The Limited Liability Agency Model with Moral Hazard*

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Abstract

We obtain a full characterization of the optimal contract for the limited liability model of agency with moral hazard using conditions that are generally satisfied in applied problems in economics and finance. We show necessary and sufficient conditions for the optimal contract to take the form of debt. The analysis is based on two conditions: the distribution of shocks has a monotone hazard rate and the shock and the agent's effort are complements. The advantage of this approach is that it is readily satisfied by several distribution and production functions in applied problems. These conditions are commonly applied in adverse selection models of agency. The two conditions replace the traditional MLRP and CDFC assumptions, which are difficult to satisfy in applied problems.

Key Words: Agency, Moral Hazard, Debt, Limited Liability.

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1 Introduction

Incentive contracts between principals and agents are fundamental for a wide range of economic and financial transactions. Principals apply agency contracts to provide incentives for employees, managers, entrepreneurs, insurance buyers, sales personnel, business representatives, independent contractors, tenant farmers, and regulated firms. When the agent's action is unobservable, the optimal agency contract is designed to maximize the joint benefits of the parties while mitigating moral hazard. Despite the importance of such contracts and their widespread usage in the economy, the form of the contract remains less well understood by researchers in economics and finance. We derive the optimal contract in an environment that corresponds to a wide range of applied problems in economics and finance.

We study a model where the outcome of a task performed by the agent depends on the effort of the agent and a random shock. The principal observes the outcome of the task, but cannot observe the agent's effort or the random shock. The principal is unable to sell the task to the agent because of the agent's limited liability constraint, which prevents the contract from attaining the first-best outcome. The second best contract effectively sells the agent the proceeds from the task in those states that provide more incentives per unit of expected return. The main result of the analysis characterizes the optimal contract based on the properties of a critical ratio that describes incentives per unit of expected return in each state. The critical ratio equals the hazard rate of the random shock times the marginal return to effort over the marginal return to the shock.

The ratio provides an intuitive and easily derived condition for characterizing the form of the optimal contract. When the critical ratio is increasing in the random shock, higher states are better in providing incentives . Therefore, when the critical ratio is increasing in the random shock, it follows that the optimal contract is debt. When the critical ratio is strictly decreasing in the random shock, lower states are more efficient at providing incentives and the optimal contract is a call option. When the critical ratio is linear in the random shock, all states are equally efficient and only in that case are linear contracts are efficient.Applying these types of conditions in a moral hazard setting yields a consistent characterization of the optimal contract based on assumptions that can be easily verified and occur in most applications in economics and finance.

Because the critical ratio is the hazard rate of the random shock multiplied by the marginal return to effort over the marginal return to the shock, determining its properties is straightforward. A suffient condition for the critical ratio to be increasing in the random shock is for both of its factors to be increasing. This property holds for a wide range of applied models in economics and finance. The hazard rate of the shock is nondecreasing for many probability distributions including the exponential family of distributions and the uniform distribution. Effort and the shock are complements when the marginal return to effort over the marginal return to the shock is increasing in the random shock. Complementarity is consistent with many types of models including additive and multiplicative outcome functions. Outcome functions such that the marginal return to effort over the marginal return to the shock is increasing in the random shock are used in models with uncertainty regarding prices, taxes, subsidies, outputs, and technology. Such outcome functions also occur in models with uncertainty regarding discount rates, depreciation rates, failure rates, and natural shocks such as weather and demographic changes. Parametric uncertainty occurs in models with demand uncertainty, supply uncertainty, and strategic uncertainty. Parametric uncertainty can takes the form of uncertain lotteries with linear probabilities and random financial valuations. Parametric uncertainty is important because it is consistent with empirical analysis in economics and finance.

The critical ratio approach introduced here for moral hazard models turns out to be analogous to familiar conditions in adverse selection models. First, requirement that the probability distribution of random shocks has a non-decreasing hazard rate corresponds to a common requirement for the distribution of agent types in adverse selection problems. Second, the requirement that the random shock and the agent's effort are complementary corresponds to the familiar single-crossing or Spence-Mirrlees condition in adverse selection problems. An important benefit of our approach is to identify fundamental connections between moral hazard and adverse selection.

In agency models with moral hazard, the principal observes the outcome that results from the agent's effort and random shocks. The standard approach is to examine the probability distribution over outcomes induced by the agent's effort. In contrast, the critical feature of our modeling approach is that we explicitly consider random shocks rather than working with an induced distribution over outcomes. By examining the interaction between random shocks and the agent's effort, we can exploit a similarity between moral hazard models and adverse selection models. The agent has private information in both types of models. The agent has private information about his action in a moral hazard model and the agent in an adverse selection model has private information about his type but not his action. The key similarity between the two models is the correspondence between the random shock in a moral hazard model and the agent's private information in an adverse selection model. The main difference between the two types of models is that the agent in a moral hazard model moves before observing the outcome of the random shock whereas the agent in an adverse selection model moves after observing his type.

We obtain necessary and sufficient conditions for the optimality of debt. Our results provide intuition for the optimality of debt contracts that does not depend on the standard Monotone Likelihood Ratio Property (MLRP) assumption but rather on the distribution of the shocks. The intuition can be extended to derive the optimal contract in scenarios where debt fails to be the optimal contract. We assume that the principal and the agent are risk neutral, which allows for the study of situations in which risk preferences do not apply, as occurs in contracting between firms. The agent has limited liability, which restricts his ability to take on risk. The agent's limited liability makes debt a powerful instrument for inducing effort under uncertainty.

This approach highlights the importance of wealth constraints, see Innes (1990) and Holmstrom and Tirole (1997). Our approach is closest to that of Innes (1990) who shows that the optimal agency contract takes the form of debt. Innes requires contracts to be chosen such that the principal's benefit is monotonic in output. This feasibility condition is desirable because it guarantees that the principal does not have an incentive to sabotage the outcome. Otherwise, the principal would wish to impede the agent's efforts so as to avoid paying the reward for effort. The monotonicity requirement also is desirable in situations where agents can shirk and costlessly reduce their performance reports. Given the monotonicity condition, Innes shows that the optimal contract must resemble debt. Our results confirm Innes' conclusion about the optimality of debt-style contracts while substantially extending the analysis of optimal contracts and providing new intuition for the results. Our results remedy a critical shortcoming in Innes' (1990) analysis. His main assumptions, including implementability and MLRP, are extremely difficult to satisfy and do not hold in most economics and finance analyses. To our knowledge, there is no example in the literature that satisfies all of these assumptions. In contrast, our main assumption that output is monotonic with respect to a random shock is readily satisfied in most economic and finance models.

Our analysis of the problem of moral hazard with explicit modeling of uncertainty has implications for various economic and finance applications. The optimality of debt-style contracts implies that such contracts perform better under uncertainty than other contractual forms. Perhaps most significantly, debt-style contracts perform better than linear sharing rules under uncertainty. The economic model of agency has its origins in labor contracts in agrarian economics, particularly sharecropping and piece-rate labor contracts, see Otsuka et al. (1992). Economic studies often assume that the contract is a linear sharing rule, appealing to the dynamic aggregation result of Holmstrom and Milgrom (1987). Debt-style contracts clearly have empirical implications. In practice, debt-style contracts are widely used and correspond to all kinds of compensation agreements with threshold effects, such as bonuses for employees.

Clearly debt-style contracts with moral hazard have important financial implications. The analysis implies that debt-style contracts are highly useful when providing financing to entrepreneurs who devote unobservable effort to establishing a firm. Our analysis provides a new "pecking order" result. Myers (1984) and Myers and Majluf (1984) establish a "pecking order" theory of financing based on adverse selection when the entrepreneur has better information about the quality of the project than does the investor. Our result extends this important insight to the moral hazard involved in financing the entrepreneur. Moral hazard provides the agent with an incentive to provide his own financing before seeking external financing. Moreover, moral hazard provides the agent with an incentive to obtain debt before seeking equity financing. This is because debt performs better than equity financing for entrepreneurs with unobservable effort because equity corresponds to a sharing rule.

Our analysis showing the greater efficiency of debt in comparison with equity also has important implications for managerial finance. Debt-style contract provide a means of reducing the manager's incentives to shirk. This addresses the vast literature on corporate finance, beginning with the work of Jensen and Meckling (1976). When managers are risk neutral and have limited liability, debt-style contracts perform better that equity in providing incentives to perform. Debt-style contracts also correspond to financial assets including securities and bonds whose features resemble options, see for example Cox and Rubinstein (1985). Real options are an important tool for analyzing investment under uncertainty, see Dixit and Pindyck (1994).The simplicity of debt-style contracts and their optimality in a broad range of environments helps to explain their wide-spread application.

Our explicit modelling of uncertainty differs from the now standard approach of working directly with the induced probability distribution on outcomes that depends on effort. Explicit recognition of uncertainty is present in Spence and Zeckhauser (1971), Ross (1973), and Harris and Raviv (1976) who apply a first-order approach with risk-averse agents. The reduced-form approach originates with Mirrlees (1974, 1976) and Holmstrom (1979). By considering uncertainty explicitly, we can obtain a characterization of the optimal contract based on the underlying form of risk and the production function.

The paper is organized as follows. Section 2 presents the basic agency model. Section 3 derives the optimal agency contract. Section 4 discusses the implications of the main results and provides formal proofs. Section 5 presents a formal proof of the results. Section 6 examines necessary conditions for debt contracts and derives optimal contracts that do not take the form of debt. Section 7 concludes the discussion.

2 The Basic Model of Agency

Consider two risk-neutral economic actors who enter into a contract. The actor designated as the principal owns a task and the actor designated as the agent performs the task. The task can represent various projects including the following: the agent performs a service under authority delegated by the principal, the agent produces a good within a firm owned by the principal, or the agent is an entrepreneur who uses financial capital provided by the principal to establish a firm.

The agent provides the production technology for the project and supplies productive effort, a. The agent owns a production technology given by

$$\Pi = \Pi(\theta, a), \tag{1}$$

where Π represents the outcome and θ is a random variable. The principal observes the outcome but cannot observe either the agent's effort a or the random variable θ. The agent chooses effort before the realization of the random variable occurs. Assume that Π is twice differentiable in a and θ. The random variable θ has a density function $f(\theta)$ with a connected support $[0, \overline{\theta}]$.

Assumption 1 *The outcome function, $\Pi(\theta,a)$, is increasing in θ.*

This assumption allows us to define $\widehat{\theta}(a, \Pi)$ as the size of the shock that satisfies

$$\Pi(\widehat{\theta}(a, \Pi), a) = \Pi, \tag{2}$$

for any effort, a, and realization of revenues, Π.

Assumption 2 *The outcome function, $\Pi(\theta,a)$, is increasing and concave in a.*

Assumption 2 guarantees the existence of a first best effort level a^{FB}, and implies that the induced distribution satisfies first order stochastic dominance in a.

The contract can be based only on the outcome Π because the agent's choice of effort a, and the realization of the random variable, θ, are not observable to the principal. The agent has a disutility of effort given by a. The contract between the principal and the agent is fully described by a function of the realized benefit,

$$w = w(\Pi). \tag{3}$$

The agent′s net benefit is given by

$$u(w, a, \Pi) = w(\Pi) - a. \tag{4}$$

Given the contract, w, the agent chooses effort to maximize his expected net benefit,

$$U(w, a) = \int_0^{\overline{\theta}} w(\Pi(a, p))f(p)dp - a. \tag{5}$$

The first-best effort level a^{FB} *is the level that satisfies* $\int \Pi_a(\theta, a)f(\theta)d\theta =$ 1.The agent's incentive compatibility constraint is:

$a \in \arg\max_a \int_0^{\overline{\theta}} w(\Pi(\theta, a))f(\theta)d\theta - a.$

The principal's cost of providing the task to the agent is $K > 0$. For any realization of Π, the principal's net benefit is $v(w, \Pi) = \Pi - w(\Pi) - K$.The principal's expected net benefit given the form of the outcome function and the distribution of the random shock equals

$$V(w, a) = \int_0^{\overline{\theta}} (\Pi(\theta, a) - w(\Pi(\theta, a)))f(\theta)d\theta - K. \tag{6}$$

The optimal contract maximizes the net benefit of the agent subject to an individual rationality constraint for the principal. This approach follows that of Innes (1990). It is the dual of the standard approach of maximizing the principal's net benefit subject to the agent's individual rationality constraint. The principal's individual rationality constraint is $V(w, a) \geq 0$.

We define a feasible contract based on several requirements.

Definition 1 *A contract w is said to be feasible if it satisfies two conditions. (a) The principal's net benefit, $v(w, \Pi)$, is non-decreasing in Π, b) The agent's payments are non-negative $w \geq 0$.*

The monotonicity requirement on the principal's net benefit, $v(w, \Pi)$, follows Innes (1990). As Innes explains, making the principal's net benefit non-decreasing in Π rules out situations in which the parties may subvert the contract. Otherwise, if the principal's return is decreasing in the outcome, the principal may attempt to sabotage the task to avoid making payments to the agent. Alternatively, an entrepreneur may costlessly borrow money to supplement the return of the firm and thereby increase his returns based on reported performance. The monotonicity requirement rules out forcing contracts that would lead to this type of behavior. The non-negativity requirement represents the fact that the agent has limited wealth, and it prevents the agent from buying the firm and achieving the first best.

Definition 2 *An effort level a is implementable if there exists a feasible contract w such that a)The effort level a is incentive compatible $a \in \arg\max U(w, a)$, and b) The principal's participation constraint holds $V(w, a) \geq 0$.*

We assume that the set of implementable effort levels is non empty. This is basically a requirement that the principal's cost, K, is not too large.

Assumption 3 *There exists an implementable effort level a.*

This assumption guarantees that the task is viable.

3 Optimal Agency Contracts

The optimal contract between the principal and the agent maximizes the agent's expected net benefit over the set of feasible contracts,

$$\max_{w(.),a} U(w,a) \;\text{ subject to}$$

$$a \in \arg\max U(w,a),$$

$$v(w,\Pi) \text{ is (weakly) increasing in } \Pi,$$

$$V(w,a) \geq 0,\; w(\cdot) \geq 0.$$

The maximization of the agent's net benefit helps to highlight the effects of the agent's limited liability. There is no loss of generality. The results are consistent with analyses of agency that maximize the principal's net benefit. Varying the agent's endowment generates other allocations of rents between the principal and the agent without changing the characterization of the optimal contract.

Our first result apply three conditions that are consistent with most applications in economics and finance. The first condition is on the distribution of the shock.

Hazard Rate Condition *The probability distribution $f(\theta)$ satisfies the hazard rate condition if $\frac{f(\theta)}{1-F(\theta)}$ is non-decreasing in θ.*

The hazard rate condition is commonly used in adverse selection models such as auctions and nonlinear pricing. A large class of distributions satisfies the hazard rate condition. The hazard rate condition as applied in studies of reliability lends itself to empirical testing and calibration[1].

The second condition applies to the production function. Examples will be given in the next section.

[1] See for example Hall and Van Keilegom (2005).

Complementarity condition *The outcome function* Π *satisfies the complementarity condition if* Π_a/Π_θ *is non-decreasing in* θ.

If the profit function Π is concave in θ , the complementarity condition is implied by the increasing differences condition $\Pi_{a\theta} > 0$ that is used monotone comparative statics analysis, see Topkis (1998) and Milgrom and Shannon (1994). Similar conditions appear frequently in the literature on adverse selection where it is referred to as the single-crossing or Spence-Mirrlees condition.

It will be shown that the optimal agency contract takes the form of debt. Letting r be the face value of the debt, the agent obtains nothing if the outcome is less than or equal to the face value of the debt. Otherwise, the agent obtains the difference between the outcome and the face value of the debt. With a debt contract, the agent chooses the effort level that solves:

$$\max_a \int_0^{\overline{\theta}} \max\{\Pi(\theta, a) - r, 0\} f(\theta) d\theta - a, \tag{7}$$

The next condition is a regularity condition that rules out pathological cases when there is a debt contract.

Weak implementability condition *There exists at most one local interior maximum in a standard debt contract.*

The condition is not stated in terms of the parameters of the model, but is easily satisfied.[2] We have been unable to find an example that satisfies the hazard rate and increasing differences conditions, but fails to satisfy the weak implementability condition.

[2] In terms of the parameters of the model, a sufficient condition for the weak implementability condition to hold is the following.

$$\int_{\widehat{\theta}(r,a)}^{\overline{\theta}} \Pi_{aaa}(\widehat{\theta}, a) f(\theta) d\theta - 2\frac{\partial \widehat{\theta}}{\partial a} \Pi_{aa}(\widehat{\theta}, a) f(\widehat{\theta}) + \frac{\partial^2 \widehat{\theta}}{\partial a^2} \Pi_{aa}(\widehat{\theta}, a) f(\widehat{\theta}) < 0 \ \forall a, r$$

This is equivalent to a concavity condition on Π.

To illustrate how easy it is to satisfy this assumption, consider the additively-separable model,

$$\Pi(\theta, a) = \theta + Q(a). \tag{8}$$

In this case the agent gets paid as long as $\theta > r - Q(a)$ and therefore the agent's problem can be expressed as

$$\max_a \int_{r-Q(a)}^{\overline{\theta}} [\theta + Q(a) - r] f(\theta) d\theta - a \tag{9}$$

Any local interior maximum satisfy the first order condition,

$$Q_a \int_{r-Q(a)}^{\overline{\theta}} f(\theta) d\theta = 1 \tag{10}$$

The second order condition for a maximum is:

$$Q_{aa} \int_{r-Q(a)}^{\overline{\theta}} f(\theta) d\theta + Q_a^2 f\left(\frac{r}{Q}\right) \leq 0 \tag{11}$$

The second order condition is satisfied as long as:

$$\frac{Q_{aa}}{Q_a^2} + \frac{f\,(r-Q)}{1 - F(r-Q)} \leq 0 \tag{12}$$

If 12 is decreasing; then the weak implementability condition is satisfied. The reason is that if two local interior maximums exist then there exists a local minimum in between two local maximums; but that is incompatible with the second derivative being decreasing in a.

The second term in (12) is the hazard rate, it is decreasing in a since the distribution satisfies the hazard rate condition. If the function Q is sufficiently concave, the first term in (12) will also be decreasing, and the weak implementability condition will hold. For example if $Q(a) = Ln(a+1)$ the condition is satisfied for any distribution of θ. Recall that these conditions are sufficient, many additive examples that do not satisfy these properties still satisfy the weak implementability condition.

To further illustrate the generality of the assumptions, consider a multiplicatively separable model,

$$\Pi(\theta, a) = \theta Q(a).$$

Under these circumstances the agent gets paid as long as $\theta > \frac{r}{Q(a)}$ and therefore the agent's problem can be expressed as:

$$\max_a \int_{r/Q(a)}^{\overline{\theta}} [\theta Q(a) - r] f(\theta) d\theta - a. \tag{13}$$

The first order condition of the problem is:

$$Q_a \int_{r/Q(a)}^{\overline{\theta}} \theta f(\theta) d\theta = 1. \tag{14}$$

The second order condition is

$$Q_{aa} \int_{r/Q(a)}^{\overline{\theta}} \theta f(\theta) d\theta + r \frac{Q_a^2}{Q^3} f\left(\frac{r}{Q}\right) \leq 0. \tag{15}$$

The second order condition can be re-written as:

$$\frac{Q_{aa} Q^3}{r Q_a^2} + \frac{f\left(\frac{r}{Q}\right)}{1 - F\left(\frac{r}{Q}\right)} \frac{1}{E\left[\theta | \theta \geq \frac{r}{Q}\right]} \leq 0. \tag{16}$$

The first term $\frac{Q_{aa}Q^3}{Q_a^2}$ is decreasing for a large family of positive concave functions such as for example, the function a^γ with any $\gamma < 1$, and logarithmic functions. The second term is the multiplication of the hazard rate times one over a conditional expectation. The hazard rate decreases in a by assumption, however the conditional expectation also decreases. The complete term is decreasing for common distributions such as the uniform.

We now present our main result. The optimal contract takes the form of a debt obligation.

Proposition 1 *If the hazard rate, complementarity, and weak implementabil-*

ity conditions hold, then the optimal contract is a standard debt contract,

$$w(\mathit{\Pi}) = \max\{\mathit{\Pi} - r, 0\} \text{ for some } r \geq 0. \tag{17}$$

The optimal contract is nondecreasing in the outcome Π and strictly increasing in the outcome Π when the payment to the agent is positive. As a result, the expected payment is strictly increasing in the underlying effort chosen by the agent.

It is important to observe that although the optimal contract takes the form of debt, it applies to any type of incentive contract with moral hazard including situations that do not involve financial obligations. The face value of the debt specified by the contract is a cut-off level such that the agent receives no payment when the outcome Π is below the cut-off and the agent receives the difference between the outcome and the cut-off value otherwise. This contract is optimal because it selects good states in which the agent receives a reward and specifics the incremental return in those states.

4 Discussion

4.1 Intuition and the Critical Ratio

The intuition for the main result can be explained as follows. Assume that the solution to the problem is continuous and differentiable. Moreover suppose that it is the case that the first order condition approach is valid, and that any effort level is implementable with some debt contract. Then, given a contract $w(\Pi)$, the effort of the agent is described by the first order condition

$$\int_0^{\overline{\theta}} w_\Pi(\Pi(\theta,a))\Pi_a(\theta,a)f(\theta)d\theta = 1 \tag{18}$$

where $w_\Pi(\Pi)$ is used as a short of $\partial w(\Pi)/\partial\Pi$ and Π_a as a short for $\partial\Pi(\theta,a)/\partial a$. The derivative of the left-hand side of equation (18) with respect to the slope of the compensation (w_Π) is $\Pi_a f(\theta)$. Provided the first-order condition approach is valid, the higher $\Pi_a f(\theta)$ is, the less we need to increase the payoff to induce a given effort level. The term $\Pi_a f(\theta)$ represents how powerful is a given state in providing incentives. Intuitively, the greater are the likelihood of a state and the marginal return to effort, the more efficient is the state in providing incentives.

The benefit of the contract to the agent is given by,

$$\int_0^{\overline{\theta}} w(\Pi(\theta,a))f(\theta)d\theta - a.$$

We can rewrite the benefit as a function of w_Π using integration by parts

$$u(w,a) = \int_0^{\overline{\theta}} w_\Pi(\Pi(\theta,a))(1-F(\theta))\Pi_\theta d\theta - a. \tag{19}$$

The derivative of the expected payoff with respect to w_Π is $(1-F(\theta))\Pi_\theta$. The higher the term $(1-F(\theta))\Pi_\theta$ is, the less we need to increase the slope of the contract w_Π to provide a given compensation level to the agent. The term $(1-F(\theta))\Pi_\theta$ represents how efficient a state is in providing compensation. Intuitively the lower the revenue and the faster the revenue increases in the state of nature θ, the more efficient the state is in providing a compensation to the agent.

The optimal contract depends on the critical ratio of incentives per unit of compensation. The critical ratio is the product of the hazard rate and the marginal product of the agent's effort over the marginal product of the random shock in producing outcomes,

$$\rho = \frac{f(\theta)}{1 - F(\theta)} \frac{\Pi_a}{\Pi_\theta}, \tag{20}$$

This ratio measures the relation between incentives and expected payoff when the slope of the contract w_Π changes. Because we have assumed that θ has non-decreasing hazard rate and since $\Pi_a/\Pi_\theta = \theta$ is increasing in θ, the ratio ρ is also increasing in the state of nature θ. Intuitively, contracts that are steeper at high revenue realizations provide more incentives and less compensation than contracts steeper at low revenue realizations. This makes debt contracts the most efficient, in the sense that among all the contracts that provide the agent with a given level of compensation, debt contracts maximize the return to investors. This is the first key characteristic of debt contracts. The second characteristic of debt contracts is that the effort level that the agent exerts is (broadly speaking) decreasing in the face value of the debt. To see why, observe that with a debt contract the expected net benefit of the entrepreneur is given by

$$\max_a U(a, R) = \int_0^{\overline{\theta}} \max\{\Pi(\theta, a) - r, 0\} f(\theta) d\theta - a,$$

which has decreasing differences in $\{\theta, a\}$ since $\partial^2 U/\partial r \partial a = -1/\Pi_{\widehat{\theta}(r,a)} < 0$. By standard monotone comparative static results, this implies that the effort is decreasing in the face value of the debt.

The argument of the proof can be summarized as follows. Take a contract $w()$ that implements a. First notice that the debt contract that implement the same effort level a maximize the payoff to the principal and therefore the principal's participation constrained is not binding under the debt contract. Because the participation constrain is not binding one can lower slightly the face value of the debt r without violating any constraint. If we lower the face value of the debt, the effort the agent exerts increases, increasing the surplus. If we lower the face value of the debt until the investor's participation constraint becomes binding again, then the investor is by definition not better off, and the agent is exerting more effort. The fact that the agent exerts more effort implies that there is more surplus, and since the investor is no better, it must be the case that the agent is strictly better.

The formal proof of this result is presented is section 5. The proof addresses four main issues. The first issue is to show that the optimal contract is continuous, the second is to show that the first order condition approach is valid, the third is to show that effort levels that are not implementable with debt contracts are never optimal and the fourth is to show the existence of an optimal contract.

4.2 Comparison with Standard Model.

The standard assumptions in the literature are the Monotone Likelihood ratio Property (MLRP) and the Convex Distribution Function condition or CDFC. The last condition can sometimes been replaced by implementability assumptions weaker than CDFC. The objective of this section is to show that our conditions (1-3) are different from standard conditions. We first show that our assumptions do not imply MLRP and later we show that CDFC or equivalent implementability conditions will not hold in this setting.

4.3 The MLRP Condition

Multiplicative examples are particularly relevant since many shocks in economics such as prices and productivity shocks affect revenues multiplicatively. For example, in Stiglitz' (1974) classic model of sharecropping, the production function has the multiplicative form

$$\Pi = \theta Q(a) \tag{21}$$

Let the probability density be given by $f(\theta) = \gamma + \alpha\theta$ with $\alpha < 1$. If $Q(a)$ is concave enough (see the discussion above) Π satisfies the assumptions of Proposition 1, but it doesn't satisfy $MLRP$.

To see that the multiplicative form does not satisfy the standard affiliation property or MLRP observe that the induced probability distribution over outcomes is as follows,

$$g(\Pi|a) = f(\Pi/Q(a)) = f(\widehat{\theta}(\Pi, a)) \tag{22}$$

Given the induced probability distribution, we can state the standard MLRP condition.

MLRP condition

$$\frac{\partial}{\partial\Pi}\left(\frac{g_a(\Pi|a)}{g(\Pi|a)}\right) > 0 \tag{23}$$

Applying the induced probability distribution on outcomes, observe that

$$\frac{g_a(\Pi|a)}{g(\Pi|a)} = -\frac{f'((\Pi/Q(a)))\Pi Q'(a)}{f(\Pi/Q(a))Q^2(a)} = -\frac{f'(\widehat{\theta}(\Pi, a))}{f(\widehat{\theta}(\Pi, a))}\widehat{\theta}(\Pi, a)\frac{Q'(a)}{Q(a)}$$

Then, the following inequality must hold,

$$\frac{\partial}{\partial\Pi}\left(\frac{g_a(\Pi|a)}{g(\Pi|a)}\right) = -\left(\frac{f'(\theta)}{f(\theta)} + \left[\theta\frac{f''(\theta)f(\theta) - (f'(\theta))^2}{f^2(\theta)}\right]\right)/Q(a) > 0$$

This inequality implies that

$$f'(\theta) + \theta f''(\theta) - \theta (f'(\theta))^2 / f(\theta) < 0 \tag{24}$$

From the form of the probability density, this inequality can be expressed as

$$\alpha - \theta\left(\alpha\right)^2 / \left(\gamma + \alpha\theta\right) < 0$$

If θ is close to zero, this expression converges to $\alpha/\gamma > 0$. So the MLRP condition fails to hold.

4.4 The CDFC Condition .

The second standard assumption in the literature is the Convex Distribution Function Condition or CDFC. The condition can be stated as follows

CDFC Condition For every contract $w(\Pi)$, the problem $\max_a \int_0^{\overline{\theta}} w(\Pi) f(\theta) d\theta - a$ is concave.

Innes (1990) imposes an implementability assumption that is stronger than our earlier weak implementability condition. CDFC implies Innes' strong implementability condition. Innes (1990) requires the following.

Strong Implementability Assumption "*When $w(\Pi)$ takes the standard debt functional form $w(\Pi) = \min\{\Pi, r\}$ for some $r > 0$, there is a unique solution to the agent's effort problem a.*"

We now show that the CDFC condition or the strong implementability condition cannot hold in the present model.

Proposition 2 The CDFC condition or the strong implementability assumption do not hold.

Proof: Because the CDFC condition is stronger than the strong implementability condition , it is sufficient to show that the strong implementability condition does not hold. Suppose to the contrary that implementabiity holds. Then, by the theorem of maximum there exists a continuous function $a(r)$. Moreover this function has support $[0, a(0)]$. This implies that any effort level in $[0, a(0)]$ is implementable with a debt contract.

Note that in a debt contract, the agent's expected utility has increasing differences in $\{r, a\}$. Therefore, by standard monotone comparative statics analysis, the function $a(r)$ must be non-increasing in r. Let $a(\widehat{r})$, be the effort level implemented by an arbitrary debt contract $\widehat{r}$ with $\widehat{r} > \Pi(\overline{\theta}, 0)$. Let $a_1 < a(\widehat{r})$ be an effort level such that $\Pi(\overline{\theta}, a_1) < \widehat{r}$. The level a_1 exists since $\widehat{r} > \Pi(\overline{\theta}, 0)$. Because $a_1 < a(\widehat{r})$ it must be implementable by some debt contract r_1 and because $a(r)$ is non-increasing we know that $r_1 \leq \widehat{r}$. Finally by definition it must be the case that

$$a_1 = \arg \max_{a \in [0,\infty)} \int_0^{\overline{\theta}} \max \{\Pi(a, \theta) - r_1, 0\} f(\theta) d\theta - a \tag{25}$$

But since $\Pi(\overline{\theta}, a_1) < \widehat{r}$ for every θ it is the case that $\max \{\Pi(a, \theta) - r_1, 0\} = 0$ and therefore it means that

$$a_1 = \arg \max_{a \in [0,\infty)} -a \tag{26}$$

This is a clear contradiction since $a = 0$ is optimal in that problem.$\square$

Jewitt (1988) observes that few distributions satisfy both the MLRP and CDFC conditions. One distribution was provided by Rogerson (1985) (attributed to Steve Matthews) and later two classes of differentiable examples were provided by Licalzi and Spaeter (2003). None of these examples satisfy all the conditions assumed in Innes(1990).[3]

[3] Innes assumes the density $g(\Pi|z)$ where Π satisfy all the following conditions:

4.5 Comparison with adverse selection

As discussed in the introduction, the methodology used to study moral hazard in this paper resembles the way we usually deal with adverse selection. To understand the connection, we present the standard model of adverse selection and show how it is related to the moral hazard model.

In the standard adverse selection, or "hidden information" model the agent has a type θ unknown by the principal. The agent can choose an output Π at a cost $c(\Pi|\theta)$. To incentivize the agent to pick the best possible output the principal designs a contract $w(\Pi)$ that determines the payment to the agent contingent on the output Π.The agent's problem is then given by

$$\max_{\Pi} w(\Pi) - c(\Pi|\theta)$$

A different interpretation is that the agent chooses the cost $c(\Pi|\theta)$ depending on his type θ and the contract w. Let $a = c(\Pi|\theta)$. Then the output that agent θ obtains if the cost of effort chosen is a is given by $\Pi(\theta, a) = c^{-1}(a|\theta)$.

The standard assumptions in the adverse selection model is $\frac{\partial^2 c(\Pi,\theta)}{\partial\Pi\partial\theta} < 0$, which implies that $\Pi_{a\theta} > 0$, since $\Pi(\theta, c(\widetilde{\Pi}|\theta)) = \widetilde{\Pi}$. This together with a concavity assumption in θ implies that $\frac{\Pi_a}{\Pi_\theta}$ is nondecreasing in θ,the complementarity requirement used in proposition 1. The problem for the agent can be re-written as

$$\max_{a} w(\Pi(a, \theta)) - a.$$

If the agent were to choose an action before he becomes aware of his own type

1) $\frac{\partial}{\partial\Pi}\left(\frac{g_z(\Pi|z)}{g(\Pi|z)}\right) > 0$;2) Unique implementation with debt (Which is implied by CDFC); 3) $g(\Pi|z) > 0 \forall\Pi, z \geq 0$. 4) $\int \Pi g(\Pi|0)d\Pi = 0$.

then the problem would be

$$\max_a \int_0^{\overline{\theta}} w(\Pi(a,\theta))f(\theta)d\theta - a,$$

This is precisely the agent's problem in the moral hazard problem. In this context the only difference between the moral hazard and the adverse selection model is that in the adverse selection version we assume the type θ is known to the agent but the principal only observes the distribution $f(\theta)$, while in the moral hazard version the type of the agent θ is unknown by both the principal and the agent. The fact that the type is unknown transforms the hidden information (the type) into a hidden action (the effort). We could also interpret the difference between the models as a problem of timing. In the moral hazard model, the agent chooses an action before observing his type, while in the adverse selection model, the agent chooses an after observing his type.

5 Non-debt optimal contracts.

The model can be easily extended to derive the optimal contract in situations that do not lead to debt contracts. In this section we restrict attention to contracts which payments are required to be non-decreasing in output for both agents.

The main idea of these section is that the form of the optimal contract depends on the critical ratio ρ representing incentives per unit of compensation as defined in equation 20 in section 3. If the critical ratio is increasing then high states are more efficient at providing incentives and the optimal contract is debt. If the critical ratio ρ is decreasing lower states are more efficient and the optimal contract is a call option finally if ρ is constant then all contracts are equally good. A nice feature of the results presented in the following propositions is that they do not require any implementability assumption.

Proposition 3 *If the critical ratio* $\rho = (\Pi_a/\Pi_\theta)f(\theta)/(1-F(\theta))$ *is decreasing, then the optimal contract is a call option* contract.

PROOF See the appendix.

Proposition 3 extends the result from Proposition 1. It shows that if the critical ratio ρ is decreasing then low states are the most efficient and it is optimal for the agent to have a call option contract.

Consider now the implications of a constant critical ratio, ρ. For example, let the production function have the additive form,

$$\Pi(\theta, a) = \theta + Q(a) \tag{27}$$

Also, let the probability distribution of the shock have the exponential form, $\theta \sim \exp(\lambda)$. In this case, both debt and call options are optimal. In fact, a constant critical ratio ρ implies that all states are equally efficient in providing incentives.

Proposition 4 *If the critical ratio* $(\Pi_a/\Pi_\theta)f(\theta)/(1-F(\theta))$ *is constant, then linear schemes are optimal.*

PROOF See the appendix

This can also be understood as a negative result. If linear incentive schemes are optimal, then it must be the case that all states are equally efficient in providing incentives. This would imply that the agent is indifferent between all the contracts that implement a given effort level a.This result is significant because of the great variety of applied work that assumes linear incentive schemes and examines the choice of linear coefficients.

In propositions 1, 3 and 4, we have seen that in general the optimal contract is the most efficient in the sense that among all contracts that implement an effort level they provide the highest expected revenue to the principal. The generalization of this principle is stated in the next proposition.

Definition 3 *A contract is of the class L if there exists $\lambda > 0$ so that $w'(\Pi) = 1$ if $\rho > \lambda$ and $w'(\Pi) = 0$ if $\rho < \lambda$.*

The next proposition shows that the L class of contracts is in general better from the principal's perspective.

Proposition 5 *Let l belong to the L class of contracts. Let a be an effort level that satisfies the IC constraint under the contract l and some arbitrary contract $w()$ then $V(w, a) \leq V(l, a)$.*

Proof See the appendix.

This proposition extends the idea that the optimal shape of the contract depends on the behavior of the critical ratio ρ.

6 Formal Proof

The proof of Proposition 1 follows from a series Lemmas. The proof first posits the optimality of a non-debt contract, and then shows that there exists a feasible debt contract that gives the agent a higher utility. Finally it proves that among debt contracts there must exists an optimal one.

The first thing to notice is that if the agent and the principal sign a debt contract of face value r; then the agent gets paid only if $\Pi(\theta, a) \geq r$. The existence and uniqueness of the first-best effort level, a^{FB}, is given by the assumption that Π is concave in a. For a contract to be feasible, the return to the investor $v(\Pi) = \Pi - w(\Pi) - K$ must be increasing in Π ; and therefore

$v(\Pi)$ has at most countably many jumps and it is differentiable $a.e.$ Let k_i be each point of discontinuity and Δ_i be the size of every discontinuity; each Δ_i is strictly positive to keep $v(\Pi)$ increasing in Π. There exists an increasing continuous function $\widetilde{v}(\Pi)$ such that the following equality holds $a.e.$

$$v(\Pi) + K = \widetilde{v}(\Pi) + \sum_{i:k_i \leq \Pi} \Delta_i. \tag{28}$$

Because the entrepreneur gets $w(\Pi) = \Pi - v(\Pi) + K$ then it is also the case that $a.e.$

$$w(\Pi) = \Pi - \widetilde{v}(\Pi) - \sum_{i:k_i \leq \Pi} \Delta_i. \tag{29}$$

Because the equality holds $a.e$ and the distribution does not have any mass points; the utility of agents is the same if two contracts are equivalent $a.e$. Therefore without loss of generality we can restrict attention to the set of functions that can be expressed as the sum of a continuous function, and a step function $w(\Pi) = \widetilde{w}(\Pi) - \sum_{i:k_i \leq \Pi} \Delta_i$ where $\widetilde{w}$ is differentiable $a.e$ and $\widetilde{w}' \leq 1$, so as to keep the payment to the investor increasing in output.

Lemma 1 *The utility of the agent $U(a, w)$ is differentiable with respect to a.*

PROOF By the definition of an integral $\int \sum_{i:k_i \leq \Pi} \Delta_i dF = \sum_i \Delta_i \left(1 - F(\widehat{\theta}(k_i, a)\right)$. The utility of the agent can therefore be written as:

$$U(w, a) = \int_\theta (\widetilde{w}(\Pi(\theta, a)))f(\theta)d\theta - \sum_i \Delta_i \left(1 - F(\widehat{\theta}(k_i, a)\right) - a. \tag{30}$$

And therefore $U_a = \int (\widetilde{w}_\Pi)\Pi_a f(\theta)d\theta + \sum_i \Delta_i f(\widehat{\theta}(k_i, a))\widehat{\theta}_a - 1$.Where we have taken the derivative inside the integral by the Lebesgue dominated convergence theorem.□

Observe that by the implicit function theorem, $\widehat{\theta}_a = -\frac{\Pi_a}{\Pi_\theta}$, and thus any contract that implements $a > 0$ must satisfy the condition:

$$U_a = \int (\widetilde{w}_\Pi)\Pi_a f(\theta)d\theta - \sum_i \Delta_i f(\widehat{\theta}(k_i, a))\frac{\Pi_a}{\Pi_\theta} - 1 = 0. \tag{31}$$

Definition 1 *Let $Z(r)$ be a set such that a belongs to $Z(r)$ if and only if a satisfies the agent IC constraint in a debt contract with face value r.*Let $a(r)$ be the greatest element in $Z(r)$.

Lemma 2 *$a(r)$ is decreasing in the face value of the debt r.*

PROOF In a debt contract the agent solves:

$$\max_a U(a,R) = \int_0^{\overline{\theta}} \max\{\Pi(\theta,a) - r, 0\} f(\theta) d\theta - a, \tag{32}$$

U has increasing differences in $\{a, -r\}$ since $\frac{\partial^2 U}{\partial a \partial r} = -\frac{1}{\Pi_{\widehat{\theta}}(r,a)} \leq 0$ and therefore the result follows from Theorem 2.8.5 in Topkis (1998).□

Lemma 2 does not imply uniqueness or continuity of the effort level a.Lemma 3 shows that different debt contracts implement different effort levels.

Lemma 3 *If $a_0 > 0 \in Z(r_0)$ and a_0 satisfies the first order condition in a debt contract r then $r = r_0$.*

PROOF If $a_0 \in Z(r_0)$ then it satisfies the first order condition

$$\int_{\theta > \widehat{\theta}(r_0, a_0)} \Pi_{a_0} f(\theta) d\theta = 1. \tag{33}$$

Since $\Pi_a > 0$ and $\widehat{\theta}(r_0, a_0)$ is increasing in r there is only one level of r that satisfies 33□

Lemma 4 *If $V(a, r_0) = \overline{V}$ and $a \in Z(r_0)$ then given any $0 \leq V' \leq \overline{V}$ there exists a $r' \leq r_0$ such that $V(a', r') = V'$ for some $a' \in Z(r')$.*

PROOF Define $\widetilde{V}(r) = \min_{\widetilde{r}:\widetilde{r} \in [r, r_0)} V(a(\widetilde{r}), \widetilde{r})$. $\widetilde{V}(r)$ is increasing and continuous in r in all the interval $[0, r_0)$. Increasing follows by definition and continuity follows because either $V(a(r), r)$ is continuous in r or if discontinuous it is decreasing in r. Moreover $\widetilde{V}(0) = 0$ and $\widetilde{V}(r_0) > V'$. By continuity there exists $r' \in [0, r_0)$ such that $\widetilde{V}(r') = V'$ which by definition implies that there exists $r'' \in [r', r_0]$ such that $V(a(r''), r'') = V'$ □

In contrast the net expected benefit of the entrepreneur is always continuous and decreasing in r since it is given by $\max_a U(a,r) - a$.

Lemma 5 *Feasible contracts always induce an effort level* $a \leq a^{FB}$.

PROOF Remember that we can restrict attention to contracts of the form $w(\Pi) = \widetilde{w}(\Pi) - \sum_{i:k_i \leq \Pi} \Delta_i$. Where $w' \leq 1$. The level of effort a is either 0 or satisfies the first order condition:

$$\int_0^{\overline{\theta}} w'(\Pi(\theta,a))\Pi_a(\theta,a)f(\theta)d\theta - \sum_i \frac{\Pi_a}{\Pi_\theta} f(\widehat{\theta}(k_i,a))\Delta_i = 1 \tag{34}$$

If the contract implements $a > a^{FB}$, then by the definition of a^{FB} we know that $\int \Pi_a f(\theta)d\theta < 1$. Moreover we know that $w' \leq 1$ and therefore $\int_0^{\overline{\theta}} w'(\Pi(\theta,a))\Pi_a(\theta,a)f(\theta)d\theta < 1$ which contradicts the fact that the first order condition is satisfied.$\square$

The utility of the agent in a debt contract is given by $\max_a U(a,r) = \max_a \int \max\{\Pi(\theta,a) - r, 0\} - a$ is decreasing and continuous in r. Let $\widetilde{r}$ be the smallest face value of the debt such that $max_a U(a,r) = 0$. Whenever $r < \widetilde{r}$;then $U(a,r) > 0$ which means that an optimal level of a must be strictly positive and therefore must be a local interior maximum. The weak implementability assumption implies that whenever $r < \widetilde{r}$ there exists a unique a optimum in the agent problem, since any optimum is interior, and there exists a unique local interior optimum.

Lemma 6 *Any effort level* $a \in [a(\widetilde{r}), a^{FB}]$ *can be induced with a debt contract.*

PROOF First observe that in $[0,\widetilde{r})$ the effort a is unique and, by the Theorem of the Maximum, it is also continuous. Moreover $\lim_{r \to \widetilde{r}^-} a(r) = a(\widetilde{r})$ since the objective function is continuous in a and r and thus $Z(r)$ is an upper semi-continuous correspondence. Therefore $a(r)$ is continuous and decreasing in $[0,\widetilde{r}]$ and the lemma follows by the intermediate value theorem.$\square$

The next Lemma is fundamental in proving the main result of the paper.

Lemma 7 *If* $a_0 \in Z(r_0)$, $a_0 > a(\tilde{r})$ *and* w *is a contract different from debt that implements* a_0; *then* $V(a_0, w) < V(a_0, r_0)$.

PROOF The proof shows that among all the contracts that satisfy the first order condition at a_0, $V(a_0, w) < V(a_0, r_0)$.

Consider the following problem:

$$\max_w V(a_0, w) \text{ s/t } U_a(a_0, w) = 1; \; v(\Pi) \text{ is non decreasing} \tag{35}$$

A solution to this problem must exist since the problem can be stated as finding the optimal payment for the principal, $\Pi - w(\Pi)$, which is weakly increasing and bounded in a bounded interval. Since increasing bounded functions in a bounded interval are compact in the relevant metric ($\mathcal{L}_1$) the problem must have a solution.[4].

Since maximizing the expected return to the principal is equivalent to minimizing it for the agent the problem can be rewritten as:

$$\min_{w, k_i, \Delta_i} \int_0^{\theta} \left(w\left(\Pi(\theta, a)\right) + \sum_{i: k_i \leq \Pi} \Delta_i \right) f(\theta) d\theta, \tag{36}$$

subject to:

$$\int_0^{\overline{\theta}} w'(\Pi(\theta, a)) \Pi_a(\theta, a) f(\theta) d\theta - \sum_i \frac{\Pi_a}{\Pi_\theta} f(\widehat{\theta}(k_i, a)) \Delta_i = 1, \tag{37}$$

$$w(\cdot) \geq 0, \tag{38}$$

$$w'(\cdot) \leq 1. \tag{39}$$

We first claim that the optimal contract has no discontinuities. To the contrary suppose that the optimal w has a $\Delta_i > 0$ at some $\Pi = k_i$

Consider the alternative contract $\widehat{w}$ that only differs from w in two ways,

1) The discontinuity at k_i decreases in δ

[4] For a proof of this see Dunford and Schwartz Corollary 11 page 294.

2) A new discontinuity of $\delta \frac{f(\widehat{\theta}(k_i,a)\Pi_a/\Pi_\theta(k_i.a)}{f(\widehat{\theta}(k_i-\varepsilon,a)\Pi_a/\Pi_\theta(k_i-\varepsilon.a)}$ is created at $\Pi(\widehat{\theta}-\varepsilon, a)$.

First observe that by construction the constraint 37 is still satisfied.

Second observe that the objective function changes in:

$$\delta\left[\frac{1-F(\widehat{\theta})}{f(\widehat{\theta})}\frac{\Pi_\theta(\widehat{\theta})}{\Pi_a(\widehat{\theta})} - \frac{1-F(\widehat{\theta}-\varepsilon)}{f(\widehat{\theta}-\varepsilon)}\frac{\Pi_\theta(\widehat{\theta}-\varepsilon)}{\Pi_a(\widehat{\theta}-\varepsilon)}\right] \tag{40}$$

This is strictly negative, since we have assumed that $\frac{f(\theta)}{1-F(\theta)}$ and Π_a/Π_θ are increasing in θ . This contradicts the assumption that the optimal contract was w; and therefore the optimal contract cannot have any discontinuities .

Therefore without loss of optimality we can restrict attention to continuous $a.e$ differentiable functions w. The problem then becomes:

$$\min_w \int_0^{\overline{\theta}} w(\Pi(\theta,a))f(\theta)d\theta \tag{41}$$

subject to

$$\int_0^{\overline{\theta}} w'(\Pi)\Pi_a f(\theta)d\theta = 1, \tag{42}$$

$$w' \leq 1, \tag{43}$$

$$w \geq 0. \tag{44}$$

Integrating by parts $\int_0^{\overline{\theta}} w(\Pi)f(\theta)d\theta = w(0) + \int_0^{\overline{\theta}} w'(\Pi)(1-F(\theta))\Pi_\theta d\theta$;

Therefore we can express the problem with the Lagrangian: [5]

$$\min_{w_0,w'} \pounds = w(0) + \int_0^{\overline{\theta}} w'(\Pi)(1-F(\theta))\Pi_\theta - \lambda f(\theta)\Pi_a)d\theta \tag{45}$$

subject to $w' \leq 1$ and $w' \geq 0$ if $w = 0$. First it is clear that it is optimal to set $w(0) = 0$.

The problem is linear in w' and the variation with respect to w' is the term $(1-F(\theta))\Pi_\theta - \lambda f(\theta)\Pi_a)$ proportional to $1/\lambda - \frac{\Pi_a}{\Pi_\theta}\frac{f(\theta)}{1-F(\theta)}$. The second term is

[5] An alternative proof using convex duality is available upon request from the authors.

the hazard rate times $\frac{\Pi_a}{\Pi_\theta}$; both increasing in θ by complementarity and the hazard rate condition. The optimal contract will have the smaller possible w' for small levels of θ and the highest possible for high levels of θ. Because for low levels of θ, $w = 0$ the optimal contract is $w' = 0$ if $\widehat{\theta}(\Pi, a) < \theta^*$ and $w' = 1$ otherwise. Because Π is increasing in θ, this is precisely the form of a debt contract.

If there exists an optimal contract, it is a debt contract r, that satisfies the first order condition. By Lemma 3, the contract must be the debt contract that implements a_0.$\square$

We are now ready to prove the main proposition of the paper.

PROOF of Proposition 1 To the contrary assume the optimal contract w is different from a debt contract and implements an effort level a_0.

i) If $a_0 \geq a(\widetilde{r})$ then by Lemma 7 there exists a debt contract r_0 that implements a_0. By Lemma 4, $V(a_0, r_0) > V(a_0, w)$. By Lemma 5, there exists $r_1 < r_0$ such that $V(a(r_1), r_1) = V(a_0, w)$. Therefore r_1 satisfies the principal's IR constraint. Finally since $a_0 < a_1 \leq a^{FB}; E(\Pi(\theta, a_0)) = V(a_0, w) + U(a_0, w) < V(a_1, r_1) + U(a_1, r_1) = E(\Pi(\theta, a_1))$ and because by definition of r_1, $V(a(r_1), r_1) = V(a_0, w)$ then $U(a_0, w) < U(a_1, r_1)$ which violates the optimality of w.

ii) If $a_0 < a(\widetilde{r})$; then consider the debt contract with face value $\widetilde{r}$. Since $a_0 < a(\widetilde{r})$, $(V(a_0, w) + U(a_0, w) < V(a(\widetilde{r}), \widetilde{r}) + U(a(\widetilde{r}), \widetilde{r})$ and since $U(a(\widetilde{r}), \widetilde{r}) = 0$, $V(a(\widetilde{r}), \widetilde{r}) > V(a_0, w)$. Moreover, by Lemma 5, there exists $r_1 < \widetilde{r}$ such that $V(a(r_1), r_1) = V(a(\widetilde{r}), \widetilde{r})$. Therefore r_1 satisfies the principal's IR constraint. Finally since $a_0 < \widetilde{r} \leq a^{FB}$; $V(a_0, w) + U(a_0, w) < V(a_1, r_1) + U(a_1, r_1)$ and because $V(a(r_1), r_1) = V(a_0, w)$ then $U(a_0, w) < U(a_1, r_1)$ which violates the optimality of w.

We have shown that if an optimal contract exists, it must be a debt contract. To show existence of an optimal debt contract, note that the face value of debt contracts can be bounded above by $\overline{r} = \Pi(a^{FB}, \overline{\theta})$ therefore the set of all debt contracts is $[0, \overline{r}]$, compact, and an optimal contract exists by the Weierstrass Theorem. $\square$

7 Conclusion

The paper develops a basic model of agency with risk neutrality and unobservable effort. Moral hazard inefficiencies arise as a consequence of the agent's limited liability. We provide sufficient and necessary conditions for the optimal contract to be debt. We characterize a critical ratio that determines the form of the optimal contract. This ratio is equal to the hazard rate of the shock times the marginal return on effort over the marginal return of the shock.

The result that the optimal contract takes the form of debt when monotonicity holds, as Innes understood, has far reaching implications. Debt-style contracts help to explain the use of performance targets and rewards in a wide variety of economic situations. Therefore, debt-style contracts are optimal for sharecropping contracts, employee performance contracts, procurement contracts, and regulatory incentives. Moreover, debt-style contracts are at the heart of financial contracts and performance rewards for entrepreneurs and managers. Debt contracts are extremely simple to design and apply, and have important properties that allow for market pricing and trading.

Our analysis emphasizes explicit uncertainty by considering shocks to the output function. By directly examining the random variable that affects the outcome, we can apply two conditions that are familiar from the adverse selection model of agency. The two conditions are that the random variable has a monotonic hazard rate, and the agent's effort and the random variable are complements. They are easily verified for most economics and finance models simply by checking the distribution of the random variable and the form of the production technology. What is most important is that these conditions hold for a very wide range of applications in economics and finance. Our approach should help researchers to derive optimal contracts within many economic and financial models. The assumptions allow for tractable economic analysis that should yield comparative statics results pertaining to uncertainty, technology, and wealth effects.

The present analysis suggests that incentive contracts can have critical performance levels rather than more complex performance schedules. It may be useful to reevaluate many standard analyses of performance incentives by explicitly modeling uncertainty. Our analysis contrasts with the standard results based on an induced probability distribution. The standard results generally apply the MLRP and CDFC conditions which tend not to be satisfied by most models in economics and finance. Such models with implicit uncertainty tend to suggest that performance rewards take the form of piece-rate compensation and linear sharing rules. The present analysis suggests instead that agency contracts can feature critical performance targets, bonus schemes, and performance guarantees.

An important aspect of our analysis is that it identifies a close connection between moral hazard and adverse selection. The uncertainty in the moral hazard setting corresponds to the unobservable type in the adverse selection setting. What is critical is the timing of the agent's decision. Before choosing his effort level, the agent does not observe the outcome of uncertainty in the moral hazard model while the agent observes his type in the adverse selection model. The agent's effort in the moral hazard setting depends on the probability distribution on states of the world, which can be interpreted as the agent's future type. In the moral hazard setting, the agent's action thus depends on anticipating his future type.

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Appendix

PROOF of Proposition 3 The proof follows by contradiction. Take a contract w that implements $\widehat{a}$ and suppose that there exists a set of values of θ S_1 with $w' > \varepsilon$ and $\rho < \lambda$ and a set S_2 with $w' < 1 - \varepsilon$ and $\rho > \lambda$.

Without loss of generality we can assume both sets are of equal measure $\int_{S_1} \theta dF(\theta) = \int_{S_2} \theta dF(\theta)$ (otherwise we can always take subsets of the original sets).We shall prove that there exists a contract that makes the agent strictly better .

Consider a contract that differs from w only in that it decreases w' in ε_1 at S_1 and increases w' in ε_2 at S_2 where ε_1 and ε_2 are such that

$$\varepsilon_1 \int_{S_1} \Pi_a f(\theta) d\theta = \varepsilon_2 \int_{S_2} \Pi_a f(\theta) d\theta$$

First notice that this contract will still implement $\widehat{a}$ since it will continue to satisfy the agent's first order condition which is necessary and sufficient. Notice also that the net expected benefit of the investor changes in

$$\varepsilon_2 \int_{S_2} \Pi_\theta (1 - F(\theta)) d\theta - \varepsilon_1 \int_{S_1} \Pi_\theta (1 - F(\theta)) d\theta$$

Which is proportional to

$$\int_{S_2} \rho \Pi_a f(\theta) d\theta - \int_{S_1} \rho \Pi_a f(\theta) d\theta$$

Which by construction is positive.

At this new contract the same effort level is induced, and the principal is strictly better off.

Take the new contract $w(\Pi)$, and define the contract $w(l, \Pi)$ as follows.

$$w_\Pi(l, \Pi) = w_\Pi(\Pi) + l(1 - w_\Pi).$$

With $l \in [0, 1]$ .Notice that $U(\ w(l, \Pi), a)$ has increasing differences in l, a which implies that the effort level is increasing in l. By the theorem of maximum we also now that a is continous in l. Finally increase l untill the principal's participation constraint becomes binding. Clearly this contract makes the agent better off.□

PROOF of Proposition 4

The proof of proposition 3 is presented after a series of lemmas.

Consider a call option contract with strike price r. The return to the agent is given by:

$$U = \int_0^{\max\{r-Q(a),0\}} [\theta + Q(a)] f(\theta) d\theta + \int_{\max\{r-Q(a),0\}}^{\overline{\theta}} r f(\theta) d\theta \tag{46}$$

Lemma 8 Effort is increasing and continous in the strike price r.

PROOF First notice that it is always the case that $Q(a) < r$, then U is concave in a, since $U_{aa} = Q''(a)F(r - Q(a)) - Q'(a)^2 f(r - Q(a))$. Therefore the first order condition $Q'(a)F(r - Q(a)) = 1$ is a sufficient and necessary condition for optimality. The effort a is therefore continous by the theorem of maximum and increasing by the implicit function theorem.

Lemma 9 Call option contracts maximize the return to investors

PROOF Consider the problem:

$$\min_w \int_0^{\overline{\theta}} w(\Pi(\theta, a)) f(\theta) d\theta \tag{47}$$

$$\int_0^{\overline{\theta}} w'(\Pi) \Pi_a f(\theta) d\theta = 1 \tag{48}$$

$$w' \leq 1 \tag{49}$$

$$w \geq 0 \tag{50}$$

Integrating by parts $\int_0^{\overline{\theta}} w(\Pi) f(\theta) d\theta = w(0, a) + \int_0^{\overline{\theta}} w'(\Pi)(1 - F(\theta)) \Pi_\theta d\theta$;

Therefore we can express the problem with the Lagrangian:

$$\min_{w_0, w'} \pounds = w(0,a) + \int_0^{\overline{\theta}} w'(\Pi)(1 - F(\theta))\Pi_\theta - \lambda f(\theta)\Pi_a) d\theta \qquad (51)$$

subject to $w' \leq 1$ and $w' \geq 0$. First it is clear that it is optimal to set $w(0,a) = 0$.

The problem is linear in w' and the variation with respect to w' is the term $(1 - F(\theta))\Pi_\theta - \lambda f(\theta)\Pi_a)$ proportional to $1/\lambda - Q_a \frac{f(\theta)}{1-F(\theta)}$. The second term is the hazard rate The optimal contract will have the smaller possible w' for higher levels of θ and the lowest possible for low levels of θ. This is the call option contract that implements a.$\square$

PROOF of Proposition 3 To the contrary assume that the optimal contract is w (different from an option contract) and it implements an effort level a_0.

Note that by Lemma 8 there exists an option contract r_0 that implements a_0. By Lemma 9 $V(a_0, r_0) > V(a_0, w)$. By Lemma 8 there exists $r_1 > r_0$ such that $V(a(r_1), r_1) = V(a_0, w)$. Therefore r_1 satisfies the principal's IR constraint. Finally since $a_0 < a_1 \leq a^{FB}$; $E(\Pi(\theta, a_0)) = V(a_0, w) + U(\ a_0, w) < V(a_1, r_1) + U(\ a_1, r_1) = E(\Pi(\theta, a_1))$ and because by definition $V(a(r_1), r_1) = V(a_0, w)$ then $U(\ a_0, w) < U(\ a_1, r_1)$ which violates the optimality of w.$\square$

PROOF of Proposition 4 First observe that with a linear contract of the form

$$w(\Pi) = l\Pi$$

the agent's problem is concave in the effort level a and therefore has a unique solution. Moreover linear incentive contracts can implement any effort level in $[0, a^{FB}]$. We will show that no contract can be better for the agent than a linear contract. To prove this suppose to the contrary that a contract $w()$ that implements $\widehat{a}$, is strictly better for the entrepreneur that any linear contract.

Let $\widehat{l}$ be the linear contract that at $\widehat{a}$ satisfies

$$\int_0^{\overline{\theta}} w'(\Pi)\Pi_a f(\theta)d\theta = \widehat{l}\int_0^{\overline{\theta}} \Pi_a f(\theta)d\theta \tag{52}$$

It is clear that the linear scheme $\widehat{l}$ implements $\widehat{a}$. By the assumption that the agent is better off under $w()$ it must be the case that

$$\int_0^{\overline{\theta}} w(\Pi)f(\theta)d\theta > \widehat{l}\int_0^{\overline{\theta}} f(\theta)d\theta \tag{53}$$

Which integrating by parts imply that

$$\int_0^{\overline{\theta}} w'(\Pi)\Pi_\theta(1 - F(\theta))d\theta > \widehat{l}\int_0^{\overline{\theta}} \Pi_\theta(1 - F(\theta))d\theta \tag{54}$$

Dividing and multiplying by $\Pi_a f(\theta)$, and remembering that $\rho = \frac{\Pi_a}{\Pi_\theta}\frac{f(a)}{(1-F(a))}$ is constant he get

$$\rho\int_0^{\overline{\theta}} w'(\Pi)(\Pi_a f(\theta))d\theta > \rho\widehat{l}\int_0^{\overline{\theta}} \Pi_a f(\theta)d\theta$$

Which contradicts (52) .□

Proof of proposition 5

It follows directly from considering the problem

$$\min_w \int_0^{\overline{\theta}} w(\Pi(\theta, a))f(\theta)d\theta \tag{55}$$

$$\int_0^{\overline{\theta}} w'(\Pi)\Pi_a f(\theta)d\theta = 1 \tag{56}$$

$$w' \leq 1 \tag{57}$$

$$w \geq 0 \tag{58}$$

Integrating by parts $\int_0^{\overline{\theta}} w(\Pi)f(\theta)d\theta = w(0, a) + \int_0^{\overline{\theta}} w'(\Pi)(1 - F(\theta))\Pi_\theta d\theta$;

Therefore we can express the problem with the Lagrangian:

$$\min_{w_0, w'} \pounds = w(0,a) + \int_0^{\overline{\theta}} w'(\Pi)(1 - F(\theta))\Pi_\theta - \lambda f(\theta)\Pi_a)d\theta \tag{59}$$

subject to $w' \leq 1$ and $w' \geq 0$. The optimal contract in this problem is of the class L. Moreover if it implements a then it satisfies the first order condition constraint, and therefore it must give the agent a smaller utility under the effort level a than any other arbitrary contract w.$\square$