups. We present a detailed analysis of the hedonic games that arise in our simple societies, and a variety of examples showing how our model can be used to understand the basic features of coalition formation driven by distributional concerns subject to vote. In spite of its simplicity, the setup is rich enough to give rise to situations where stability may not be achieved. Yet, we are able to identify a rich and interesting class of societies that we call three-way clustered, whose core stable organizational structures we fully characterize. Furthermore, for a subclass of these societies such structures are proven to be unique and non-segregated.

Our highly stylized model allows for different interpretations. In all its apparent simplicity, it allows us to touch upon a variety of topics that are at the forefront of today's economic research. What we offer is a very compact view of the forces that may drive the different members of the same society to stick to one group and to dissociate from others.

An interpretation of our model approaches it to the work on country formation and secessions (Alesina and Spolaore (1997), Le Breton et al. (2004)), except that we abstract from geography and instead highlight the role of distributional issues as a driving force that shapes different types of societies. Stability issues will be central in our case, as they are in that related literature: one of our goals is to characterize core stable country configurations (organizational structures). Under this country formation interpretation, our model is suggestive of different phenomena that have been recently highlighted in the literature, regarding the differences in characteristics among advanced societies. We prove that different countries may adopt different distributional criteria and still coexist, as in the literature on "varieties of capitalism" (Acemoglu et al (2012), Hall and Soskice (2001)). Of course, our static analysis cannot fully encompass all the dynamic and incentive aspects of a more complex setup, but it is significant that this important stylized fact arises from such a simple model as ours. Moreover, we can provide a highly suggestive comparative static analysis pointing at incentive issues. We present examples where changes in the population and/or in the threshold size required for organizations to become productive have consequences on the country's ability to compete for highly qualified individuals.

Other interpretations of our model reflect on the formation of organizations or jurisdictions within a community. Leading interpretations include the establishment of decentralized regions, public university systems, cooperative firms or partnerships within regulated professions. In that respect, our contribution is related to the literature on the endogenous formation of institutions in broad terms (Caplin and Nalebuff (1994)), or the more specific ones analyzing the choice of regional tax systems, local public goods, clubs and sorting (Tiebout (1959), Schelling (1969), Epple and Romer (1991), Ellickson et al (1999), Hansen and Kessler (2001), Puy (2007), Damiano et al (2010), among others). However, our results are different, both on technical grounds and also regarding some basic conclusions. For example, they are in contrast with the usual conclusion that stability pushes agents to form segregated groups. Our model is one where segregation may or may not arise, and the structural characteristics of the distributions leading to non-segregation can be clearly traced to the fundamentals of the productivity distribution.

Our model is deliberately very simple. Thanks to that, we are able to analyze the structure of core stable partitions in finite societies, in contrast to much of the preceding literature that resorts to alternative equilibrium concepts in the spirit of Nash stability, and mostly concentrates in the study of societies with a continuum of agents. The reader will also appreciate that, in spite of our stark formulation, the analysis of stability in the hedonic games that arise is indeed complex.

Different authors have discussed stability issues for hedonic games in a variety of contexts, none of which matches our case. Preceding papers include the work of Farrell and Scotchmer (1988), Banerjee et al (2001), Bogomolnaia and Jackson (2002), Alcalde and Romero-Medina (2006), Papai (2011), Iehlé (2007), Pycia (2012). All of them apply to domains of preferences that do not match our specifications. In particular Pycia (2012) develops a model that encompasses many applications, including cases of joint production. Its basic implication is that in order to achieve stability the preferences of agents over sets must be aligned and generate segregation. Yet, our model admits divergences among agents and shows that non-segregated societies can belong to stable coalitional structures. The reason is that Pycia's domain of preferences is larger than ours. Specifically, when specialized to our case, his domain of preferences allows agents to rank groups of identically productive types in any arbitrary manner, while we require a natural form of anonymity. Hence, we provide a completely new set of existence results.

After this Introduction, and a formal presentation of the model in Section 2, the paper can be read in two sequences. Sections 3, 4 and 5 contain the analysis of the hedonic games that our agents will face, and concentrate on the characterization of core stable organizational structures. Section 6 contains examples and applications of the model. The general reader may want to jump to Section 6 after Section 3, being reassured that all the examples we present there fall into the categories that we have studied, from a more technical point of view, along the intermediate Sections 4 to 6. We close the paper in Section 7, where we discuss extensions and variants of the model, and the kind of questions we'd like to keep addressing with them.

2. The model. Organizational Structures

Let $N = \{1, 2, ..., n\}$ be a set of n individuals characterized by their individual potential productivities $\lambda = (\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n \geq 0$. Subsets of N are called *groups* or *organizations* interchangeably. Individuals can only become productive if they work within a group $G \subseteq N$ of size at least v. Groups of smaller size produce nothing, while groups of size v or larger produce the sum of their members' productivities. A *society* is represented by a triple (N, λ, v).

We refer to a group of cardinality less than v as being *unproductive*.

We denote the average productivity of a group $G \subseteq N$ by $\bar{\lambda}_G$, and by λ_G the vector of productivities of the agents in G.

If a productive group is formed, its total production must be distributed among the agents of the group. Agents prefer to get a higher than a lower pay. Lexicographically, if they must choose among organizations that will pay them the same, they prefer those whose average productivity is higher. [1]

Productive groups internally decide, by majority voting, whether to distribute their product in an egalitarian or in a meritocratic manner. That is, whether all agents in the organization G get the same reward, $\bar{\lambda}_G$, or each one is rewarded by its productivity, λ_i. There is no way to commit a priori to any of these two principles. A majority in group G will favor meritocracy if the agent who is ranked median in the order of productivities is more productive than the average of the group. Otherwise, the majority will be for egalitarianism. Ties are broken in the following way: if there are more than one median agent, ties are broken in favor of the agent with the highest productivity. If the productivity of the median agent is equal to the mean productivity, we consider that the group is meritocratic.

Since agents know the rules and also the productivities of all others, they can anticipate what rewards they will get from joining any given group. They will thus play a hedonic game (Drèze and Greenberg (1980)), where outcomes are partitions of agents into groups. A natural prediction is that stable partitions will arise from playing these games. The following definitions formalize the stability concept that we use in this paper.

Definition 1. *Given a society (N, λ, v), an organizational structure is a partition of N denoted by π. Two organizational structures, π and π', are equivalent if for all $G \in \pi$ there is $G' \in \pi'$ such that $\lambda_G = \lambda_{G'}$ and viceversa. A group G is segregated if given i and j in G with $\lambda_i < \lambda_j$, and $k \in N$ such that $\lambda_i \leq \lambda_k \leq \lambda_j$, $k \in G$. An organizational structure is segregated if all the groups in the partition are segregated.*

Definition 2. *An organizational structure is blocked by a group B if all members in B are strictly*

[1] We adopt this lexicographic specification of preferences as the simplest way to represent the fact that, in addition to the material reward, individuals may also value other dimensions of the participation in a group, like prestige. Other specifications that reflect richer trade-offs between individual rewards and the "quality" are possible, but make the model less tractable.
Most of our resutls apply to setups with preferences without the lexicographic component (see our concluding remarks).

better off in B than in the group they are assigned in the organizational structure. An organizational structure is core stable *if there are no groups that block it.*

Simple sufficient conditions assuring the existence of core stable organizational structures are easy to describe. For any distribution of productivities guaranteeing that segregated groups are meritocratic, any organization of society into segregated groups of minimal size is core stable. This is the case for example, under a uniform or concave distribution[2], that is, for any three consecutive agents i, j, k with $\lambda_i \leq \lambda_j \leq \lambda_k$, $\lambda_k - \lambda_j \leq \lambda_j - \lambda_i$.

In general, core stability may not always be possible to get, as shown by the following example.

Example 1. *A society with no core stable organizational structures.*
Let $N = \{1, 2, 3, 4, 5\}$, $v = 3$, *and* $\lambda = (100, 84, 84, 84, 60)$. *Let us see first that in any core stable organizational structure the high productivity agent cannot belong to a productive meritocratic group. This is because in this example the grand coalition is the highest mean meritocratic group containing the high productivity agent. But the organizational structure composed by the grand coalition is not core stable because the medium type agents on their own can form a meritocratic group with a higher average productivity. Now, let us see that in any core stable organizational structure the high productivity agent cannot belong to a productive egalitarian group. Like before, we only need to check that the organizational structure where the productive group is the egalitarian group with the highest mean productivity is blocked. This productive group is the one formed by the high type agent with two of the medium type agents. This organizational structure is blocked by the meritocratic group formed by the high type agent, the third medium type agent and the low type agent. Finally, note that the high type agent cannot belong to an unproductive group either, because the productive one formed by the high type and two medium types blocks any organizational structure where the productive group does not contain the high type. Thus, there is no core stable organizational structure.*

Our next example shows two important and independent points. The first one is that in a core stable organizational structure different reward systems may coexist. The second one is that a core stable organizational structure may be non-segregated.

Example 2. *A society with stable non-segregated organizations where different reward systems coexist.*

[2] In the dual case, where all segregated groups are egalitarian, the organization of society into segregated groups of minimal size does no longer assure core stability.

Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $v = 5$, *and* $\lambda = (100, 100, 75, 75, 75, 75, 75, 75, 75, 45)$. *Let* $G_1 = \{1, 3, 4, 5, 10\}$ *and* $G_2 = \{2, 6, 7, 8, 9\}$. *Note that* G_1 *is meritocratic and* G_2 *is egalitarian. Let us see that the organizational structure* $\pi = (G_1, G_2)$ *is core stable. Note that the medium type agents in* G_2 *can only improve if a high type is added to the group or if a medium type is substituted by a high type. But since the other high type not in* G_2 *is already in a meritocratic group, he does not have incentives to form the potential blocking group. The two high types cannot be together in a meritocratic group, and any other agent needs high types to improve. That implies that* π *is a core stable organizational structure. Note that high and medium productivity agents are split between the two groups. Any other core stable organizational structure is equivalent to this one.*

At this point, it is worth comparing the consequences of our model with those that one would obtain under alternative criteria to choose the distributional rules. For ease of comparison, we assume that $n = kv$, for any integer k. If agents could freely bargain about the payoffs they will get when joining a group, the core stable organizational structure would be the segregated one where the v most productive agents get together, the next v most productive agents form a second group, and so on, with all groups adopting the meritocratic distribution. If, on the other extreme, agents were forced to adopt a fixed distributional rule (either meritocracy or egalitarianism), a core stable organizational structure would also always exist and coincide with the segregated partition into groups of size v.

Hence, the ability of societies to choose by vote between our two distributional criteria is what gives rise to the differential traits of our results. In our societies stability becomes an issue, that we must carefully discuss. It is also the case that rich, non-segregated organizational structures can arise. And this, in turn, allows for a richer analysis of societies where voting has distributional consequences.

Our paper now takes two complementary routes. Sections 3, 4, and 5 analyze in detail the structure of core stable organizations under different assumptions, and provide characterization results. Section 6 adds new economic implications to those that we have introduced in the previous sections, provides comparative static results, and may be read separately. Section 7 conludes with comments on our assumptions and suggestions for further work.

3. Core stable organizational structures and the weak top group property

The analysis of hedonic games is never a trivial task. In particular, finding out when a hedonic game may give rise to stable coalitional structures requires a careful understanding of the interactions between the interests of agents and their ability to sustain the groups they belongs to, or to challenge those they wish to join. Our model restricts the class of games that we need to consider, thus allowing for more specific results than those available in the literature on general hedonic games. In this and the next two sections we provide characterizations of societies for which core stable organizational structures exist. We begin here by considering a condition that is sufficient (but typically not necessary) for the existence of core stable organizational structures in general hedonic games: *the weak top group property* (Banerjee et al, 2001). We prove (Proposition 1) that identifying weak top groups in our model, when they exist, is an easy task. In addition to its intrinsic interest, this result is used in subsequent sections, when searching for potential candidates to form core stable organizational structures. Then we turn to a particular case of our model, one simple enough to generate interesting intuitions: societies where only one productive group can be formed. For these simple societies, the weak top group property has additional bite. It allows us to provide a full characterization of those "small societies" for which there are core stable organizational structures (Proposition 2), and to identify those cases where existence is guaranteed (Proposition 3).

Definition 3. *A group $W \subseteq G \subseteq N, W \neq \emptyset$, is a weak top group of G if it has an ordered partition $(S_1, ..., S_l)$ such that (i) any agent in S_1 weakly prefers W to any subset of G, and (ii) for any $k > 1$, any agent in S_k needs cooperation of at least one agent in $\cup_{m<k} S_m$ in order to form a strictly better group than W. A game satisfies the* weak top group property *if for any group $G \subseteq N, G \neq \emptyset$, there exists a weak top group W of G.*

If the weak top group property is satisfied, a core stable organizational structure, $(G_1, ..., G_m)$ always exists and can be constructed by sequentially selecting weak top groups from the population: G_1 is the weak top group of N, G_2 the weak top group of $N \backslash G_1$, and so on. [3]

We can now show that in our model, weak top groups, if they exist, must have a very specific and simple form. This fact will greatly simplify our discussion of stability, and is therefore an

[3] Stronger conditions can be found in the literature that guarantee core stable organizational structures. For example, the Top Group Property (TGP), requires that any group G of agents contains a subgroup that is the best subset of G for all of its members (Banerjee et al, 2001). The TGP is a relaxation of the common ranking property introduced by Farrell and Scotchmer (1988). Under those conditions the core is nonempty and it has a unique element.

important step to be repeatedly used in our proofs.

Before discussing the form of weak top groups, let's introduce the notion of a congruent group (Le Breton et al (2008)).

Definition 4. *A group $C \subseteq G \subseteq N, C \neq \emptyset$, is a congruent group of G if for all $i \in C$, and for all $S \subset G$ such S is a strictly better group than C for i, there is an agent $j \in S \cap C$ such that C is a strictly better group than S for this agent j.*

Note that any weak top group of G is a congruent group of G.

We can now state our characterization result for weak top groups of G.

Proposition 1. *Let $M_+(G)$ be the set of meritocratic groups of G with the greatest mean, and let $E_+(G)$ be the set of egalitarian groups of G with the greatest mean. A group W is weak top group of G if and only if it is a congruent group of G, and either belongs to $M_+(G)$ or to $E_+(G)$.*

Proof. The proof consists of two parts.

Part 1: Weak top groups of G are congruent groups of G and must belong to either $M_+(G)$ or to $E_+(G)$.

If W is a weak top group of G then it is a congruent group of G.

Next we show that if G has a weak top group, W, then $\bar{\lambda}_W \geq \bar{\lambda}_S$ for all $S \subseteq G \backslash W$. Suppose on the contrary that there is a group $S \subseteq G \backslash W$ such that $\bar{\lambda}_W < \bar{\lambda}_S$. Suppose first that there is an agent $i \in W$ such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$. Let $S' = S \cup \{i\}$. Since $\lambda_i < \bar{\lambda}_S$, the mean productivity of group S' will be bigger than the productivity of i, $\lambda_i < \bar{\lambda}_{S'}$. Thus, agent i, independently of the regime will be better off in S' than in W, in contradiction with W being a weak top group. If there is no agent $i \in W$ such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, we distinguish two cases: in the first one we suppose that W is egalitarian and in the second we suppose that W is meritocratic.

If W is egalitarian, since no agent $i \in W$ exists such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, then an agent $i \in W$ exists such that $\bar{\lambda}_S \leq \lambda_i$. Let $S' = S \cup \{i\}$, note first that $\bar{\lambda}_W < \bar{\lambda}_{S \cup \{i\}} \leq \lambda_i$. So, independently of the regime of S', agent i will be better off in S' than in W, in contradiction with W being a weak top group.

If W is meritocratic, since no agent $i \in W$ exists such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, the median productivity of W is above $\bar{\lambda}_S$. Let $\lambda_{med}(W)$ be this median productivity. Let $i \in W$ such that $\lambda_i < \bar{\lambda}_W$. Suppose first that there is an agent $j \in S$ such that $\lambda_i < \lambda_j \leq \lambda_{med}(W)$. Let $W' = (W \backslash \{i\}) \cup \{j\}$. Note that since the productivities of agents i and j are both below the median productivity of W, replacing in W agent i by agent j does not change the median but increases the average. Thus, all agents in $W' \cap W$ are better off in W' than in W, in contradiction with W being a weak top group. Finally, if there is no an agent $j \in S$ such that $\lambda_i < \lambda_j \leq \lambda_{med}(W)$, then there is an agent $j \in S$ such that $\lambda_j < \lambda_i < \bar{\lambda}_W$. Let $S' = (S \backslash \{j\}) \cup \{i\}$, $\bar{\lambda}_{S'} > \bar{\lambda}_S > \bar{\lambda}_W > \lambda_i$. Thus, independently of the regime of S', agent i will be better off in S' than in W, in contradiction with W being a weak top group.

Suppose now that the weak top group is meritocratic but does not belong to $M_+(G)$. Note first that $W \cap M_+ = \emptyset$ for all $M_+ \in M_+(G)$, because otherwise, all agents in $W \cap M_+$ would strictly prefer M_+ to W contradicting that W is a weak top group. Since $W \cap M_+ = \emptyset$, our previous reasoning applies, and therefore $\bar{\lambda}_W \geq \bar{\lambda}_M$. But then $W \in M_+(G)$, a contradiction. The same argument applies if W is an egalitarian group.

Part 2: If a set in $M_+(G)$ or in $E_+(G)$ is a congruent group of G then it is a weak top group of G.

Suppose $M_+ \in M_+(G)$ is a congruent group of G. If M_+ is a segregated group with the best productivity agents in G, it is clearly a weak top group of G. If it is not of the preceding form, suppose that M_+ is not a weak top group of G. Since it is congruent but not weak top, there is no subgroup of agents in M_+ for which M_+ is the best group. This implies that the most productive agent in G is not in M_+, and for the most productive agent in M_+ there is an egalitarian group E which is preferred to M_+. But then all agents in $E \cap M_+$ would be better off in E, in contradiction with M_+ being congruent.

Suppose finally that $E_+ \in E_+(G)$ is a congruent group of G. If E_+ is a segregated group with the best productivity agents in G, it is clearly a weak top group of G. If it is not of the preceding form, suppose that E_+ is not a weak top group of G. Since it is congruent but not weak top, there is no a subgroup of agents in E_+ for which E_+ is the best group. But note that for the less productive agent in this group E_+ is always its best set, a contradiction. ■

We are now ready to establish two simple results regarding "small" societies. One is a characterization result: when v is such that the population admits only one productive group, the weak top group property is necessary and sufficient for the existence of core stable organizational structures. The other shows existence of core stable organizational structures for a particular case.

Proposition 2. *A society composed of $n < 2v$ individuals has core stable organizational structures if and only if N has a weak top group.*

Proof. Sufficiency is clear: just partition the society into the weak top group of N and leave the other agents together in an unproductive group.

Necessity follows from the fact that if a partition $\pi = (P, N \backslash P)$ is in the core, $P \in M_+(N)$ or $P \in E_+(N)$. Since π cannot be blocked, there is no group $S \subseteq N$ such that all $i \in S \cap P$ are better off in S than in P. Thus, P is congruent and by Proposition 1 it is a weak top group of N. ■

A direct application of Proposition 2 is the following.

Proposition 3. *In societies where $n < 3v/2$, a weak top group of N always exists. Therefore, there are always core stable organizational structures.*

Proof. Let $T = \{1, ..., v\}$. If T is meritocratic, it is trivially a weak top group of N and thus the core is not empty. Let us see that if T is egalitarian it is also a weak top group of N. Note first that all agents with productivity below the mean are in their best group. Only agents above the mean could improve. But, since the group is egalitarian, the mean is above the median and thus the group that can improve has a cardinality smaller than $v/2$. But the unproductive group $I = \{v + 1, .., n\}$ also has a cardinality smaller that $v/2$. Thus, there is no way of forming a group that can improve upon T. ■

Note, however, that existence of core stable organizational structures is not guaranteed when $3v/2 < n < 2v$, as we have shown in Example 1, where neither the meritocratic group with the greatest mean (the group of the medium productivity agents), nor any of the egalitarian groups with the greatest mean (the high productivity plus two medium productivity agents) are weak top groups of N.

Finally, let us make clear that when $n \geq 2v$, the weak top property is not a necessary condition for the existence of core stable organizational structures as the following example shows. This remark leads us to analyze, in the next section, societies where $n \geq 2v$.

Example 3. $N = 6$, $\lambda = (100, 84, 84, 84, 84, 60)$, $v = 3$. *The meritocratic group with the greatest mean is the group formed by the medium productivity agents. This group is not a weak top group because its members would be better off in the egalitarian group with a high productivity agent. The egalitarian group with the greatest mean is the one composed by a high productivity agent and two medium productivity agents. This group is not weak top either because this high productivity agent will be better off in the meritocratic group formed by the remaining medium and low productivity agents. However, $\pi = (G_1, G_2)$, where G_1 is a meritocratic group with three* *medium type agents and G_2 is a meritocratic group with one high one medium and one low type is a core stable organizational structure.*

We now turn to examine populations where productivities are limited to no more that three types.

4. Societies with three types of agents

We have seen in the preceding section that the weak top property has specific implications for our model, but is not strong enough to provide a full characterization of core stability, except for small societies where only one productive group can be formed. In this section and the next we provide full characterizations for societies of any size. These results are obtained at the prize of restricting the domain of societies under consideration. In the present section, we study societies where N is composed of three types of agents, who differ in productivities: the high, the medium and the low. This three-way partition of society is certainly limitative, but also a reference case. It is not only here, where types refer to productivities, but in many other contexts. People are classified by social status into the elite, the middle and the lower class; countries are classified into developed, developing and less developed, etc... In fact, all the basic intuitions that one gets from the model can be grasped through the analysis of this three-type case. Better than that, our analysis in the present Section can be extended in the following Section 5 to the much wider family of societies where non-identical agents can be divided into three clusters, provided that members in the same cluster satisfy a number of defining characteristics. Hence, the present section is presented in a double spirit, both as the study of an important class of benchmark societies, and as the building block for a considerable extension to three-way clustered societies.

In a three type society (N, λ, v) a generic type is denoted by j, $j \in \{h, m, l\}$, and productivities are $\lambda_h > \lambda_m > \lambda_l$. Let H, M, and L be the sets of all high, medium and low type agents respectively, and let n_H, n_M and n_L be the cardinality of these sets. Without loss of generality we assume agents to be ordered from 1 to n. In that order high type agents come first, then medium ones, and finally low ones. Clearly the order of individuals of the same type is arbitrary and will have no effect on our results. Under this convention, the set $T = \{1, ..., v\}$ will be the one containing the first v agents in terms of productivity. Note that because of this arbitrariness, any two organizational structures which only differ in the numbering of individuals of the same type are structurally equivalent in the sense that if one is core stable the other will also be. In what follows when we refer to uniqueness of core stable organizational structures, we mean that they are all structurally equivalent. Finally, for any $G \subset N$, $T(G)$ denotes the first v agents in G, and $H(G)$, $M(G)$, and $L(G)$ denote respectively the high, medium and low type agents in G.

From now on, we discuss the characteristics of societies where core stable organizational structures exist and also the form that these structures take under different conditions. We start by a remark regarding one or two types societies.

Remark 1. *In degenerate cases where there are only individuals of one or two types, the existence of core stable organizational structures is guaranteed. The case with only one type is trivial. In societies composed by two types of agents, say h and l, if $n_H \geq v$, the organizational structure (H, L) is trivially core stable. If $n_H < v$, the reader may check that the organizational structure $(T, N \backslash T)$ is core stable.*

We now turn attention to societies with three distinct types of agents. We shall distinguish between two types of societies, that we call structured and unstructured, and prove that the limits between the two indeed determine whether or not core stability can be attained. We can prove that core stable organizational structures will exist in a society if and only if it is structured.

Since the definition of a structured society is complex, we start by describing its characteristics from two different perspectives.

First, regarding the type of groups that may be part of core stable structures. We'll prove that such structures must either contain T or some meritocratic group G with high type agents. Although this does not provide a full description of the whole structure, it points at a salient group in it. We'll say that core stable partitions must be structured around T or around some meritocratic G, meaning that one of these sets has to be part of the partition and that the rest of society must be able to accommodate the further requirements imposed by overall stability. As a result, stability requires in all cases that some of the high type agents are part of a group where they get their best possible treatment. They will either be all part of the best egalitarian group, when no stable partition can be structured around any meritocratic group containing high types, or else some of them will manage to structure a stable organization around a meritocratic group, where they get paid their full productivity, even if sometimes at the expense of other high type agents.

Second, we can look at the requirements that separate these two types of societies. In order to be unstructured, a society must have a rather special distribution of types. In particular, it must satisfy at least the following requirements: (i) It must be that the number of high type agents is less than $v/2$. Otherwise, they could form a meritocratic group including all of them, and let the remaining members of society, which will now be of at most two types, to organize in a stable manner. (ii) In addition, unstructured societies must contain a number of middle types that is bounded above and below, so that $v \leq n_H + n_M < 2v$. This is because a very small middle class, when coupled with a small high class, cannot de-stabilize a partition structured around T, while a large enough middle class will leave room for T to structure a stable partition again, this time thanks to the fact that the remaining middle type agents not in T will be able to achieve the highest mean meritocratic group, the one formed by medium type agents alone. In the case $n_H + n_M < 3v/2$, unstructured societies must contain a "sufficient" number of low types to allow high type agents to challenge a partition structured around T with a meritocratic group. Finally, (iii) unstructured societies are not able to satisfy medium type agents. Any partition structured around a meritocratic group G with high type agents can always be challenged by some of the medium type agents.

The formal definition of a structured society is as follows.

Definition 5. *A three type society is structured if at least one of the following three conditions holds:*

1. N has a weak top group.

2. Either $n - n_L \geq 2v$, or $n - n_L < 3v/2$ and for all G meritocratic such that all $i \in G \cap T$ are better off in G than in T, $0 \leq n - 2v < \#L(G)$.

3. There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ such that:

(a) $\bar{\lambda}_{G_1} \geq \bar{\lambda}_G$ for all meritocratic group $G \subset (G_1 \cup H(G_2) \cup G_3)$ where $G_2 = T(N \backslash G_1)$ and $G_3 = N \backslash (G_1 \cup G_2)$, and

(b) Either $\#(H \cup M) \backslash G_1 = v$ or $\#(M \cup H(G_2)) < v$, $M \subset G_1$ and $\bar{\lambda}_{T(M \cup (N \backslash G_1))} < \lambda_m$.

A three type society is unstructured if it is not structured, that is, if none of the above conditions holds.

Note that condition 1 is a limited version of the weak top group condition. Recall that the latter is a sufficient condition for the existence of core stable structures in general hedonic games. Here we only need to require the existence of a weak top group of N, the set of all agents. Also remark that, in view of Proposition 1, this condition is an easy one to check. Given its transparency, we do not elaborate any further regarding it. Condition 2, then, specifies that a society may still be structured, this time around T, in the absence of a weak top group for N, provided the set of middle productivity agents is "small enough" or "large enough", in the sense of point (ii) in our preceding discussion. Finally, condition 3 provides a specific set of restrictions that give meaning to our remark (iii) above: even when conditions 1 and 2 do not hold, it may still be that society can be structured in a stable way around a meritocratic G, provided the medium type agents cannot do better.

Thus, the definition of a structured society is "nested" in the following sense: Condition 1 is a sufficient condition for existence of stable organizational structures. If condition 1 does not hold, condition 2 is sufficient for the existence of stable organizational structures, and finally, if neither condition 1 nor condition 2 hold, condition 3 is sufficient for the existence of stable organizational structures. Furthermore, if none of the conditions hold the core is empty. The following proposition formally states these results.

Proposition 4. *There exist core stable organizational structures for a three type society if and only if the society is structured.*

Before giving the full detailed proof, we provide an overview of its main elements that we hope is helpful for the reader.

We shall first show that if there is a weak top group for N, existence is guaranteed. Then we'll turn to consider societies that have no weak top group, but that can be structured around T. First of all note that the non existence of weak top groups implies that: T is egalitarian ($n_H < v/2$); there exist meritocratic groups that contain high type agents; and $n - n_L > v$. Otherwise, T would be a weak top group. Thus, $N \backslash T$ contains at most two types of agents, medium and low, which π must organize in an stable way. A group potentially blocking π must be a meritocratic group. Since there are no weak top groups, a meritocratic group G exists such that all $i \in G \cap T$ are better off in G than in T. If G does not block π, it must be because some of the agents in $(N \backslash T) \cap G$ are placed in π in a group they prefer to G. This is assured if the medium type agents in $N \backslash T$ are sufficiently many to form a productive homogeneous group and that is their organization in π. If medium type agents in $N \backslash T$ are too few to create a meritocratic group, then π must organize them in an egalitarian group with low type agents, and some of the later are necessary to form G.

Next, consider a society that has no weak top group and is not structured around T, but it is structured around a meritocratic group G containing high type agents. If G excludes some of the medium type agents, then π must place the medium types in $N \backslash G$ in a group they prefer over the group than can be created by they joining G. Hence, if G contains all the high types, the medium types in $N\backslash G$ must be in a homogeneous productive group; and if G excludes some high types, π must organize the medium and high types in $N\backslash G$ in an egalitarian productive group, that excludes all low types in $N\backslash G_1$. These high type agents in the egalitarian group, by joining G, cannot generate a meritocratic group. Furthermore, the group created by adding high and low types to G, if meritocratic, must have lower average productivity than $\bar{\lambda}_G$. On the other hand if G contains all the medium type agents, then it must be that (i) $n_M < v$ because otherwise M blocks π, (ii) G cannot contain all the high types, because there is no weak top groups; and (iii) π must place high type agents in $N\backslash G$ in a productive group. Hence, π organizes the high types in $N\backslash G$ in an egalitarian group of size v with low types, and the remaining low types are either unproductive or in an homogeneous low type agents group. And finally, conditions must hold to assure that π cannot be blocked: i) if a meritocratic group $\tilde{G}$ with a greater average that G exists, it must be that π places some of the agents in $(N\backslash G) \cap \tilde{G}$ in a group they like better than $\tilde{G}$, and ii) any egalitarian group of medium type agents in G and high type agents in $N\backslash G$ must be unproductive.

To check the necessary condition we show that when neither condition 1 nor 2 nor 3 hold, in a core stable organization structure the high type agents cannot be in a meritocratic group, neither in a egalitarian group, neither in an umproductive group, which is a contradiction.

Proof of Proposition 4.

Part 1: Structured societies have core stable organizational structures.

For each condition assuring a structured society we describe how to construct a core stable organizational structure.

(i) Suppose condition 1 holds, i.e. there exist weak top groups in N. We first argue that there will always be one weak top group W such that $N\backslash W$ contains only two types. This is because

- if $n_H \geq v$, then H is weak top (in fact top), and therefore $N\backslash H$ contains two types of agents, medium and low.

- if $n_H < v$, and T is meritocratic, T is weak top and $N\backslash T$ contains at most two types of agents, medium and low.

- if $n_H < v$, T is egalitarian and weak top, then $N\backslash T$ contains at most two types of agents, medium and low.

- if $n_H < v$, T is egalitarian but not weak top, then any weak top group W must be meritocratic with highest mean. W must contain some high type agents, because all agents in a meritocratic group without high type agents will gain from adding one high type, whether this enlarged set is egalitarian or meritocratic. In addition, W must contain all medium type agents, because if one of them was left out, adding that agent would increase the group mean while keeping meritocracy. Then $N\backslash W$ contains at most two types of agents, high and low.

Let us now construct a core stable structure. Take a weak top group W such that $N\backslash W$ contains only two types. We have just shown that this is always possible. Let W be one of the groups in the organizational structure. By Remark 1 we know that the two-type society $N\backslash W$ has a core stable organizational structure. The groups in that structure plus W are core stable in our initial society.

(ii) Suppose that condition 1 does not hold but condition 2 holds. Since condition 1 does not hold, $T = \{1, .., v\}$ is egalitarian, thus $n_H < v/2$.

If $n - n_L \geq 2v$, high and medium types alone can form two productive groups. Let $G_1 = T$, $G_2 = M\backslash T$, and $G_3 = L$. Clearly (G_1, G_2, G_3) is a core stable organizational structure.

If $n - n_L < 2v$, then $n - n_L < 3v/2$ and $0 \leq n - 2v < \#L(G)$ for every meritocratic group G such that all $i \in G \cap T$ are better off in G than in T. Let $G_1 = T$, $G_2 = \{v+1, .., 2v\}$ (G_2 is an egalitarian group given that $n - n_L < 3v/2$), and $G_3 = N\backslash (G_1 \cup G_2)$ a group of low types. Again (G_1, G_2, G_3) is a core stable organizational structure. This is because the potential blocking group of this structure is a meritocratic group G that contains low type agents. But since low type agents in G_2 are in an egalitarian group, they cannot be part of the blocking, and since $n - 2v < \#L(G)$, for any of those potential meritocratic groups blocking π, low type agents in G_3 are not enough to form the potential blocking group G.

(iii) Last, suppose that condition 1 and 2 fail but condition 3 holds.

First of all note that, because of the failure of 1 and 2, $n_H < v/2$ and $n_H + n_M < 2v$.

Second, because 3 holds, there exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N\backslash G_1) \geq v$ satisfying (a) and (b). Let $\pi = (G_1, G_2, G_3)$ where $G_2 = T(N\backslash G_1)$ and $G_3 = N\backslash(G_1 \cup G_2)$.

If $\#(H \cup M)\backslash G_1 = v$, G_2 is either an egalitarian group with high and medium types or just a meritocratic group with medium type agents if all high type agents are in G_1, and G_3 is a group of low types. If $\#(H \cup M)\backslash G_1 \neq v$, all the medium type agents are in G_1, G_2 is an egalitarian group with high and low types and G_3 is a group of low type agents if any. In both cases, conditions a and b guarantee that π cannot be blocked.

Part 2: Unstructured societies have no core stable organizational structures.

Assume that neither 1 nor 2 nor 3 hold and that a core stable organization structure π exists. Let $G \in \pi$ such that $G \cap H \neq \emptyset$. We show that G cannot be meritocratic, nor egalitarian, nor unproductive, which is a contradiction.

Assume G is meritocratic.

Since condition 1 does not hold, there are no weak top groups in N. Then $n_H < v/2$, because otherwise T would be a meritocratic group and it would be a weak top group of N. Thus, if G is a meritocratic group it must include three types of agents. Since there is no weak top group, $\#N\backslash G \geq v$, because otherwise, if the remaining agents are in an unproductive group, π can be blocked. Apart from G, no other productive group $G' \in \pi$ with three types can be meritocratic. Otherwise the medium type agents in the group with lower average productivity can switch to that other group. This generates a meritocratic new group with a greater average productivity that blocks π. So, if π contains another productive group G' with three types, that G' must be egalitarian and it must contain all the high type agents in $(H \cup M)\backslash G$. If $\overline{\lambda}_{G'} > \lambda_m$, replacing a low type in G' by one of the medium types in G increases the average and keeps egalitarianism, and this later group blocks π. But if $\bar{\lambda}_{G'} \leq \lambda_m$ we contradict that π is core stable as well - since switching one of the medium types from G' to G increases the average in G and keeps meritocracy. Thus, agents in $N\backslash G$ can only be organized in two-types groups, and the high types in $N\backslash G$ are in an egalitarian group. Note also that medium type agents cannot be in a group with just low type agents, because by joining G they increase the mean while keeping meritocracy, and this new group will block π. Thus, π contains $G_2 = T(N\backslash G)$, which is either egalitarian with high and medium types, or meritocratic with just medium type agents (if all high type agents are in G), or egalitarian with high and low types if G contains all the medium agents. In any case, the remaining agents, $N\backslash(G \cup G_2)$ are low type agents.

Since condition 3 does not hold, either (a) or (b) fails.

If (a) fails, a meritocratic group $G' \subset (G \cup H(G_2) \cup G_3)$ where $G_2 = T(N\backslash G)$ and $G_3 = N\backslash(G \cup G_2)$ exists with $\bar{\lambda}_{G'} > \bar{\lambda}_G$. Since only high type agents in G_2 are potentially part of this meritocratic group, G' blocks π.

If (b) fails, then $\#(H \cup M)\backslash G \neq v$. Since, as we argue above, π cannot place medium type agents in a group with just low type agents, then $\#(H \cup M)\backslash G < v$. Thus, all medium type agents are in G, and π organizes $N\backslash G$ with an egalitarian group with high and low types and a group of low type agents alone. If $\#(M \cup H(G_2)) \geq v$, then the group of cardinality v with high types not in G and medium type agents is egalitarian (or meritocratic if only contains medium type agents) and blocks π. If $\#(M \cup H(G_2)) < v$, the average productivity of $T(M \cup (N\backslash G))$ is greater than λ_m, which implies that $T(M \cup (N \backslash G))$ is an egalitarian group which blocks π.

Because of all the above points, high type agents cannot be in a meritocratic group.

Assume next that G is egalitarian. Then, since there are no weak top groups and high type agents cannot be in a meritocratic group, it must be that $G = T$. Since condition 2 does not hold, $n - n_L < 2v$ and either $n - n_L \geq 3v/2$ or there exist a meritocratic group G^* such that all $i \in G^* \cap T$ are better off in G^* than in T and $n - 2v \geq \#L(G^*)$.

In the first case, any organizational structure containing T, where agents in $N \backslash T$ are organized in a stable way, is such that $T(N \backslash T)$ is a meritocratic group with medium and low types, and the remaining agents are just low type agents. Since T is not weak top, a meritocratic group G' exist such that all $i \in G' \cap T$ are better off in G' than in T. This meritocratic group contains high type agents in T and medium and low types in $N \backslash T$. Medium type agents in $N \backslash T$ are in a meritocratic group and low type agents are also in meritocratic groups or alone. Then G' blocks π because (1) high type agents in $G' \cap T$ are better off in G' than in T, and (2) medium and low types in G' are better off than in their respective groups because G' has a greater mean.

In the second case, $T(N \backslash T)$ is egalitarian, and the low agents in $T(N \backslash T)$ cannot be used to block π with a meritocratic group. But, since condition 2 fails, then a meritocratic group can be constructed that blocks π. This is because the remaining low types not in T neither in $T(N \backslash T)$ are enough to construct G^*.

To conclude, assume G is unproductive. But $h \in G$ is very welcome in any meritocratic group (even if that changes the regime), and if there are no meritocratic groups, T blocks π.

Hence, there are no core stable organizational structures. ■

Remark 2. *(a) Note that when $n < 2v$, conditions 2 and 3 in the definition of a structured society never hold because they involve restrictions that only apply when more than one group can form. Hence, if $n < 2v$ a society is structured if and only if N has weak top groups. This remark leads us directly to the necessary and sufficient condition for the existence of core stable organizational structures that we already discussed in Proposition 2.*
(b) Also note that, in a structured society that fails to satisfy conditions 1 and 2, the unique stable organization structures are non-segregated. They are structured around a meritocratic group G that may or may not contain all high type agents. If G leaves some high type agents out, these must be organized in an egalitarian group. If G contains all the high type agents, there must be enough medium type agents out of G to form a productive group by themselves.

Proposition 4 not only provides an existence result for general three type societies. It also opens the door to study a particular case about which we know a lot, and that provides us with an important tool to generate significant examples in our next section. This is when societies contain $n = 2v$ agents, that is, just enough individuals to form two productive groups.

An important special case: $n = 2v$.

The first important fact about these societies is that they always have some core stable organizational structure. In fact, going further in our analysis, we can prove that the set of such stable structures is always small, containing only one or at most two of them. We can be even more precise, and establish the characteristics of the groups that can achieve stability, depending on the distribution of productivities.

In what follows we present these results in detail. But first we must introduce a property that societies may or may not satisfy, and that turns out to be determinant of the kind of stable groups that can form.

Definition 6. *A society is maximally mixed meritocratic if* $n_H < v/2$, $n_L \leq v/2$, *and* $(\lambda_h + \lambda_m + n_L\lambda_l)/(n_L + 2) \leq \lambda_m$.

In maximally mixed meritocratic societies we can always construct a meritocratic group of cardinality v that contains agents of all three types, all agents of the low type and the highest number of high types allowing for all the preceding characteristics to hold. We call this a *maximally mixed meritocratic group*, and denote it by $M3$. This group can be constructed as follows. Start with all n_L low types, one medium type and one high type. This starting group may not be productive, but the mean of its $\lambda's$ is below λ_m. Next add as many high types as possible while keeping the mean of the $\lambda's$ below λ_m. And finally, if the group is not yet productive, fill the set with medium types until reaching size v.

Remark that an organizational structure that contains a group with the characteristics of $M3$ is non-segregated. Also note that in societies that are not maximally mixed, some group may still have the structure required in the definition of $M3$. What is characteristic of maximally mixed meritocratic societies is that they guarantee the existence of $M3$ sets which, in addition, do belong to core stable structures. We shall see this in Proposition 6. We first establish the existence of core stable organization structures in Proposition 5.

Proposition 5. *Societies where* $2v = n$ *are always structured. Hence, they always have core stable organizational structures.*

Proof. If the society has a weak top group in N, then clearly society is structured. Suppose that a society where $n = 2v$ does not have any weak top group. Then the following conditions must hold:

(i)The set of high type agents are not a majority in any group of cardinality v,that is, $n_H < v/2$. Because otherwise T would be meritocratic and consequently weak top.

(ii) T has to be egalitarian. Because if T is meritocratic, it is weak top.

(iii) There are meritocratic groups with high types. Because if not, T would be weak top.

Furthermore, since $n = 2v$, $n - n_L < 2v$. Then either $n - n_L < 3v/2$, or $n - n_L \geq 3v/2$. In the first case, condition 2 will hold since $n - 2v < \#L(G)$ for all G meritocratic such that all $i \in G \cap T$ are better off in G than in T. If $n - n_L \geq 3v/2$, then condition 2 does not hold and we have to see that condition 3 in this case always holds. First of all, recall that since conditions 1 and 2 fail, and $n = 2v$, then $n_H < v/2$, $n_L \leq v/2$ (because $n - n_L \geq 3v/2$), and $n_M > v$. Since there are meritocratic groups with high types and $n_H < v/2$, those meritocratic groups must contain three types of agents. Thus, $(\lambda_h + \lambda_m + n_L\lambda_l)/(n_L + 2) \leq \lambda_m$, because otherwise it is impossible to construct a meritocratic group with the three types. That is, we must be in a maximally mixed meritocratic society where $M3$ can be constructed. Part (a) of condition 3 does not apply because it involves cases where $n > 2v$. Part (b) of condition 3 holds because $M3$ has cardinality v and contains all the low type agents, thus $\#(H \cup M)\backslash M3 = v$. Thus, if condition 1 and 2 do not hold, condition 3 always does and $(M3, N\backslash M3)$ is a core stable organizational structure. ■

Our next proposition establishes that maximally mixed meritocratic societies have a unique and non-segregated core stable organizational structure.

Proposition 6. *In societies with $n = 2v$ that are maximally mixed meritocratic there is a unique core stable organizational structure,* $(M3, N\backslash M3)$.

Proof. The constructive argument in the proof of Proposition 4 shows that $(M3, N\backslash M3)$ is a core stable organizational structure. $M3$ contains three types of agents and is meritocratic. $N\backslash M3$ contains only medium type agents or a combination of medium and high types and is egalitarian. Let us see that $(M3, N\backslash M3)$ is the unique core stable organizational structure.

We first show that no structure with only one productive group can be core stable. For this to happen, the productive group would have to be weak top. Candidates to be weak top groups are $G \in E_+(N)$, or $G \in M_+(N)$.

If $G \in E_+(N)$, G has size v and cannot be part of an organizational structure with only one productive group. This is because in a maximally mixed meritocratic society, $n_H < v/2$, $n_L \leq v/2$, and consequently $n_M > v$. These conditions imply that T is egalitarian, $T \in E_+(N)$, and any other $G \in E_+(N)$ has the same structure as T.

If $G \in M_+(N)$, G only contains medium type agents. This is because any meritocratic group with high type agents has the mean below the productivity of the medium type agents, and $n_M > v$. But groups containing only medium type agents are never weak top, because its members always prefer to add high types to their group.

Let us now prove that for all other organizational structures with two productive groups $(G_1, G_2) \neq (M3, N \backslash M3)$ there is always a group that blocks (G_1, G_2).

(i) If G_1 and G_2 are both meritocratic, both groups have three types of agents or one of them three types and the other two types, medium and low. In any case, adding the medium type agents to the group with greater mean forms a meritocratic group with increased mean that blocks (G_1, G_2).

(ii) If G_1 and G_2 are both egalitarian then none of them is T, because $N \backslash T = B$ is meritocratic. Thus, T blocks (G_1, G_2).

(iii) If G_1 is meritocratic and G_2 is egalitarian, $G_2 \neq T$. Because otherwise, $G_1 = B$ and then $M3$ blocks (T, B). G_2 cannot have three types of agents, because by replacing low types in G_2 by medium types, the mean increases while keeping egalitarianism. This new group will block (G_1, G_2). Thus, G_2 can only contain two types of agents. Since $n_H < v/2$, $n_L \leq v/2$, G_2 contains only high and medium types. Since G_2 is different from $N \backslash M3$ it must contain more high type agents. But then, given the construction of $M3$, we can replace medium type agents in G_1 by high type agents while keeping meritocracy and increasing the mean, and this new group will block (G_1, G_2).

Thus, $(M3, N \backslash M3)$ is the unique core stable organizational structure. ■

Although for non maximally mixed meritocratic societies we cannot guarantee uniqueness of the core stable organizational structures, we prove that there are at most two core stable organizational structures.

Proposition 7. *In societies with $n = 2v$ that are not maximally mixed meritocratic, (T, B) is always a core stable organizational structure and there exists at most another core stable one.*

The proof is presented in the Appendix.

We close the section by recapitulating what we have learned about the case $n = 2v$, and highlighting some of the main findings that hold in these societies but also basically extend to more general cases.

One first lesson refers to segregation. For societies that are maximally mixed meritocratic, stability implies non-segregation, as proven in Propositions 5 and 6. For societies that are not, we can assert for sure that stability holds for the segregated structure (T, B), but this is sometimes compatible with the existence of a second stable structure which may be non-segregated.

A second set of remarks refer to the combinations of reward schemes that are compatible within core stable organizational structures. In societies that are maximally mixed meritocratic, at least one of the groups in a stable structure must be meritocratic, while the second group may adopt meritocracy or egalitarianism. In societies where (T, B) is stable, each one of the two sets can adopt any of the two distributional criteria. Moreover, note that in this case the resulting distributional criteria are determined by the number of agents of each type that belong to each of the two sets, and not on the exact values of their productivities.

All of these facts are exploited in Section 6, where we illustrate the implications of our model through examples, some of which involve comparative static remarks. The (almost) uniqueness results in the present section provide the grounds for the use of comparative statics, which could be blurred in cases where multiple stable structures could arise.

5. Three-way clustered societies

We now propose an extension of our previous existence and characterization results to societies with an arbitrary number of types, but whose members are clustered into at most three distinct groups of agents with "similar" and "sufficiently differentiated" productivities.

Definition 7. *A society $\mathcal{S} = (N, \lambda, v)$ is three clustered if there exists a partition of N into three groups $\{H, M, L\}$ (clusters)*[4] *with the following properties:*

C1. For all $h \in H$, $m \in M$, and $l \in L$, $\lambda_h > \lambda_m > \lambda_l$.

C2. For any $J \in \{H, M, L\}$, all segregated productive subgroups of J are meritocratic.

C3. For any J, $J' \in \{H, M, L\}$, $J \neq J'$ such that $\lambda_i < \lambda_j$ for all $i \in J$, $j \in J'$, and for any $S_J \subseteq J$ and $S_{J'} \subseteq J'$ $\lambda_i < \bar{\lambda}_{S_J \cup S_{J'}} < \lambda_j$ for all $i \in S_J$, for all $j \in S_{J'}$.

[4] When this does not lead to confusion and in order to avoid repetitions we may sometimes refer to those agents belonging to the same cluster as being of the same type. Notice however that unlike in the preceding section this loose way to speak does to imply that two members of a cluster are identical.

C4. For all $S_H \subseteq H$, $S_L \subseteq L$ and $j \in M$ and $S_M \subseteq M$, if $\bar{\lambda}_{S_H \cup \{j\} \cup S_L} < \lambda_j$ (resp $> \lambda_j$), then $\bar{\lambda}_{S_H \cup S_M \cup S_L} < \lambda_i$ (resp $> \lambda_i$) for all $i \in S_M$.

Condition $C1$ just requires that clusters must be formed by agents whose productivities are correlative in the natural order, and thus allows to properly speak about the high, the medium and the low cluster. All the agents with the same productivity must belong to the same cluster. Condition $C2$ is an intracluster condition. It always holds if for example productivities of the agents in a cluster are uniformly distributed or have a concave distribution, that is, for any three consecutive agents $i, j, k \in J$ with $\lambda_i \leq \lambda_j \leq \lambda_k$, $\lambda_k - \lambda_j \leq \lambda_j - \lambda_i$. Conditions $C3$ and $C4$ are intercluster conditions. Condition $C3$ requires that there should be enough "distance" between any two clusters. Condition $C4$ requires that the average of productivities for any set containing elements of the three clusters should be "strictly between" clusters. That is, either it belongs to the interval $(\min_{j \in S_H} \lambda_j, \max_{j \in S_M} \lambda_j)$ or to the interval $(\min_{j \in S_M} \lambda_j, \max_{j \in S_L} \lambda_j)$.

The following notation will be useful in what follows. Given a society (N, λ, v) and any set $G \subseteq N$ of cardinality n_G, k_G denotes the maximal number of productive groups of size v in G and $r_G = n_G - k_G v$. Subsets of G are denoted S_G. The partition of the first $k_G v$ elements of G into k_G segregated minimal size productive groups is denoted by $(S_G^1 ... S_G^{k_G})$: that is, $S_G^1 = T(G)$ and $S_G^k = T(G \backslash \cup_{k=1}^{k-1} S_G^q)$.

Remark 3. *Our definition allows for three clustered societies which are degenerate in the sense that some of the clusters may be empty. In these cases, it is easy to prove that core stable organizational structures exist. When only one cluster in non empty, only the intracluster condition $C2$ is operative. And then, the segregated partition of the $k_N v$ most productive agents into k_N meritocratic groups of size v, along with an unproductive group formed by the r_N less productive agents is trivially core stable.*

In societies with two non-empty clusters, say H and L, let R_H be the set that contains the last r_H agents in cluster H and at most the $v - r_H$ most productive agents in cluster L. If $n_L < v - r_H$, R_H is an unproductive group and the structure $\{\{S_H^k\}_{k=1}^{k_H}, R_H\}$ is core stable. If $n_L \geq v - r_H$, let $\hat{L}$ be the remaining agents in the low cluster, that is, $\hat{L} = L \backslash R_H$. Then $\{\{S_H^q\}_{q=1}^{k_H}, R_H, \{S_{\hat{L}}^q\}_{q=1}^{k_{\hat{L}}}, U\}$, where U is an unproductive group formed by the $r_{\hat{L}}$ less productive agents is a core stable structure.

We now turn to the non degenerate case with three non empty clusters. Our first result refers to the distribution of agents from the high cluster within any core stable organization.

Proposition 8. *If a three cluster society $\mathcal{S}$ has a core stable structure, then at most $v-1$ agents in H belong to sets containing agents from other clusters.*

Proof. Let $\hat{\pi}$ be a core stable organizational structure for society $\mathcal{S}$. Denote by S_H the subgroup of the high cluster whose agents are assigned in $\hat{\pi}$ to sets containing individuals from other clusters. Refer to sets containing agents from at least two clusters as mixed groups. Assume that $\#S_H \geq v$. If all mixed groups containing agents from S_H are egalitarian, by condition $C3$ the high type members receive a payoff below their productivity. In this case, the group S_H, which has a greater average, will block $\hat{\pi}$ independently of its regime. If some of the mixed groups containing agents from S_H are meritocratic, we distinguish two cases:

(i) Suppose that there is at least a productive subgroup of S_H which is meritocratic. Then, this subgroup constitutes a blocking group of $\hat{\pi}$, because it is meritocratic and has a greater mean that any of the other groups in $\hat{\pi}$ containing agents from S_H.

(ii) Suppose all productive subgroups of S_H are egalitarian. Consider the meritocratic group in $\hat{\pi}$ containing agents from S_H with the greatest mean. Call this group G. Let $j \in G$ be the agent in G not in S_H with the greatest productivity in G. Form the group $G' = S_H \cup \{j\}$. The group G' is meritocratic because agents in S_H form a majority and, by $C3$, the average of the group is between the productivity of the less productive agent in S_H and λ_j. If $G' \neq G$, then G' is a blocking group of $\hat{\pi}$. If $G' = G$, suppose first that some agents of the high cluster not in S_H are organized in an egalitarian group. This implies that some of those agents are receiving less than their productivity. Add those agents to G'. The new group is meritocratic with a greater mean than G', and will block $\hat{\pi}$. Suppose now that all agents outside S_H are organized in meritocratic groups. Since S_H form an egalitarian group, it is non-segregated nor are some of the groups with high types outside S_H. Order the groups in $H \backslash S_H$ so that the first one is the one that contains the highest productivity agent, the second the one which contain the highest productivity agent among the remaining agents, and so on. Consider the first group in this order which is non-segregated and let i be the agent with the greatest productivity in that group. Form the segregated productive group of cardinality v that contains i as the highest productivity agent. Note that to form this group we could use agents in S_H. Clearly this new meritocratic group will block $\hat{\pi}$.

All the above arguments imply that at most $v-1$ agents in H belong to sets containing agents from other clusters. ■

In view of Proposition 8 it is important to understand the characteristics of core stable organizational structures in societies with at most $v-1$ agents in the high cluster.

We first define a condition that is necessary and sufficient for the existence of core stable organizational structures for such societies. It is a natural extension of the notion of structured societies for three types of agents.

Definition 8. *A non degenerate three clustered society with $n_H < v$ is structured if the following holds:*

1. N has a weak top group.

2. For all G meritocratic such that all $i \in G \cap T$ are better off in G than in T, either the society $(N \backslash T, \lambda_{N \backslash T}, v)$ has a core stable structure, π_1, such that $\#\{i \in M \backslash T \mid$ (payoff of i in $\pi_1) < \lambda_i\} < \#M(G)$, or $\#\{i \in L \backslash T \mid$ (payoff of i in $\pi_1) \leq \lambda_i\} < \#L(G)$.

3. There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ such that:

(a) $\bar{\lambda}_{G_1} \geq \bar{\lambda}_G$ for all meritocratic groups $G \subset (G_1 \cup H(G_2) \cup G_3)$ where $G_2 = T(N \backslash G_1)$ and $G_3 = L \backslash (G_1 \cup G_2)$.

(b) Either the society $((H \cup M) \backslash G_1, \lambda_{H \cup M) \backslash G_1}, v)$ has a core stable structure with segregated groups all of them productive, or $\#(M \cup H(G_2)) < v$, $M \subset G_1$ and $\bar{\lambda}_{T(M \cup (N \backslash G_1))} < \lambda_m$.

Proposition 9. *A non degenerate three clustered society with $n_H < v$ has a core stable organizational structure if and only if it is structured.*

The proof is similar to that of Proposition 4 and is presented in the Appendix.

Finally, we provide two results regarding core stability in societies with v or more agents in the high cluster. One is a necessary condition and the other a sufficient condition for existence. Both are based on our previous results.

For this purpose, we introduce some additional notation.

Given a non degenerated three cluster society $\mathcal{S}$, let $\mathbf{C}^H$ be the set of core stable structures for (H, λ_H, v), the subsociety formed by the high cluster agents. For any $\pi \in \mathbf{C}^H$, let U^π be the set of unproductive agents in π and let $\mathcal{S}^\pi = (U^\pi \cup M \cup L, \lambda_{U^\pi \cup M \cup L}, v)$. That is: we take those high type agents, U^π, that would be in an unproductive group within a stable organization π of the high cluster, and consider the subsociety, $\mathcal{S}^\pi$, that they would form along with agents in the medium and low clusters.

Proposition 10. *If a three cluster society $\mathcal{S}$ has a core stable structure, then there exists $\pi \in \mathbf{C}^H$ such that subsociety $\mathcal{S}^{\pi}$ is structured.*

Proof. The proof is a direct consequence of Propositions 8 and 9 and the fact that subpartitions of a stable organization must be stable within their subsociety. ■

Recall that by Remark 3 the subsociety (H, λ_H, v) has at least one core stable structure, namely the segregated partition. Denote it by $\pi^s = \{\{S_H^k\}_{k=1}^{k_H}, R_H\}$. With this notation, the sufficient condition reads as follows.

Proposition 11. *Consider a three clustered society $\mathcal{S}$. If the subsociety $\mathcal{S}^{\pi^s} = (R_H \cup M \cup L, \lambda_{R_H \cup M \cup L}, v)$ is structured then $\mathcal{S}$ has a core stable organization.*

Proof. Take $\pi^s \in \mathbf{C}^H$, that is $\pi^s = \{\{S_H^k\}_{k=1}^{k_H}, R_H\}$, and let $\pi(\mathcal{S}^{\pi^S})$ be a core stable structure of $\mathcal{S}^{\pi^c}$. Let us see that $\pi(\mathcal{S}) = (\{S_H^k\}_{k=1}^{k_H}, \pi(\mathcal{S}^{\pi^c}))$ is a core stable structure of $\mathcal{S}$. If a set G blocks $\pi(\mathcal{S})$ it must contain agents from $H \backslash R_H$ and agents from $M \cup L$. But, given conditions $C3$ and $C4$, the high type agents in a mixed group are always worse off than in a meritocratic group with just high type agents (as they are in $\{S_H^k\}_{k=1}^{k_H}$). This holds because the average of productivities in a mixed group is always smaller than the productivity of the less productive agent in the high type cluster. Thus, $\pi(\mathcal{S})$ is core stable. ■

For this general case we do not reach a full characterization result because there is some gap between the necessary and sufficient condition for existence. The necessary condition is not sufficient because the productive groups in the core stable partition of the high cluster may not match consistently with the core stable partition of the rest of society to form an overall stable organization. The sufficient condition is not necessary because the segregated partition of the high cluster need not be the only form to organize those agents within a core stable organization of the whole society.

6. On the size and competitiveness of stable organizations

This section continues with the presentation of examples that, along with those already proposed in Section 2, give us a measure of the different issues that we can tackle within our present model.

Recall that our Examples 1 and 2 already show that the choice of reward schemes by majority voting creates the possibility that different regimes coexist in stable arrangements, and that the choice of egalitarianism or meritocracy is not tied to the average productivity level of the different groups that coexist. We have seen more on this in Section 4 for the case of three agent types. This conclusion is interesting under several possible interpretations of the model. In particular, when interpreted as one of country formation, it generates an important stylized fact in the debate on varieties of capitalism: countries with a similar level of productivity do not need to share the same distributional principles.

We have also already remarked that instability may arise, as a consequence of the inability of individuals to commit to a given distributional principle. And that the groups that may form within stable organizational structures can sometimes be segregated, and at other times non-segregated. Again, these remarks apply to any possible interpretation of the model, but may be particularly relevant when it is interpreted as one where communities are formed, or individuals are sorted.

In this section we provide new examples that emphasize the role of the parameters defining our societies in shaping stable organizational structures, and the consequences of changing these parameters. In that case, our remarks, even if they apply in general, may be of special relevance when we think of groups as being different institutions within a society whose government may try to influence the group formation process. We do not have a formal model of government here, but we may assume that what it can control is the minimal size of organizations and the total number of agents who are eligible to form them. For example, if we think of a university system within a country, the government may determine who is qualified to become a professor, and what is the minimal size of the faculty to form a university. Similar requirements on group size and qualifications apply if we interpret the groups to be formed as partnerships in regulated professions.

Our first remark concerns the lack of ability on the part of the government to control the effective size of emerging organizations, even if they may decide on the parameters v and n. Given a minimal size v and a total number of agents $n = kv$, which is therefore sufficient to form k groups, it is possible that no organizational structure with groups of size v can achieve stability, even in contexts where other core stable structures exist. Stability may require larger units. This is shown in the following example.

Example 4. *A case where $n = kv$, and yet no partition of agents into groups of of size v can achieve stability.*

Let $N = \{1, 2, 3, 4, 5, 6\}$, $v = 3$, and $\lambda = (50, 40, 40, 35, 25, 10)$. Let (P, U) be an organizational structure where $P = \{1, 2, 3, 5\}$ and it is meritocratic and $U = \{4, 6\}$ and it is an unproductive group. (P, U) cannot be blocked because P is the meritocratic group with the highest mean *and the only agent that could improve without using anyone from P is agent* 4 *but* $\{4, 5, 6\}$ *is meritocratic. The egalitarian group with the greatest mean in* $E = \{1, 2, 3\}$, $N \backslash E$ *is meritocratic. The organization* $(E, N \backslash E)$ *is blocked by* $G = \{1, 4, 6\}$ *which is a meritocratic group with a greatest mean than* $N \backslash E$. *Any organization with two meritocratic groups or one meritocratic and one unproductive group is blocked by* P, *any organization with two egalitarian groups or one egalitarian and one unproductive group is blocked by* E. *It can be checked that any other organization is blocked by* P. *Thus,* (P, U) *is the unique core stable organizational structure.*

Now we turn to a remark regarding the potential consequences of changing v, the minimal size of organizations. In particular, we note that in contexts where stability is not guaranteed, changes in v can be either stabilizing or de-stabilizing.

Example 5. *Changes in* v *can be either stabilizing or de-stabilizing.*
Let $N = \{1, 2, ..., 7\}$ *and* $\lambda = (100, 84, 84, 84, 84, 60, 60)$. *Suppose that initially* $v = 4$.
Note that medium type agents can form a group by their own with a payoff of 84. The egalitarian group with the greatest mean is blocked by the meritocratic group containing the high, one medium and two low type agents. Any meritocratic G *with the high type is blocked by the four medium agents together. No organizational structure is stable.*
But, if $v = 3$, (G_1, G_2, U), *with* $G_1 = \{1, 2, 6\}$ *and* $G_2 = \{3, 4, 5\}$ *both meritocratic and* $U = \{7\}$ *unproductive is a core stable organizational structure, because the high type is in a meritocratic group and he cannot increase the mean above* 84 *while keeping meritocracy.*

The above remarks have referred to changes in v while keeping the eligible population constant. Now we consider the case where v is reduced, and the total number of available agents is also reduced, so as to keep fixed the maximum number of productive institutions. This is relevant to analyze, in a simple way, some possible consequences of budget cuts. Before we present examples of the type of conclusions we can reach, let us define a second form of stability, other than the one associated to the core.

Remark that in order to define a hedonic game we fix the set of potential players who may be part of the groups that eventually form. In our university story, these would be the people in the country who can eventually join a university, because they meet all the credentials. Typically, these people may have outside options. We shall assume that the best agents in our set are highly demanded by other instances (other jobs, other countries), and that they will only stay for long in the group they join if they are paid at least their productivity. It is natural, in the presence of outside options, to assume that the best will only stabilize in one of our organizations if they join a meritocratic group. This suggests a second (external) stability requirement, that we can call competitive stability: the best (however we define them) must be rewarded meritocratically.

Our comparative static analysis proves that even societies that enjoy core stability may easily shift from being competitively stable to competitively unstable, due to changes in our basic parameters. To the extent that these changes may be generated by or controlled through public policies, keeping the best agents within the system may require finely tuned actions, in order to guarantee that competitive stability remains when circumstances do change.

To illustrate our point, consider the following example.

Example 6. *Competitive stability*

Let $N = \{1, ..., 14\}$, $v = 7$, $\lambda = (10, 10, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.

A possible arrangement, which is both core and competitive stable, because the best university would be meritocratic, consists of having one group with all the high and medium types, plus two lows, and the other containing low types only. Assume now that society must reduce the size of universities by two faculty members each. It would seem natural to fire the two worse people of each group. The groups resulting from these actions (two high and three medium types in the best university, all lows in the other), would still be in the core of the corresponding game with 10 *candidates and* $v = 5$. *Yet, this organization structure is no longer competitively stable, since now the worst university will adopt egalitarianism as a norm! In that case, firing two medium agents, rather than two low ones, would preserve competitive stability.*

Similar and apparently anomalous phenomena would arise as the potential result of other parametric changes. For example, if the low type members would upgrade their qualifications close to the medium type, say from 1 to 6, competitive stability would also be lost.

7. Concluding Remarks

We have presented a very simple model of group formation where people are driven to cooperate by a minimal size requirement, and choose their reward schemes by majority. This very simple model is able to generate a variety of interesting stylized facts that are under examination in different strands of literature, through more complex formulations. We do not claim that the features of our model can be immediately transposed to reality. But they certainly show that one can get a head start in explaining several intriguing phenomena with a minimal set of tools.

We also want to emphasize that the model is simple to describe, but complex to analyze. Our existence and characterization results are hard to get, even after restricting attention to a subclass of societies with three types of agents.

Before discussing possible extensions, let us comment on the sensitivity of our results to other possible specifications of the model. The reader may wonder whether the assumption that ties among different groups that provide the same reward are broken in favor of the highest mean productivity group plays any essential role. We claim that our main results would be very similar if those ties were left unbroken[5]. Indeed, this assumption makes agents' preferences a bit more demanding and restricts the set of potential core stable organizational structures in some profiles, relative to those that would arise if ties would not be broken. But the frontiers that we establish between societies admitting stability or not remain essentially the same with one exception. Specifically, existence in our three clustered societies without the tie breaking would be guaranteed whenever $n = kv$ for all natural numbers k, while in our case the result is only true when $k = 2$.

A second assumption in our model is that agents must chose between only two reward systems. We could have derived the same results by enlarging the set of potential choices to admit any convex combination of these two principles, since in fact agents will always chose one of the two extreme points in that continuum. Our reward systems can be seen as resulting from a model of tax choice where a proportional tax t is levied and its proceeds are equally distributed: egalitarianism corresponds to the case $t = 1$ and meritocracy arises when $t = 0$, since again voters will always favor one of these two extreme cases as their best choice.

Our model admits many potential extensions. A natural one is to model the externality resulting from cooperation with other agents in other ways. Here the monetary reward is only supplemented lexicographically with some preference to belong to a group with the highest mean, given the same payment. But we could think of stronger impacts to be received from cooperating with others, ones where the prestige of working along with highly productive agents may lead to accept lower pays that the ones one can get in less productive groups. Exploring the combinations between material and subjective rewards would certainly be a next step in understanding the interaction between group formation and the choice of distributional criteria.

Other features of our model that could be extended are those relating to the technology of individual and of joint production, to the incentives to exert effort, or to the voting system. Here

[5] Technical discussion on this case is available upon request.

agents either contribute zero, when they belong to small groups, or a fixed productivity in groups larger than v. In a larger picture, one could incorporate complementarities among agents, returns to size, and the possibilities that individuals contributions to productions are affected by their actual rewards. All of these extensions seem promising, and none of them appears to be trivial. But we trust our main messages to be robust.

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8. Appendix

Proof of Proposition 7

Case 1. Assume $n_H \geq v/2$.

In this case, $T \in M_+(N)$, and therefore is a weak top group. Thus, (T, B) is a core stable organizational structure. Furthermore, we first show that there may exist a second core stable structure if (i) $G \in E_+(N)$, $\#G > v$, and G is a weak top group, or (ii) $G \in E_+(N)$, $\#G = v$, and $N \backslash G$ is also egalitarian[6].

If (i), since G is a weak top group, $(G, N \backslash G)$ is core stable and only G is productive.

If (ii), since both G and $N \backslash G$ are egalitarian, $(G, N \backslash G)$ is a core stable organizational structure with two productive groups. To see that, note that no group can block $(G, N \backslash G)$ because such group would have to be meritocratic and thus formed by agents that are receiving less than their productivity in $(G, N \backslash G)$. Given that both groups in $(G, N \backslash G)$ are egalitarian, those agents are the ones whose productivity is above the mean of the group, and since the mean is above the median, they are less than $v/2$ in each group. Hence they cannot form a productive group blocking $(G, N \backslash G)$.

Let us see that, apart from this possible second core stable structure, there can be no other.

In structures $(P, N \backslash P)$ where only P is productive, if P is not weak top, there will exist a productive group G such that all $i \in P \cap G$ will be better off in G than in P. Since all $i \in (N \backslash P) \cap G$ are getting zero in $N \backslash P$, they will also be better off in G. Thus, G will block $(P, N \backslash P)$. Hence, P has to be weak top, and the unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from (T, B) and the one considered in case (ii) will be unstable.

(1) If G_1 and G_2 are meritocratic, it is blocked by T which is also meritocratic.

(2) If G_1 is meritocratic and G_2 is egalitarian we distinguish two cases.

- If all the high type agents are in G_1, G_2 can only contain medium and low types, and since it is egalitarian $\bar{\lambda}_G < \lambda_m$. But then, adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1 which blocks (G_1, G_2).

[6] This last situation can only happen if $v_L > v/2$. To see this, note that, since T is meritocratic, high type agents have to be distributed between G and $N \backslash G$. Furthermore, let us see that all medium type agents have to be in G. If $\bar{\lambda}_G < \lambda_m$, the median agent is a low type agent, and $N \backslash G$ has to contain three types. Adding a high, a medium, and a low type agent to G from $N \backslash G$ will create a new egalitarian group of higher mean, contradicting that $G \in E_+(N)$. If $\bar{\lambda}_G \geq \lambda_m$, adding a high and a medium type to G from $N \backslash G$ will create a new egalitarian group of higher mean. Again, this contradicts that $G \in E_+(N)$. Thus, G contains all the medium type agents. Therefore, for $N \backslash G$ to be egalitarian, $v_L > v/2$.

- If the high type agents are split between G_1 and G_2, we can add all missing high type agents to G_1 and drop enough non high types in G_1 to create a new group of size v. This new group will still be meritocratic, have a higher mean than G_1, and block (G_1, G_2).

(3) If G_1 and G_2 are egalitarian, neither G_1 nor G_2 are in $E_+(N)$. Thus, any egalitarian group $G \in E_+(N)$ will block (G_1, G_2).

Case 2. Assume $n_H < v/2$ and $n_L > v/2$.

In this case, T can be either egalitarian or meritocratic. In both cases, as proven in Proposition 5, (T, B) is a core stable organizational structure.

Case 2a. Suppose first that T is meritocratic.

Since $n_H < v/2$, T has three types of agents and consequently B is the meritocratic group with just low types, which implies that $n_L > v$.

As in Case 1, a second core stable structure may exist if (i) $G \in E_+(N)$, $\#G > v$, and G is a weak top group, or (ii) $G \in E_+(N)$, $\#G = v$, and $N \backslash G$ is also egalitarian[7]. The same argument as in Case 1 applies.

Let us see that, apart from this possible second core stable structure, there can be no other.

As explained in Case 1, structures $(P, N\backslash P)$ where only P is productive are core stable if and only if P is weak top. The unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from (T, B) and the one considered in case (ii) will be unstable.

The arguments in (a) and (b) in Case 1 apply here.

In the case that G_1 is meritocratic and G_2 is egalitarian, G_2 must contain at least two types of agents. If G_2 contains high type agents, replacing a low type agent in G_1 by a high type agent will create a new meritocratic group G (because T is meritocratic) of higher mean than G_1 which blocks (G_1, G_2). The same kind of argument will apply if G_2 does not contains high type agents but contains medium type agents.

Case 2b. Suppose that T is egalitarian. Since $n_H < v/2$, T has two or three types of agents and consequently B is either egalitarian with medium and low types ot meritocratic with only low type agents.

We first show that there may exist a second core stable structure if (i) $G \in M_+(N)$, $\#G > v$,

[7] Note that since T is meritocratic, this situation can only happen if G contains all the high type agents and $v - v_H$ low type agents and it should be such that adding a medium type changes the regime. This structure only exists if $v_H = v/2 - 1$ and $v_M < v/2$.

and G is a weak top group, or (ii) if $G \in M_+(N)$, $\#G = v$, $N\backslash G$ is egalitarian, $(N\backslash G) \cap M = \emptyset$, and the mean productivity of the group is below λ_m.

If (i), since G is a weak top group, $(G, N\backslash G)$ is core stable and only G is productive.

If (ii), since the mean productivity of $N\backslash G$ is below λ_m, $(G, N\backslash G)$ is core stable. There is no possibility of blocking because a potential blocking group should contain medium type agents. Since they are in a meritocratic group with the greatest mean, they will only participate in an egalitarian group with mean above their productivity. But this is not possible.

Let us see that, apart from this possible second core stable structure, there can be no other.

As explained in Case 1, structures $(P, N\backslash P)$ where only P is productive are core stable if and only if P is weak top. The unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from (T, B) and the one considered in case (ii) will be unstable.

(a) If G_1 and G_2 are both egalitarian, it is blocked by T which is also egalitarian.

(b) If G_1 and G_2 are both meritocratic, and neither G_1 nor G_2 are in $M_+(N)$, any meritocratic group $G \in M_+(N)$ will block (G_1, G_2). If one of them belongs to $M_+(N)$ (let us say $G_1 \in M_+(N)$), since $n_H < v/2$ and $n_L > v/2$, both G_1 and G_2 contains medium type agents. Suppose that $\bar{\lambda}_{G_1} \geq \bar{\lambda}_{G_2}$: then adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1 which blocks (G_1, G_2).

(c) If G_1 is meritocratic and G_2 is egalitarian, G_1 may contain agents of two or three types. In the first case they must be medium and low types with a majority of medium types. Thus, G_2 contains low type and high type agents and (possibly) medium types. In any case, $T \in E_+(N)$ blocks (G_1, G_2). If G_1 contains three types, G_2 can contain two or three types (with low and medium types for sure in both cases). If $\bar{\lambda}_{G_2} < \lambda_m$, adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1, which blocks (G_1, G_2). If $\bar{\lambda}_{G_2} > \lambda_m$, replacing a low type in G_2 with a medium type from G_1 creates a new egalitarian group of higher mean than G_2, which blocks (G_1, G_2).

Case 3. Assume that $n_H < v/2$, $n_L \leq v/2$, and $(\lambda_h + \lambda_m + n_L\lambda_l)/(n_L + 2) > \lambda_m$.

In this case T only contains high types and medium type agents and it is egalitarian, B contains only medium and low type agents and is meritocratic. Condition $(\lambda_h + \lambda_m + n_L\lambda_l)/(n_L + 2) > \lambda_m$ implies that high type agents cannot be part of a meritocratic group, thus T is weak top and (T, B) is core stable.

Note that the meritocratic group with the greatest mean in this case is M, which is not a weak top group. Thus, no other organizational structure with only one productive group can be core stable.

Finally, let us show that any structure (G_1, G_2) with two productive groups different from (T, B) will be unstable.

(a) Note that G_1 and G_2 cannot be both meritocratic, since there is no meritocratic group that contains high type agents.

(b) If G_1 and G_2 are both egalitarian, it is blocked by T which is also egalitarian.

(c) If G_1 is meritocratic and G_2 is egalitarian, G_1 can only contain medium and low types or only medium type agents, but since this group is different from B, G_2 must contain low type agents also. Note that since G_2 is egalitarian and low types do not constitute a majority, $\bar{\lambda}_{G_2} > \lambda_m$.

Replacing in G_2 low type agents by medium type agents from G_1 will create a new egalitarian group of higher mean than G_2 which blocks (G_1, G_2). ■

Proof of Proposition 9. *Part 1: Structured societies* with $n_H < v$ *have core stable organizational structures.*

For each condition assuring a structured society we describe how to construct a core stable organizational structure.

(i) Suppose condition 1 holds, i.e. there exist weak top groups in N. Let W be one of those weak top groups. Note first that $N \backslash W$ only contains agents from at most two clusters. This is because either $W = T$ and then $N \backslash T \subset M \cup L$, or W is a meritocratic group with agents from the three clusters. In the latter case, since W is a meritocratic group with maximal average productivity it is necessary that $M \subset W$ and then $N \backslash W \subset H \cup L$. Hence, by Remark 3, the two-type society $N \backslash W$ has a core stable organizational structure. The groups in that structure plus W constitute a core stable organizational structure for N.

(ii) Suppose that condition 1 does not hold but condition 2 does. Since condition 1 does not hold, T is egalitarian, and $(N \backslash T, \lambda_{N \backslash T}, v)$ is a two cluster society. Hence by Remark 3, $(N \backslash T, \lambda_{N \backslash T}, v)$ has a core stable organizational structure π_1. Then $\pi = \{T, \pi_1\}$ is a core stable organization of N because any group G potentially blocking π must be meritocratic and include agents from every cluster, and either some $i \in M \cap G$ is worse off in G than in π_1 (if $\#\{i \in M \backslash T \mid$ (payoff of i in $\pi_1) < \lambda_i\} < \#M(G))$, or else some $i \in L \cap G$ is worse off in G than in π_1 (if $\#\{i \in L \backslash T \mid$ (payoff of i in $\pi_1) < \lambda_i\} < \#L(G))$.

(iii) Last, suppose that conditions 1 and 2 fail but condition 3 holds.

There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ satisfying a and b. Without loss of generality suppose that $i \in G_1 \cap M$ are the agents with the lowest productivity in M (note that if this is not the case, we can always replace each of the medium type agents in G_1 by one less productive medium type agent without changing the above characteristics of G_1). Also without loss of generality, suppose that all $i \in G_1 \cap L$ are consecutive with the greater productivities in L compatible with G_1 being meritocratic. Suppose first that society $((H \cup M) \backslash G_1, \lambda_{(H \cup M) \backslash G_1}, v)$ has a core stable structure with segregated groups, all of them productive. Let $\pi((H \cup M) \backslash G_1)$ be this structure. Consider the following organizational structure of N : the first group is G_1, then all the groups in $\pi((H \cup M) \backslash G_1)$ and finally the core stable structure of the remaining low type agents, $\pi(L \backslash G_1)$. This structure is stable given conditions (a) and (b). Otherwise, if such core structure $\pi((H \cup M) \backslash G_1)$ does not exist, we consider the structure formed by G_1 that contains all the medium type agents (recall that since $\#(M \cup H(G_2)) < v$, medium type agents cannot form a productive group on their own), by $T(N \backslash G_1)$ that contains high and low types, and finally by the core stable structure of the remaining low type agents, $\pi(L \backslash (G_1 \cup T(N \backslash G_1))$. Again, conditions (a) and (b) guarantee that this is a core stable organizational structure for N.

Part 2: Unstructured societies with $n_H < v$ have no core stable organizational structures.

Assume that neither 1 nor 2 nor 3 hold and that a core stable organization structure π exists. Let $G \in \pi$ such that $G \cap H \neq \emptyset$. We show that G cannot be meritocratic, nor egalitarian, nor unproductive, which is a contradiction.

Assume G is meritocratic, let us see that the negation of conditions 1 and 3 lead to a contradiction.

Since condition 1 does not hold, there are no weak top groups in N. Then $n_H < v/2$, because otherwise T would be a meritocratic group and it would be a weak top group of N. Thus, if G is a meritocratic group it must include agents from the three clusters (by $C3$). Since there are no weak top groups, then $\#N \backslash G \geq v$, because otherwise, the remaining agents are in an unproductive group and π can be blocked. Apart from G, no other productive group $G' \in \pi$ with agents from the three clusters can be meritocratic. Otherwise, given $C4$, an $i \in M$ in the group with lower average productivity could switch to the other and increase the average productivity while keeping meritocracy, and this new group would block π. So, if π contains another productive group G' with three types, it must be egalitarian and it must contain all $i \in H \backslash G$. If $\overline{\lambda}_{G'} > \lambda_m$ for some $m \in M$, replacing an agent from L in G' by one from M in G increases the average and keeps egalitarianism, and this later group blocks π (given that $C4$ implies that $\bar{\lambda}_{G'} > \lambda_j$ for all $j \in G' \cap M$). But if $\bar{\lambda}_{G'} \leq \lambda_m$ for some $m \in M$, we contradict that π is core stable as well - since switching one of the agents in M from G' to G increases the average in G and keeps meritocracy. Thus, agents in $N \backslash G$ can only be organized in groups with agents from one or two clusters, and all $i \in H \backslash G$ are in an egalitarian group. Note also that an agent $i \in M \cap (N \backslash G)$ cannot be in a group that does not contain agents from H, because by joining G they increase the mean while keeping meritocracy, and this new group will block π. If $M \backslash G \neq \emptyset$, π contains $G_2 = T(N \backslash G)$ which is egalitarian with agents from H and M, or meritocratic with just agents form M (if $H \subset G$). If there are still more agents in M, they are organized in segregated meritocratic groups with just medium type agents. Note that they cannot be organized in egalitarian groups because the agents in those groups that receive a payoff below their productivity by joining G will increase the mean while keeping meritocracy. The rest of society is composed by agents from L. If $M \backslash G = \emptyset$, π contains $G_2 = T(N \backslash G)$ which is egalitarian with $T(N \backslash G) \subset H \cup L$, and again the remaining society is composed by agents from L. Since condition 3 does not hold, either (a) or (b) fails:

-If (a) fails, a meritocratic group $G' \subset (G \cup H(G_2) \cup G_3)$ where $G_2 = T(N \backslash G)$ and $G_3 = L \backslash (G_1 \cup G_2)$ exists with $\bar{\lambda}_{G'} > \bar{\lambda}_G$. Note that G' blocks π.

-If (b) fails, the society $((H \cup M) \backslash G_1, \lambda_{H \cup M) \backslash G_1}, v)$ cannot be organized in a segregated stable way with all groups productive for any meritocratic group G_1. Since, as we argued above, π cannot place $i \in M$ in groups without agents from H, it must be that $\#(H \cup M) \backslash G < v$. Thus, $M \subset G$, and π organizes $N \backslash G$ with an egalitarian group $E \subset H \cup L$ such that $H \backslash G \subset E$, and the rest of low type agents are organized in an stable way. If $\#(M \cup H(G_2)) \geq v$, then the group of cardinality v containing all $i \in H \backslash G$ and some medium type agents is egalitarian (or meritocratic if it only contains medium type agents) and blocks π. If $\#(M \cup H(G_2)) < v$, the average productivity of $T(M \cup (N \backslash G))$ is greater than λ_m for some $m \in M$, which implies that $T(M \cup (N \backslash G))$ is an egalitarian group which blocks π.

All the above points imply that a meritocratic G containing high type agents cannot be part of a core stable organizational structure of N.

Assume next that G is egalitarian. Let us see that the negation of conditions 1 and 2 leads to a contradiction.

Since there are no weak top groups and high type agents cannot be in a meritocratic group, it must be that $G = T$. Since condition 2 does not hold, any possible stable organization of the society $(N\backslash T, \lambda_{N\backslash T}, v)$ is such that $\#\{i \in M\backslash T \mid (\text{payoff of } i \text{ in } \pi_1) < \lambda_i\} \geq \#M(G)$, or $\#\{i \in L\backslash T \mid (\text{payoff of } i \text{ in } \pi_1) \leq \lambda_i\} \geq \#L(G)$. Thus, the medium and low types necessary to form the meritocratic group that would challenge T are available. This group will block π.

To conclude, assume G is unproductive. But $h \in G$ is very welcome in any meritocratic group (even if that changes the regime), and if there are no meritocratic groups, T blocks π.

Hence, there are no core stable organizational structures. ■