A General Framework for Rational Learning in Social Networks

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Abstract

This paper provides a formal characterization of the process of rational learning in social networks. A finite set of agents select an option out of a choice set under uncertainty in infinitely many periods observing the history of choices of their neighbors. Choices are made based on a common behavioral rule. We find that if learning ends in finite time and the choice correspondence is union consistent, then every action selected by any agent once learning ends is optimal for all his neighbors. Local indifference across neighbors, however, does not in general imply global indifference across all agents in the network. We further provide sufficient conditions for the existence of a finite time for every state of the world such that every action chosen by an agent from that time period onward is optimal for all his neighbors in the limit. If only common knowledge of rationality rather than common knowledge of strategies is assumed, the validity of the aforementioned results depends on the network structure. If the network is complete, the result of local indifference across neighbors once learning ends still holds, while it can fail in incomplete networks. Our results have direct implications for the literature on social learning, knowledge and consensus, and coordination games.

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1 Introduction

Social networks have a very important function as a source of information. Individuals constantly communicate with their social peers and use the information obtained through their interactions when forming opinions and making decisions. Within the economic literature, the importance of social networks is widely recognized. The relevant role of social networks for employment outcomes[1], technology adoption[2], models of collective political action[3], and bargaining outcomes[4] has been established[5]. However, there is a lack of understanding of the formal learning process of rational individuals in incomplete networks[6] and how behavior and opinions of individuals evolve over time if they interact repeatedly. This paper fills the current gap in the literature and contains the following contributions. First, we provide a general model of repeated interactions of rational individuals in social networks under uncertainty. Second, we formally characterize the learning process and show that if learning ends for all agents, local indifference across neighbors holds (Theorem 1). Once learning ends the action that an agent selects is optimal for all his neighbors in the network. We provide an example to show that the local indifference does not extend to global indifference in incomplete networks. Next we provide sufficient conditions for the existence of a finite time such that from that period on every action that an agent selects is optimal for all her neighbors in the limit (Theorem 2). Furthermore we analyze the case where only common knowledge of rationality, instead of common knowledge of strategies, is given and show that in a complete network the local indifference among neighbors once learning ends still holds (Theorem 3). For incomplete networks, however, this local indifference can fail. We provide an example to highlight the failure of local indifference in an incomplete network

Our general framework is the following: a finite set of agents faces uncertainty represented by a measurable space $(\Omega, \mathcal{F})$. We assume a partitional information structure where agent's initial private information regarding the state of the world is given by the realized cell of their partition. After observing the cell of their partition, agents have to select an option out of a common choice set A in each of infinitely many periods. The agents are organized in a social network which is assumed to be a connected undirected graph. The network structure determines the observability of the history of actions; each agent observes the history of actions of his neighbors in the social

[1] See for example Boorman (1975), Calvo-Armengol and Jackson (2004), Iaonnides and Loury (2004), Munshi (2003), and Topa (2001).

[2] See for example Besley and Case (1994), Foster and Rosenzweig (1995), Munshi (2004), and Udry and Conley (2001)

[3] See for example Chwe (2000). Within the sociology literature Opp and Gern (1993), and Snow, Zurcher and Ekland-Olson (1980) explored the importance of social networks for political participation.

[4] See Manea (2008), and Wang and Wen (1998).

[5] An extensive survey on the relevancy of social networks from an economic perspective is given by Jackson (2008).

[6] A network is incomplete if there exists at least one pair of agents that are not neighbors of each other.

network[7]. Let us state two examples that are captured by the framework described so far.

Suppose all agents share a common prior belief over the state space, have partitional information, and are concerned with the likelihood of an event, like, for example, the probability of a certain presidential candidate being elected. Suppose that in each stage agents announce their posterior probability of the candidate being elected observing the history of previous announcements of their neighbors. In this case, the choice in each stage equals the posterior probability, and the choice set is given by the unit interval.

Another example is an expected utility setting. Suppose agents share a common prior belief over the state space and a common utility function u, where the utility of agent i depends on his action and the realized state of the world. In each period, agents have to select an action out of the set of actions observing the history of actions of their neighbors. We assume that agents behave myopically in the sense of being non-forward looking. This avoids potential strategic behavior and guarantees that agents select the action in each period that maximizes their expected utility in that given period. The assumption is common in the literature and can be rationalized[8] by assuming that each node of the network houses a non-atomic continuum of identical agents, and only the distribution of actions is observed.

In both of the examples mentioned, the optimal action is a function of the available information. We will analyze a more general version of the examples provided, where agents act according to a common behavioral rule described by a choice correspondence. The choice correspondence maps information sets[9] to the action space, $C : \mathcal{F} \rightrightarrows A$. To each information set, the common choice correspondence assigns a subset of the choice set: all choices that are optimal given the information. Note that if two agents face the same information set, they have the same set of optimal actions. This follows from the assumption that the choice correspondence is common, i.e. identical for all agents.

We need to also define the strategies of players. The strategies of players assign a single optimal action to each information set out of the set of optimal actions prescribed by the choice correspondence for the given information set. Strategies are assumed to be common knowledge.

We pose one condition on the choice correspondence, which is known as **Union Consistency** or the **Sure Thing Principle** in the literature on knowledge and consensus[10]. Consider a collection of disjoint information sets. A choice correspondence is union consistent under the following condition: if the intersection of the set of optimal actions across all information sets in the collection is not empty, then the intersection of the set of optimal actions equals the set of optimal actions assigned

[7] Agent j is a neighbor of agent i if they are connected by an edge in the network

[8] See Gale and Kariv (2003).

[9] An information set is the smallest subset of Ω that a given agent knows to contain the true state of the world.

[10] See for example Bacharach (1985), Cave (1983), and Krasucki (1996).

to the union over all information sets in the collection. In other words, if an action is optimal for all information sets in the collection then it also has to be optimal for the information set equaling the union over all sets in the collection. Furthermore, any action that is not optimal for all sets in the collection cannot be optimal under the union over all sets in the collection. Union consistency is satisfied in the probability announcement example as long as the partitions are countable. In the expected utility setting, union consistency holds if the utility function is bounded and measurable for each action, and all elements of the join[11] have positive probability[12]. In cases where the join has elements of probability zero, union consistency can fail in the expected utility setting. We provide a general condition (**UC′**) to capture those cases. Please see the supplementary Appendix for an extensive treatment of the special case of the expected utility setting, as well as the posterior announcements example[13]. Also see the supplementary Appendix for an example of a union consistent choice correspondence that can not be expressed in an expected utility setting, thus highlighting the higher degree of generality of our framework.

The main contributions of the paper are the following. This paper is the first to provide a full characterization of the rational[14] learning process in incomplete social networks. The characterization is based on the learning process in Geanakoplos and Polemarchakis (1982) and Geanakoplos and Sebenius (1983). In their setting, two agents repeatedly select an action or announce a posterior belief and the history of actions is common knowledge. In an incomplete network where the history of choices of all agents is not common knowledge, there are significant complexities involved in the learning process compared to the complete network case that Geanakoplos et al. analyzed. The main difficulty arises from the fact that for any pair of neighbors in the network, the private observables determining the information set of agent i from the perspective of his neighbor j do not only contain a static component, the true cell of i's partition, but a dynamic component as well, the history of choices of neighbors of i that are not neighbors of j.

Having obtained a full characterization of the learning process, we can extend and cast new light on existing results in the literature on social learning and knowledge and consensus, as well as give answers to questions that have not been addressed before. The results we achieve as a consequence of the formal and complete characterization of the learning process are presented in three theorems.

Theorem 1 gives a result regarding local indifference. If learning ends in a finite period t^* for all

[11] The coarsest common refinement of all partitions.

[12] See Proposition 1 in the supplementary Appendix. The supplementary Appendix can be downloaded on the authors website www.depot.northwestern.edu/~mmu834/indexjm.html

[13] Note that the posterior announcement example can be captured by the expected utility setting with quadratic loss functions.

[14] In the general setting without probabilities we denote the learning process as rational. In the special cases with a probability space we refer to Bayesian learning.

agents in the network, then the choice agent i makes in any period $t \geq t^*$ is also optimal for all his neighbors. We provide two corollaries of the main result. If agents share a choice function rather than a correspondence, then all agents in the network select the same action once learning ends. Theorem 1 implies a generalization of Geanakoplos and Polemarchakis (1982) "We Can't Disagree Forever" result. In a connected social network with finite partitions and where agents communicate their posterior belief of an uncertain event to their neighbors in each period, all agents will converge to the same posterior in finite periods. Within the setting of posterior announcements we ask the natural question: which network structure leads to optimal aggregation of private information[15]? Contrary to one's intuition, complete networks do not dominate incomplete networks in terms of quality of information aggregation. We provide an example of an incomplete network converging to the pooled information posterior while the complete network with the same underlying information parameters does not.

As the indifference outcome is restricted to pairs of neighbors, in networks with a diameter larger than one, we have an interesting result in relation to the social learning literature. We provide an example of a connected social network where a pair of homogeneous agents that are not neighbors of each other select distinct actions once learning ends while not being indifferent among them.

It is directly apparent that the convergence of posterior beliefs has important implications for coordination games when pre-play communication within the social network is feasible. We provide an example of a game of regime change, where through introduction of a message stage prior to agents deciding whether to attack the status quo or not, the set of equilibrium outcomes is reduced to either a perfectly coordinated attack among all agents or uniform inaction.

Theorem 2 characterizes sufficient conditions under which there exists a finite time period $t^*(\omega)$ for each state of the world such that every action an agent chooses from that finite period on will be eventually optimal for all his neighbors. In order to be able to analyze asymptotic behavior, we introduce the concept of a dominant set. A set B is dominant if for all sequences of sets $\left\{B^t\right\}_{t=1}^{\infty}$ converging to B there exists a finite t' such that any choice that is optimal for a set B^t with $t \geq t'$ is also optimal in the limit set B. In the expected utility framework with a bounded and measurable utility function and finite actions, every set of positive probability is a dominant set[16]. Additionally we introduce condition (**D**), which requires every set containing a dominant set to be a dominant set. In the special case of the expected utility setting with finite actions, condition (**D**) is fulfilled as every set that contains a set with positive probability has positive probability as well. The precise result of Theorem 2 is the following: If all elements of the join of partitions

[15] A network aggregates information optimally if the posterior beliefs of agents converge to the pooled information posterior, i.e. the belief if the true cell of the join where common knowledge.

[16] Please see Proposition 2 in the supplementary Appendix.

are dominant sets, the choice set is finite, and the choice correspondence is union consistent and complies with property (**D**), then for every state of the world there exists a finite time $t^*(\omega)$ such that every choice that player i makes in periods $t \geq t^*(\omega)$ will be optimal for each of her neighbors in their limit information set.

Theorem 3 uncovers a new result in the literature. We consider an environment where strategies are not common knowledge. Instead, only common knowledge of rationality is assumed, i.e. common knowledge of the choice correspondence agents base their decisions on. Theorem 3 states that if the network is complete[17], the choice correspondence is union consistent, and learning ends in a finite time period t^*, then the action chosen by agent i in period $t \geq t^*$ is optimal for all other agents. Generally, the optimality of one's neighbors choices fails for incomplete networks. We provide an example of an incomplete three player network where the optimality of actions across neighbors fails to hold.

The rest of the paper is organized as follows. In the next section, we give a brief overview of the related literature. In section 3, we introduce our general framework. In section 4, we present a simple example to provide intuition about the learning process and characterize the general learning process. Section 5 presents Theorem 1, corollary results, an example of failure of global consensus, as well as an example of superior information aggregation in an incomplete network compared to a complete network with the same underlying information parameters. In section 6, we analyze the asymptotic case, introduce the concept of a dominant set, and present Theorem 2. In section 7, we analyze the case of common knowledge of rationality versus common knowledge of strategies. We present Theorem 3 and provide an example of an incomplete network where the optimality of choices among neighbors fails. Section 8 provides a game of regime change as an illustration of the importance of our results on the set of equilibrium outcomes of coordination games if pre-play communication in the social network is feasible. Section 9 concludes. The Appendix presents proofs omitted in the main text. In the supplementary Appendix[18], we provide an example of a union consistent choice correspondence that is incompatible with expected utility representation. Furthermore, we formally establish the expected utility setting and the probability announcement setting as special cases of our general model. We also present the general condition ($\mathbf{UC'}$) that captures the case where the join has elements of probability zero, and state the implication for Theorem 1 and Theorem 3 under the more general condition ($\mathbf{UC'}$).

[17] A network is complete if every agent is neighbor of every other agent.

[18] The supplementary Appendix can be downloaded at www.depot.northwestern.edu/~mmu834/indexjm.html

2 Related Literature

There is a vast literature on social learning which can be separated into two main categories depending on whether learning is Bayesian or myopic. Following the categorization of Gale and Kariv (2003), the Bayesian social learning literature can further be separated into sequential social learning models and social network models. In sequential social learning models, a countably infinite set of agents select a single action in an exogenous sequence observing the history of choices of all their predecessors. Agents have common values, and the utility each one achieves is a function of his individual action and the state of nature. Each agent observes a private signal conditioning on the true state of nature. This line of research was started by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992), and significantly extended by Smith and Sorensen (2000), who show that in finite time a cascade almost surely arises. An informational cascade describes a situation where agents select the same action as their predecessor independent of their private information. Smith and Sorensen (2000) prove that under unbounded private beliefs, the optimal action is almost surely chosen. Acemoglu, Dahleh, Lobel and Ozdaglar (2008) in their recent contribution generalize the sequential social learning model by relaxing the assumption that agents observe the actions of all their predecessors. In their model, agents observe only the previous actions of agents in their neighborhood, where the neighborhood of each agent is stochastically generated and its realization is private information. They find that if private beliefs are unbounded, and the stochastic process generating the neighborhoods has expanding observations, then individual actions converge to the optimal action in probability. Our results are aligned with the resulting convergence of actions in the sequential social learning literature in some cases. We show that if agents select actions repeatedly and learning ends in finite time, perfect uniformity among all agents is reached if the common behavioral rule is a choice function. On the other hand, we provide a counter example for uniformity and consensus even under common knowledge of strategies. In networks with a diameter of at least two and an underlying choice correspondence, pairs of homogeneous agents can select different actions forever and the indifference of actions across those agents fails as long as the shortest path among them is larger than one.

Social network models are characterized by simultaneous actions in countable periods where agents observe only the actions of their neighbors. Gale and Kariv (2003) were the first to analyze Bayesian learning in social network models, and their paper is most closely related to ours. In their model, a finite set of agents share a common payoff function, a common prior over the state space and receive a private signal at the beginning of the game. Agents are organized in a social network represented by a directed graph. In each period, players select an expected utility maximizing action out of a finite set observing the history of actions of their neighbors. Gale and Kariv apply the

Martingale Convergence Theorem as well the Imitation Principle[19] to show that for any subsets of the state space in which two neighbors select two different actions infinitely often, both players will be indifferent among those actions in the limit for almost every state in the set. Their main result is similar in nature to our Theorem 1 and Theorem 2. The main differences from our framework are in the assumptions regarding (1) the social network, (2) utility, and (3) cardinality of the choice set. While Gale and Kariv allow for directed graphs as their underlying network, we assume an undirected graph that implies the following symmetry: if agent i is a neighbor of j then agent j is a neighbor of i. In terms of the assumptions on the underlying graph, their set-up is more general, but our results are extendable to a directed graph setting for each pair of agents that are both neighbors of each other. (2) Gale and Kariv use an expected utility setting where agents share a common prior belief and a common utility function, which constitutes a special case of our general approach with a common choice correspondence. (3) While they assume a finite action space, our results in Theorem 1 and Theorem 3 hold for countably finite, countably infinite and uncountable choice sets.

Other than the points mentioned, the main difference between Gale and Kariv (2003) and our paper lies in the analytical approach that we use to achieve the results. We formally characterize the learning process and prove our results directly from its properties, while they rely on the Martingale Convergence Theorem and the Imitation Principle.

Within the non-Bayesian stream of literature, Bala and Goyal (1998,2001), DeMarzo, Vayanos and Zwiebel (2003), Golub and Jackson (2007) and most recently Acemoglu, Ozdaglar and ParandehGheibi (2009) study learning in social networks. Bala and Goyal (1998) establish a payoff equalization result in a non-Bayesian setting. They show that asymptotically each agent achieves the same payoff as his neighbor. De Marzo, Vayanos and Zwiebel (2003), Golub and Jackson (2007), and Acemoglu, Ozdaglar and ParandehGheibi (2009) provide conditions under which the beliefs of all agents in the social network converge.

Our analysis is also closely related to the literature on communication, consensus and common knowledge started with Aumann (1976). He showed that if two individuals have partitional information, share a common prior and their posterior beliefs are common knowledge, then their posteriors are equal. Geanakoplos and Polemarchakis (1982) generalized Aumann's result by proving that if two individuals repeatedly communicate their posterior probability of an uncertain event, their posterior beliefs will converge in finite steps if the partitions of both individuals are finite. Cave (1983) and Bacharach (1985) generalized Geanakoplos and Polemarchakis' result from consensus on conditional probabilities to union consistent decision functions mapping subsets of the state

[19] The Imitation Principle relies on the idea that an agent can always imitate the actions of his neighbor and be at least as well off on average.

space into a decision set. The papers mentioned so far analyzed the case of complete networks where the announcements are common knowledge. Parikh and Krasucki (1990) analyzed the case of a finite set of agents that communicate according to a communication protocol. In each period, a pair of agents is selected and one of them announces his message to the other. They showed that if the decision function is convex and the protocol fair, i.e. every agent is a receiver and a sender in infinitely many periods and receives information from every other agent, then consensus is reached in finitely many periods. Krasucki (1996) then strengthened the above result by proving that a union consistent function is sufficient for consensus in finite steps if the communication protocol contains no cycles. The most recent contribution to this literature is Menager (2006). She shows that if agents communicate their expected utility maximizing action according to a pairwise communication protocol, and if for each possible information set a single action is optimal, then consensus is reached for any fair protocol in finitely many steps. Our Theorem 1 generalizes the existing consensus results in this stream of literature. All previous contributions were based on the assumption of pairwise communication, decision functions, as well as finite partitions, while we consider simultaneous communication in incomplete networks, a decision correspondence, and finite, countable, and uncountable partitions. In contrast to the usual global consensus result obtained in this literature, only local consensus or indifference is guaranteed in incomplete networks under a choice correspondence. Our asymptotic result of Theorem 2 is novel in the literature, as all previous contributions considered only the finite partition case. While the union consistency condition we use constitutes a generalization of the union consistency condition for decision functions used in the literature, our conditions (**D**), (**UC**$'$) as well as the concept of a dominant set are novel to the literature. The analysis of common knowledge of rationality rather than common knowledge of strategies and the result in Theorem 3 are new insights as well.

3 The model

3.1 Synopsis

There is a finite set of agents $M = \{1, ..., m\}$ that face uncertainty represented by a measurable space $(\Omega, \mathcal{F})$, where Ω is the state space and $\mathcal{F}$[20] a sigma algebra of subsets of Ω. Each agent i has private information about the realized state given by his partition $\mathcal{P}^i$. If the realized state of the world is ω, then i knows that a state in $\mathcal{P}^i(\omega)$ has occurred. The set of partitions of all players $\{\mathcal{P}^i\}_{i \in M}$ is commonly known. Time is discrete, $t \in \mathbb{N}$, and at the beginning of the first period agents observe the true cell of their partition.

[20] We assume $\mathcal{F}$ to be equal to the power set of the join of partitions

Players form part of a social network G and in each period t all players simultaneously select an action out of a set choice A. Agents observe the history of actions of their neighbors in G and make inference regarding the realized state of nature based on the true cell of their partition and the history they observe. The information set of an agent i denotes the smallest subset of Ω that i knows to be true. A set $E \subset \Omega$ is true, if the true state ω is contained in E, $\omega \in E$.

3.2 The social network G

The social network is represented by an undirected graph G. A graph is a pair of sets $G = (M, E)$ such that $E \subset [M]^2$. The elements of M are nodes of the graph and the elements of E are the edges of the graph.

Node i in G represents player i. The neighborhood of agent i, N_i contains all agents that are connected to i by an edge in G

$$N_i = \{m \in M : im \in E\}$$

We assume that the social network G is common knowledge among all agents.

An undirected graph has the following symmetric property, if agent j is contained in agent i's neighborhood then agent i is contained in agent j's neighborhood. The common neighborhood of two players i and j is denoted by N_{ij} and consists of the set of agents that are neighbors of both i and j

$$N_{ij} = N_i \cap N_j$$

A graph G is connected if for all nodes i and j there exists a sequence of nodes $k_1, ..., k_l$ where $k_1 = i$ and $k_l = j$ such that $k_{f+1} \in N_{k_f}$ for $f = 1, .., l - 1$. A graph is complete if for all nodes $i, j \in M$ we have $i \in N_j$.

3.3 The common choice correspondence and strategies

Agents select actions based on a common choice correspondence C,

$$C : \mathcal{F} \rightrightarrows A$$

To each information set $I \in \mathcal{F}$ the choice correspondence assigns a subset of A, the set of actions that are optimal given information set I. A pure strategy s_i for player i is a function that assigns to each information set I a single element of $C(I)$[21], $s_i : \mathcal{F} \rightarrow A$ such that $s_i(I) \in C(I)$ for all $I \in \mathcal{F}$. The choice correspondence gives a set of optimal actions for each information set and the strategy s_i selects one of them. We assume that the strategies of all players are common knowledge.

The history of play of all agents at time t is denoted as $h^t = (a^{t-1}, ..., a^1)$, $a^k \in A^m$ for $k = 1, ..., t-1$. The history that player i observes is denoted as h_i^t and the history of actions that both player i and j observe is denoted as h_{ij}^t and contains the history of choices of both i and j.

We require the choice correspondence C to be **union consistent**[22]. C is union consistent if for every collection of disjoint sets $\mathcal{B}$ we have

$$(\mathbf{UC}) \bigcap_{B \in \mathcal{B}} C(B) \neq \emptyset \Rightarrow \bigcap_{B \in \mathcal{B}} C(B) = C\left(\bigcup_{B \in \mathcal{B}} B\right)$$

Union consistency requires that the set of actions that are optimal for all sets in the collection $\mathcal{B}$ is equal to set of actions assigned to the union over all sets in $\mathcal{B}$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{F}$ be collections of disjoint sets such that

$$\bigcup_{B \in \mathcal{B}_1} B = \bigcup_{B \in \mathcal{B}_2} B$$

Please note that union consistency is equivalent to the following condition which we will denote as **pairwise consistency**

$$\textbf{(PC)} \bigcap_{B \in \mathcal{B}_1} C(B) \neq \emptyset \text{ , } \bigcap_{B \in \mathcal{B}_2} C(B) \neq \emptyset \;\Rightarrow \bigcap_{B \in \mathcal{B}_1} C(B) = \bigcap_{B \in \mathcal{B}_2} C(B)$$

This equivalent statement of union consistency will be used to derive our results. For a proof of the equivalence please see the Appendix.

4 The learning process

Agents progressively learn over time through the inferences they make from the history of choices of their neighbors. Compared to a complete network analysis like in Geanakoplos and Polemarchakis (1982), the added difficulty in an incomplete network is that the privately observable component contributing to the information set of a given player from perspective of one of his neighbors j is not

[21] The strategy s_i can additionaly be indexed by time periods allowing agent i to select different actions over time even if his information set remains constant.

[22] The union consistency condition for decision functions was introduced independently by Bacharach (1985) and Cave (1983).

given only by a static component, $\mathcal{P}^i(\omega)$, but contains a dynamic component as well, the history of choices of neighbors of i that are not neighbors of j. In an incomplete network there might be no set of agents such that their history of play is common knowledge among all agents while in a complete network the history of all choices is common knowledge and the privately observable component of the information set of each player consists only of the cell of his partition

We assume fully rational agents that make all possible inferences based on the history they observe. Their inference consists of direct inference regarding the realized partition cell of their neighbors as well as, over time, indirect inference regarding the realized cell of all other agents. Before we introduce the formal learning process we present a simple example for the learning process in an incomplete three agent network to provide some intuition.

4.1 Learning in an incomplete network with three agents and two actions

There are three agents α, β, and γ organized in an incomplete network where β is the center agent observing the history of actions of both α and γ, while α and γ observe only the history of actions of β, $N_\alpha = \{\beta\}$, $N_\beta = \{\alpha, \gamma\}$, and $N_\gamma = \{\beta\}$. The state space consists of four states, $\Omega = \{A, B, C, D\}$, and the partitions are given by $\mathcal{P}^\alpha = \{AB, CD\}$, $\mathcal{P}^\beta = \{ABCD\}$, and $\mathcal{P}^\gamma = \{AC, BD\}$.

A	B
C	D

Figure 1: State space of example 1

Thus agent α observes the rows of the matrix while agent γ observes the columns. Let the common prior probability measure over the state space be uniform. In every period agents have to choose one of two actions, $\mathcal{A} = \{a, d\}$. The utility function is given by

$$u(a, \omega) = \begin{cases} 1 \text{ if } \omega = A \\ 0 \text{ otherwise} \end{cases} \quad u(d, \omega) = \begin{cases} 1 \text{ if } \omega = D \\ 0 \text{ otherwise} \end{cases}$$

Agent α will select action a if $\mathcal{P}^\alpha(\omega) = \{AB\}$ is realized, and action d otherwise. Agent β is indifferent between both actions and agent γ selects action a when observing $\mathcal{P}^\gamma(\omega) = \{AC\}$ and action d otherwise. Let the strategies of all players assign action d in cases of indifference.

Suppose $\omega = A$ is realized. The first period information sets of the agents are $I^1_\alpha(\mathcal{P}^\alpha(A)) = \{AB\}$, $I^1_\beta(\mathcal{P}^\beta(A)) = \{ABCD\}$, and $I^1_\gamma(\mathcal{P}^\gamma(A)) = \{AC\}$. Agents α and γ select action a in the first stage while agent β, being indifferent, selects action d. As agent β has no initial private information and agent α as well as γ observe only the action chosen by β, the second stage information sets of both α and γ are identical to their first period information set, $I^2_\alpha(\mathcal{P}^\alpha(A), h^2_\alpha(A)) = \{AB\}$, and $I^2_\gamma(\mathcal{P}^\gamma(A), h^2_\gamma(A)) = \{AC\}$. Agent β on the other hand observes the first period choices of both agents α and γ and makes inference regarding their realized partition cells. The second stage information set of agent β is given by

$$I^2_\beta(\mathcal{P}^\beta(A), h^2(A)) = \{AB\} \cap \{AC\} = A$$

Prior to making inference based on β's second period choice both other agents have to consider what the possible second stage information sets of agent β are. Among agent α and β it is common knowledge that agent α's information set is given by $\{AB\}$ while the information set of agent γ is private information of agent β, as α did not observe agent γ's first period choice. Thus agent α has to consider the possibility that γ selected action a in the first period as well as the possibility of him having selected action d. It is common knowledge among α and β that agent β's true second stage information set is contained in

$$\mathcal{I}^2_\beta(h^2_{\alpha\beta}(A); A) = \{\{AB\} \cap \{AC\}; \{AB\} \cap \{BD\}\} = \{A, B\}$$

Similar reasoning for agent γ who does not observe agent α's first period action yields the set of possible second stage information sets of β from perspective of agent γ

$$\mathcal{I}^2_\beta(h^2_{\beta\gamma}(A); A) = \{\{AB\} \cap \{AC\}; \{CD\} \cap \{AC\}\} = \{A, C\}$$

Note that agent β selects action a if he observes $\omega = A$ and action d if he observes $\omega = B, C$ according to his strategy. As the information sets of α and γ are identical to their first period information sets they select the same action, $a^2_\alpha = a^2_\gamma = a$. Agent β selects action a as well. Based on the second period action of agent β both agents α and γ can infer his second stage information set and thus learn the true state

$$I^3_\alpha(\mathcal{P}^\alpha(A), h^3_\alpha(A)) = I^3_\beta(\mathcal{P}^\beta(A), h^3(A)) = I^3_\gamma(\mathcal{P}^\gamma(A), h^3_\gamma(A)) = A$$

At the beginning of stage three the true state of the world is common knowledge and all players select action a in each following period.

We chose this simple example to showcase the indirect inference agents make on the realized partition cells of all agents in the network based on the history of choices they observe. The sets $\mathcal{I}^t_i(h^t_{ij}(\omega); \omega)$ play the crucial role in the learning process and we give the formal characterization in the following subsection.

4.2 The general learning process

Let us now present the formal learning process and introduce the notation we will use throughout the paper. At the beginning of stage $t = 1$ the information set of each agent i is given by the true cell of his partition. For period $t = 1$ we have

$$I_i^1(\omega) = \mathcal{P}^i(\omega)$$

It is common knowledge among any pair of agents i and j that the true cell of their meet[23] $\mathcal{P}^i \wedge \mathcal{P}^j(\omega)$ is realized. We will denote the smallest event that is common knowledge to be true among two agents in period $t = 1$ as $CK_{ij}^1(\omega)$

$$CK_{ij}^1(\omega) = \mathcal{P}^i \wedge \mathcal{P}^j(\omega)$$

Let r_i denote a cell in i's partition, $r_i \in \mathcal{P}^i$. The set of possible first stage information sets of agent i based on the common information $CK_{ij}^1(\omega)$ of i and j is given by

$$\mathcal{I}_i^1(\omega; j) = \{r_i \in \mathcal{P}^i : r_i \cap CK_{ij}^1(\omega) = r_i\}$$

Based on his first stage information set and strategy player i selects an action, a_i^1. His neighbor j can make the following inference regarding player i's realized partition cell

$$\mathcal{D}_i^1(a_i^1; \omega) = \left\{r_i \in \mathcal{I}_i^1(\omega; j) : a_i^1 = s_i(r_i)\right\}$$

As the action selected by i has to be consistent with the strategy of agent i, his neighbor j learns from the first period action a_i^1 that the true partition cell of agent i is contained in $\mathcal{D}_i^1(a_i^1, \omega)$. The set $\mathcal{D}_i^1(a_i^1, \omega)$ consists of all cells of agent i's partition that are a subset of $CK_{ij}^1(\omega)$ and induce him to select a_i^1 based on his strategy.

After observing the first period choices of his neighbors and making inference regarding the realized cells of their partitions, player i takes the intersection of the true cell of his partition with the sets $\cup\mathcal{D}_l^1(a_l^1, \omega)$ over all his neighbors $l \in N_i$ to compute his second stage information set. Agent i's information set in period $t = 2$ is denoted as $I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega))$, where

$$I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^1(a_l^1; \omega)$$

[23] The meet of a set of partitions is the finest common coarsening of all partitions.

In period t the information set of agent i is given by

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, h_{il}^{t-1}(\omega); \omega)$$

Any pair of neighbors i and j share a common history $h_{ij}^t(\omega)$ at the outset of period t given by the history of choices up to period t of i and j, as well as the choices of all agents l that are both neighbors of i and j. The set $\mathcal{I}_i^t(h_{ij}^t(\omega); \omega)$ consists of possible information sets of player i in period t that are consistent with the common observables of agent i and j. $\mathcal{I}_i^t(h_{ij}^t(\omega); \omega)$ is common knowledge among them and contains the true information set of player i. We have

$$\mathcal{I}_i^t(h_{ij}^t(\omega); \omega) = \left\{ I_i^t(r_i, \hat{h}_i^t) : \begin{array}{l} \hat{h}_{ij}^t = h_{ij}^t(\omega) \\ \exists I_i^{t-1} \in \mathcal{D}_i^{t-1}(a_i^{t-1}, h_{ij}^{t-1}(\omega); \omega) \text{ s.t. } I_i^t(r_i, \hat{h}_i^t) \subset I_i^{t-1} \end{array} \right\}$$

All information sets of player i that are commonly considered possible among i and j have to be consistent with the common observables in period t, the common history of actions of i and j as well as the inference on agent i's information set in period $t-1$ based on his action in period $t-1$.

Player i chooses an action $a_i^t \in A$ in period t according to his strategy which leads to a refinement of the set $\mathcal{I}_i^t(h_{ij}^t(\omega); \omega)$. We have

$$\mathcal{D}_i^t(a_i^t, h_{ij}^t(\omega); \omega) = \left\{ I_i^t \in \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) : a_i^t = s_i(I_i^t) \right\}$$

where $\mathcal{D}_i^t(a_i^t, h_{ij}^t(\omega); \omega)$ is common knowledge among i and j as it relies only on variables that are commonly known among them. At the beginning of stage $t+1$ agent i processes the information contained in the choices his neighbors made in period t terminating in his private information set in period $t+1$ given by

$$I_i^{t+1}(\mathcal{P}^i(\omega), h_i^{t+1}(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega)$$

The alert reader might have realized that the inference made by agent i regarding the information set of his neighbor j occurs out of a set of commonly known possible information sets $\mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$ even though player i's private information might lead to an exclusion of some elements of $\mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$, all those that have an empty intersection with his information set $I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$.

The information sets of all agents based on the learning process defined above equal the information sets based on an alternative learning process where inferences are made out of a subset of $\mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$ given by excluding all elements that i privately knows not to be realized. The intuition behind the equivalence is that if a certain element is excluded based on private observables but still contained in $\mathcal{D}_j^{t-1}(a_j^{t-1}, h_{ij}^{t-1}(\omega); \omega)$ then the intersection of that element with the true information set of player i in the next stage will still be empty. Thus agent i makes inference on his neighbors only based on common indicators, i.e. a_l^t and h_{il}^t, but when processing them across agents, he takes all his privately available indicators, i.e. h_i^t and $P^i(\omega)$, into account. Let the information sets based on the alternative learning process be denoted by $\check{I}_i^t()$. For a formal definition of the alternative learning process please see the Appendix.

Proposition 1 *For every player $i \in M$, for every period $t \in \mathbb{N}$ and every state of the world ω*

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \check{I}_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$$

For the proof please see the Appendix. Let us next state a very important property of the rational learning process in networks.

Proposition 2 *For all $i \in M$ and all periods t, $\left(\mathcal{P}^i(\omega), h_i^t(\omega)\right) \neq \left(\mathcal{P}^i(\omega'), \hat{h}_i^t(\omega')\right)$ implies*

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) \cap I_i^t(\mathcal{P}^i(\omega'), \hat{h}_i^t(\omega')) = \emptyset$$

Proposition 2 states that any two information sets of player i that rely on unequal observables, the true cell of i's partition or the history that i observes, have an empty intersection. This directly implies that the set of possible information sets of agent i, given the common observables of i and his neighbors j, is a collection of disjoint sets. This feature of $\mathcal{I}_i^t(h_{ij}^t(\omega); \omega)$ enables us to apply the union consistency condition.

Due to the heavy notation we provide a table on the last page with definitions and descriptions of all relevant sets used throughout the paper.

5 The finite learning case

We now present the first main result. We show that all actions chosen by agent i once learning ends are optimal for his neighbor j and the information set that agent j observes[24].

[24] Within our framework only pure strategies are considered. Our result is not robust against expanding the strategy space to mixed strategies in the expected utility framework. The optimality of choices across neighbors once learning ends can fail if mixed strategies are played.

Theorem 1 *If the choice correspondence C is union consistent and for a given state ω there exists a finite t' such that for all $t \geq t'$*

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))$$

and

$$\cup \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) = \cup \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$$

for all $i \in M$ and $j \in N_i$, then

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)))$$

for all $t \geq t'$

Theorem 1 states a local indifference result, the indifference among actions chosen once learning ends among pairs of neighbors. The following two Lemmas together with Proposition 2 form the main components of the proof.

Lemma 1 *For a given state ω, and $i \in M$, $j \in N_i$, if*

$$\cup \mathcal{I}_i^{t'+1}(h_{ij}^{t'+1}(\omega); \omega) = \cup \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$$

then for $a_i^{t\prime} = s_i(I_i^{t'}((P^i(\omega), h^{t'}(\omega))$

$$\mathcal{D}_i^{t'}(a_i^{t'}, h_{ij}^{t'}(\omega); \omega) = \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$$

Lemma 2 *For all states of the world ω, time periods t and agents $i \in M$, $j \in N_i$*

$$\cup \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) = \cup \mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$$

The proofs of the Lemmas are provided in the Appendix. We now present the proof of Theorem 1.

Proof. Take any pair of neighbors $i \in M$, $j \in N_i$. As $\cup \mathcal{I}_i^{t'+1}(h_{ij}^{t'+1}(\omega); \omega) = \cup \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$, we have by Lemma 1

$$s_i\left(I_i^{t'}(\mathcal{P}^i(\omega), h^{t'}(\omega))\right) = a_i^{t'} \in C(I_i^{t'}) \ \forall I_i^{t'} \in \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$$

Similarly for agent j

$$s_j\left(I_j^{t'}(\mathcal{P}^j(\omega), h^{t'}(\omega))\right) = a_j^{t'} \in C(I_j^{t'})\ \forall I_j^{t'} \in \mathcal{I}_j^{t'}(h_{ij}^{t'}(\omega);\omega)$$

Proposition 2 implies that $\mathcal{I}_i^{t'}(h_{ij}^{t'};\omega)$ is a collection of disjoint sets. By Lemma 2 we have

$$\cup\mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega);\omega) = \cup\mathcal{I}_j^{t'}(h_{ij}^{t'}(\omega);\omega)$$

Union consistency of the choice function implies pairwise consistency which gives

$$s_j(I_j^{t'}(\mathcal{P}^j(\omega), h_j^{t'}(\omega))) = a_j^{t'} \in C(I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega)))$$

and

$$s_i\left(I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))\right) = a_i^{t'} \in C(I_j^{t'}(\mathcal{P}^j(\omega), h_j^{t'}(\omega)))$$

As individual learning ends in t' the same is true for any later period $t > t'$ ■

There are a number of corollary results to Theorem 1. The first Corollary considers the special case where there is a common choice function $C : \mathcal{F} \to A$ that assigns a single element of A to each information set in $\mathcal{F}$. It is intuitively directly apparent, that as only a single action is optimal for each information set any pair of neighbors have to choose the same action once learning ends. If the network is connected and every pair of neighbors select the same action, then all agents select the same action. Thus under a common choice function perfect uniformity of choice occurs once learning ceases. Formally the result is stated in Corollary 1 below.

Corollary 1 *If G is connected, the choice function $C : \mathcal{F} \to A$ is union consistent and for a given state ω there exists a finite t' such that for all $t \geq t'$ we have*

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))$$

and

$$\cup\mathcal{I}_i^t(h_{ij}^t(\omega);\omega) = \cup\mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega);\omega)$$

for all $i \in M$ and $j \in N_i$, then there exists an $a^* \in A$ such that for all agents $i \in M$ and all periods $t \geq t'$

$$a^* = C(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)))$$

Proof. C being a function and Theorem 1 imply

$$\begin{aligned} a_i^{t'} &= a_j^{t'} = C(I_i^{t'}) \\ a_j^{t'} &= a_i^{t'} = C(I_j^{t'}) \end{aligned}$$

for all pair of neighbors i, j. G being connected implies that for every pair of agents $i, j \in M$ there exists a sequence of agents $k_1, ..., k_l$ where $k_1 = i$ and $k_l = j$ such that $k_{f+1} \in N_{k_f}$ for $f = 1, .., l-1$, where

$$C(I_i^{t'}) = C(I_{k_1}^{t'}) = C(I_{k_2}^{t'}) = ... = C(I_{k_l}^{t'}) = C(I_j^{t'})$$

Thus there exists an $a^* \in A$ such that

$$a^* = a_i^{t'} = C(I_i^{t'})$$

for all agents $i \in M$. As the information set remains constant from period t' on, i.e. $I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega)) = I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$ for all $t > t'$, we have

$$a^* = a_i^t = C(I_i^t)$$

for all $t > t'$ ■

Corollary 2 states a sufficient condition for learning to end in finite time. If the partitions of all agents are finite, then learning ends in finite time.

Corollary 2 *If the partitions $\mathcal{P}^i$ are finite for all $i \in M$ and the choice correspondence C is union consistent, then there exists a finite $t'(\omega)$ for each state of the world such that for all $t \geq t'(\omega)$ we have*

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)))$$

for all $i \in M$ and $j \in N_i$.

The finiteness of partitions implies that the join is finite and as a consequence the power set of the join is finite as well. As an information set consists of intersections of subsets of the partitions of all players, it is contained in the power set of the join. For individual learning not to end in finite time there would have to exist an infinite sequence of information sets such that each set is a strict subset of his predecessor in the sequence contradicting the fact that the power set of the join is finite.

Corollaries 1 and 2 have an interesting implication for the "Agreeing to Disagree" literature. Geanakoplos and Polemarchakis (1982) have shown that two agents who repeatedly communicate back and forth their updated posterior belief of an event E, will converge on one posterior in finite time, if partitions are finite and agents share a common prior. Our two Corollaries imply a generalization of their result to finite players in a connected social network. In order to show this, we need to add more structure to our model. Let the common choice function

$$C : \mathcal{F} \to [0, 1]$$

be defined as

$$C(I) = \frac{p\,(I \cap E)}{p(I)} \quad \forall I \in \mathcal{F}$$

where p is a common prior probability measure. If the thus defined choice function is union consistent[25], then Corollary 1 and Corollary 2 imply that in a connected social network the posterior beliefs of all agents converge in finite time if the partitions of all agents are finite.

In the following subsections we provide two examples that highlight features of the Bayesian learning process in connected networks that are novel to the literature. The first example concerns a setting where agents share a common prior and announce their posterior belief of an uncertain event in each period. We consider the following question: which network structure leads to optimal information aggregation? A network aggregates individual information in an optimal manner if the posterior beliefs of all agents converge to the pooled information posterior[26]. On first thought one's intuition is that a complete network where all agents observe the announcements of everybody should dominate incomplete networks in terms of quality of information aggregation. We show by example that this is generally not the case.

Our second example concerns the set of actions chosen in the network once learning ends. For a choice correspondence, Theorem 1 implies only local indifference across neighbors, not global indifference across all members of the social network. We find that in networks with a diameter larger than one and with an underlying choice correspondence the global consensus and optimality of actions chosen once learning ends generally fails to hold for all pairs of agents. This casts new light on the literature on consensus and knowledge which just considers decision functions[27].

[25] In the supplementary Appendix we show that the finiteness of partitions together with the countable additivity property of probability measures implies union consistency for the above defined common choice function.

[26] The posterior of event E conditioning on the true cell of the join.

[27] See for example Krasucki (1996) and Menager (2006) who prove global consensus across all agents under pairwise communication and decision functions.

5.1 An example for quality of information aggregation in complete versus incomplete networks

There are three players $M = \{1, 2, 3\}$, and the state space Ω is given by a compact subset of $\mathbb{R}^2$. We assume a uniform probability measure p over Ω. For ease of understanding please see the graphical representation of the state space in Figure 2 below.

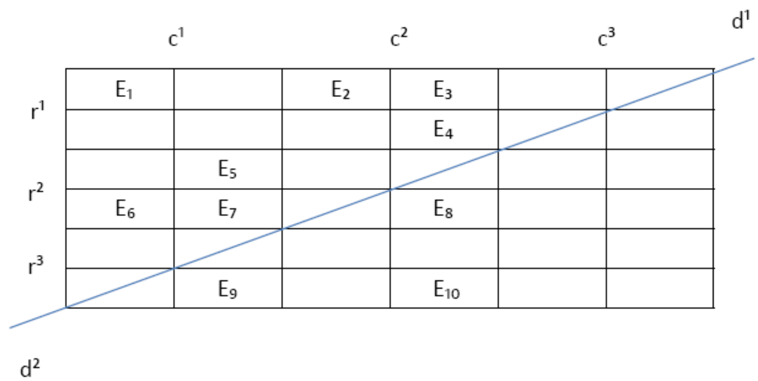

Figure 2: The state space Ω of example 2

Player 1 partitions Ω horizontally, the cells of his partitions are given by the union of the first two rows, r^1, the union of row three and four, r^2, and the remaining two rows, r^3, $\mathcal{P}^1 = \{r^1, r^2, r^3\}$ Player 2 partitions the state space diagonally, the cells of his partition are the upper triangle, d^1, and the lower triangle d^2, $\mathcal{P}^2 = \{d^1, d^2\}$. Player 3 partitions the state space vertically. The cells of his partition are given by the union of the first two columns, c^1, the union of columns three and four, c^2, and the union of the remaining two columns, c^3, $\mathcal{P}^3 = \{c^1, c^2, c^3\}$.

The event E whose likelihood we are concerned with is given by $E = \bigcup_{i=1}^{10} E_i$. Suppose that E_1 is realized. Let us consider the complete network case first, $N_i = \{j, k\}$ for $i, j, k \in M$. The players observe r^1, d^1 and c^1 respectively which leads to first stage announcements of $q_1^1 = \frac{1}{3}$, $q_2^1 = \frac{7}{18}$ and $q_3^1 = \frac{5}{12}$. At the beginning of period $t = 2$ the information sets of the players are given by

$$\begin{aligned} I_1^2(r^1, h^2(\omega)) &= r^1 \cap \left(c^1 \cup c^2\right) \\ I_2^2(d^1, h^2(\omega)) &= \left(r^1 \cup r^2\right) \cap \left(c^1 \cup c^2\right) \cap d^1 \\ I_3^2(c^1, h^2(\omega)) &= \left(r^1 \cup r^2\right) \cap c^1 \end{aligned}$$

Leading to second stage announcements of $q_1^2 = \frac{1}{2}$, $q_2^2 = \frac{1}{2}$ and $q_3^2 = \frac{1}{2}$. As

$$q_1^2(I_1^2(r^1, h^2(\omega))) = q_1^2(I_1^2(r^2, h^2(\omega))) = \frac{1}{2}$$

and

$$q_3^2(I_3^2(s^1, h^2(\omega))) = q_3^2(I_3^2(s^2, h^2(\omega))) = \frac{1}{2}$$

no further information is aggregated. No player learns the cell of the join of their partitions and the final common posterior equals $q = \frac{1}{2}$ which is unequal to the pooled information posterior of $\frac{1}{4}$.

Next let us analyze what occurs in the same example but with an incomplete network. Let 2 be the center player in a star network, i.e. $N_1 = N_3 = \{2\}$ and $N_2 = \{1, 3\}$. Again suppose that $\omega \in E_1$ is realized. The first stage information sets and probability announcements mirror the complete network case leading to $q_1^1 = \frac{1}{3}$, $q_2^1 = \frac{7}{18}$ and $q_3^1 = \frac{5}{12}$.

At the beginning of stage $t = 2$ the branch players have less observables than in the complete case, as they only observe the announcement of the center player. The second stage information sets are given by

$$\begin{aligned}
I_1^2(r^1, h_1^2(\omega)) &= r^1 \cap d^1 \\
I_2^2(d^1(\omega), h^2(\omega)) &= \left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right) \\
I_3^2(c^1, h_3^2(\omega)) &= d^1 \cap c^1
\end{aligned}$$

There is common knowledge among player 1 and 2 that player 2 faces an information set within $\mathcal{I}_2^2(h_{12}^2(\omega); \omega)$ where

$$\mathcal{I}_2^2(h_{12}^2(\omega); \omega) = \left\{\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right), \left(r^1 \cup r^2\right) \cap d^1 \cap c^3\right\}$$

The posterior probabilities depending on the information set of agent 2 are

$$\begin{aligned}
q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= \frac{1}{2} \\
q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap c^3\right) &= 0
\end{aligned}$$

Thus at the beginning of stage three agent 1 learned the second stage information set of agent 2. Based on the common observables of agents 1 and 2 the information set of agent 1 is an element of $\mathcal{I}_1^2(h_{12}^2(\omega); \omega)$ where

$$\mathcal{I}_1^2(h_{12}^2(\omega); \omega) = \left\{r^1 \cap d^1, r^2 \cap d^1\right\}$$

leading to the following probability announcements

$$\begin{aligned} q_1^2(r^1 \cap d^1) &= \frac{2}{5} \\ q_1^2(r^2 \cap d^1) &= \frac{1}{2} \end{aligned}$$

Thus player 2 learns the true cell of agent 1 through his second stage announcements. Based on the common observables of agent 2 and 3 the set of possible information sets of agent 1 from perspective of agent 2 is either

$$\mathcal{I}_1^2(h_{12}^2(\omega);\omega) = \left\{r^1 \cap d^1, r^2 \cap d^1\right\}$$

if agent 1 announced a probability of $\frac{1}{3}$ in the first stage. Otherwise the information set of agent 1 is commonly known among 1 and 2 to be equal to

$$I_1^2(r^3, h_1^2(\omega)) = r^3 \cap d^1$$

For all three possible information sets agent 1 announces a different probability as

$$q_1^2(r^3 \cap d^1) = 0$$

Thus it is common knowledge among all agents that player 2 learned the true cell of agent 1 at the outset of stage 3. Based on the common observables of agents 2 and 3 it is common knowledge that agent 2's second stage information set is contained in

$$\mathcal{I}_2^2(h_{23}^2(\omega);\omega) = \left\{\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right), r^3 \cap d^1 \cap \left(c^1 \cup c^2\right)\right\}$$

leading to the following second stage announcements of agent 2

$$\begin{aligned} q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= \frac{1}{2} \\ q_2^2\left(r^3 \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= 0 \end{aligned}$$

Thus agent 3 learns the second stage information set of agent 2 through his second stage announcement. Based on the common observables of player 2 and player 3 it is common knowledge that the second stage information set of agent 3 is contained in

$$\mathcal{I}_3^2(h_{23}^2(\omega);\omega) = \left\{d^1 \cap c^1, d^1 \cap c^2\right\}$$

leading to the following second stage announcements of agent 2

$$\begin{aligned} q_2^2(d^1 \cap c^1) &= \frac{2}{5} \\ q_1^2(d^1 \cap c^2) &= \frac{1}{2} \end{aligned}$$

Thus player 2 learns the true cell of agent 3 through his second stage announcement. Based on the common observables of agent 1 and 2 it is common knowledge that the set of possible information sets of agent 3 from perspective of agent 2 is equal to

$$\mathcal{I}_3^2(h_{23}^2(\omega);\omega) = \left\{d^1 \cap c^1, d^1 \cap c^2\right\}$$

if agent 3 announces a posterior of $q_3^1 = \frac{5}{12}$ in the first period. If agent 3 announced a posterior of $q_3^1 = 0$ in the first stage it is common knowledge among 2 and 3 that the true information set of agent 3 equals

$$I_3^2(c^3, h_3^2(\omega)) = d^1 \cap c^3$$

leading to

$$q_2^2(d^1 \cap c^3) = 0$$

All three possible second stage information sets of agent 3 lead to different posterior announcements. Thus it is common knowledge among all agents that agent 2 knows the true cell of agent 3 at the beginning of stage three if the true cell of agent 2 is equal to d^1.

Based on the considerations above it is common knowledge among all agents that agent 2 knows the true cell of the partition of agent 1 as well as of agent 3 at the beginning of stage three[28]. Agents announce their second stage posteriors $q_1^2 = \frac{2}{5}$, $q_2^2 = \frac{1}{2}$ and $q_3^2 = \frac{2}{5}$ leading to the following third stage information sets

$$\begin{aligned} I_1^3(r^1, h_1^3(\omega)) &= r^1 \cap d^1 \cap \left(c^1 \cup c^2\right) \\ I_2^3(d^1, h^3(\omega)) &= r^1 \cap c^1 \\ I_3^3(c^1, h_3^3(\omega)) &= \left(r^1 \cup r^2\right) \cap d^1 \cap c^1 \end{aligned}$$

and announcements $q^3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$. It is common knowledge among all players that player 2 knows the true cell of the join at the beginning of stage 3. Thus after observing the third period announcement of agent 2 the other two agents will announce the third period posterior of agent 2 from period $t = 4$ on forward. The third period announcement of agent 2 is given by

$$q_2^3(r^1 \cap c^1) = \frac{1}{4}$$

[28] For $\omega \in d^1$

which equals the pooled information posterior. The third period announcements of agent 1 and 3 are $q_1^3 = \frac{1}{2}$ and $q_3^3 = \frac{1}{2}$ respectively. For all periods $t \geq 4$ we have

$$q_1^t(r^1 \cap c^1) = q_2^t(r^1 \cap c^1) = q_3^t(r^1 \cap c^1) = \frac{1}{4}$$

Thus for $\omega \in E_1$ the incomplete network aggregates information optimally while the complete network does not[29].

5.2 An example for failure of global indifference

We consider a network consisting of three agents, α, β and γ. They are organized in a line network, $N_\alpha = \{\beta\}$, $N_\beta = \{\alpha, \gamma\}$, and $N_\gamma = \{\beta\}$. The state space consists of nine states $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the probability measure p over Ω is uniform. Agents α and γ have private information while agent β has not. The partitions are given by $\mathcal{P}^\alpha = \{123, 456, 789\}$, $\mathcal{P}^\beta = \{123456789\}$, and $\mathcal{P}^\gamma = \{147, 258, 369\}$. The state space is graphically displayed in figure 3

1	2	3
4	5	6
7	8	9

Figure 3: The state space Ω in example 3

Agent α observes the rows of the matrix while agent γ observes the columns. In every period agents have to choose one of three actions, $A = \{a, b, c\}$. The utility function is given by

$$u(a, \omega) = \begin{cases} 1 \text{ if } \omega = 1, 2, 4, 5, 6, 8 \\ 0 \text{ otherwise} \end{cases} \quad u(b, \omega) = \begin{cases} 1 \text{ if } \omega = 1, 2, 4, 5, 6, 8, 9 \\ 0 \text{ otherwise} \end{cases}$$

$$u(c, \omega) = \begin{cases} 1 \text{ if } \omega = 2, 4, 5, 6, 8, 9 \\ 0 \text{ otherwise} \end{cases}$$

As agents maximize their expected utility in each period conditioning on their information, agent α selects either action a or b when observing $\mathcal{P}^\alpha(\omega) = \{123\}$, action a, b, or c when observing $\mathcal{P}^\alpha(\omega) = \{456\}$, and action b or c when $\mathcal{P}^\alpha(\omega) = \{789\}$ is realized. Agent β does not have private information in the first period and thus selects action b. Agent γ on the other hand selects action a or b if $\mathcal{P}^\gamma(\omega) = \{147\}$ is realized, action a, b or c if he observes $\mathcal{P}^\gamma(\omega) = \{258\}$, and action b or c if he observes $\mathcal{P}^\gamma(\omega) = \{369\}$.

[29] For all states $\omega \in (r^1 \cup r^2) \cap d^1 \cap (c^1 \cup c^2)$ this holds true.

Let us assume the following strategies for the players: agents α and γ select a whenever a is optimal and action c whenever c is optimal and a is not, while agent β selects action b whenever it is optimal[30].

Suppose $\omega = 3$ is realized. Agent α selects action a according to his strategy, agent β selects action b, while agent γ selects action c according to his strategy. At the outset of stage two the information sets of α and γ are identical to their information sets in period one as agent β had no initial private information to reveal through his first period action.

Agent β on the other hand makes inference through the actions chosen by agents α and γ and his second stage information set is $I^2_\beta(\mathcal{P}^\beta(3), h^2_\beta(3)) = \{36\}$. Agent α knows his own first period choice but not the action chosen by agent γ. It is common knowledge among α and β that β's second stage information set is contained in

$$\mathcal{I}^2_\beta(h^2_{\alpha\beta}(3); \omega = 3) = \{1245, 36\}$$

For both information sets in $\mathcal{I}^2_\beta(h^2_{\alpha\beta}(3); \omega = 3)$ agent β selects action b according to his strategy implying that the information set of agent α in the third stage to be equal to α's information set in the second stage. The set of possible information sets of agent β based on the common observables of β and γ is given by

$$\mathcal{I}^2_\beta(h^2_{\beta\gamma}(3); \omega = 3) = \{36, 9\}$$

For both information sets in $\mathcal{I}^2_\beta(h^2_{\beta\gamma}(3); \omega = 3)$ agent β selects action b following his strategy and thus the third stage information set of agent γ is identical to her second stage information set. The set of possible second stage information sets of player α based on the common observables of α and β is given by

$$\mathcal{I}^2_\alpha(h^2_{\alpha\beta}(3); \omega = 3) = \{123, 456\}$$

and agent α selects action a for both sets in $\mathcal{I}^2_\alpha(h^2_{\alpha\beta}(3); \omega = 3)$ yielding no further insight for agent β. The set of possible second stage information sets of agent γ based on common observables among γ and β is given by

$$\mathcal{I}^2_\gamma(h^2_{\beta\gamma}(3); \omega = 3) = \{369\}$$

thus agent γ can not transmit further information. In this example learning ends in the second stage and agent α selects action a in every period while agent γ selects action c in each period while action a is not optimal for agent γ and action c is not optimal for agent α.

The example shows that the result of local indifference across agents does not imply global indifference, if we assume an underlying choice correspondence. This is an interesting insight in

[30] This does not provide a complete description of the strategy of agent β but is all we require in the given example.

relation to the usual uniformity result in the sequential social learning literature which generally assumes common priors and common utility functions. We show that in a setting where agents select actions repeatedly the result of uniformity of actions once learning ends does not generally hold. On a local level, among neighbors, any action that an agent selects once learning ends is optimal for his neighbor, but on a global level, across all agents in the network, this property can fail.

6 An asymptotic result

The result of indifference among neighbors between their equilibrium choices has been shown to hold once learning ends. So far we have established that learning ends in finite time in the case of finite partitions. Next we will present a limit result. For that we need to introduce the concept of a dominant set under a choice correspondence C

Definition 1 $B \in \mathcal{F}$ *is a* ***dominant set under*** C *if for all sequences* $\{B^t\}_{t=1}^{\infty}$ *in* $\mathcal{F}$ *such that* $B^{t+1} \subset B^t$ *and* $\bigcap_{t=1}^{\infty} B^t = B$ *there exists a finite* t^* *and*

$$C(B^t) \subset C(B)$$

for all $t \geq t^*$.

A set B is dominant under choice correspondence C if for every sequence of sets that converge to B there exists a finite time period such that from that period onward every action which is optimal for information set B^t is also optimal given the limit information set. In the expected utility framework with a common prior, bounded and measurable common utility function, and finite actions every set with positive probability is a dominant set. Please see Proposition 2 and the corresponding proof in the supplementary Appendix.

We need to impose one additional condition on the choice correspondence C.

(**D**) If $B \in \mathcal{F}$ is a dominant set under C, then every set $E \in \mathcal{F}$ that contains B is a dominant set under C

In the expected utility setting described above property (**D**) holds as every set that contains a set of positive probability has positive probability. Before stating the theorem let us define the limit information set $I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))$ as

$$I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)) = \bigcap_{t=1}^{\infty} I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$$

Remember that $\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$ denotes the set of possible information sets of agent i in period t based on the common observables of i and his neighbor j. $\left\{\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)\right\}_{t\in\mathbb{N}}$ denotes a sequence of collections of sets. Let the limit of the sequence be denoted as $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ and defined as

$$\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega) = \left\{I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) : \; I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \in \mathcal{I}_i^t(h_{ij}^t(\omega);\omega) \;\forall t\right\}$$

Note that the limit set $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ is not empty as the true information set of player i, $I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$ is contained in $\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$ for every period t.

Theorem 2 *If all elements of the join* $\bigvee_{j\in M} \mathcal{P}^j$ *are dominant sets under* C*, the choice correspondence is union consistent and complies with condition (D), and the action set* A *is finite, then for every* $\omega \in \Omega$ *and for any pair of neighbors* $i \in M$*,* $j \in N_i$ *we have the following two conclusions:*

1. *There exist non-empty sets* $A_{ij}^\infty(\omega) \subset A$ *such that*

$$A_{ij}^\infty(\omega) \subset C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))) \cap C(I_j^\infty(\mathcal{P}^j(\omega), h_j^\infty(\omega)))$$

2. *There exists a finite* $t^*(\omega)$ *such that for all* $t \geq t^*(\omega)$

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

Theorem 2 says that if the stated conditions are satisfied, then for each pair of neighbors in the network there exists a non-empty set of actions that will be optimal in the limit information set for both agents. Furthermore for each state of the world there exists a finite time $t^*(\omega)$ such that any action that agent i selects from that period onward will be optimal for all his neighbors j in their limit information set. In other words, every action agent i selects from a finite moment in time onward, will be eventually optimal for all neighbors of i.

The proof appears in the Appendix. The following Lemmas are used[31].

[31] For the proofs of Lemma 3-5 please see the Appendix.

Lemma 3 *$I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega'))$, $I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega'')) \in \mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ implies*

$$I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \cap I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega'')) = \emptyset$$

Lemma 4 *If $I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \in \mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ then for each period t*

$$s_i\left(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))\right) = s_i\left(I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))\right)$$

Lemma 5 *For every state ω and all $i \in M$, $j \in N_i$*

$$\cup\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega) = \cup\mathcal{I}_j^\infty(h_{ij}^\infty(\omega);\omega)$$

Let us outline the steps of the proof. First we establish that every limit information set $I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))$ is a dominant set relying on the assumption that every cell of the join is a dominant set together with condition (**D**). As the action set is finite there exists a set of actions $A_i^\infty(\omega)$ for player i in state ω that are selected infinitely often by i. We use the property of the limit set being dominant to show that every action in $A_i^\infty(\omega)$ is optimal for agent i in his limit information set. By Lemma 4, for any pair of information sets $I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))$, $I_i^t(\mathcal{P}^i(\omega''), h_i^t(\omega''))$ that converge to limit information sets in $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$, the strategy assigns the same actions for all periods t, $s_i\left(I_i^t(\mathcal{P}^i(\omega''), h_i^t(\omega''))\right) = s_i\left(I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))\right)$, which implies that all actions in $A_i^\infty(\omega)$ are optimal for all limit information sets in $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$. Lemma 3 shows that $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ is a collection of disjoint sets. We apply the same reasoning for neighbor j of agent i and use Lemma 5 in combination with the union consistency property of the choice correspondence to establish that there exists a non-empty set of actions $A_i^\infty(\omega) \cup A_j^\infty(\omega) = A_{ij}^\infty(\omega)$ that are optimal in the limit information set for both agent i and his neighbor j.

In order to establish the claim of the existence of a finite time for each stage of the world such that any action that an agent selects from that time onward is optimal for all his neighbors, we exploit the finiteness of the action space to show that there exists a finite time such that from that time onward each agent selects only those actions that he selects infinitely often. By the first result of Theorem 2 those are optimal for all his neighbors.

7 Common knowledge of rationality versus common knowledge of strategies

So far we have assumed that the strategies of all agents are common knowledge, i.e. there is common knowledge which action any agent selects out of the set of actions assigned by the choice correspondence for every information set in $\mathcal{F}$. In this section we will make the assumption that only rationality of agents is commonly known, i.e. it is common knowledge that agents select an option according to the common choice correspondence. We will show that whether Theorem 1 holds in this setting depends on the underlying network structure.

Let us start by giving some intuition for why Theorem 1 might fail in this setting. Take any pair of neighbors i, j and consider the inference that player j makes based on player i's first period choice. We have

$$\mathcal{D}_i(a_i) = \left\{r_i \in \mathcal{P}^i : \ a_i \in C(r_i)\right\}$$

Whenever there is a cell in player i's partition to which the choice correspondence assigns more than a single action, we have that the collection of sets $\left\{D_i^1(a_i)\right\}_{a_i \in A}$ for a_i such that $D_i^1(a_i) \neq \emptyset$ does not constitute a partition of Ω because some elements have a non-empty intersection. Whenever an agent is making inference on the information set of his neighbor this can cause difficulties. Next we introduce the learning process in a complete network under common knowledge of rationality. We then state a positive result of global optimality of actions chosen in complete networks once learning ends, before we provide an example where the optimality across neighbors fails in an incomplete network.

7.1 The learning process in a complete network under common knowledge of rationality

In period $t = 1$ the smallest event that is common knowledge to be realized among all players is the true cell of the meet of all players

$$CK^1(\omega) = \bigwedge_{i \in M} \mathcal{P}^i(\omega)$$

For a given first period choice a_i^1 of player i it becomes common knowledge among all agents that the realized cell of agent i is contained in set $\mathcal{D}_i^1(a_i^1;\omega)$ where

$$\mathcal{D}_i^1(a_i^1;\omega) = \left\{r_i \in \mathcal{P}^i : a_i^1 \in C(r_i), \ r_i \cap CK^1(\omega) = r_i\right\}$$

When observing a_i^1 it is common knowledge among all agents that player i's realized partition cell has the property that a_i^1 is optimal given the realized cell. At the beginning of period $t = 2$ it is common knowledge among all agents that $CK^2(h^2;\omega)$ is realized where

$$CK^2(h^2;\omega) = \bigcap_{i \in M} \cup \mathcal{D}_i^1(a_i^1;\omega)$$

Agent i makes inference regarding the realized cells of all agents based on their first period actions and processes the inferences by taking the intersection of the cell of his partition with the intersection of the set of possible information sets across all other agents. The information set of player i is given by

$$I_i^2(\mathcal{P}^i(\omega), h^2) = \mathcal{P}^i(\omega) \cap CK^2(h^2; \omega)$$

The common inference that is made through the second period choice of player i is given by set $\mathcal{D}_i^2(a_i^2, h^2; \omega)$ where

$$\mathcal{D}_i^2(a_i^2, h^2; \omega) = \left\{ r_i \in \mathcal{P}^i : a_i^2 \in C(r_i \cap CK^2(h^2; \omega)) \right\}$$

At the outset of period t we have

$$CK^t(h^t; \omega) = \bigcap_{i \in M} \cup \mathcal{D}_i^{t-1}(a_i^{t-1}, h^{t-1}; \omega)$$

The private information of player i at time t is given by

$$I_i^t(\mathcal{P}^i(\omega), h^t) = \mathcal{P}^i(\omega) \cap CK^t(h^t; \omega)$$

and the inference agents make regarding agent i's cell based on his choice in period t is given by $\mathcal{D}_i^t(a_i^t, h^t; \omega)$

$$\mathcal{D}_i^t(a_i^t, h^t; \omega) = \left\{ r_i \in \mathcal{P}^i : a_i^t \in C(r_i \cap CK^t(h^t; \omega)) \right\}$$

The information set of agent i in period $t+1$ depending on the choices of all other agents in period t is given by

$$I_i^t(\mathcal{P}^i(\omega), h^t) = \mathcal{P}^i(\omega) \cap \bigcap_{i \in M} \cup \mathcal{D}_i^t(a_i^t, h^t; \omega)$$

The information set of an agent is given by the intersection of his true partition cell with set $CK^t(h^t; \omega))$ which is common knowledge among all agents.

7.2 Theorem 3

Theorem 3 provides a positive result in the setting of common knowledge of rationality.

Theorem 3 *If G is complete and the choice correspondence union consistent, then common knowledge of rationality is sufficient for the following result: If there exists a time $t'(\omega)$ such that $CK^t(h^t; \omega) = CK^{t'}(h^{t'}; \omega)$ for all $t \geq t'$ then private learning ends for all players and*

$$a_j^t \in C(I_i^t(\mathcal{P}^i(\omega), h^t))$$

for all $i, j \in M$ *and* $t \geq t'$.

The Theorem states that if common learning ends in stage t', then from that stage on, every action chosen in the network is optimal for all agents. The driving force of the result is that the history of choices of all agents is common knowledge leading to a set of states which is common knowledge among all players in each period. The information set of each agent consists of the intersection of the true cell of his partition with the set $CK^t(h^t;\omega)$.

The proof appears in the Appendix. It relies on the fact that the information set of each player i for any period t is the intersection of the true cell of his partition with the smallest set of states that is commonly known to contain the true state, $CK^t(h^t;\omega)$. Thus the set of information sets of player i that are commonly known to be feasible at time t partition the set $CK^t(h^t;\dot{\omega})$. The fact that learning ends in period t' implies that there exists an action $a_i^{t'}$ that is optimal for all possible information sets of player i. The union consistency property of the choice correspondence then implies the result.

7.3 An example for failure of indifference across neighbors

Let us consider an example of an incomplete network with three agents where there exists a pair of neighbors i, j such that i and j select a distinct action once learning ends and the action chosen by i is not optimal for j and vice versa.

There are three agents α, β and γ. Agents α and γ are neighbors of β but not of each other, i.e. $N_\alpha = \beta$, $N_\beta = \{\alpha, \gamma\}$, $N_\gamma = \beta$. The state space is finite and given by $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The σ-algebra $\mathcal{F}$ equals the power set of Ω. For ease of understanding we present Ω graphically by the matrix below

1	2	3
4	5	6
7	8	9

Figure 4: The state space of example 4

Agent α partitions the state space horizontally observing the rows of the matrix, $\mathcal{P}^\alpha = \{123, 456, 789\}$. Agent β has no private information, $\mathcal{P}^\beta = \{123456789\}$, while agent γ partitions the state space vertically observing the columns of the matrix, $\mathcal{P}^{\gamma} = \{147, 258, 369\}$. The choice set A consists of two elements $A = \{b, c\}$. The choice correspondence C prescribes the following choices for the possible first period information sets

$$\begin{aligned} C(123) &= C(147) = b \\ C(456) &= C(258) = \{b, c\} \\ C(\Omega) &= C(369) = C(789) = c \end{aligned}$$

Let the state $\omega = 3$ be realized. For $\omega = 3$ player α observes the first row and selects b, player β selects c, and player γ observes the third column and selects c. The information sets at the beginning of the second period for players are $I_{\alpha}^{2}(\mathcal{P}^{\alpha}(3), h_{\alpha}^{2}) = \{123\}$, $I_{\beta}^{2}(\mathcal{P}^{\beta}(3), h_{\beta}^{2}) = \{2356\}$ and $I_{\gamma}^{2}(\mathcal{P}^{\gamma}(3), h_{\gamma}^{2}) = \{369\}$.

Neither player α nor γ can make any inference based on β's first period choice, but player β makes inference based on the choice of the other two players. The second stage information sets that are common knowledge to be possible for agent i among agents i and j based on the common observables of agents i and j are

$$\begin{aligned} \mathcal{I}_{\beta}^{2}(h_{\alpha\beta}^{2}; 3) &= \{1245, 2356\} \\ \mathcal{I}_{\alpha}^{2}(h_{\alpha\beta}^{2}; 3) &= \{123, 456\} \\ \mathcal{I}_{\beta}^{2}(h_{\beta\gamma}^{2}; 3) &= \{2356, 5689\} \\ \mathcal{I}_{\gamma}^{2}(h_{\beta\gamma}^{2}; 3) &= \{258, 369\} \end{aligned}$$

Let the choice correspondence C be such that

$$C(1245) = C(2356) = C(5689) = C(4578) = c$$

As the information sets of agent α and γ in period $t = 2$ are identical as in period $t = 1$, they select the same action as in period one, $a_{\alpha}^{2} = b, a_{\gamma}^{2} = c$. Agent β selects action c as it is the single action prescribed by his choice correspondence for his second stage information set $I_{\beta}^{2}(\mathcal{P}^{\beta}(3), h_{\beta}^{2}) = \{2356\}$. In order to determine the learning process from period $t = 3$ on the branch player i has to consider the sets $\mathcal{I}_{j}^{2}(h_{j\beta}^{2}; \omega)$ and $\mathcal{I}_{\beta}^{2}(h_{j\beta}^{2}; \omega)$ that are consistent with the common history $h_{i\beta}^{2}$ in order to determine the set $\mathcal{I}_{\beta}^{3}(h_{j\beta}^{3}; \omega)$ as well as with $h_{i\beta}^{2}$ consistent sets $\mathcal{I}_{j}^{3}(h_{j\beta}^{3}; \omega)$ and $\mathcal{I}_{\beta}^{3}(h_{j\beta}^{3}; \omega)$. Let us first consider the analysis of agent α. As agent α does not know the first period choice of γ the following sets are consistent with the history $h_{\alpha\beta}^{2}$.

$$\begin{aligned} \mathcal{I}_{\beta}^{2}(h_{\beta\gamma}^{2}; \omega) &= \{1245, 4578\}, \mathcal{I}_{\gamma}^{2}(h_{\beta\gamma}^{2}; \omega) = \{147, 258\} \text{ if } a_{\gamma}^{1} = b \\ \mathcal{I}_{\beta}^{2}(h_{\beta\gamma}^{2}; \omega) &= \{2356, 5689\}, \mathcal{I}_{\gamma}^{2}(h_{\beta\gamma}^{2}; \omega) = \{258, 369\} \text{ if } a_{\gamma}^{1} = c \end{aligned}$$

As the choice correspondence assigns action c to all possible second stage information sets of player β for all histories $h^2_{\beta\gamma}$ that are consistent with $h^2_{\alpha\beta}$, agent γ can not infer any additional information from agent β's second period choice, a fact that is common knowledge among α and β. Let us now consider the analysis of agent γ. He does not know the first period choice of agent α thus he considers the following sets which are consistent with $h^2_{\beta\gamma}$

$$\begin{aligned}
\mathcal{I}^2_\beta(h^2_{\alpha\beta};\omega) &= \{1245, 2356\}, \mathcal{I}^2_\alpha(h^2_{\alpha\beta};\omega) = \{123, 456\} \text{ if } a^1_\alpha = b \\
\mathcal{I}^2_\beta(h^2_{\alpha\beta};\omega) &= \{4578, 5689\}, \mathcal{I}^2_\alpha(h^2_{\alpha\beta};\omega) = \{456, 789\} \text{ if } a^1_\alpha = c
\end{aligned}$$

Independent of his first period choice, agent γ can not make additional inference based on player β's second period action as the choice correspondence prescribes action c for all possible information sets of agent β. Effective second stage learning can only occur for player β and only if one of his neighbors selects a second period action that is unequal to his first period action. This would reveal the middle cell of α's or γ's partition respectively. But given that $\omega = 3$ has occurred player α and γ have a single optimal action given their information set. Thus the private information sets in the stage $t = 3$ are identical to stage $t = 2$. The common information sets $\mathcal{I}^3_\beta(h^3_{\beta i};3)$ contain the elements of the second stage information set and an additional element capturing the possibility of an action switch, while the inference made is identical in each stage as $\omega = 3$ and each agent has a single optimal action.

In this example the private information sets as well as the union over the pairwise information sets remain constant for all periods $t \geq 2$ and agent α selects action b for all periods t. Action b is not optimal for his neighbor β, while β selects action c, which is not optimal for agent α. Thus we have shown that without common knowledge of strategies, the local indifference result of Theorem 1 can fail in an incomplete network.

8 Illustration of results: A game of regime change

Games of regime change are coordination games in which all players have the option to attack the status quo. Whether attacking leads to an abandonment of the status quo may depend on the number of agents that attack, as well as the underlying state of nature. This type of game is commonly used to model phenomena as bank runs, revolution against the political status quo, as well as currency attacks.

We consider a game of regime change purely to illustrate the relevancy of Theorem 1 for coordination games when agents can communicate prior to taking an action. Let us consider the following game. There is a finite set of agents $M = \{1, ..., m\}$ organized in a social network $G = (M, E)$ who simultaneously decide whether to attack the status quo, $a_i = r$, or remain inactive, $a_i = \neg r$.

Uncertainty is described by a probability space $(\Omega, \mathcal{F}, p)$ where Ω is the state space, $\mathcal{F}$ a σ-algebra of subset of Ω, and p a common prior. All agents have private information regarding the realized state of nature given by their finite partition $\mathcal{P}^i$. The common utility function $u_i : A^m \times \Omega \to \mathbb{R}$ is given by

$$\begin{aligned} u_i(r, a_{-i}, \omega) &= \begin{cases} 1 & \text{if } \omega \in E \text{ and } a_j = r \ \forall j \in M \setminus i \\ -1 & \text{otherwise} \end{cases} \\ u_i(\neg r, a_{-i}, \omega) &= 0 \end{aligned}$$

The individual decision of player i to attack, leads to an overthrow of the status quo and a beneficiary outcome only if all other agents attack as well and the status quo is weak, $\omega \in E$. If an agent takes part in an uncoordinated attack or a coordinated attack[32] on a strong status quo, the status quo prevails, and the agent is penalized.

Let us analyze the set of pure strategy Bayesian Nash equilibria. A strategy s_i of a player is a mapping of partition cells into actions, $s_i : \mathcal{P}^i \to \{r, \neg r\}$. An equilibrium is a strategy tuple $s = (s_1, ..., s_m)$ such that for all player i

$$E_p[u_i(s_i(P), s_{-i}, \omega) \,|P] \geq E_p[u_i(a_i, s_{-i}, \omega) \,|P] \ \ \forall P \in \mathcal{P}^i, \ \forall a_i \in \{r, \neg r\}$$

As perfect coordination is required among all agents for the status quo to be overcome, a strategy tuple that involves inaction for each agent and each of his partition cells is an equilibrium for all $(\Omega, \mathcal{F}, p)$ and set of partitions $\left\{\mathcal{P}^i\right\}_{i \in M}$.

For an attack to occur with positive probability it is necessary that there exists a non empty subset of Ω such that the status quo being weak is common $\frac{1}{2}$-belief on that subset. In other words, if and only if there exists a collection of sets $\left\{\bar{\mathcal{P}}^i\right\}_{i \in M}$ where $\bar{\mathcal{P}}^i \neq \emptyset$ is a subset of $\mathcal{P}^i$ and

$$\frac{p\left(P^i \cap E \cap \bigcap_{j \neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)} \geq \frac{1}{2} \quad \forall P^i \in \bar{\mathcal{P}}^i, \forall i \in M$$

then there exists an equilibrium s such that $s_i(P^i) = r$ if and only if $P^i \in \bar{\mathcal{P}}^i$[33]. In our game of regime change there is a variety of possible equilibrium outcomes depending on the underlying parameters. One might observe uncoordinated attacks, uniform inaction, as well as coordinated attacks that are either successful or fail.

[32] By coordinated attack we mean a simultaneous attack of all agents.
[33] Please see the Appendix for the proof.

From an ex ante perspective the probability of a perfectly coordinated attack is strictly smaller than one whenever there exists an agent i with a cell P^i of his partition with $p(P^i) > 0$ such that

$$\frac{p\left(P^i \cap E\right)}{p(P^i)} < \frac{1}{2}$$

as it is always optimal for agent i to remain inactive when observing P^i, independent of the choices of all other agents. Having introduced the game of regime change and its equilibrium let us now introduce a communication stage that precedes the action stage where all agents can exchange information regarding the state space within their neighborhood.

Formally the communication stage is modeled in the following way. In discrete steps agents simultaneously send a message $m \in \mathcal{M}$ observing the history of messages of their neighbors. The message space takes the form that it separates information sets with a posterior probability of the status quo being weak of smaller than half, from information sets with a posterior probability of the status quo being weak of larger or equal to half. Furthermore let the common message function $f : \mathcal{F} \to \mathcal{M}$ be union consistent. Let us give two examples of a message space that complies with the stated conditions. For one, the message being send could be simply the posterior belief of event E given the information set observed. In that case

$$f(I) = \frac{p\left(I \cap E\right)}{p(I)}$$

Alternatively agents could communicate either to be willing to attack, $m = w$, or unwilling to attack, $m = \neg w$, where agents are willing to attack whenever the posterior belief of the status quo being weak is larger or equal to half[34].

As the common message function is union consistent and partitions are finite Corollaries 1 and 2 imply that all agents will send the same message within a finite number of steps and reach an information set which remains constant from that period on. As messages are send truthfully, in a finite number of steps all agents will either be willing or all unwilling, a fact which is common knowledge among all agents, allowing for perfect coordination.

Our analysis shows that if agents in a game of regime change can communicate with their neighbors and partitions are finite, there are only two possible equilibrium outcomes[35], coordinated attack among all agents or uniform inaction.

[34] As perfect coordination is required for an attack and as the result of convergence of messages in finite steps is common knowledge agents have no incentive to send a non-truthful message.

[35] We are restricting attention to pure strategy equilibria.

9 Conclusion

Our analysis is motivated by the essential role social networks play in many economic and social settings and the necessity of a thorough understanding of the underlying learning dynamics. We provide a general framework for rational learning in social networks and formally characterize the learning process. This general framework and the characterization of the learning process allow us to generalize existing results in the literature, as well as give answers to questions that have not been addressed before.

We present three main results. Our first result shows the local consensus achieved in social networks once learning ends. If agents have a common underlying choice correspondence and learning ends in finite time, then any action agent i selects once learning ends, is optimal for all his neighbors. This local optimality across neighbors is shown generally not to hold globally.

Our second theorem establishes an asymptotic result. We provide a sufficient condition for the existence of a finite time for each state of the world, such that any choice agent i makes from that time period forward will eventually become optimal for all of his neighbors.

For our third result we consider an environment without common knowledge of strategies. If the strategies of agents are not common knowledge, then the optimality across neighbors once learning ends depends on the network structure. If the network is complete, the optimality property holds, if it is incomplete, the optimality across neighbors can fail.

The theoretical results we provide have significance for several streams in the economic literature. We contribute to the social learning and networks literature by providing a general framework for repeated interaction in social networks and by formally characterizing the learning process. Our three theorems represent new insights in the literature on knowledge and consensus started by Aumann (1976). We illustrate the importance for coordination games when pre-play communication is feasible, through our example of a game of regime change.

Furthermore, our results have important implications for communication networks in firms and institutions. We find that connected local committees are sufficient to achieve perfect coordination. This is relevant whenever introducing more communication links in a network is costly. Our example about information aggregation in networks also shows that a complete network does not generally lead to better information aggregation than an incomplete network. This is a very interesting area for future research. A general characterization of the optimal communication network depending on the underlying information parameters would be highly desirable.

Appendix

Proof of equivalence of Union Consistency and Pairwise Consistency

Proof. Let $\mathcal{B}_1, \mathcal{B}_2$ be collections of disjoint sets in $\mathcal{F}$. Suppose C satisfies (UC). We have

$$\bigcap_{B \in \mathcal{B}_1} C(B) = C\left(\bigcup_{B \in \mathcal{B}_1} B\right)$$

and

$$\bigcap_{B \in \mathcal{B}_2} C(B) = C\left(\bigcup_{B \in \mathcal{B}_2} B\right)$$

As the union over all sets in $\mathcal{B}_1$ is equal to the union over all sets in $\mathcal{B}_2$,

$$\bigcup_{B \in \mathcal{B}_1} B = \bigcup_{B \in \mathcal{B}_2} B$$

the choice correspondence assigns the same set of actions

$$C\left(\bigcup_{B \in \mathcal{B}_1} B\right) = C\left(\bigcup_{B \in \mathcal{B}_2} B\right)$$

implying by union consistency

$$\bigcap_{B \in \mathcal{B}_1} C(B) = \bigcap_{B \in \mathcal{B}_2} C(B)$$

Suppose C satisfies (PC). Let $\mathcal{B}_1$ be a collection of disjoint sets such that

$$\bigcap_{B \in \mathcal{B}_1} C(B) \neq \emptyset$$

and let $\mathcal{B}_2$ consist of one set

$$\mathcal{B}_2 = \bigcup_{B \in \mathcal{B}_1} B$$

thus

$$\bigcap_{B \in \mathcal{B}_2} C(B) = C(\mathcal{B}_2) = C\left(\bigcup_{B \in \mathcal{B}_1} B\right)$$

by (PC)

$$\bigcap_{B \in \mathcal{B}_1} C(B) \neq \emptyset \, , \bigcap_{B \in \mathcal{B}_2} C(B) \neq \emptyset \;\Rightarrow\; \bigcap_{B \in \mathcal{B}_1} C(B) = \bigcap_{B \in \mathcal{B}_2} C(B)$$

which implies

$$\bigcap_{B \in \mathcal{B}_1} C(B) = C\left(\bigcup_{B \in \mathcal{B}_1} B\right)$$

■

Proof of Proposition 1

The alternative learning process

In the first period we have

$$\check{I}_i^1(\omega) = I_i^1(\omega) = \mathcal{P}^i(\omega)$$

and

$$\check{\mathcal{I}}_i^1(\mathcal{P}^j(\omega)) = \{r_i \in \mathcal{P}^i : r_i \cap \check{I}_j^1(\omega) \neq \emptyset\}$$

The inference agent j makes based on agent i's first period action is given by

$$\check{\mathcal{D}}_i^1(a_i^1; \mathcal{P}^j(\omega)) = \{r_i \in \check{\mathcal{I}}_i^1(\mathcal{P}^j(\omega)) : a_i^1 = s_i(r_i)\}$$

The information set of agent i in period $t = 2$ is given by

$$\check{I}_i^2(\mathcal{P}^i(\omega), h_i^2(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \check{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$$

In period t the information set of agent i is given by

$$\check{I}_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \check{\mathcal{D}}_l^{t-1}(a_l^{t-1}; \mathcal{P}^i(\omega), h_i^{t-1}(\omega))$$

The set of information sets of agent i considered possible by his neighbor j based on j's private observables is given by

$$\check{\mathcal{I}}_i^t(\mathcal{P}^j(\omega), h_j^t(\omega)) = \left\{ \check{I}_i^t(r_i, \hat{h}_i^t) : \begin{array}{l} \hat{h}_{ij}^t = h_{ij}^t(\omega) \\ \exists \check{I}_i^{t-1} \in \check{\mathcal{D}}_i^{t-1}(a_i^{t-1}; \mathcal{P}^j(\omega), h_j^{t-1}(\omega)) \text{ s.t. } \check{I}_i^t(r_i, \hat{h}_i^t) \subset \check{I}_i^{t-1} \\ \check{I}_i^t(r_i, \hat{h}_i^t) \cap \check{I}_j^t(\mathcal{P}^j(\omega), h_j^t(\omega)) \neq \emptyset \end{array} \right\}$$

Based on the action a_i^t selected by agent i in period t his neighbor j makes the following inference regarding i's realized information set

$$\check{\mathcal{D}}_i^t(a_i^t; \mathcal{P}^j(\omega), h_j^t(\omega)) = \{\check{I}_i^t(r_i, \hat{h}_i^t) \in \check{\mathcal{I}}_i^t(\mathcal{P}^j(\omega), h_j^t(\omega)) : a_i^t = s_i(\check{I}_i^t(r_i, \hat{h}_i^t))\}$$

The information set of agent i in period $t+1$ is then given by

$$\breve{I}_i^{t+1}(\mathcal{P}^i(\omega), h_i^{t+1}(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$$

Proof of Proposition 1

Proof. We use an induction argument for the proof. Suppose that for $t = 1$

$$\mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^1(a_l^1; \omega) \neq \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$$

which implies

$$\bigcap_{l \in N_i} \cup \mathcal{D}_l^1(a_l^1; \omega) \neq \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$$

Thus there exists a player $l \in N_i$ such that $\mathcal{D}_l^1(a_l^1; \omega) \neq \breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$. Let $\tilde{r}_l \in \mathcal{D}_l^1(a_l^1; \omega)$ and $\tilde{r}_l \notin \breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$ which by definition of $\breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$ implies

$$\tilde{r}_l \cap \mathcal{P}^i(\omega) = \emptyset$$

and thus implies

$$\tilde{r}_l \cap \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^1(a_l^1; \omega) = \emptyset$$

This holds true for all $l \in N_i$ and $\tilde{r}_l \in \mathcal{D}_l^1(a_l^1; \omega) \smallsetminus \breve{\mathcal{D}}_l^1(a_l^1; \mathcal{P}^i(\omega))$ establishing the base case for $t = 1$. For the inductive step let us assume that

$$\mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, h_{il}^{t-1}(\omega); \omega) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^{t-1}(a_l^{t-1}; \mathcal{P}^i(\omega), h_i^{t-1}(\omega))$$

holds in period t which implies

$$\breve{\mathcal{I}}_l^t(\mathcal{P}^i(\omega), h_i^t(\omega)) \subset \mathcal{I}_l^t(h_{il}^t(\omega); \omega) \ \forall l \in N_i$$

Suppose now that the condition is not fulfilled in period $t+1$

$$\mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) \neq \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$$

which implies

$$\bigcap_{l \in N_i} \cup \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) \neq \bigcap_{l \in N_i} \cup \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$$

Thus there exists a player $l \in N_i$ such that $\mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) \neq \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$. Select an $\tilde{I}_l^t(.)$

such that

$$\tilde{I}_l^t() \in \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) \smallsetminus \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$$

which implies

$$\tilde{I}_l^t(.) \cap \hat{I}_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \emptyset$$

and by the induction hypothesis

$$\tilde{I}_l^t(.) \cap I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \emptyset$$

By definition of the sets $\mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega)$ we have

$$I_i^{t+1}(\mathcal{P}^i(\omega), h_i^{t+1}(\omega)) \subset I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$$

where

$$I_i^{t+1}(\mathcal{P}^i(\omega), h_i^{t+1}(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega)$$

Thus we have

$$\tilde{I}_l^t(.) \cap \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) = \emptyset$$

This holds true for all $l \in N_i$ and for all $\tilde{I}_l^t(.) \in \mathcal{D}_l^t(a_l^t, h_{il}^t(\omega); \omega) \smallsetminus \breve{\mathcal{D}}_l^t(a_l^t; \mathcal{P}^i(\omega), h_i^t(\omega))$ establishing the induction step and concluding the proof ■

Proof of Proposition 2

Proof. Let us first consider the case where $\mathcal{P}^i(\omega) = \mathcal{P}^i(\omega')$ and $h_i^t(\omega) \neq \hat{h}_i^t(\omega')$. We use a proof by induction. First we show that $h_i^2(\omega) \neq \hat{h}_i^2(\omega')$ implies $I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega)) \cap I_i^2(\mathcal{P}^i(\omega'), \hat{h}_i^2(\omega')) = \emptyset$. By definition of $I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega))$ we have

$$I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \mathcal{D}_l^1(a_l^1; \omega)$$

$\mathcal{P}^i(\omega) = \mathcal{P}^i(\omega')$ implies $CK_{il}(\omega) = CK_{il}(\omega')$ leading to $\mathcal{I}_l^1(\omega; i) = \mathcal{I}_l^1(\omega'; i)$. Thus $\left\{\mathcal{D}_l^1(a; \omega)\right\}_{a \in A} = \left\{\mathcal{D}_l^1(a; \omega')\right\}_{a \in A}$ for a such that $\mathcal{D}_l^1(a; \omega) \neq \emptyset$ form identical partitions of $\mathcal{I}_l^1(\omega; i)$ which implies for $a_l^1 \neq \hat{a}_l^1$

$$\mathcal{D}_l^1(a_l^1; \omega) \cap \mathcal{D}_l^1(\hat{a}_l^1; \omega') = \emptyset$$

$h_i^2(\omega) \neq \hat{h}_i^2(\omega')$ implies that there exists an agent $l \in N_i$ such that $a_l^1 \neq \hat{a}_l^1$ which implies

$$I_i^2(\mathcal{P}^i(\omega), h_i^2(\omega)) \cap I_i^2(\mathcal{P}^i(\omega'), \hat{h}_i^2(\omega')) = \emptyset$$

Suppose now that for all $h_i^{t-1}(\omega) \neq \hat{h}_i^{t-1}(\omega')$ we have

$$I_i^{t-1}(\mathcal{P}^i(\omega), h_i^{t-1}(\omega)) \cap I_i^{t-1}(\mathcal{P}^i(\omega'), \hat{h}_i^{t-1}(\omega')) = \emptyset$$

Take any $h_i^t(\omega) \neq \hat{h}_i^t(\omega')$. There are two cases to consider. The first case is where the history of actions of the previous period is unequal, $h_i^{t-1}(\omega) \neq \hat{h}_i^{t-1}(\omega')$. By the induction hypothesis we have

$$I_i^{t-1}(\mathcal{P}^i(\omega), h_i^{t-1}(\omega)) \cap I_i^{t-1}(\mathcal{P}^i(\omega'), \hat{h}_i^{t-1}(\omega')) = \emptyset$$

This together with the fact that the information set of player i in period t is a subset of the information set in period $t-1$, i.e.

$$\begin{aligned} I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) &\subset I_i^{t-1}(\mathcal{P}^i(\omega), h_i^{t-1}(\omega)) \\ I_i^t(\mathcal{P}^i(\omega'), \hat{h}_i^t(\omega')) &\subset I_i^{t-1}(\mathcal{P}^i(\omega'), \hat{h}_i^{t-1}(\omega')) \end{aligned}$$

yields the desired result

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) \cap I_i^t(\mathcal{P}^i(\omega'), \hat{h}_i^t(\omega')) = \emptyset$$

Now let us consider the case where $h_i^t(\omega) \neq \hat{h}_i^t(\omega')$ and $h_i^{t-1}(\omega) = \hat{h}_i^{t-1}(\omega')$. This implies that there are some agents l in i's neighborhood that select a different action in period $t-1$ under $h_i^t(\omega)$ than under $\hat{h}_i^t(\omega')$, i.e. $a_l^{t-1} \neq \hat{a}_l^{t-1}$ for some $l \in N_i$. The equality of histories until time $t-1$, $h_i^{t-1}(\omega) = \hat{h}_i^{t-1}(\omega')$ implies

$$\mathcal{I}_l^{t-1}(h_{il}^{t-1}(\omega); \omega) = \mathcal{I}_l^{t-1}(\hat{h}_{il}^{t-1}(\omega'); \omega')$$

for all $l \in N_i$. The collection of sets $\left\{\mathcal{D}_l^{t-1}(a, h_{il}^{t-1}(\omega); \omega)\right\}_{a \in A_l^{t-1}(h_{il}^{t-1}(\omega);\omega)}$, where

$$A_l^{t-1}(h_{il}^{t-1}(\omega); \omega) = \left\{a \in A : \exists\ I_l^{t-1} \in \mathcal{I}_l^{t-1}(h_{il}^{t-1}(\omega); \omega) \text{ s.t. } s_l(I_l^{t-1}) = a\right\}$$

constitutes a partition of $\mathcal{I}_l^{t-1}(h_{il}^{t-1}(\omega); \omega)$. Thus for any $a_l^{t-1} \neq a_l'^{t-1}$ we have

$$\mathcal{D}_l^{t-1}(a_l^{t-1}, h_{il}^{t-1}(\omega); \omega) \cap \mathcal{D}_l^{t-1}(a_l'^{t-1}, h_{il}^{t-1}(\omega'); \omega') = \emptyset$$

As the information set $I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$ is given by

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = \mathcal{P}^i(\omega) \cap \bigcap_{l \in N_i} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, h_{il}^{t-1}(\omega); \omega)$$

and there are some $l \in N_i$ with $a_l^{t-1} \neq \hat{a}_l^{t-1}$ we have

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) \cap I_i^t(\mathcal{P}^i(\omega'), \hat{h}_i^t(\omega')) = \emptyset$$

Furthermore it is directly apparent that for $\mathcal{P}^i(\omega) \neq \mathcal{P}^i(\omega')$ we have

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) \cap I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) = \emptyset$$

as $P^i(\omega) \neq P^i(\omega')$ implies $P^i(\omega) \cap P^i(\omega') = \emptyset$ concluding the proof ■

Proof of Lemma 1

Proof. Take any pair of information sets $I_i^{t'}$, $\hat{I}_i^{t'} \in \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$. Suppose

$$s_i(I_i^{t'}) \neq s_i(\hat{I}_i^{t'})$$

If $I_i^{t'}$ is the true information set of player i, he selects the action $a_i^{t'} = s_i(I_i^{t'})$ leading to $\mathcal{D}_i^{t'}(a_i^{t'}, h_{ij}^{t'}(\omega); \omega)$ with

$$\hat{I}_i^{t'} \notin \mathcal{D}_i^{t'}(a_i^{t'}, h_{ij}^{t'}(\omega); \omega)$$

As $\mathcal{I}_i^{t'+1}(h_{ij}^{t'+1}(\omega); \omega)$ contains only elements $I_i^{t'+1}$ such that there exists an $I_i^{t'} \in \mathcal{D}_i^{t'}(a_i^{t'}, h_{ij}^{t'}(\omega); \omega)$ with $I_i^{t'+1} \subset I_i^{t'}$ and by Proposition 2 $\mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega)$ is a disjoint collection of sets we have

$$\cup \mathcal{I}_i^{t'+1}(h_{ij}^{t'+1}(\omega); \omega) \cap \hat{I}_i^{t'} = \emptyset$$

which yields a contradiction to

$$\cup \mathcal{I}_i^{t'}(h_{ij}^{t'}(\omega); \omega) = \cup \mathcal{I}_i^{t'+1}(h_{ij}^{t'+1}(\omega); \omega)$$

■

Proof of Lemma 2

Proof. By definition

$$\mathcal{I}_i^t(h_{ij}^t(\omega); \omega) = \left\{ I_i^t(r_i, \hat{h}_i^t) : \begin{array}{l} \hat{h}_{ij}^t = h_{ij}^t(\omega) \\ \exists I_i^{t-1} \in \mathcal{D}_i^{t-1}(a_i^{t-1}, h_{ij}^{t-1}(\omega); \omega) \text{ s.t. } I_i^t(r_i, \hat{h}_i^t) \subset I_i^{t-1} \end{array} \right\}$$

taking the union over all elements of $\mathcal{I}_i^t(h_{ij}^t(\omega); \omega)$ yields

$$\begin{aligned} & \cup \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) \\ = \; & \cup \mathcal{D}_i^{t-1}(a_i^{t-1}, h_{ij}^{t-1}(\omega); \omega) \cap \left(\cup \mathcal{D}_j^{t-1}(a_j^{t-1}, h_{ij}^{t-1}(\omega); \omega) \right) \cap \bigcap_{l \in N_{ij}} \left(\bigcup_{\hat{h}_{il}^{t-1} \in H_{il}^{t-1}(h_{ij}^{t-1}(\omega))} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, \hat{h}_{il}^{t-1}; \omega) \right) \end{aligned}$$

where

$$H_{il}^{t-1}(h_{ij}^t(\omega)) = \left\{ H_{il}^{t-1} : \hat{h}_{il}^{t-1} \text{ consistent with } h_{ij}^{t-1}(\omega) \ \right\}$$

Equally

$$\begin{aligned} & \cup \mathcal{I}_j^t(h_{ij}^t(\omega);\omega) \\ = \ & \cup \mathcal{D}_i^{t-1}(a_i^{t-1}, h_{ij}^{t-1}(\omega);\omega) \cap \left(\cup \mathcal{D}_j^{t-1}(a_j^{t-1}, h_{ij}^{t-1}(\omega);\omega) \right) \cap \bigcap_{l \in N_{ij}} \left(\bigcup_{\hat{h}_{il}^{t-1} \in H_{il}^{t-1}(h_{ij}^{t-1}(\omega))} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, \hat{h}_{il}^{t-1};\omega) \right) \end{aligned}$$

implying

$$\cup \mathcal{I}_i^t(h_{ij}^t(\omega);\omega) = \cup \mathcal{I}_j^t(h_{ij}^t(\omega);\omega)$$

■

Proof of Corollary 2

Proof. If finite partitions imply that learning ends in finite time we have the stated result through application of Theorem 1. First we show that finite partitions imply

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))$$

for some t' and all $t > t'$.

Let us assume that there is no t' such that for all $t > t'$ we have the desired equality of information sets. This implies the existence of an infinite sequence of information sets $\{I_i^{t_1}, I_i^{t_2}, ...\}$ such that

$$I_i^{t_{k+1}}(\mathcal{P}^i(\omega), h_i^{t_{k+1}}(\omega)) \subset I_i^{t_k}(\mathcal{P}^i(\omega), h_i^{t_k}(\omega)) \text{ and } I_i^{t_{k+1}}(\mathcal{P}^i(\omega), h_i^{t_{k+1}}(\omega)) \neq I_i^{t_k}(\mathcal{P}^i(\omega), h_i^{t_k}(\omega)) \forall k \in \mathbb{N}$$

For all players i and time t information set $I_i^t\left(\mathcal{P}^i(\omega), h_i^t(\omega)\right)$ is element of the power set of the join of all partitions $2^{\left(\bigvee_{j \in M} \mathcal{P}^j\right)}$. As $\mathcal{P}^i$ are finite for all $i \in M$ and M is finite, so is $2^{\left(\bigvee_{j \in M} \mathcal{P}^j\right)}$. Thus there can be no infinite sequence $\left\{I_i^{t_k}(.)\right\}_{k=1}^{\infty}$ with the above property and there exists a finite t' such that

$$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega)) = I_i^{t'}(\mathcal{P}^i(\omega), h_i^t(\omega))$$

for all $i \in M$, $t \geq t'$.

For finite partitions and finite M the set $\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$ consists of finitely many elements. For every time t every set $I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$ in $\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$ converges to a set $I_i^{t'}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))$ in finite time as we have shown above. Thus there exists a finite time t^* with

$$\mathcal{I}_i^{t^*}(h_{ij}^{t^*}(\omega);\omega) = \mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$$

for all $i \in M$, $j \in N_i$ and $t > t^*$ thus implying

$$\cup\mathcal{I}_i^{t^*}(h_{ij}^{t^*}(\omega);\omega) = \cup\mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$$

concluding the proof ■

Proof of Lemma 3

Proof. Take any two distinct limit information sets $I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega'))$, $I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega''))$ that are contained in $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$. For $\mathcal{P}^i(\omega') \neq \mathcal{P}^i(\omega'')$ we have by Proposition 2

$$I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \cap I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega'')) = \emptyset \ \forall t$$

implying

$$I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \cap I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega'')) = \emptyset$$

Suppose $\mathcal{P}^i(\omega)$ is equal to $\mathcal{P}^i(\omega')$ and $h_i^\infty(\omega) \neq h_i^\infty(\omega')$. The inequality of histories then implies that there exists a $\hat{t}$ such that $h_i^{\hat{t}}(\omega) \neq h_i^{\hat{t}}(\omega')$. Proposition 2 then yields

$$I_i^{\hat{t}}(\mathcal{P}^i(\omega'), h_i^{\hat{t}}(\omega')) \cap I_i^{\hat{t}}(\mathcal{P}^i(\omega''), h_i^{\hat{t}}(\omega'')) = \emptyset$$

For each period t and each state ω' by definition of the limit set

$$I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \subset I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))$$

which implies

$$I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \cap I_i^\infty(\mathcal{P}^i(\omega''), h_i^\infty(\omega'')) = \emptyset$$

■

Proof of Lemma 4

Proof. Take any period t and any information set $I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \in \mathcal{I}_i^t(h_{ij}^t(\omega);\omega)$ that converges to a limit set $I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega'))$ in $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$. Suppose

$$\hat{a} = s_i\left(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))\right) \neq s_i\left(I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))\right) = a'$$

which implies by Proposition 2

$$I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \cap \mathcal{D}_i^t(\hat{a}, h_i^t(\omega); \omega) = \emptyset$$

By definition $\cup \mathcal{I}_i^{t+1}(h_{ij}^{t+1}(\omega); \omega)$ is a subset of $\cup \mathcal{D}_i^t(a_i^t, h_i^t(\omega); \omega)$ implying

$$I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \cap \cup \mathcal{I}_i^{t+1}(h_{ij}^{t+1}(\omega); \omega) = \emptyset$$

As $I_i^{t+1}(\mathcal{P}^i(\omega'), h_i^{t+1}(\omega')) \subset I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))$ we have

$$I_i^{t+1}(\mathcal{P}^i(\omega'), h_i^{t+1}(\omega')) \cap \mathcal{I}_i^{t+1}(h_{ij}^{t+1}(\omega); \omega) = \emptyset$$

yielding a contradiction with $I_i^{\infty}(\mathcal{P}^i(\omega'), h_i^{\infty}(\omega')) \in \mathcal{I}_i^{\infty}(h_{ij}^{\infty}(\omega); \omega)$ ■

Proof of Lemma 5

Proof. The definition of $\mathcal{I}_i^{\infty}(h_{ij}^{\infty}(\omega); \omega)$ implies

$$\cup \mathcal{I}_i^{\infty}(h_{ij}^{\infty}(\omega); \omega) = \bigcap_{t \in \mathbb{N}} \cup \mathcal{I}_i^t(h_{ij}^t; \omega)$$

By Lemma 2 we have in every stage t

$$\cup \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) = \cup \mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$$

which implies

$$\cup \mathcal{I}_i^{\infty}(h_{ij}^{\infty}(\omega); \omega) = \bigcap_{t \in \mathbb{N}} \cup \mathcal{I}_j^t(h_{ij}^t; \omega) = \cup \mathcal{I}_j^{\infty}(h_{ij}^{\infty}(\omega); \omega)$$

■

Proof of Theorem 2

Proof. Let us start by proving the first claim of the Theorem: If all elements of the join $\bigvee_{j \in M} \mathcal{P}^j$ are dominant sets under C, C is union consistent and complies with (D), and A is finite, then for every $\omega \in \Omega$ and for any pair of neighbors $i \in M$, $j \in N_i$ there exists a non-empty sets $A_{ij}^{\infty}(\omega) \subset A$ such that

$$A_{ij}^{\infty}(\omega) \subset C(I_i^{\infty}((\mathcal{P}^i(\omega), h_i^{\infty}(\omega))) \cap C(I_j^{\infty}((\mathcal{P}^j(\omega), h_j^{\infty}(\omega)))$$

If $\bigvee_{j \in M} \mathcal{P}^j(\omega)$ is a dominant set for all $\omega \in \Omega$ and property (D) holds for the choice correspondence, then for all states $\omega \in \Omega$ the limit information set $I_i^{\infty}(\mathcal{P}^i(\omega), h_i^{\infty}(\omega))$ is a dominant set as it contains the true cell of the join,

$$\bigvee_{j \in M} \mathcal{P}^j(\omega) \subset I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))$$

Let us define the set $A_i^\infty(\omega)$ as

$$A_i^\infty(\omega) = \left\{a \in A : a = s_i\left(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))\right) \text{ for infinite periods } t\right\}$$

$A_i^\infty(\omega)$ contains all actions that agent i selects infinitely often in state ω. As the set of actions is finite, the set $A_i^\infty(\omega)$ is not empty. Next we will establish that the set $A_i^\infty(\omega)$ is contained in the set of optimal actions for every $I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega')) \in \mathcal{I}_i^\infty(h_{ij}^\infty(\omega); \omega)$. Each information set $I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))$ is a dominant set which implies the existence of a finite time $t_i(\omega)$ such that for all $t \geq t_i(\omega)$

$$C\left(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))\right) \subset C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

Take any action $a' \in A_i^\infty(\omega)$. As a' is selected in infinitely periods, for every time period t there exists a time period $t' > t$ such that

$$a' = s_i\left(I_i^{t\prime}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))\right) \in C\left(I_i^{t\prime}(\mathcal{P}^i(\omega), h_i^{t\prime}(\omega))\right)$$

As $t_i(\omega)$ is finite there exists a $t' > t_i(\omega)$ such that

$$a' = s_i\left(I_i^{t\prime}(\mathcal{P}^i(\omega), h_i^{t'}(\omega))\right) \in C\left(I_i^{t\prime}(\mathcal{P}^i(\omega), h_i^{t\prime}(\omega))\right)$$

$t' > t_i(\omega)$ then implies

$$a' \in C\left(I_i^{t\prime}(\mathcal{P}^i(\omega), h_i^{t\prime}(\omega))\right) \subset C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

Thus we have

$$A_i^\infty(\omega) \subset C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

For any two information sets $I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega))$, $I_i^\infty(\mathcal{P}^i(\omega'), h_i^\infty(\omega'))$ in the set $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega); \omega)$ we have by Lemma 4 for every period t

$$s_i\left(I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))\right) = s_i\left(I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega'))\right)$$

which implies that all elements of $A_i^\infty(\omega)$ are optimal in each limit information set in $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega); \omega)$,

$$A_i^\infty(\omega) \subset C(I_i^\infty) \ \forall I_i^\infty \in \mathcal{I}_i^\infty(h_{ij}^\infty(\omega); \omega)$$

Applying the same reasoning for agent $j \in N_i$ yields

$$A_j^\infty(\omega) \subset C(I_j^\infty) \; \forall I_j^\infty \in \mathcal{I}_j^\infty(h_{ij}^\infty(\omega);\omega)$$

Lemma 3 states that $\mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)$ is a collection of disjoint sets. Lemma 5 establishes the following equality

$$\cup \mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega) = \cup \mathcal{I}_j^\infty(h_{ij}^\infty(\omega);\omega)$$

Thus we can use the union consistency property of the choice correspondence which implies

$$A_i^\infty(\omega) \cup A_j^\infty(\omega) \subset \bigcap_{I_i^\infty \in \mathcal{I}_i^\infty(h_{ij}^\infty(\omega);\omega)} C(I_i^\infty)$$

and thus for the true limit information set

$$A_{ij}^\infty(\omega) = A_i^\infty(\omega) \cup A_j^\infty(\omega) \subset C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

concluding the proof for the first claim.

Let us next prove the second claim of the Theorem: for every state of the world ω there exists a finite $t^*(\omega)$ such that for all $t \geq t^*(\omega)$ and all pairs of neighbors $i \in M$, $j \in N_i$

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

Above we have established that all actions $a \in A_i^\infty(\omega) \cup A_j^\infty(\omega)$ are optimal for both player i and j in their limit information sets. Take any action a' not contained in $A_i^\infty(\omega) \cup A_j^\infty(\omega)$. Let us consider player j. $A_j^\infty(\omega)$ denotes the set of actions that agent j selects infinitely often. Take any action a' that is not element of $A_j^\infty(\omega)$. As a' is not selected infinitely often there exists a time $t_{a'}(\omega)$ such that a' is not selected in state ω by player j for any stage succeeding stage $t_{a'}(\omega)$, for all $t > t_{a'}(\omega)$

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) \neq a'$$

As the action set A is finite there are only finitely many actions a' that are not contained in $A_j^\infty(\omega)$ thus the following maximum is defined

$$t_j^*(\omega) = \max_{a' \in A \setminus A_j^\infty(\omega)} t_{a'}(\omega)$$

For all periods after $t_j^*(\omega)$ agent j selects only actions in $A_j^\infty(\omega)$ which are all optimal for agent i

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

for all $t > t_j^*(\omega)$. As the set of players is finite the following maximum is defined

$$t^*(\omega) = \max_{i \in M} t_i^*(\omega)$$

and for all pairs agents $i \in M$ and their neighbors $j \in N_i$

$$s_j(I_j^t(\mathcal{P}^j(\omega), h_j^t(\omega))) = a_j^t \in C(I_i^\infty(\mathcal{P}^i(\omega), h_i^\infty(\omega)))$$

for all $t > t^*(\omega)$ concluding the proof ■

Proof of Theorem 3

Proof. By assumption common learning ends in period t'

$$CK^{t'+1}(h^{t'+1};\omega) = CK^{t'}(h^{t'};\omega)$$

The set of cells of i's partition that are commonly known to be feasible at t' is given by

$$\left\{r_i \in \mathcal{P}^i : r_i \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

Suppose there is an $r_i' \in \mathcal{P}^i$ such that

$$r_i' \in \left\{r_i \in \mathcal{P}^i : r_i \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

and

$$r_i' \notin \mathcal{D}_i^{t'}(a_i^{t'}, h^{t'};\omega)$$

By definition

$$CK^{t'+1}(h^{t'+1},\omega) = \bigcap_{i \in M} \cup \mathcal{D}_i^{t'}(a_i^{t'}, h^{t'};\omega)$$

$r_i' \notin \mathcal{D}_i^{t'}(a_i^{t'}, h^{t'};\omega)$ then contradicts

$$CK^{t'+1}(h^{t'+1},\omega) = CK^{t'}(h^{t'},\omega)$$

Thus action $a_i^{t'}$ has to be optimal for each of agent i's possible information sets in period t'

$$a_i^{t'} \in C(r_i \cap CK^{t'}(h^{t'};\omega))$$

for all

$$r_i \in \left\{r_i \in \mathcal{P}^i : r_i \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

Note that the collection of sets $\left\{r_i \cap CK^{t'}(h^{t'};\omega)\right\}_{r_i \in \mathcal{D}_i^{t'}(a_i^{t'},h^{t'};\omega)}$ constitutes a partition of $CK^{t'}(h^{t'};\omega)$. Similar reasoning for agent j yields

$$a_j^{t'} \in C(r_j \cap CK^{t'}(h^{t'};\omega))$$

for all

$$r_j \in \left\{r_j \in \mathcal{P}^j : r_j \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

Union consistency then implies

$$a_j^{t'} \in C(r_i \cap CK^{t'}(h^{t'};\omega))$$

for all

$$r_i \in \left\{r_i \in \mathcal{P}^i : r_i \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

As the true cell $\mathcal{P}^i(\omega)$ is contained in

$$\left\{r_i \in \mathcal{P}^i : r_i \cap CK^{t'}(h^{t'};\omega) \neq \emptyset\right\}$$

we have

$$a_j^{t'} \in C(I_i^{t'}(\mathcal{P}^i(\omega), h^{t'}))$$

concluding the proof ■

Proof of equilibrium in the game of regime change

Proof. Let us first establish that if $\{\mathcal{P}^i\}_{i\in M}$ with the above property exists, the strategy s with $s_i(P^i) = r$ if and only if $P^i \in \bar{\mathcal{P}}^i$, is indeed an equilibrium. Take any agent i and suppose that all other players follow the strategy. Let $P^i \in \bar{\mathcal{P}}^i$. Agent j attacks whenever his partition cell P^j is element of $\bar{\mathcal{P}}^j$. Thus all other agents will attack whenever $\omega \in \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j$. A successful attack requires the government to be weak meaning $\omega \in E$. So a decision to attack of agent i will lead to an abandonment of the status quo whenever

$$\omega \in E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j$$

The updated belief of agent i of that event given his private information P^i is given by

$$\frac{p\left(P^i \cap E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)}$$

Attacking thus yields a expected utility of

$$E_p[u_i(a, s_{-i}, \omega)\,|P^i] = \frac{p\left(P^i \cap E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)} \times 1 - \left(1 - \frac{p\left(P^i \cap E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)}\right) \times 1$$

while choosing to be inactive always yields a utility of 0. Thus it is optimal for agent i to attack whenever

$$\frac{p\left(P^i \cap E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)} \geq \frac{1}{2}$$

which is the case for all $P^i \in \bar{\mathcal{P}}^i$.

Now suppose there exists no collection of sets $\{\bar{\mathcal{P}}^i\}_{i\in M}$, $\bar{\mathcal{P}}^i \neq \emptyset$ for all i such that the above property holds. This implies that there exists an agent i such that for every collection of sets

$\left\{\hat{\mathcal{P}}^j\right\}_{j\in M\smallsetminus i}$ where $\hat{\mathcal{P}}^j \subset \mathcal{P}^j$ and $\hat{\mathcal{P}}^j \neq \emptyset$ we have for all $P^i \in \mathcal{P}^i$

$$\frac{p\left(P^i \cap E \cap \bigcap_{j\neq i} \cup \bar{\mathcal{P}}^j\right)}{p(P^i)} < \frac{1}{2}$$

which implies

$$\frac{p\left(P^i \cap E\right)}{p(P^i)} < \frac{1}{2}$$

leading agent i to remain inactive for each of the cells of his partition. As perfect coordination is required all agents will remain inactive and there exists no equilibrium where $s_i(P^i) = r$ for some $i \in M$ ■

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Cheat Sheet

Union Consistency: Let $\mathcal{B}$ be a collection of disjoint sets

$$\bigcap_{B\in\mathcal{B}} C(B) \neq \emptyset \;\Rightarrow\; \bigcap_{B\in\mathcal{B}} C(B) = C\left(\bigcup_{B\in\mathcal{B}} B\right)$$

Let $\mathcal{B}_1, \mathcal{B}_2$ be a collection of disjoint sets such that

$$\bigcup_{B\in\mathcal{B}_1} B = \bigcup_{B\in\mathcal{B}_2} B$$

Pairwise Consistency:

$$\bigcap_{B\in\mathcal{B}_1} C(B) \neq \emptyset \,, \bigcap_{B\in\mathcal{B}_2} C(B) \neq \emptyset \;\Rightarrow\; \bigcap_{B\in\mathcal{B}_1} C(B) = \bigcap_{B\in\mathcal{B}_2} C(B)$$

$I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$	$= \mathcal{P}^i(\omega) \cap \bigcap_{l\in N_i} \cup \mathcal{D}_l^{t-1}(a_l^{t-1}, h_{il}^{t-1}(\omega)\,; \omega)$
	The information set of player i in period t as a function of his private observables, the history of choices of his neighbors $h_i^t(\omega)$ and the true cell of his partition $\mathcal{P}^i(\omega)$
$\mathcal{I}_i^t(h_{ij}^t(\omega)\,; \omega)$	$= \left\{ I_i^t(r_i, \hat{h}_i^t) : \begin{array}{l} \hat{h}_{ij}^t = h_{ij}^t(\omega) \\ \exists I_i^{t-1} \in \mathcal{D}_i^{t-1}(a_i^{t-1}, h_{ij}^{t-1}(\omega); \omega) \text{ s.t. } I_i^t(r_i, \hat{h}_i^t) \subset I_i^{t-1} \end{array} \right\}$
	Set of possible information sets of player i in period t that are consistent with the common observables of i and his neighbor j, where the common observables are given by the common history $h_{ij}^t(\omega)$ as well as the meet $\mathcal{P}^i \wedge \mathcal{P}^j(\omega)$
$D_i^t(a_i^t, h^t(\omega); \omega)$	$= \left\{ I_i^t \in \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) : a_i^t = s_i(I_i^t) \right\}$
	Set of information sets in $\mathcal{I}_i^t(h_{ij}^t(\omega)\,; \omega)$ such that the strategy of player i assigns action a_i^t

$I_i^{\infty}(\mathcal{P}^i(\omega), h_i^{\infty}(\omega))$	$= \bigcap_{t=1}^{\infty} I_i^t(\mathcal{P}^i(\omega), h_i^t(\omega))$
	The limit information set of player i
$\mathcal{I}_i^{\infty}(h_{ij}^t(\omega); \omega)$	$= \left\{ I_i^{\infty}(\mathcal{P}^i(\omega'), h_i^{\infty}(\omega')) : \ I_i^t(\mathcal{P}^i(\omega'), h_i^t(\omega')) \in \mathcal{I}_i^t(h_{ij}^t(\omega); \omega) \ \forall t \right\}$
	Set of possible limit information sets of player i that are consistent with the common observables of i and his neighbor j