Perfect Public Ex-Post Equilibria of Repeated Games with Uncertain Outcomes*

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Abstract

This paper studies repeated games with imperfect public monitoring where the players are uncertain both about the payoff functions and about the relationship between the distribution of signals and the actions played. To analyze these games, we introduce the concept of perfect public ex-post equilibrium (PPXE), and show that it can be characterized with an extenstion of the techniques used to study perfect public equilibria. We then develop identifiability conditions that are sufficient for a folk theorem; these conditions imply that there are PPXE in which the payoffs are approximately the same as if the monitoring structure and payoff functions were known.

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1 Introduction

The role of repeated play in facilitating cooperation is one of the main themes of game theory. Past work has shown that reciprocation can lead to more cooperative equilibrium outcomes even if there is *imperfect public monitoring*, so that players do not directly observe their opponents' actions but instead observe noisy public signals whose distribution depends on the actions played. This work has covered a range of applications, from oligopoly pricing (e.g. Green and Porter (1984) and Athey and Bagwell (2001)), repeated partnerships (Radner, Myerson, and Maskin (1986)) and relational contracts (Levin (2003)). These applications are accompanied by a theoretical literature on the structure of the set of equilibrium payoffs and its characterization as the discount factor approaches 1, most notably Abreu, Pearce, and Stachetti (1986), Abreu, Pearce, and Stachetti (1990, hereafter APS), Fudenberg and Levine (1994, hereafter FL), Fudenberg, Levine, and Maskin (1994, hereafter FLM), and Fudenberg, Levine, and Takahashi (2007). All of these papers assume that the players know the distribution of public signals as a function of the actions played. In some cases this assumption seems too strong: For example, the players in a partnership may know that high effort makes good outcomes more likely, but not know the exact probability of a bad outcome when all agents work hard. This paper allows for such uncertainty, and also allows for uncertainty about the underlying payoff functions.

Specifically, we study repeated games in which the state of the world, chosen by Nature at the beginning of the play, influences the distribution of public signals and/or the payoff functions of the stage game. The effect of the state on the payoff functions can be direct, and can also be an indirect consequence of the effect of the state on the distribution of signals. For example, in a repeated partnership, the players will tend to have higher expected payoffs at a given action profile at states where high output is most likely, so even if the payoff to high output is known, uncertainty about the probability of high output leads to uncertainty about the expected payoffs of the stage game. While the study of uncertain monitoring structures is new, there is a substantial literature on repeated games with unknown payoff functions and perfectly observed actions, notably Aumann and Hart (1992), Aumann and Maschler (1995), Cripps and Thomas (2003), Gossner and Vieille (2003), Wiseman (2005), Hörner and Lovo (2008), Wiseman (2008), and Hörner, Lovo, and Tomala (2008). Our work makes two extensions to this literature- first to the case of unknown payoff functions and imperfectly observed actions but a known monitoring technology, and from there to the case where the monitoring structure is itself unknown.

Because actions are imperfectly observed, the players' posterior beliefs need not coincide in later periods, even when they share a common prior on the distribution of states.[1] This complicates the verification of whether a given strategy profile is an equilibrium, and thus makes it difficult to provide a characterization of the entire equilibrium set. Instead, we consider a subset of Nash equilibria, called *perfect public ex-post equilibria* or *PPXE*. A strategy profile is a PPXE if it is public- i.e. it depends only on publicly available information- and if its continuation strategy constitutes a Nash equilibrium given any state and given any history. In a PPXE, a player's best reply does not depend on her belief, so that the equilibrium set has a recursive structure and the analysis is greatly simplified. As with ex-post equilibrium, PPXE are robust to variations in beliefs about the underlying uncertainty- a PPXE for a given prior distribution is a PPXE for an arbitrary prior.[2]

PPXE is closely related to the "belief-free" equilibria used by Hörner and Lovo (2008) and Hörner, Lovo, and Tomala (2008) in their analyses of games with perfectly observed actions and incomplete information. This equilibrium concept allows players to condition on their type, while PPXE does not, as it requires public strategies. Nevertheless, PPXE can be used to analyze incomplete-information games; here it amounts to a "pooling equilibrium" as players are not allowed to

[1]Cripps and Thomas (2003), Gossner and Vieille (2003), and Wiseman (2005) study symmetric-information settings where actions and payoffs are perfectly observed, so players always have the same beliefs, and this difficulty does not arise. In Aumann and Hart (1992), Aumann and Maschler (1995), Hörner and Lovo (2008), Wiseman (2008), and Hörner, Lovo, and Tomala (2008), players receive private signals about the payoff functions and so can have different beliefs. (In Wiseman (2008) the players privately observe their own realized payoff each period, in the other papers the players do not observe their own realized payoffs, and the private signals are the players' initial information or "type.")

[2]See Bergmann and Morris (2007) for a discussion of various definitions of ex-post equilibrium. Miller (2007) analyzes a different sort of ex-post equilibrium: he considers repeated games of adverse selection, where players report their types each period, as in section 8 of FLM, and adds the restriction that announcing truthfully should be optimal regardless of the announcements of the other players.

condition on their type. We say more about the comparison of these equilibrium concepts in Section 7. In Fudenberg and Yamamoto (2009), we define the notion of a "Bayesian perfect public ex-post equilibrium," which allows players to condition on their initial private information in addition to the public history, and develop an analog of the linear programming characterization of limit equilibrium payoffs. This equilibrium concept reduces to the belief-free equilibrium of Hörner and Lovo (2008) and Hörner, Lovo, and Tomala (2008) when actions are perfectly observed. The PPXE concept is also related to belief-free equilibria in repeated games with private monitoring, as in Piccione (2002), Ely and Välimäki (2002), Ely, Hörner, and Olszewski (2005), Yamamoto (2007), Kandori (2008), and Yamamoto (2009).[3] However, unlike the belief-free equilibria in those papers, PPXE does not require that players be indifferent, and so it is not subject to the robustness critiques of Bhaskar, Mailath, and Morris (2008); this is what motivates our choice of a different name for the concept.

To characterize the limit of the set of PPXE payoffs as the discount factor goes to 1, we extend the FL linear programming characterization of the limit payoffs of PPE. That is, we show in Section 3 that the limit of the set of payoff vectors to PPXE as the discount factor goes to 1 is the intersection of the "maximal half-spaces" in various directions, where each component $\lambda_i(\omega)$ of the direction vector λ corresponds to the weight attached to player i's payoff in state ω. The main new feature is that in a PPXE, the equilibrium payoffs are allowed to vary with the state, and can do so even if the state does not influence the expected payoffs to each action profile- for example there can be PPXE where player 1 does better in state ω_1 and player 2 does better in state ω_2. Thus PPXE can involve a form of "utility transfer" across states. For this reason, the "maximal half space" in these "cross-state directions" can be the whole space, while in FL the maximal half space in each direction is bounded by the feasible set.

In Section 4, we use this characterization to prove an "ex-post" folk theorem, asserting that for any map from states to payoff vectors that are feasible and individually rational in that state, there is a PPXE whose payoffs in each state approximate the target map as the discount factor tends to 1. This theorem uses individual and pairwise full rank conditions as in FLM, and adds the assumption

[3]Belief-free equilibria and the use of indifference conditions have also been applied to repeated games with random matching (Takahashi (2008), Deb (2008)).

that for every pair (i,ω) and $(j,\tilde{\omega})$ of individuals and states, there is a profile α that has "statewise full rank," which means roughly that the observed signals reveal the state regardless of whether i or j (but not both!) unilaterally deviate from α.

Because our proof of the folk theorem uses the LP characterization, it does not explicitly construct equilibrium strategies. To give some insight into the mechanics of how PPXE work, Section 5 presents an explicit construction in two related examples. These examples also help illustrate the role of information conditions in ensuring that PPXE exist, and show how the ex-post folk theorem can apply even though the game does not have a static ex-post equilibrium.

As in FLM, a weaker, "static-threats," version of the folk theorem holds under milder informational conditions. Section 6.1 shows that pairwise full rank can be replaced by the condition of "pairwise identifiability," which can be satisfied with a smaller number of signals and that statewise full rank can be relaxed to "statewise identifiability." Both of these identifiability conditions are equivalent to their full-rank analogs when individual rank conditions are satisfied, but in general they are weaker and can be satisfied in models with fewer signals relative to the size of the action spaces. Even the statewise identifiability condition is stronger than needed, as shown in Section 6.2. In particular, when the individual full rank conditions are satisfied, statewise identifiability requires more signals than in FLM, but statewise distinguishability can be satisfied without a larger signal space. Very roughly speaking, the key is that for every pair of players i, j and pair of states ω, $\tilde{\omega}$, there be a strategy profile whose signal distribution distinguishes between the two states regardless of the deviations of player j, and such that continuation payoffs can give a large reward to player i in state ω without increasing player i's incentive to deviate and without affecting player j's payoff in state $\tilde{\omega}$.

2 Unknown Signal Structure and Perfect Public Ex-Post Equilibria

2.1 Model

Let $\boldsymbol{I} = \{1, \cdots, I\}$ represent the set of players. At the beginning of the game, Nature chooses the state of the world ω from a finite set $\Omega = \{\omega_1, ..., \omega_O\}$. Assume that players cannot observe the true state ω, and let $\mu \in \triangle\Omega$ denote the players' common prior over ω.[4] For now we assume that the game begins with symmetric information: Each player's beliefs about ω correspond to the prior. We relax this assumption in Section 7.

Each period, players move simultaneously, and player $i \in \boldsymbol{I}$ chooses an action a_i from a finite set A_i. Given an action profile $a = (a_i)_{i \in \boldsymbol{I}} \in A \equiv \times_{i \in \boldsymbol{I}} A_i$, players observe a public signal y from a finite set Y according to the probability function $\pi^\omega(a) \in \triangle Y$; we call the function π^ω the "monitoring technology." Player i's realized payoff is $u_i(a_i, y, \omega)$, so that her expected payoff conditional on $\omega \in \Omega$ and on $a \in A$ is $g_i^\omega(a) = \sum_{y \in Y} \pi_y^\omega(a) u_i(a_i, y, \omega)$; $g^\omega(a)$ denotes the vector of expected payoffs associated with action profile a.[5]

In the infinitely repeated game, players have a common discount factor $\delta \in (0,1)$. Let (a_i^τ, y^τ) be the realized pure action and observed signal in period τ, and denote player i's private history at the end of period $t \geq 1$ by $h_i^t = (a_i^\tau, y^\tau)_{\tau=1}^t$. Let $h_i^0 = \emptyset$, and for each $t \geq 1$, let H_i^t be the set of all h_i^t. Likewise, a public history up to period $t \geq 1$ is denoted by $h^t = (y^\tau)_{\tau=1}^t$, and H^t denotes the set of all h^t. A strategy for player i is defined to be a mapping $s_i : \bigcup_{t=0}^\infty H_i^t \to \triangle A_i$. Let S_i be the set of all strategies for player i, and let $S = \times_{i \in \boldsymbol{I}} S_i$. Note that the case of a known public monitoring structure corresponds to a single possible state, $\Omega = \{\omega\}$.

[4]Because our arguments deal only with ex-post incentives, they extend to games without a common prior. However, as Dekel, Fudenberg, and Levine (2004) argue, the combination of equilibrium analysis and a non-common prior is hard to justify.

[5]If u_i depends on ω, it might seem natural to assume that players do not observe the realized value of u_i as the game is played; otherwise players might learn the state from observing their realized payoff. Since we restrict attention to ex-post equilibria, where players' belief about the state do not matter, we do not need to impose this restriction.

We define the set of feasible payoffs in a given state ω to be

$$V(\omega) \equiv \text{co}\{(g^{\omega}(a))|a \in A\} = \{g^{\omega}(\eta)|\eta \in \Delta(A)\};$$

where $\Delta(A)$ is the set of all probability distributions over A: As in the standard case of a game with a known monitoring structure, the feasible set is both the set of feasible average discounted payoffs in the infinite-horizon game when players are sufficiently patient and the set of expected payoffs of the stage game that can be obtained when players use of a public randomizing device to implement distribution η over the action profiles.

Next we define the set of feasible payoffs of the overall game to be

$$V \equiv \times_{\omega \in \Omega} V(\omega),$$

so that a point $v \in V = (v(\omega_1), \cdots, v(\omega_O)) = ((v_1(\omega_1), \cdots, v_I(\omega_1)), \cdots, (v_1(\omega_O), \cdots, v_I(\omega_O)))$.

Note that a given $v \in V$ may be generated using different action distributions $\eta(\omega)$ in each state ω. If players observe ω at the start of the game and are very patient, then any payoff in V can be obtained by a state-contingent strategy of the infinitely repeated game. Looking ahead, there will be equilibria that approximate payoffs in V if the state is *identified* by the signals, so that players learn it over time. Note also that, even if players have access to a public randomizing device, the set of feasible payoffs of the stage game is the smaller set

$$V^U = \{g^{\omega}(\eta)|\eta \in \Delta(A)\}_{\omega \in \Omega},$$

because play in the stage game must be a constant independent of ω.

2.2 Perfect Public Ex-Post Equilibria

This paper studies a special class of Nash equilibria called *perfect public ex-post equilibria* or PPXE; this is an extension of the concept of perfect public equilibrium that was introduced by FLM. Given a public strategy profile $s \in S$ and a public history $h^t \in H^t$, let $s|_{h^t}$ denote its continuation strategy profile after h^t.

Definition 1. A strategy $s_i \in S_i$ is *public* if it depends only on public information, i.e., for all $t \geq 1$, $h_i^t = (a_i^{\tau}, y^{\tau})_{\tau=1}^t \in H_i^t$, and $\tilde{h}_i^t = (\tilde{a}_i^{\tau}, \tilde{y}^{\tau})_{\tau=1}^t \in H_i^t$ satisfying $y^{\tau} = \tilde{y}^{\tau}$ for all $\tau \leq t$, $s_i(h_i^t) = s_i(\tilde{h}_i^t)$. A strategy profile $s \in S$ is *public* if s_i is public for all $i \in \boldsymbol{I}$.

Definition 2. A strategy profile $s \in S$ is a *perfect public ex-post equilibrium* if for every $\omega \in \Omega$ the profile is a perfect public equilibrium of the game with known monitoring structure π^{ω}.[6]

Given a discount factor $\delta \in (0,1)$, let $E(\delta)$ denote the set of PPXE payoffs, i.e., $E(\delta)$ is the set of all vectors $v = (v_i(\omega))_{(i,\omega)\in \boldsymbol{I}\times\Omega} \in \boldsymbol{R}^{I\times|\Omega|}$ such that there is a PPXE $s \in S$ satisfying

$$(1-\delta)E\left[\sum_{t=1}\delta^{t-1}g_i^{\omega}(a^t)\middle| s,\omega\right] = v_i(\omega)$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$. Note that $v \in E(\delta)$ specifies the equilibrium payoff for all players and for all possible states. Note also that the set of PPXE can be empty, in contrast to the case of perfect public equilibria of games with a known state.[7]

The notion of minmax payoff extends to PPXE in a natural way. Let $\underline{v}_i(\omega) = \min_{\alpha_{-i}} \max_{a_i} g_i^{\omega}(a_i, \alpha_{-i})$ be the minmax payoff for player i in state ω, and let

$$V^* \equiv \{v \in V | \forall i \in \boldsymbol{I}\ \forall \omega \in \Omega\ \ v_i(\omega) \geq \underline{v}_i(\omega)\}$$

be the subset of the feasible payoff state where each player receives at least her minmax payoff in each state. Then $E(\delta) \subseteq V^*$, since any perfect public equilibrium of the game with known monitoring structure π must give each player i payoff at least $\underline{v}_i(\omega)$.

By definition, any continuation strategy of a PPXE is also a PPXE, so the set of payoffs of PPXE equals the set of continuation payoffs of PPXE. This recursive structure facilitates the use of dynamic programming techniques to characterize the equilibrium payoff set.

Definition 3. For $\delta \in (0,1)$ and $W \subseteq \boldsymbol{R}^{I\times|\Omega|}$, a pair $(\alpha, v) \in (\times_{i\in \boldsymbol{I}} \triangle A_i) \times \boldsymbol{R}^{I\times|\Omega|}$ of an action profile and a payoff vector is *ex-post enforceable with respect to* δ

[6]That is, s is a public strategy, and for every $\omega \in \Omega$, and any public history $h^t \in H^t$. the continuation strategy profile $s|_{h^t}$ constitutes a Nash equilibrium of the "continuation game" corresponding to $\{h^t, \omega\}$. In this continuation game, players know tha the state is ω, and because all opponents are using public strategies, each player can compute the expected payoff to any of their strategies (public or private) even though $\{h^t, \omega\}$ is not the root of a proper subgame.

[7]With a known state, repeated play of a static Nash equilibrium is a perfect public equilibrium of the repeated game. Similalry, repeated play of a static ex-post equilibrium is a PPXE, but static ex-post equilibria need not exist.

and W if there is a function $w : Y \to W$ such that

$$v_i(\omega) = (1-\delta)g_i^{\omega}(\alpha) + \delta \sum_{y \in Y} \pi_y^{\omega}(\alpha) w_i(y,\omega)$$

for all $i \in \boldsymbol{I}$ and $\omega \in \Omega$, and

$$v_i(\omega) \geq (1-\delta)g_i^{\omega}(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) w_i(y,\omega)$$

for all $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $a_i \in A_i$.

For each $\delta \in (0,1)$, $W \subseteq \boldsymbol{R}^{I \times |\Omega|}$, and $\alpha \in \times_{i \in \boldsymbol{I}} \triangle A_i$, let $B(\delta, W, \alpha)$ denote the set of all payoff vectors $v \in \boldsymbol{R}^{I \times |\Omega|}$ such that (α, v) is ex-post enforceable with respect to δ and W. Let $B(\delta, W)$ be a union of $B(\delta, W, \alpha)$ over all $\alpha \in \times_{i \in \boldsymbol{I}} \triangle A_i$.

To prove our main results, we will use the fact that various useful properties of PPE extend to PPXE.

Definition 4. A subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is *ex-post self-generating with respect to* δ if $W \subseteq B(\delta, W)$.

Proposition 1. *If a subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is bounded and ex-post self-generating with respect to δ, then $W \subseteq E(\delta)$.*

Proof. See Appendix. The proof is very similar to APS; the key is that when W is ex-post self-generating, the continuation payoffs $w(y)$ used to enforce a vector $v \in V \subset \boldsymbol{R}^{I \times |\Omega|}$ have the property that for each $y \in Y$, the vector $w(y) \in \boldsymbol{R}^{I \times |\Omega|}$ can in turn be ex-post generated using a single next-period action α (independent of ω) so that the strategy profile constructed by "unpacking" the ex-post generation conditions does not directly depend on ω. *Q.E.D.*

Definition 5. A subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is *locally ex-post generating* if for each $v \in W$, there exist $\delta_v \in (0,1)$ and an open neighborhood U_v of v such that $W \cap U_v \subseteq B(\delta_v, W)$.

Proposition 2. *If a subset W of $\boldsymbol{R}^{I \times |\Omega|}$ is compact, convex, and locally ex-post generating, then there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta}, 1)$.*

Proof. See Appendix; this is a straightforward generalization of FLM. *Q.E.D.*

3 Characterizing E(δ)

3.1 Using Linear Programming to Bound $E(\delta)$

In this subsection, we provide a bound on the set of PPXE payoffs that holds for any discount factor; the next subsection shows that this bound is tight as the discount factor converges to one.

Consider the following linear programming problem. Let $\alpha \in \times_{i \in \boldsymbol{I}} \triangle A_i$, $\lambda \in \boldsymbol{R}^{I \times |\Omega|}$, and $\delta \in (0,1)$.

$$\text{(LP-Average)} \quad k^*(\alpha, \lambda, \delta) = \max_{v \in \boldsymbol{R}^{I \times |\Omega|}, w: Y \to \boldsymbol{R}^{I \times |\Omega|}} \lambda \cdot v \qquad \text{subject to}$$

$$\text{(i)} \quad v_i(\omega) = (1-\delta) g_i^{\omega}(\alpha) + \delta \sum_{y \in Y} \pi_y^{\omega}(\alpha) w_i(y, \omega)$$

$$\text{for all } i \in \boldsymbol{I} \text{ and } \omega \in \Omega,$$

$$\text{(ii)} \quad v_i(\omega) \geq (1-\delta) g_i^{\omega}(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) w_i(y, \omega)$$

$$\text{for all } i \in \boldsymbol{I},\ \omega \in \Omega, \text{ and } a_i \in A_i,$$

$$\text{(iii)} \quad \lambda \cdot v \geq \lambda \cdot w(y) \qquad \text{for all } y \in Y.$$

If there is no (v,w) satisfying the constraints, let $k^*(\alpha, \lambda, \delta) = -\infty$. If for every $K > 0$ there is (v,w) satisfying all the constraints and $\lambda \cdot v > K$, then let $k^*(\alpha, \lambda, \delta) = \infty$.

Here condition (i) is the "adding-up" condition, condition (ii) is ex-post incentive compatibility, and condition (iii) requires that the continuation payoffs lie in half-space corresponding to direction vector λ and payoff vector v. Note that when $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) \neq 0$ for some $\omega \neq \tilde{\omega}$, condition (iii) allows "utility transfer" across states. This utility transfer is the most significant way that LP-average differs from the linear program in FL, so we will discuss it in more detail below.

As we show in Lemma 1 (a), the value $k^*(\alpha, \lambda, \delta)$ is independent of δ, so that we denote it by $k^*(\alpha, \lambda)$. For each $\lambda \in \boldsymbol{R}^{I \times |\Omega|} \setminus \{0\}$ and $k \in \boldsymbol{R}$, let $H(\lambda, k) = \{v \in \boldsymbol{R}^{I \times |\Omega|} | \lambda \cdot v \leq k\}$. For $k = \infty$, let $H(\lambda, k) = \boldsymbol{R}^{I \times |\Omega|}$. For $k = -\infty$, let $H(\lambda, k) = \emptyset$.

Then, let

$$k^*(\lambda) = \sup_{\alpha} k(\alpha, \lambda),$$
$$H^*(\lambda) = H(\lambda, k^*(\lambda)),$$

and

$$Q = \bigcap_{\lambda \in \boldsymbol{R}^{I \times |\Omega|} \setminus \{0\}} H^*(\lambda).$$

Lemma 1.

(a) *$k^*(\alpha, \lambda, \delta)$ is independent of δ.*

(b) *If $(\lambda_i(\omega))_{i \in \boldsymbol{I}} \neq 0$ for some ω and $(\lambda_i(\tilde{\omega}))_{i \in \boldsymbol{I}} = 0$ for all $\tilde{\omega} \neq \omega$, then $k^*(\lambda) \leq \sup_{\alpha} \lambda \cdot g(\alpha)$.*

(c) *If $\lambda_i(\omega) < 0$ for some (i, ω) and $\lambda_j(\tilde{\omega}) = 0$ for all $(j, \tilde{\omega}) \neq (i, \omega)$ then $k^*(\lambda) \leq \lambda_i(\omega) \underline{v}_i(\omega)$.*

(d) *Consequently $Q \subseteq V^*$.*

Proof. Part (a) follows from the fact that the constraint set in (iii) is a half-space: Suppose that (v, w) satisfies constraints (i) through (iii) in LP-Average for $(\alpha, \lambda, \delta)$. For $\tilde{\delta} \in (0, 1)$, let

$$\tilde{w}(y) = \frac{\tilde{\delta} - \delta}{\tilde{\delta}(1 - \delta)} v + \frac{\delta(1 - \tilde{\delta})}{\tilde{\delta}(1 - \delta)} w(y).$$

Then $(v, \tilde{w})$ satisfies constraints (i) through (iii) in LP-Average for $(\alpha, \lambda, \tilde{\delta})$, so that the set of feasible v in LP-Average is independent of δ, and thus so is $k^*(\alpha, \lambda, \delta)$.

Let Λ^* be the set of $\lambda \in \boldsymbol{R}^{I \times |\Omega|}$ such that $(\lambda_i(\omega))_{i \in \boldsymbol{I}} \neq 0$ for some $\omega \in \Omega$ and $(\lambda_i(\tilde{\omega}))_{i \in \boldsymbol{I}} = 0$ for all $\tilde{\omega} \neq \omega$. Since parts (b) and (c) consider a single state ω they follow from FL Lemma 3.1. Thus $\bigcap_{\lambda \in \Lambda^*} H^*(\lambda) \subseteq V^*$, and part (d) follows from $Q \subseteq \bigcap_{\lambda \in \Lambda^*} H^*(\lambda)$. *Q.E.D.*

Since we already know that $E(\delta) \subseteq V^*$, part (d) of this lemma shows that Q is "not too big": it doesn't contain any payoff vector we can rule out on *a priori* grounds. The next lemma shows that Q is "big enough" to contain all the payoffs of PPXE.

Lemma 2. *For every* $\delta \in (0,1)$, $E(\delta) \subseteq E^*(\delta) \subseteq Q$, *where* $E^*(\delta)$ *is the convex hull of* $E(\delta)$.

Proof. It is obvious that $E(\delta) \subseteq E^*(\delta)$. Suppose $E^*(\delta) \not\subseteq Q$. Then, since the score is a linear function, there is $v \in E(\delta)$ and λ such that $\lambda \cdot v > k^*(\lambda)$. In particular, since $E(\delta)$ is compact, there exist $v^* \in E(\delta)$ and λ such that $\lambda \cdot v^* > k^*(\lambda)$ and $\lambda \cdot v^* \geq \lambda \cdot \tilde{v}$ for all $\tilde{v} \in E^*(\delta)$. By definition, v^* is enforced by $(w(y))_{y \in Y}$ such that $w(y) \in E(\delta) \subseteq E^*(\delta) \subseteq H(\lambda, \lambda \cdot v^*)$ for all $y \in Y$. But this implies that $k^*(\lambda)$ is not the maximum score for direction λ, a contradiction. *Q.E.D.*

To help explain the role of cross-state utility transfers, we will show that the conclusion of Lemma 2 does not hold if constraint (iii) is replaced by the uniform-over-states version

$$\text{(iii}'\text{)} \quad \sum_{i \in \boldsymbol{I}} \lambda_i(\omega) v_i(\omega) \geq \sum_{i \in \boldsymbol{I}} \lambda_i(\omega) w_i(y, \omega) \qquad \text{for all } \omega \in \Omega \text{ and } y \in Y.$$

The resulting "uniform" LP problem corresponds to a form of "ex-post" enforceability on half-spaces, and it is fairly intuitive that the resulting set of payoffs should be attainable by PPXE for large discount factors. However, this condition is too restrictive to capture all of the payoffs of PPXE, as shown by the combination of the following claim and the example that follows it.

Claim 1. *In the LP-Uniform problem formed by replacing (iii) in LP-Average with (iii′), the solution* $k^U(\alpha, \lambda, \delta) \leq \lambda \cdot g(\alpha)$ *for each* α *and* λ. *Therefore,* $k^U(\lambda, \delta) \equiv \sup_\alpha k^U(\alpha, \lambda, \delta) \leq \sup_\alpha \lambda \cdot g(\alpha)$, *and the computed set* Q^U *is a subset of payoffs* V^U *that can be attained with actions that are independent of the state.*

Proof. Inspection of the constraints in the LP-Uniform problem shows that it is equivalent to solving a separate LP problem for each state $\omega \in \Omega$ in isolation. As FL show, a solution to the LP problem for given (α, ω) cannot exceed $\sum_{i \in \boldsymbol{I}} \lambda_i(\omega) g_i^\omega(\alpha)$. Therefore, $k^U(\alpha, \lambda, \delta)$, the maximal score in LP-Uniform for a given α, is at most $\sum_{\omega \in \Omega} \sum_{i \in \boldsymbol{I}} \lambda_i(\omega) g_i^\omega(\omega) = \lambda \cdot g(\alpha)$, so $\sup_\alpha k^U(\alpha, \lambda, \delta) \leq \sup_\alpha \lambda \cdot g(\alpha)$. *Q.E.D.*

In contrast, the following example shows how PPXE can generate payoffs outside of V^U.

Example 1. There are two players, $\boldsymbol{I} = \{1,2\}$, and two possible states, $\Omega = \{\omega_1, \omega_2\}$. In every stage game, player 1 chooses an action from $A_1 = \{U,D\}$, while player 2 chooses an action from $A_2 = \{L,R\}$. Their expected payoffs $g_i^{\omega}(a)$ are as follows.

	L	R
U	2,2	0, 1
D	0,0	1, 1

	L	R
U	1,1	0, 0
D	1,0	2, 2

Here, the left table shows expected payoffs for state ω_1, and the right table shows payoffs for state ω_2. Suppose that the set of possible public signals is $Y = A \times \Omega$, and that the monitoring technology is such that $\pi_y^{\omega}(a) = \varepsilon > 0$ for $y \neq (a, \omega)$, and $\pi_y^{\omega}(a) = 1 - 7\varepsilon$ for $y = (a, \omega)$.

Note that (U,L) is a static Nash equilibrium for each state. Hence, playing (U,L) in every period is a PPXE, yielding the payoff vector $((2,2),(1,1))$. Likewise, playing (D,R) in every period is a PPXE, yielding the payoff vector $((1,1),(2,2))$. "Always (U,L)" Pareto-dominates "always (D,R)" for state ω_1, but is dominated for state ω_2. Note that these equilibrium payoff vectors are in the set V^U. Let $Y(\omega_1)$ be the set $\{y = (a, \omega) \in Y | \omega = \omega_1\}$, and $Y(\omega_2)$ be the set $\{y = (a, \omega) \in Y | \omega = \omega_2\}$. Consider the following strategy profile:

- In period one, play (U,L).
- If $y \in Y(\omega_1)$ occurs in period one, then play (U,L) afterwards.
- If $y \in Y(\omega_2)$ occurs in period one, then play (D,R) afterwards.

After every one-period public history $h^1 \in H^1$, the continuation strategy profile is a PPXE. Also, given 6 any state $\omega \in \Omega$, nobody wants to deviate in period one, since (U,L) is a static Nash equilibrium and players cannot affect the distribution of the continuation play. Therefore, this strategy profile is a PPXE; its payoff vector converges to $v^* = ((2-4\varepsilon, 2-4\varepsilon),(2-4\varepsilon, 2-4\varepsilon))$ as $\delta \to 1$. Observe that $v^* \notin V^U$ if $\varepsilon \in (0, \frac{1}{8})$. In particular, this equilibrium approximates the efficient payoff vector $((2,2),(2,2))$ as the noise parameter ε goes to zero.

The idea of this construction is that players wait one period to learn the state of the world, and once they learn the true state, they adjust their continuation strategy accordingly. When players observe $y \in Y(\omega_1)$ and learn that ω_1 is more likely, they choose "always (U,L)" to achieve an efficient payoff $(2,2)$ for ω_1 (while it gives an inefficient outcome $(1,1)$ for ω_2). Likewise, when players observe $y \in Y(\omega_2)$ and learn that ω_2 is more likely, they choose "always (D,R)" to achieve an efficient payoff $(2,2)$ for ω_2. Notice that when players observe $y \in Y(\omega_1)$ in period one, they choose the continuation payoff vector $w(y) = ((2,2),(1,1))$, which yields more payoffs than v^* for ω_1 but less payoffs for ω_2. This is the sense in which the PPXE allows "utility transfers" across states.

The uniform feasibility constraint (iii′) in LP-Uniform problem rules out this equilibrium because it does not allow utility transfer across states. For example, for $\lambda = (1,1),(1,1))$, $k^U(\lambda) = 6$ while $\lambda \cdot v^* = 8 - 16\varepsilon$.

Example 1 is misleadingly simple, because there is an ex-post equilibrium of the static game, and for this reason there is a PPXE for all discount factors. It is also very easy to construct equilibria that approximate efficient payoffs in this example: simply specify that (U,L) is played for T periods, and then either (U,L) or (D,R) is played forever afterwards, depending on which state is more likely. In section 5, we present an example where there is no static ex-post equilibrium, and hence no PPXE for a range of small discount factors,[8] but where the folk theorem still applies.

3.2 Computing the Limit of $E(\delta)$ as Players Become Patient

Now we show that the set $E(\delta)$ of PPXE payoffs expands to equal all of Q as the players become sufficiently patient, provided that a full-dimensionality condition is satisfied.

Definition 6. A subset W of $\boldsymbol{R}^{I\times|\Omega|}$ is *smooth* if it is closed and convex; it has a nonempty interior; and there is a unique unit normal for each point on its boundary.[9]

Lemma 3. *If* $\dim Q = I \times |\Omega|$*, then for any smooth strict subset* W *of* Q*, there is* $\overline{\delta} \in (0,1)$ *such that* $W \subseteq E(\delta)$ *for* $\delta \in (\overline{\delta},1)$.

[8]To see this note that the equilibrium payoffs of a PPXE for given discount factor δ must lie in the convex hull of the payoffs to strategies of the discounted repeated game, and that this set V^δ will be close to V^U when the discount factor is close to zero.

[9]A sufficient condition for each boundary point of W to have a unique unit normal is that the boundary of W is a C^2-submanifold of $\boldsymbol{R}^{I\times|\Omega|}$.

Proof. From lemma 1 (d), Q is bounded, and hence W is also bounded. Then, from Proposition 2, it suffices to show that W is locally ex-post generating, i.e., for each $v \in W$, there exist $\delta_v \in (0,1)$ and an open neighborhood U_v of v such that $W \cap U_v \subseteq B(\delta_v, W)$.

First, consider v on the boundary of W. Let λ be normal to W at v, and let $k = \lambda \cdot v$. Since $W \subset Q \subseteq H^*(\lambda)$, there exist α, $\tilde{v}$, and $(\tilde{w}(y))_{y \in Y}$ such that $\lambda \cdot \tilde{v} > \lambda \cdot v = k$, $(\alpha, \tilde{v})$ is enforced using continuation payoffs $(\tilde{w}(y))_{y \in Y}$ for some $\tilde{\delta} \in (0,1)$, and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$. For each $\delta \in (\tilde{\delta}, 1)$ and $y \in Y$, let

$$w(y,\delta) = \frac{\delta - \tilde{\delta}}{\delta(1-\tilde{\delta})} v + \frac{\tilde{\delta}(1-\delta)}{\delta(1-\tilde{\delta})} \left(\tilde{w}(y) - \frac{v - \tilde{v}}{\tilde{\delta}} \right).$$

By construction, (α, v) is enforced by $(w(y,\delta))_{y \in Y}$ for δ, and there is $\kappa > 0$ such that $|w(y,\delta) - v| < \kappa(1-\delta)$. Also, since $\lambda \cdot \tilde{v} > \lambda \cdot v = k$ and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$, there is $\varepsilon > 0$ such that $\tilde{w}(y) - \frac{v-\tilde{v}}{\tilde{\delta}}$ is in $H(\lambda, k - \varepsilon)$ for all $y \in Y$, thereby

$$w(y,\delta) \in H\left(\lambda, k - \frac{\tilde{\delta}(1-\delta)}{\delta(1-\tilde{\delta})} \varepsilon \right)$$

for all $y \in Y$. Then, as in the proof of Theorem 3.1 by FL, it follows from the smoothness of W that $w(y,\delta) \in \text{int}W$ for sufficiently large δ, i.e., (α, v) is enforced with respect to int$W$. To enforce u in the neighborhood of v, use α and a translate of $(w(y,\delta))_{y \in Y}$.

Next, consider v in the interior of W. Choose λ arbitrarily, and let α and $(w(y,\delta))_{y \in Y}$ be as in the above argument. By construction, (α, v) is enforced by $(w(y,\delta))_{y \in Y}$. Also, $w(y,\delta) \in \text{int}W$ for sufficiently large δ, since $|w(y,\delta) - v| < \kappa(1-\delta)$ for some $\kappa > 0$ and $v \in \text{int}W$. Thus, (α, v) is enforced with respect to int$W$ when δ is close to one. To enforce u in the neighborhood of v, use α and a translate of $(w(y,\delta))_{y \in Y}$, as before. *Q.E.D.*

These two lemmas establish the following proposition.

Proposition 3. *If* $\dim Q = I \times |\Omega|$, *then* $\lim_{\delta \to 1} E(\delta) = Q$.

It is possible that $\dim Q < I \times |\Omega|$, so that this proposition does not apply, but that $\lim_{\delta \to 1} E(\delta) \neq \emptyset$. A trivial example of this occurs when the state ω has no effect on either the monitoring structure or the payoffs, so that it cannot possibly be observed, but is simply a nuisance parameter. In this $E(\delta)$ is a subset of the space V^U of payoff that can be generated with actions that are independent of the state, so $Q \subseteq E(\delta)$ has dimension at most I. In this particular case, the solution is obviously to ignore the state and characterize the perfect public equilibria of the game where (any) ω is known; these equilibria correspond to the ful set of PPXE of the game with the noise parameter added. More generally, the full-dimension conditions could fail due to the imperfect observability of ω, but ω might matter for the payoff functions. In this case one might be able to characterize $\lim_{\delta\to 1} E(\delta)$ using an extension of the iterative algorithm in Fudenberg, Levine, and Takahashi (2007), but this remains a topic for future research.

4 A Perfect Ex-Post Folk Theorem

In this section we give sufficient conditions for a folk theorem to hold in PPXE. This theorem shows that any map from states of the world to payoffs that are feasible and individually rational in that state can be approximated by equilibrium payoffs as the discount factor goes to 1, and in particular by payoffs of a PPXE. More formally, our folk theorem gives conditions under which $\lim_{\delta\to 1} E(\delta) = V^*$.[10] When this is true, so that efficient payoffs can be approximated by PPXE, the players have little reason to play other sorts of equilibria or to try to change the monitoring structure. Conversely, when the set of PPXE is empty, or when all PPXE are far from efficient but there are efficient sequential equilibria, the PPXE restriction might be less compelling.

Since we have already shown that $Q \subseteq V^*$ and that $\lim_{\delta\to 1} E(\delta) = Q$ under the full-dimension condition, it remains to show that $V^* \subseteq Q$, which is equivalent to showing that $k^*(\lambda) \geq \max_{v\in V^*} \lambda \cdot v$ for each direction λ. Our sufficient conditions are actually stronger than that: they will imply that $k^*(\lambda) = \infty$ for directions λ with non-zero components in two or more states. Conversely, the folk theorem fails if there is a λ such that $k^*(\lambda) < \max_{v\in V^*} \lambda \cdot v$; we use this fact in Example 4 below.

For each $i \in \boldsymbol{I}$, $\alpha \in \times_{i\in \boldsymbol{I}} \triangle A_i$, and $\omega \in \Omega$, let $\Pi_{(i,\omega)}(\alpha)$ represent a matrix with rows $(\pi_y^\omega(a_i, \alpha_{-i}))_{y\in Y}$ for all $a_i \in A_i$.

[10]Recall that $V^* \equiv \{v \in V | \forall i \in \boldsymbol{I}\ \forall \omega \in \Omega\ \ v_i(\omega) \geq \underline{v}_i(\omega)$.

Definition 7. Profile α has *individual full rank for* (i, ω) if $\Pi_{(i,\omega)}(\alpha)$ has rank equal to $|A_i|$. Profile α has *individual full rank* if it has individual full rank for all players and all states.

Let $\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$ be a matrix constructed by stacking matrices $\Pi_{(i,\omega)}(\alpha)$ and $\Pi_{(j,\tilde{\omega})}(\alpha)$.

Definition 8. For each $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$, profile α has *pairwise full rank for* (i, ω) *and* (j, ω) if $\Pi_{(i,\omega)(j,\omega)}(\alpha)$ has rank equal to $|A_i| + |A_j| - 1$.

Definition 9. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, profile α has *statewise full rank for* (i, ω) *and* $(j, \tilde{\omega})$ if $\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$ has rank equal to $|A_i| + |A_j|$.

Note that both pairwise full rank and statewise full rank imply individual full rank. Note also that the pairwise full rank conditions require as many signals as in FLM, and the statewise full rank conditions require at most twice as many signals. (Statewise full rank requires only one more signal than FLM if all players have the same number of actions; it requires twice as many signals if one player has more than two actions and all the other players have only two.)

One way of thinking about the statewise full rank condition is that it guarantees that the observed signals will reveal the state, regardless of the play of player i in state ω and the play of player j (possibly equal to i) in state $\tilde{\omega}$, assuming that everyone else plays according to α. This condition is more restrictive than necessary for the existence of a strategy that allows the players to learn the state: For that it would suffice that there be a single profile α where the distributions on signals are all distinct, which requires only two signals.[11] On the other hand, the condition is less restrictive than the requirement that the state is revealed to an outside observer even if a pair of players deviates. For example, statewise full rank is consistent with a signal structure where a joint deviation by players 1 and 2 could conceal the state from the outside observer, as in a two-player game with $A_1 = A_2 = \{L, R\}$ and $\pi_y^{\omega}(L, R) = \pi_y^{\tilde{\omega}}(R, L)$. Intuitively, since equilibrium conditions only test for unilateral deviations, the statewise full rank condition is

[11] Note that the learnability of the state does not requires that the signal distributions correponding to each state be linearly independent. This is because the players only need to distinguish between a finite set of signal distributions, and not between all possible convex combinations of them.

sufficient for the existence of an equilibrium where the players eventually learn the state. In Section 6, we introduce the more complicated but substantially weaker condition of statewise distinguishability, and show that it is sufficient for a static-threat version of the folk theorem.

The following proposition establishes an ex-post folk theorem. Note that the set of assumptions of this proposition is generically satisfied if $|Y| \geq 2|A_i|$ for all $i \in \boldsymbol{I}$.

Condition 1. Every pure action profile has individual full rank.

Condition 2. For each (i,ω) and (j,ω) satisfying $i \neq j$, there is an action profile α that has pairwise full rank for (i,ω) and (j,ω).

Condition 3. For each (i,ω) and $(j,\tilde{\omega})$ satisfying $\omega \neq \tilde{\omega}$, there is an action profile α that has statewise full rank.

Proposition 4. *Suppose that Conditions 1 through 3 hold. Then, for any smooth strict subset W of V^*, there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta},1)$.*

The following lemmas are useful in this proof.

Lemma 4. *Suppose that Condition 2 holds. Then, there is an open and dense set of profiles each of which has pairwise full rank for all (i,ω) and (j,ω) satisfying $i \neq j$.*

Proof. Analogous to that of Lemma 6.2 of FLM. *Q.E.D.*

Lemma 5. *Suppose that Condition 1 holds. Then, for any $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $\varepsilon > 0$, there is a profile $\underline{\alpha}^{\omega}$ such that $\underline{\alpha}_i^{\omega} \in \arg\max_{\alpha_i} g_i^{\omega}(\alpha_i, \underline{\alpha}_{-i}^{\omega})$; $|g_i^{\omega}(\underline{\alpha}^{\omega}) - \underline{v}_i(\omega)| < \varepsilon$; and $\underline{\alpha}^{\omega}$ has individual full rank for all $(j,\tilde{\omega}) \neq (i,\omega)$.*

Proof. Analogous to that of Lemma 6.3 of FLM. *Q.E.D.*

Lemma 6. *Suppose that a profile α has statewise full rank for (i,ω) and $(j,\tilde{\omega})$ satisfying $\omega \neq \tilde{\omega}$ and that α has individual full rank for all players and states. Then, $k^*(\alpha,\lambda) = \infty$ for a direction λ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) \neq 0$.*

Remark 1. Because $k^*(\alpha,\lambda) \leq \lambda \cdot g(\alpha)$ in the known-monitoring-structure case of FL, this lemma shows a key difference between that setting and the uncertain monitoring structure case we consider here. The idea is that under statewise full rank, the continuation payoffs in such half-spaces can give player i a very large payoff in state ω by giving player j a very low payoff in that state, while reversing this transfer in state $\tilde{\omega}$.

Remark 2. The proof of this lemma is complicated, so we illustrate it here with a simple example. Assume $A_i = \{a'_i, a''_i\}$ and $A_j = \{a'_j, a''_j\}$, and consider LP-Average problem for a direction λ such that $\lambda_i(\omega) = \lambda_j(\tilde{\omega}) = 1$ and all other components of λ are zero. Constraints (i) and (ii) for $(l, \overline{\omega}) \in \boldsymbol{I} \times \Omega \setminus \{(i,\omega),(j,\tilde{\omega})\}$ can be satisfied by some choice of $(w_l(y,\overline{\omega}))_{y\in Y}$ because of individual full rank, and constraint (iii) is vacuous for these coordinates. So the LP problem reduces to finding $(w_i(y,\omega))_{y\in Y}$ and $\left(w_j(y,\tilde{\omega})\right)_{y\in Y}$ to solve

$$k^*(\alpha,\lambda,\delta) = \max_{v,w} v_i(\omega) + v_j(\tilde{\omega})$$

subject to

$$v_i(\omega) = (1-\delta)g_i^{\omega}(\alpha) + \delta \sum_{y\in Y} \pi_y^{\omega}(\alpha) w_i(y,\omega),$$

$$v_j(\tilde{\omega}) = (1-\delta)g_j^{\tilde{\omega}}(\alpha) + \delta \sum_{y\in Y} \pi_y^{\tilde{\omega}}(\alpha) w_j(y,\tilde{\omega}),$$

$$v_i(\omega) \geq (1-\delta)g_i^{\omega}(a_i,\alpha_{-i}) + \delta \sum_{y\in Y} \pi_y^{\omega}(a_i,\alpha_{-i}) w_i(y,\omega), \ \forall a_i \in A_i$$

$$v_j(\tilde{\omega}) \geq (1-\delta)g_j^{\tilde{\omega}}(a_j,\alpha_{-j}) + \delta \sum_{y\in Y} \pi_y^{\tilde{\omega}}(a_j,\alpha_{-j}) w_j(y,\tilde{\omega}), \ \forall a_j \in A_j$$

$$v_i(\omega) + v_j(\tilde{\omega}) \geq w_i(y,\omega) + w_j(y,\tilde{\omega}), \ \forall y \in Y.$$

We claim that $k^*(\alpha,\lambda,\delta) = \infty$ if α has statewise full rank. It suffices to show that for any sufficiently large $v_i(\omega)$ and $v_j(\tilde{\omega})$, there exist $(w_i(y,\omega), w_j(y,\tilde{\omega}))_{y\in Y}$ that satisfy the first four constraints with equalities and

$$w_i(y,\omega) + w_j(y,\tilde{\omega}) = 0, \ \forall y \in Y.$$

Eliminate this last equation by solving for $w_j(y,\tilde{\omega})$. Then the coefficient matrix for the set of the remaining four equations is

$$\begin{pmatrix} (\pi_y^{\omega}(a'_i,\alpha_{-i}))_{y\in Y} \\ (\pi_y^{\omega}(a''_i,\alpha_{-i}))_{y\in Y} \\ (\pi_y^{\tilde{\omega}}(a'_j,\alpha_{-j}))_{y\in Y} \\ (\pi_y^{\tilde{\omega}}(a''_j,\alpha_{-j}))_{y\in Y} \end{pmatrix}$$

The statewise full rank condition guarantees that this matrix has rank four, so the system has a solution for any $(v_i(\omega), v_j(\tilde{\omega}))$, and thus $k^*(\alpha,\lambda) = \infty$. Intuitively, this construction makes $w_i(y,\omega)$ large for signals y that are more likely under in state ω than in state $\tilde{\omega}$ and makes $w_i(y,\omega)$ negative for signals that are more likely under $\tilde{\omega}$, while keeping player i indifferent between all actions in state ω, and player j indifferent in state ω'. This would not be possible if the signal distribution were the same at the two states, or more generally if the above matrix is were singular.

This example only explains why the k^* can be made aribitrarily large when exactly two components of λ are non-zero. And a similar idea applies even if λ has other nonzero components. For example, suppose that $\lambda_i(\omega) = \lambda_j(\tilde{\omega}) = \lambda_l(\overline{\omega}) = 1$ and other components are zero. First, choose $(v_i(\omega), v_j(\tilde{\omega}), w_i(\omega), w_j(\tilde{\omega}))$ as in the above example, so that constraints (i) and (ii) for (i,ω) and $(j,\tilde{\omega})$ are satisfied, $v_i(\omega)$ and $v_j(\tilde{\omega})$ are enough large, and $w_i(y,\omega) + w_j(y,\tilde{\omega}) = 0$ for all $y \in Y$. What remains is to find $w_l(\overline{\omega})$ that satisfy constraints (i) and (ii) for $(l,\overline{\omega})$ and the feasibility constraint

$$v_i(\omega) + v_j(\tilde{\omega}) + v_l(\overline{\omega}) \geq w_i(y,\omega) + w_j(y,\tilde{\omega}) + w_l(y,\overline{\omega}), \ \ \forall y \in Y.$$

But note that $w_i(y,\omega) + w_j(y,\tilde{\omega}) = 0$ and $v_i(\omega) + v_j(\tilde{\omega})$ can be arbitrarily large; so the left-hand side can be infinitely large while the right-hand side is $w_l(y,\overline{\omega})$. This shows that the feasibility constraint impose no restriction in choosing $w_l(y,\overline{\omega})$, and thanks to the individually full rank condition, we can find $w_l(y,\overline{\omega})$ that satisfies constraints (i) and (ii).

Proof of Lemma 6. Let (i,ω) and $(j,\tilde{\omega})$ be such that $\lambda_i(\omega) \neq 0$, $\lambda_j(\tilde{\omega}) \neq 0$, and $\tilde{\omega} \neq \omega$. Let α be a profile that has statewise full rank for all (i,ω) and $(j,\tilde{\omega})$ satisfying $\omega \neq \tilde{\omega}$.

First, we claim that for every $K > 0$, there exist $(z_i(y,\omega), z_j(y,\tilde{\omega}))_{y \in Y}$ such that

$$\sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) z_i(y,\omega) = \frac{K}{\delta \lambda_i(\omega)} \tag{1}$$

for all $a_i \in A_i$,

$$\sum_{y \in Y} \pi_y^{\tilde{\omega}}(a_j, \alpha_{-j}) z_j(y,\tilde{\omega}) = 0 \tag{2}$$

for all $a_j \in A_j$, and

$$\lambda_i(\omega) z_i(\omega) + \lambda_j(\tilde{\omega}) z_j(y,\tilde{\omega}) = 0 \tag{3}$$

for all $y \in Y$. To prove that this system of equations indeed has a solution, eliminate (3) by solving for $z_j(y, \tilde{\omega})$. Then, there remain $|A_i| + |A_j|$ linear equations, and its coefficient matrix is $\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$. Since statewise full rank implies that this coefficient matrix has rank $|A_i| + |A_j|$, we can solve the system.

Next, for each $(l, \overline{\omega}) \in \boldsymbol{I} \times \Omega$, we choose $(\tilde{w}_l(y, \overline{\omega}))_{y \in Y}$ so that

$$(1-\delta)g_l^{\overline{\omega}}(a_l, \alpha_{-l}) + \delta \sum_{y \in Y} \pi_y^{\overline{\omega}}(a_l, \alpha_{-l}) \tilde{w}_l(y, \overline{\omega}) = 0 \tag{4}$$

for all $a_l \in A_l$. Note that this system has a solution, since α has individual full rank. Intuitively, continuation payoffs $\tilde{w}(y)$ are chosen so that players are indifferent over all actions and their payoffs are zero.

Let $K > \max_{y \in Y} \lambda \cdot \tilde{w}(y)$, and choose $(z_i(y, \omega))_{y \in Y}$ and $(z_j(y, \tilde{\omega}))_{y \in Y}$ to satisfy (1) through (3). Then, let

$$w_l(y, \overline{\omega}) = \begin{cases} \tilde{w}_i(y, \omega) + z_i(y, \omega) & \text{if} \quad (l, \overline{\omega}) = (i, \omega) \\ \tilde{w}_j(y, \tilde{\omega}) + z_j(y, \tilde{\omega}) & \text{if} \quad (l, \overline{\omega}) = (j, \tilde{\omega}) \\ \tilde{w}_l(y, \overline{\omega}) & \text{otherwise} \end{cases}$$

for each $y \in Y$. Also, let

$$v_l(\overline{\omega}) = \begin{cases} \frac{K}{\lambda_i(\omega)} & \text{if} \quad (l, \overline{\omega}) = (i, \omega) \\ 0 & \text{otherwise} \end{cases} .$$

We claim that this (v, w) satisfies constraints (i) through (iii) in LP-Average. It follows from (4) that constraints (i) and (ii) are satisfied for all $(l, \overline{\omega}) \in (\boldsymbol{I} \times \Omega) \setminus \{(i, \omega), (j, \tilde{\omega})\}$. Also, using (1) and (4), we obtain

$$\begin{aligned}
&(1-\delta)g_i^{\omega}(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) w_i(y, \omega) \\
=&(1-\delta)g_i^{\omega}(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) \left(\tilde{w}_i(y, \omega) + z_i(y, \omega)\right) \\
=&\left((1-\delta)g_i^{\omega}(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y^{\omega}(a_i, \alpha_{-i}) \tilde{w}_i(y, \omega)\right) + \frac{K}{\lambda_i(\omega)} \\
=&\frac{K}{\lambda_i(\omega)}
\end{aligned}$$

for all $a_i \in A_i$. This shows that (v, w) satisfies constraints (i) and (ii) for (i, ω). Likewise, from (2) and (4), (v, w) satisfies constraints (i) and (ii) for $(j, \tilde{\omega})$. Furthermore, using (3) and $K > \max_{y\in Y} \lambda \cdot \tilde{w}(y)$,

$$\begin{aligned}\lambda \cdot w(y) &= \lambda \cdot \tilde{w}(y) + \lambda_i(\omega) z_i(y,\omega) + \lambda_j(\tilde{\omega}) z_j(y,\tilde{\omega}) \\ &= \lambda \cdot \tilde{w}(y) < K = \lambda \cdot v\end{aligned}$$

for all $y \in Y$, and hence constraint (iii) holds.

Therefore, $k^*(\alpha,\lambda) \geq \lambda \cdot v = K$. Since K can be arbitrarily large, we conclude $k^*(\alpha,\lambda) = \infty$. *Q.E.D.*

Lemma 7. *Suppose that a profile α has pairwise full rank for all (i,ω) and (j,ω) satisfying $i \neq j$. Then, $k^*(\alpha,\lambda) = \lambda \cdot g(\alpha)$ for direction λ such that $(\lambda_i(\omega))_{i\in \boldsymbol{I}}$ has at least two non-zero components for some ω while $\lambda_j(\tilde{\omega}) = 0$ for all $j \in \boldsymbol{I}$ and $\tilde{\omega} \neq \omega$.*

Proof. It follows from Lemma 1 (b) that $k^*(\lambda,\alpha) \leq \lambda \cdot g(\alpha)$. Thus, in what follows, we establish that $k^*(\lambda,\alpha) \geq \lambda \cdot g(\alpha)$. To do so, we need to show that there exist continuation payoffs $(w(y))_{y\in Y}$ in $H(\lambda, \lambda \cdot g(\alpha))$ that enforce $(\alpha, g(\alpha))$.

As in the proof of Lemma 6, for each $i \in \boldsymbol{I}$ and $\tilde{\omega} \neq \omega$, there exist $(w_i(y,\tilde{\omega}))_{y\in Y}$ such that

$$v_i(\tilde{\omega}) = (1-\delta) g_i^{\tilde{\omega}}(a_i,\alpha_{-i}) + \delta \sum_{y\in Y} \pi_y^{\tilde{\omega}}(a_i,\alpha_{-i}) w_i(y,\tilde{\omega})$$

for all $a_i \in A_i$. Moreover, it follows from Lemmas 4.3, 5.3, and 5.4 of FLM that there exist $(w_i(y,\omega))_{(i,y)}$ such that

$$v_i(\omega) = (1-\delta) g_i^{\omega}(a_i,\alpha_{-i}) + \delta \sum_{y\in Y} \pi_y^{\omega}(a_i,\alpha_{-i}) w_i(y,\omega)$$

for all $i \in \boldsymbol{I}$ and $a_i \in A_i$, and

$$\lambda \cdot w(y) = \sum_{i\in \boldsymbol{I}} \lambda_i(\omega) w_i(y,\omega) = \sum_{i\in \boldsymbol{I}} \lambda_i(\omega) v_i(\omega) = \lambda \cdot v.$$

Obviously, the specified continuation payoffs are in $H(\lambda, \lambda \cdot g(\alpha))$ and enforce $(\alpha, g(\alpha))$, as desired. *Q.E.D.*

Lemma 8. *Suppose that α has individual full rank for all $(j,\tilde{\omega}) \neq (i,\omega)$ and has the best-response property for player i and for state ω. Then, $k^*(\alpha,\lambda) = \lambda \cdot g(\alpha)$ for a direction λ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) = 0$ for all $(j,\tilde{\omega}) \neq (i,\omega)$.*

Proof. This is a straightforward generalization of Lemmas 5.1 and 5.2 of FLM. *Q.E.D.*

Proof of Proposition 4. From Lemma 3, it suffices to show that $Q = V^*$. To do so, we will compute the maximum score $k^*(\lambda)$ for each direction λ.

Case 1. Consider λ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) \neq 0$ for some $\tilde{\omega} \neq \omega$ and i possibly equal to j. In this case, players can transfer utilities across different states ω and $\tilde{\omega}$ while maintaining the feasibility constraint, and this construction allows $k^*(\alpha, \lambda, \delta) > \lambda \cdot g(\alpha)$, as Example 1 shows. In particular, from Condition 3 and 6 we obtain $k^*(\lambda) = \infty$ for this direction λ.

Case 2. Consider λ such that $(\lambda_i(\omega))_{i \in \boldsymbol{I}}$ has at least two non-zero components for some ω while $\lambda_i(\tilde{\omega}) = 0$ for all i$\in \boldsymbol{I}$ and $\tilde{\omega} \neq \omega$. It follows from Lemmas 4 and 7 that $k^*(\lambda) = \sup_\alpha k^*(\lambda, \alpha) = \max_{v \in V} \lambda \cdot v$.

Case 3. Consider λ such that $\lambda_i(\omega) \neq 0$ for some (i, ω) and $\lambda_j(\tilde{\omega}) = 0$ for all $(j, \tilde{\omega}) \neq (i, \omega)$. Suppose first that $\lambda_i(\omega) > 0$. Since every pure action profile has individual full rank, $a^* \in \arg\max_{a \in A} g_i^\omega(a)$ also has individual full rank. Therefore, from Lemma 8,

$$k^*(\lambda) \geq k^*(a^*, \lambda) = \lambda_i(\omega) g_i^\omega(a^*) = \max_{v \in V} \lambda \cdot v.$$

On the other hand, from Lemma 1 (b), $k^*(\lambda) \leq \max_{v \in V} \lambda \cdot v$. Hence, we have $k^*(\lambda) = \max_{v \in V} \lambda \cdot v$.

Next, suppose that $\lambda_i(\omega) < 0$. It follows from Lemmas 5 and 8 that for every $\varepsilon > 0$, there is a profile $\underline{\alpha}^\omega$ such that $|k^*(\underline{\alpha}^\omega, \lambda) - \lambda_i(\omega)\underline{v}_i(\omega)| < \varepsilon$. Since Lemma 3.2 by FL asserts that $k^*(\lambda) \leq \lambda_i(\omega)\underline{v}_i(\omega)$, we have $k^*(\lambda) = \lambda_i(\omega)\underline{v}_i(\omega)$.

Combining these cases, we obtain $Q = V^*$. *Q.E.D.*

Because it relies on local generation and Proposition 3, Proposition 4 does not explicitly construct PPXE strategies. To help illustrate how PPXE work, the next section gives an explicit construction of a PPXE. The example also shows how the folk theorem can apply even though the set of PPXE is empty for small discount factors.

5 Explicit Examples of PPXE

Suppose that there are two players, $\boldsymbol{I} = \{1,2\}$, and two states, $\Omega = \{\omega_1, \omega_2\}$. In every stage game, player 1 chooses U or D while player 2 chooses L or R. Players' expected payoffs $g_i^\omega(a)$ are as follows.

ω_1	L	R
U	10,−4	1, 1
D	1,1	0, 0

ω_2	L	R
U	0,0	1, 1
D	1,1	10, −4

Note that these are the payoff matrices in Example 2 of Hörner and Lovo (2008). Note also that the minimax payoffs $((\underline{v}_1(\omega_1), \underline{v}_2(\omega_1)), (\underline{v}_1(\omega_2), \underline{v}_2(\omega_2)))$ are $((1, \frac{1}{6}), (1, \frac{1}{6}))$, and that the set V^* has non-empty interior: $\dim V^* = \dim V^*(\omega_1) + \dim V^*(\omega_2) = 2 + 2 = 4$, where $V^*(\omega)$ is the set of feasible and individually rational payoffs given a state $\omega \in \Omega$. This stage game does not have a static ex-post equilibrium, so we know that regardless of the monitoring structure it does not have a PPXE for a range of discount factors near 0.

We will now consider two repeated games with these payoff matrices and differing monitoring structures.

Example 2a. First, as in Hörner and Lovo (2008), assume that actions are observable, but states (and rewards) are not. That is, $Y = A$ and $\pi_y^\omega(a) = 1$ if $y = a$. In this case, it is easy to adapt the argument of Hörner and Lovo (2008) to show that there is no PPXE. In a PPXE, a player's equilibrium payoff conditional on ω cannot fall below the minimax payoff for that ω. In particular, player 2's equilibrium payoff conditional on ω_1 must be positive. This implies that the outcome $(10, -4)$ realizes at most the fifth of the time, and hence player 1's equilibrium payoff conditional on ω_1 is at most $\frac{14}{5}$. Likewise, player 1's equilibrium payoff conditional on ω_2 is at most $\frac{14}{5}$. However, if player 1 randomizes $(.5U, .5D)$ independent of the state, she earns at least 3 in one of the states. Therefore, there is no PPXE for any discount factor.

Note that with this monitoring structure, the statewise full rank condition is not satisfied. Indeed, since players directly observe the actions but not the states, the matrices $\Pi_{(i,\omega_1)}(\alpha)$ and $\Pi_{(i,\omega_2)}(\alpha)$ are identical, meaning that no profile α has statewise full rank for (i, ω_1) and (i, ω_2).

Example 2b. Now suppose that the set of possible public signals is $Y = A \times \Omega$, and that the monitoring technology is perfect: $\pi_y^\omega(a) = 1$ if $y = (a, \omega)$, and $\pi_y^\omega(a) = 0$ otherwise. This monitoring structure satisfies all of our full rank conditions, so the perfect ex-post folk theorem applies, and in particular a PPXE exists. We will now explicitly construct a PPXE whose payoffs converge to the efficient frontier in each state.

Let $Y(\omega_1)$ be the set $\{y = (a, \omega) \in Y | \omega = \omega_1\}$, and $Y(\omega_2)$ be the set $\{y = (a, \omega) \in Y | \omega = \omega_2\}$. To make our exposition as simple as possible, we assume that players can observe two additional public signals x_1 and x_2 from $U[0,1]$ at the beginning of every stage game, but this is not essential.

Our equilibrium strategy profile is implemented by an automaton with six phases. As usual, each of these six phases is represented by its target payoff vector.

$$\begin{aligned}
&\text{Phase 1: } \left(\left(2, \frac{4}{9}\right), \left(\frac{8(1-\delta)}{9} + 2\delta, \frac{8(1-\delta)}{9} + \frac{4\delta}{9}\right) \right), \\
&\text{Phase 2: } \left((1,1), \left((1-\delta) + 2\delta, (1-\delta) + \frac{4\delta}{9}\right) \right), \\
&\text{Phase 3: } \left(\left(1, \frac{1}{6} + \frac{25(1-\delta)}{108}\right), \left((1-\delta) + 2\delta, \frac{4\delta}{9} - \frac{14(1-\delta)}{6}\right) \right), \\
&\text{Phase 4: } \left(\left(\frac{8(1-\delta)}{9} + 2\delta, \frac{8(1-\delta)}{9} + \frac{4\delta}{9}\right), \left(2, \frac{4}{9}\right) \right), \\
&\text{Phase 5: } \left(\left((1-\delta) + 2\delta, (1-\delta) + \frac{4\delta}{9}\right), (1,1) \right), \\
&\text{Phase 6: } \left(\left((1-\delta) + 2\delta, \frac{4\delta}{9} - \frac{14(1-\delta)}{6}\right), \left(1, \frac{1}{6} + \frac{25(1-\delta)}{108}\right) \right).
\end{aligned}$$

Note that the target payoff in Phase 1 attains a Pareto-efficient outcome $(2, \frac{4}{9})$ for state ω_1, and that payoffs in Phases 2 and 3 approximate the minimax payoffs to players 1 and 2, respectively, for state ω_1. Likewise, for state ω_2, the target payoff in Phase 4 attains a Pareto-efficient outcome $(2, \frac{4}{9})$, while the payoffs in Phase 5 and 6 approximate the minimax payoffs to players 1 and 2, respectively.

The play for Phases 1 through 3 is specified as follows.

- In Phase 1, mix (U, L) and (D, L) with probability $\frac{1}{9}$ and $\frac{8}{9}$, using public randomization x_2. If $y \in Y(\omega_1)$ and player 1 unilaterally deviates, then go to Phase 2. If $y \in Y(\omega_1)$ and player 2 unilaterally deviates, then go to Phase 3. If $y \in Y(\omega_2)$ and nobody deviates, then go to Phase 4. If $y \in Y(\omega_2)$ and player 1 unilaterally deviates, then go to Phase 5. If $y \in Y(\omega_2)$ and player 2 unilaterally deviates, then go to Phase 6. Otherwise, stay.

- In Phase 2, play (U,R) to punish player 1 for ω_1. If $y \in Y(\omega_2)$ and nobody deviates, then go to Phase 4. If $y \in Y(\omega_2)$ and player 1 unilaterally deviates, then go to Phase 5. If $y \in Y(\omega_2)$ and player 2 unilaterally deviates, then go to Phase 6. Otherwise, stay.

- In Phase 3, play $(\frac{1}{6}U + \frac{5}{6}D, R)$ to punish player 2 for ω_1. If $y = ((U,R),\omega_1)$, then stay. If $y = ((D,R),\omega_1)$, then mix Phase 1 and Phase 3 with probability p and $1-p$ where $p = \frac{1-\delta}{\delta}$, using public randomization x_1. If $y \in Y(\omega_1)$ and player 2 deviates, then stay. If $y = ((U,R),\omega_2)$, then go to Phase 4. If $y = ((D,R),\omega_2)$, then mix Phases 4 and 5 with probability $1-p$ and p where $p = \frac{9(1-\delta)}{\delta}$, using public randomization x_1. If $y \in Y(\omega_2)$ and player 2 deviates, then go to Phase 6.

Roughly speaking, given that the true state is ω_1, the play in Phases 1 through 3 corresponds to a subgame-perfect equilibrium yielding the Pareto-efficient payoff $(2, \frac{4}{9})$. Phase 1 is a regular phase, so that players mix (U,L) and (D,L) with probability $\frac{1}{9}$ and $\frac{8}{9}$ to achieve the target payoff $(2, \frac{4}{9})$. Players remain Phase 1, as long as nobody deviates (and $y \in Y(\omega_1)$). If player 1 deviates, then they move to Phase 2, in which players minimaxes player 1's payoff by (U,R). Since (U,R) is a Nash equilibrium for state ω_1, players do not have to move, as long as $y \in Y(\omega_1)$. Likewise, if player 2 deviates in Phase 1, then players move to Phase 3 to punish player 2. Here, player 1 needs to mix between U and D to minimax player 2, and hence the continuation play must be chosen so that player 1 is indifferent between these two actions. Specifically, when player 1 chooses D and so gets a lower payoff, players move to Phase 1 with small probability to reward player 1.[12]

It is easy to see that all the incentive constraints in state ω_1 are satisfied, provided that δ is sufficiently large. To be a PPXE, the constructed strategies also need to satisfy the incentive constraints in state ω_2. Our solution is very simple;

[12]This use of future reward to induce randomization is similar to a construction in Fudenberg and Maskin (1986).

in Phase 1 or Phase 2, if $y \in Y(\omega_2)$ occurs and nobody deviates today, then go to Phase 4; if $y \in Y(\omega_2)$ occurs and player 1 unilaterally deviates, then go to Phase 5; and if $y \in Y(\omega_2)$ occurs and player 2 unilaterally deviates, then go to Phase 6. Since Phases 5 and 6 minimax players 1 and 2 respectively, players do not want to deviate today for sufficiently large δ, even if the true state is ω_2. In Phase 3, we need to make player 1 indifferent between U and D, so players need to mix Phases 4 and 5 as before. Thus, the target payoff vectors for Phases 1 through 3 are enforceable for each state $\omega \in \Omega$, as long as δ is close to one.

The play for Phases 4 through 6 is specified in a similar way.

- The play in Phase 4 is defined as a "mirror image" of that in Phase 1. Specifically, mix (D,R) and (U,R) with probability $\frac{1}{9}$ and $\frac{8}{9}$, using public randomization x_2. If $y \in Y(\omega_2)$ and player 1 unilaterally deviates, then go to Phase 5. If $y \in Y(\omega_2)$ and player 2 unilaterally deviates, then go to Phase 6. If $y \in Y(\omega_1)$ and nobody deviates, then go to Phase 1. If $y \in Y(\omega_1)$ and player 1 unilaterally deviates, then go to Phase 2. If $y \in Y(\omega_1)$ and player 2 unilaterally deviates, then go to Phase 3. Otherwise, stay.

- Likewise, the plays in Phases 5 and 6 are defined as those in Phases 2 and 3, respectively. We omit a precise description.

The play in Phases 4 through 6 corresponds to a subgame-perfect equilibrium yielding the Pareto-efficient payoff $(2, \frac{4}{9})$, when the true state is ω_2. Thus one can see that all the incentive constraints for ω_2 are satisfied for sufficiently large δ. Also, the incentive constraints for ω_1 are satisfied as before, so that the target payoff vectors in Phase 4 through 6 are enforceable for each state $\omega \in \Omega$ as long as δ is close to one. Summing up, we can conclude that the specified strategy profile is a PPXE. Note that this equilibrium is almost efficient as $\delta \to 1$, since the target payoff vectors in Phase 1 and 4 approximate $((2, \frac{4}{9}), (2, \frac{4}{9}))$, which is on the Pareto frontier of V.

Because the stage game does not have a static ex-post equilibrium, the strategies in each period must provide future incentives to prevent current deviations. And for such forward-looking strategies to be a PPXE, the strategies in each period must prescribe continuation play and continuation punishments that would be optimal in each state ω, even those that past signals have ruled out. This was not the case in Example 1, where we considered a PPXE in which players play a static ex-post equilibrium in every period. In that construction, there was no need for intertemporal incentives, so the PPXE we constructed could ignore all the signals from period 2 on.

6 Weaker Sufficient Conditions for Weaker Folk Theorems

6.1 Ex-Post Threat Folk Theorem

Of course our folk theorems give only sufficient conditions for the folk theorem, and the full folk theorem can hold even if the stated conditions fail.[13] In this section we present a few alternative theorems that use weaker informational conditions to prove "ex-post-threats" folk theorems, meaning that the theorems only ensure the attainability of payoffs that Pareto-dominate the payoffs of a static ex-post equilibrium. Consequently, these theorems assume that a static ex-post equilibrium exists. This is always true when the state only matters for the monitoring structure but has no impact on the expected payoffs (that is $g^{\omega}(a) = g(a)$), and it is also satisfied for generic payoff functions g when the state has a sufficiently small impact on the payoff function. Several of our other assumptions in this section seem more likely to be satisfied if the uncertainty is "small," though that is not $(j, \tilde{\omega})$, as shown by Example 3, and we have not tried to prove formal results along those lines.

Definition 10. For each $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$, a profile α is *pairwise identifiable for* (i, ω) *and* (j, ω) if $\operatorname{rank}\Pi_{(i,\omega)(j,\omega)}(\alpha) = \operatorname{rank}\Pi_{(i,\omega)}(\alpha) + \operatorname{rank}\Pi_{(j,\omega)}(\alpha) - 1$.

This is exactly the FLM definition of pairwise identifiability. (Recall that pairwise full rank is equivalent to the combination of individual full rank and pairwise identifiability.)

Definition 11. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, a profile α is *state-wise identifiable for* (i, ω) *and* $(j, \tilde{\omega})$ if $\operatorname{rank}\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha) = \operatorname{rank}\Pi_{(i,\omega)}(\alpha) + \operatorname{rank}\Pi_{(j,\tilde{\omega})}(\alpha)$.

[13] For example, as in FLM the linear independence conditions implicit in the full rank conditions can be weakened to only consider convex combinations of the pure actions.

Note that statewise full rank is the combination of individual full rank and statewise identifiability. Thus when individual full rank is satisfied, statewise identifiability requires just as many signals as statewise full rank, in contrast to the statewise distinguishability condition in the next subsection.

We say that α is ex-post enforceable if it is ex-post enforceable with respect to $\boldsymbol{R}^{I\times|\Omega|}$ and δ for some $\delta \in (0,1)$. This is equivalent to α being enforceable with respect to $\boldsymbol{R}^{I}$ and δ for each information structure π^{ω} in isolation.

Condition 4. If a pure action profile $a \in A$ gives a Pareto-efficient payoff vector for some $\omega \in \Omega$, then it is ex-post enforceable and for each $i \in \boldsymbol{I}$ and $j \neq i$, a is pairwise identifiable for (i,ω) and (j,ω).

Condition 5. If a pure action profile $a \in A$ gives a Pareto-efficient payoff vector for some $\tilde{\omega} \in \Omega$, then it gives a Pareto-efficient payoff vector for every $\omega \in \Omega$, and for each $i \in \boldsymbol{I}$, $j \neq i$, and $\omega \in \Omega$, a is pairwise identifiable for (i,ω) and (j,ω).

Condition 5 is typically satisfied if $u_i(y,a_i,\omega)$ is independent of (or very insensitive to) ω and the various distributions π^{ω} are sufficiently similar.

Lemma 9. *If $u_i(y,a_i,\omega)$ is independent of ω and Condition 5 holds, then Condition 4 holds.*

Proof. Because each player's payoff depends only on their own action and the realized signal, Lemma 6.1 of FLM applied to each state ω in isolation implies that profile a is enforceable for each ω. *Q.E.D.*

Condition 6. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, there is a profile that is ex-post enforceable and statewise identifiable for (i,ω) and $(j,\tilde{\omega})$.

Intuitively, it is this condition that will allow the players to "learn the state" in a PPXE. It can be replaced by the less restrictive but harder to check condition of statewise distinguishability, as we present in the next subsection.

Proposition 5. *Suppose Condition 2 or Condition 4 holds. Suppose also that Condition 6 holds. Assume that there is a static ex-post equilibrium α^0, and let $V^0 \equiv \{v \in V | \forall i \in \boldsymbol{I}\ \forall \omega \in \Omega\ \ v_i(\omega) \geq g_i^{\omega}(\alpha^0)\}$. Then, for any smooth strict subset W of V^0, there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta},1)$.*

This proposition is established by the following lemmas that determine the maximal score k^* in various directions. The next lemma says that score of a static ex-post equilibrium can be enforced in any direction; this score will be used to generate the score in directions that minimize a player's payoff.

Lemma 10. *Suppose that there is a static ex-post equilibrium α^0. Then, $k^*(\alpha^0,\lambda) \geq \lambda \cdot g(\alpha^0)$ for any direction λ.*

Proof. Let $v_i(\omega) = w_i(y,\omega) = g_i^{\omega}(\alpha^0)$ for all $i \in \boldsymbol{I}$, $\omega \in \Omega$, and $y \in Y$. Then, this (v,w) satisfies constraints (i) through (iii) in LP-Average, and $\lambda \cdot v = \lambda \cdot g(\alpha^0)$. Hence, $k^*(\alpha^0,\lambda) \geq \lambda \cdot g(\alpha^0)$. *Q.E.D.*

Lemma 11.

(a) *Suppose that Condition 4 holds, and let a be a profile that gives a Pareto-efficient payoff for some $\omega \in \Omega$. Then, $k^*(a,\lambda) = \lambda \cdot g(a)$ for direction λ such that $(\lambda_i(\omega))_{i\in\boldsymbol{I}}$ has at least two non-zero components while $\lambda_j(\tilde{\omega}) = 0$ for all $j \in \boldsymbol{I}$ and $\tilde{\omega} \neq \omega$.*

(b) *Suppose that Condition 4 holds. Then, $k^*(\lambda) = \max_{v\in V} \lambda \cdot v$ for direction λ such that $\lambda_i(\omega) > 0$ and $\lambda_j(\tilde{\omega}) = 0$ for all $(j,\tilde{\omega}) \neq (i,\omega)$.*

Proof. Part (a). Lemma 1 (b) shows that the maximum score in direction λ is at most $\lambda \cdot g(a)$. Because a is a pure action profile, and it is enforceable for all ω and pairwise identifiable from Condition 4, it is enforceable on hyperplanes corresponding to λ from Theorem 5.1 of FLM, so the score $\lambda \cdot g(a)$ can be attained.

Part (b). Let a be a Pareto-efficient profile that maximizes player i's payoff in state ω. By Condition 4, this is ex-post enforceable, and since the profile has the best-response property in state ω, lemma 5.2 of FLM implies it is enforceable on λ. *Q.E.D.*

Lemma 12. *Suppose that Condition 6 holds. Then, $k^*(\lambda) = \infty$ for direction λ such that there exist $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) \neq 0$.*

Proof. See Appendix. As in Lemma 5.5 of FLM, the idea is that if profile a is enforceable, then any action $a_i' \neq a_i$ leading to the same distribution of public signals as a_i cannot increase player i's payoff and so can be ignored. Statewise identifiability condition assures that once we delete these redundant actions, the matrix $\Pi_{(i,\omega)(j,\tilde{\omega})}$ corresponding to the remaining actions satisfies statewise full rank. Therefore, as in Lemma 6, we can choose continuation payoffs to attain infinitely large maximal score while maintaining exact indifference among the remaining actions. These continuation payoffs also deter deviations to the deleted actions. *Q.E.D.*

6.2 Relaxing Statewise Identifiability

When individual full rank holds, statewise identifiability implies statewise full rank, which can require that there be twice as many signals as required by the FLM folk theorem. The following, more complex, condition can be satisfied with far fewer signals.

Definition 12. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, a profile α *statewise distinguishes* (i,ω) *from* $(j,\tilde{\omega})$ if there is $A_i^* \subseteq A_i$ such that

(i) $\text{supp}\alpha_i \subseteq A_i^*$;

(ii) $\text{rank}\Pi_{(i,\omega,A_i^*)}(\alpha) + \text{rank}\Pi_{(j,\tilde{\omega})}(\alpha) = \text{rank}\Pi_{(i,\omega,A_i^*)(j,\tilde{\omega})}(\alpha) = \text{rank}\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$ where $\Pi_{(i,\omega,A_i^*)}(\alpha)$ is the submatrix of $\Pi_{(i,\omega)}(\alpha)$ that includes only the rows corresponding to actions $a_i \in A_i^*$, and $\Pi_{(i,\omega,A_i^*)(j,\tilde{\omega})}(\alpha)$ is the matrix with the rows of $\Pi_{(i,\omega,A_i^*)}(\alpha)$ and $\Pi_{(j,\tilde{\omega})}(\alpha)$;

(iii) for each $a_i \in A_i^* \setminus \text{supp}\alpha_i$, the vector $(\pi_y^\omega(a_i,\alpha_{-i}))_{y\in Y}$ is not a linear combination of $(\pi_y^\omega(a_i',\alpha_{-i}))_{y\in Y}$ for $a_i' \in A_i^* \setminus \{a_i\}$; and

(iv) for each $a_i \notin A_i^*$, there exist $(\kappa^\omega(a_i'))_{a_i'\in A_i^*}$ and $(\kappa^{\tilde{\omega}}(a_j))_{a_j\in A_j}$ such that

$$\sum_{a_i' \in \text{supp}\alpha_i} \kappa^\omega(a_i') \leq 1$$

and for all $y \in Y$,

$$\pi_y^\omega(a_i,\alpha_{-i}) = \sum_{a_i'\in A_i^*} \kappa^\omega(a_i')\pi_y^\omega(a_i',\alpha_{-i}) + \sum_{a_j\in A_j} \kappa^{\tilde{\omega}}(a_j)\pi_y^{\tilde{\omega}}(a_j,\alpha_{-j}).$$

To understand this definition, note first that it is asymmetric: it can be that α statewise distinguishes (i,ω) from $(j,\tilde{\omega})$ but that α does not statewise distinguish $(j,\tilde{\omega})$ from (i,ω). Next note that when $A_i^* = A_i$, the third and fourth clauses of the definition are vacuously satisfied, and the matrix $\Pi_{(i,\omega,A_i^*)(j,\tilde{\omega})}(\alpha)$ is the same as $\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$, so the conditions of the definition are then equivalent to statewise identifiability. However, allowing A_i^* to be a strict subset of A_i allows clause (i) of the definition to be satisfied when there are too few signals for statewise identifiability, as in the example below. The third clause of the definition says that at state ω, actions $a_i \in A_i^* \setminus \text{supp}\alpha_i$ can be distinguished from actions in A_i^*. This condition is a weaker condition than individual full rank for player i at state ω, which requires that all actions can be distinguished; in particular, the third clause is vacuous when $A_i^* = \text{supp}\alpha_i$. Just as individual full rank for player i implies that all deviation by player i can be deterred, clause (iii) implies that there are continuation payoffs that deter deviations to actions in $A_i^* \backslash \text{supp}\alpha_i$. Clause (iv) implies that the continuation payoffs can give player i an arbitrarily large reward in state ω without increasing player i's incentive to play $a_i \notin A_i^*$, and without affecting player j's payoff in state $\tilde{\omega}$.[14]

In the proof of Lemma 13, we show that if profile α statewise distinguishes (i,ω) from $(j,\tilde{\omega})$, then the A_i^* can be chosen so that $A_i^* = \text{supp}\alpha_i \bigcup (A'_{ij} \bigcap A_i)$, where $A'_{ij} \subseteq A_i \bigcup A_j$ provides a basis for the space spanned by

$$\{(\pi_y^{\omega}(a_i,\alpha_{-i}))_{y\in Y} | a_i \in A_i\} \bigcup \{(\pi_y^{\tilde{\omega}}(a_j,\alpha_{-j}))_{y\in Y} | a_j \in A_j\}$$

This says that actions in A'_{ij} can generate all of the distributions that player i could cause in state ω or that player j could cause in $\tilde{\omega}$.

The following partnership-game example illustrates how the folk theorem still holds when statewise identifiability is replaced by statewsie distinguishability.

Example 3. There are two players, two states, and three outcomes $Y = \{H,M,L\}$.Each player i's action space is $A_i = \{C_i, D_i\}$, and the monitoring structure π is as fol-

[14] To see this, note that if we increase player i's expected continuation payoff conditional on $a_i \in \text{supp}\alpha_i$ by K, then player i's expected continuation payoff when he deviates to $a_i \notin A_i^*$ increases by $\sum_{a_i' \in \text{supp}\alpha_i} \kappa^{\omega}(a_i')$ times K. Since clause (iv) says that the coefficient $\sum_{a_i' \in \text{supp}\alpha_i} \kappa^{\omega}(a_i')$ cannot exceed one, this change in continuation payoffs does not tempt player i to deviate to $a_i \notin A_i^*$. Also, this does not affect player j's payoff in state $\tilde{\omega}$, as clause (ii) assures that player j's actions can be distinguished from actions in α_i.

lows. (We display the probability of the outcomes H and M; their sum is always less than 1, with the remaining probability is assigned to L.)

$$
\begin{gathered}
(\pi_H^{\omega_1}(C_1,C_2),\pi_M^{\omega_1}(C_1,C_2)) = (o_H+p_H+q_H, o_M+p_M+q_M)\\
(\pi_H^{\omega_1}(D_1,C_2),\pi_M^{\omega_1}(D_1,C_2)) = (o_H+q_H, o_M+q_M)\\
(\pi_H^{\omega_1}(C_1,D_2),\pi_M^{\omega_1}(C_1,D_2)) = (o_H+p_H, o_M+p_M)\\
(\pi_H^{\omega_1}(D_1,D_2),\pi_M^{\omega_1}(D_1,D_2)) = (o_H, o_M)\\
(\pi_H^{\omega_2}(C_1,C_2),\pi_M^{\omega_2}(C_1,C_2)) = (o_H+p_H+q_H, o_M+p_M+q_M)\\
(\pi_H^{\omega_2}(D_1,C_2),\pi_M^{\omega_2}(D_1,C_2)) = (o_H+p_H, o_M+p_M)\\
(\pi_H^{\omega_2}(C_1,D_2),\pi_M^{\omega_2}(C_1,D_2)) = (o_H+q_H, o_M+q_M)\\
(\pi_H^{\omega_2}(D_1,D_2),\pi_M^{\omega_2}(D_1,D_2)) = (o_H, o_M)
\end{gathered}
$$

Note that the baseline probabilities of H and M if both players choose D are independent of the state, and that in each state the monitoring structure is additive: the change in probabilities induced by player $i's$ changing from C_i to D_i is the same regardless of the action of the other player. Moreover, in this example the uncertainty is symmetric in the state: In state ω_1, if player 1 chooses C_1 instead of D_1, then the probabilities of H and M increase by p_H and p_M, while player 2's choice of C_2 increases the probabilities by q_H and q_M; in state ω_2, the roles are reversed.

The payoffs are

$$u_i(C_i,y) = r_i(y) - e_i \quad \text{and} \quad u_i(D_i,y) = r_i(y)$$

for each $i \in \boldsymbol{I}$ and $y \in Y$. We assume that for each $i \in \boldsymbol{I}$,

$$
\begin{gathered}
r_i(H) > r_i(M) > r_i(L),\\
e_i > p_H(r_i(H) - r_i(L)) + p_M(r_i(M) - r_i(L)),\\
e_i > q_H(r_i(H) - r_i(L)) + q_M(r_i(M) - r_i(L)).
\end{gathered}
$$

Here the left-hand side of the second inequality is the cost of player 1's choice of C_1 for state ω_1 (or the cost of player 2's choice of C_2 for state ω_2), and the right-hand side is an increase in player 1's benefit from the project when he chooses C_1 instead of D_1 for state ω_1 (or an increase in player 2's benefit when he chooses C_2 for state ω_2). Since the left-hand side is greater than the right-hand side, we conclude that D_1 strictly dominates C_1 for state ω_1, and D_2 strictly dominates C_2 for state ω_2. Likewise, the third inequality asserts that D_1 strictly dominates C_1 for state ω_2, and D_2 strictly dominates C_2 for state ω_1. Thus, D_i strictly dominates C_i for each state. Moreover, we assume that for each $i \in \boldsymbol{I}$,

$$e_i < p_H(u_1(H)+u_2(H)-u_1(L)-u_2(L))+p_M(u_1(M)+u_2(M)-u_1(L)-u_2(L))$$

and

$$e_i < q_H(u_1(H)+u_2(H)-u_1(L)-u_2(L))+q_M(u_1(M)+u_2(M)-u_1(L)-u_2(L)).$$

This means that choosing C_i instead of D_i always increases the total surplus. Summing up, the payoff matrix of the stage game corresponds to a prisoner's dilemma for each sate; hence, V^* has a non-empty interior and (D_1, D_2) is a static ex-post equilibrium.

Note that individual full rank is satisfied, and that pairwise full rank is satisfied at every profile and every state if

$$\begin{pmatrix} p_H & p_M \\ q_H & q_M \end{pmatrix}$$

has full rank. For example, the matrix $\Pi_{(1,\omega_1)(2,\omega_1)}(D_1, C_2)$ is represented by

$$\begin{pmatrix} o_H+q_H & o_M+q_M & 1-(o_H+q_H+o_M+q_M) \\ o_H+p_H+q_H & o_M+p_M+q_M & 1-(o_H+p_H+q_H+o_M+p_M+q_M) \\ o_H+q_H & o_M+q_M & 1-(o_H+q_H+o_M+q_M) \\ o_H & o_M & 1-(o_H+o_M) \end{pmatrix},$$

and this matrix has rank three if the above two-by-two matrix has a full rank. Therefore, the profile (D_1, C_2) has pairwise full rank for $(1, \omega_1)$ and $(2, \omega_1)$. On the other hand, statewise identifiability is not satisfied at any profile, as there are only three signals, while four signals would be needed to satisfy statewise identifiability and individual full rank.

Condition 7. For each $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$, there is a profile that is ex-post enforceable and statewise distinguishes (i, ω) from $(j, \tilde{\omega})$.

In the example, (D_1, C_2) statewise distinguishes all eight of the relevant comparisons. Consider for example distinguishing $(1, \omega_1)$ from $(2, \omega_2)$. Let $A_1^* = \{D_1\}$, then

$$\Pi_{(1,\omega_1,A_1^*)}(\alpha) = (\pi_y^{\omega_1}(D_1, C_2))_{y \in Y},$$

$$\Pi_{(2,\omega_2)}(\alpha) = \begin{pmatrix} (\pi_y^{\omega_2}(D_1, C_2))_{y \in Y} \\ (\pi_y^{\omega_2}(D_1, D_2))_{y \in Y} \end{pmatrix},$$

and

$$\Pi_{(1,\omega_1,A_1^*)(2,\omega_2)}(\alpha) = \begin{pmatrix} (\pi_y^{\omega_1}(D_1, C_2))_{y \in Y} \\ (\pi_y^{\omega_2}(D_1, C_2))_{y \in Y} \\ (\pi_y^{\omega_2}(D_1, D_2))_{y \in Y} \end{pmatrix},$$

and so the equalities in the second clause of the definition are satisfied. The third clause is vacuous. The fourth clause requires that

$$\pi_y^{\omega_1}(C_1, C_2) = \kappa^{\omega_1}(D_1)\pi_y^{\omega_1}(D_1, C_2) + \kappa^{\omega_2}(D_2)\pi_y^{\omega_2}(D_1, D_2) + \kappa^{\omega_2}(C_2)\pi_y^{\omega_2}(D_1, C_2),$$

with $\kappa^{\omega_1}(D_1) \leq 1$. Substituting for the distributions, this requires that

$$\begin{aligned} & (o_H + p_H + q_h, o_M + p_M + q_m) \\ = & \kappa^{\omega_1}(D_1)(o_H + q_H, o_M + q_M) + \kappa^{\omega_2}(D_2)(o_H, o_M) + \kappa^{\omega_2}(C_2)(o_H + p_H, o_M + p_M) \end{aligned}$$

so we can take $\kappa^{\omega_1}(D_1) = \kappa^{\omega_2}(C_2) = 1$, $\kappa^{\omega_2}(D_2) = -1$.[15]

On the other hand, the condition does not hold for any pair (i, ω) versus $(j, \tilde{\omega})$ at (C_1, C_2). Intuitively, this is because the distribution of signals at (C_1, C_2) does not reveal the state. Formally, there aren't enough signals for the definition to be satisfied with $A_i^* = \{C_i, D_i\}$, and if $A_i^* = C_i$, then rank $\Pi'' = 2$ while $\text{rank}\Pi_{(i,\omega)(j,\tilde{\omega})}(C_1, C_2) = 3$.

Now we state the ex-post threat folk theorem with statewise distinguishability.

Proposition 6. *Suppose Condition 2 or Condition 4 holds. Suppose also that Condition 7 holds. Assume that there is a static ex-post equilibrium α^0, and let $V^0 \equiv \{v \in V | \forall i \in \boldsymbol{I}, \forall \omega \in \Omega \ v_i(\omega) \geq g_i^{\omega}(\alpha^0)\}$. Then, for any smooth strict subset W of V^0, there is $\overline{\delta} \in (0, 1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta}, 1)$.*

[15] Likewise, the profile $\alpha = (D_1, C_2)$ statewise distinguishes (i, ω) from $(j, \tilde{\omega})$ for any $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$. To see this, when $i \neq j$, let $A_i^* = \text{supp}\alpha_i$, $\kappa^{\omega}(a_i) = 1$ for $a_i \in \text{supp}\alpha_i$, $\kappa^{\tilde{\omega}}(a_j) = -1$ for $a_j \in \text{supp}\alpha_j$, and $\kappa^{\tilde{\omega}}(a_j) = -1$ for $a_j \notin \text{supp}\alpha_j$. When $i = j$, let $A_i^* = \text{supp}\alpha_i$, $\kappa^{\omega}(a_i) = 0$ for $a_i \in \text{supp}\alpha_i$, $\kappa^{\tilde{\omega}}(a_j) = 0$ for $a_j \in \text{supp}\alpha_j$, and $\kappa^{\tilde{\omega}}(a_j) = 1$ for $a_j \notin \text{supp}\alpha_j$. Then, we can confirm that all the clauses in the definition of statewise distinguishability are satisfied.

The proof is almost the same as that of Proposition 5. The only difference is to use Lemma 13 instead of Lemma 12 to show that $k^*(\lambda) = \infty$ for cross-state and nonnegative λ.

Lemma 13. *Suppose that a profile α is ex-post enforceable and statewise distinguishes (i,ω) from $(j,\tilde{\omega})$. Then, $k^*(\alpha,\lambda) = \infty$ for direction λ such that $\lambda_i(\omega) > 0$ and $\lambda_j(\tilde{\omega}) \neq 0$.*

Proof. See Appendix. The difference from Lemma 12 is that even after we delete all the redundant actions, the resulting matrix $\Pi_{(i,\omega)(j,\tilde{\omega})}$ may not satisfy statewise full rank.[16] To fix this problem, we also delete all $a_i \notin A_i^*$. Then the matrix $\Pi_{(i,\omega)(j,\tilde{\omega})}$ corresponding to the remaining actions has full rank, so we can choose continuation payoffs to make the maximal score infinitely large while maintaining exact indifference among the remaining actions. The fourth clause of the definition of statewise distinguishability assures that player i does not want to deviate to $a_i \notin A_i^*$ under these continuation payoffs. *Q.E.D.*

Note that all the assumptions in Proposition 6 are satisfied in Example 3. Indeed, every profile α has pairwise full rank for every state, and the profile (D_1, C_2) statewise distinguishes all four of the relevant comparisons. Therefore, any payoff vector in V^0 can be achieved by PPXE for sufficiently large δ.

Statewise distinguishability is only a sufficient condition for the folk theorem, but the following example suggests that some condition like statewise distinguishability may be needed. Here in the absence of incentive problems it would be possible to learn the state, but nevertheless the folk theorem fails, because of the lack of statewise distinguishability.

Example 4. This is a slightly modified version of the partnership game of Example 3. As before, there are two players, two actions $A_i = \{C_i, D_i\}$, two states, and

[16] To see this, consider $a_i' \notin A_i^*$. Under statewise distinguishability, it might be that playing $a_i' \notin A_i^*$ is statistically distinguished from $i's$ deviations, but not from player j's. This implies that a_i' survives the deletion process in the proof of Lemma 12, and the resulting matrix $\Pi_{(i,\omega)(j,\tilde{\omega})}$ does not have full rank

three outcomes $Y = \{H, M, L\}$. The monitoring structure π is different:

$$\begin{gathered}
(\pi_H^{\omega_1}(C_1,C_2), \pi_M^{\omega_1}(C_1,C_2)) = (o_H + p_H + q_H, o_M + p_M + q_M) \\
(\pi_H^{\omega_1}(D_1,C_2), \pi_M^{\omega_1}(D_1,C_2)) = (o_H + q_H, o_M + q_M) \\
(\pi_H^{\omega_1}(C_1,D_2), \pi_M^{\omega_1}(C_1,D_2)) = (o_H + p_H, o_M + p_M) \\
(\pi_H^{\omega_1}(D_1,D_2), \pi_M^{\omega_1}(D_1,D_2)) = (o_H, o_M) \\
(\pi_H^{\omega_2}(C_1,C_2), \pi_M^{\omega_2}(C_1,C_2)) = (o_H + p_H + \beta q_H, o_M + p_M + \beta q_M) \\
(\pi_H^{\omega_2}(D_1,C_2), \pi_M^{\omega_2}(D_1,C_2)) = (o_H + \beta q_H, o_M + \beta q_M) \\
(\pi_H^{\omega_2}(C_1,D_2), \pi_M^{\omega_2}(C_1,D_2)) = (o_H + p_H, o_M + p_M) \\
(\pi_H^{\omega_2}(D_1,D_2), \pi_M^{\omega_2}(D_1,D_2)) = (o_H, o_M)
\end{gathered}$$

where $\beta \in (0,1]$. Note that if player 1 chooses C_1 instead of D_1, then the probabilities of H and M increase by p_H and p_M, independent of the state. The contribution of player 2's choice of C_2 is discounted by β in state ω_2: if player 2 chooses C_2 instead of D_2, then the probabilities of H and M increase by q_H and q_M in state ω_1, but they increase only by βq_H and βq_M in state ω_2.

Note that if $\beta = 1$ the outcome distributions in the two states are identical, so as we discussed at the end of section 4, the folk theorem as we have defined it fails, as does clause (ii) of statewise distinguishability.. If $\beta < 1$, the states have different outcome distributions, so can be identified by repeated observation. However, as we explain below, statewise distinguishability fails, and the PPXE payoffs are bounded away from efficiency.

As in Example 3, the payoffs are

$$u_i(C_i, y) = r_i(y) - e_i \quad \text{and} \quad u_i(D_i, y) = r_i(y)$$

for each $i \in \boldsymbol{I}$ and $y \in Y$. We again make assumptions on the r_i so that the stage game payoffs in each state correspond to a prisoner's dilemma: D_i is a dominant strategy, so (D_1, D_2) is a static ex-post equilibrium, (C_1, C_2) is efficient, and V^* has a non-empty interior.[17]

[17]The conditions on the payoffs are: $r_i(H) > r_i(M) > r_i(L)$; $e_1 > p_H(r_1(H) - r_1(L)) + p_M(r_1(M) - r_1(L))$; $e_2 > q_H(r_2(H) - r_2(L)) + q_M(r_2(M) - r_2(L))$; $e_1 < p_H(u_1(H) + u_2(H) - u_1(L) - u_2(L)) + p_M(u_1(M) + u_2(M) - u_1(L) - u_2(L))$; and $e_2 < \beta q_H(u_1(H) + u_2(H) - u_1(L) - u_2(L)) + \beta q_M(u_1(M) + u_2(M) - u_1(L) - u_2(L))$.

Individual full rank and the pairwise full rank are satisfied at every profile and every state, if the matrix

$$\begin{pmatrix} p_H & p_M \\ q_H & q_M \end{pmatrix}$$

has full rank, as in Example 3. However, no profile can statewise distinguish $(1,\omega_1)$ from $(2,\omega_2)$. To see this, note first that if $\mathrm{supp}\alpha_1 = \{C_1, D_1\}$, then α cannot statewise distinguish $(1,\omega_1)$ from $(2,\omega_2)$. Indeed, if $\mathrm{supp}\alpha_1 = \{C_1, D_1\}$, the first clause of the definition requires that $A_1^* = \{C_1, D_1\}$, but then $\mathrm{rank}\Pi_{(1,\omega_1,A_1^*)}(\alpha) + \mathrm{rank}\Pi_{(2,\omega_2)}(\alpha) = 2+2 > 3 = \mathrm{rank}\Pi_{(1,\omega_1,A_1^*)(2,\omega_2)}(\alpha)$.

Note next that α such that $\mathrm{supp}\alpha_1 = \{a_1\}$ cannot distinguish $(1,\omega_1)$ from $(2,\omega_2)$. If it could, then either $A_1^* = \{a_1\}$ or $A_1^* = \{C_1, D_1\}$. If $A_1^* = \{a_1\}$ then since

$$\Pi_{(1,\omega_1,A_1^*)(2,\omega_2)}(\alpha) = \begin{pmatrix} (\pi_y^{\omega_1}(a_1,\alpha_2))_{y\in Y} \\ (\pi_y^{\omega_2}(a_1,C_2))_{y\in Y} \\ (\pi_y^{\omega_2}(a_1,D_2))_{y\in Y} \end{pmatrix}$$

and

$$(\pi_y^{\omega_1}(a_1,\alpha_2))_{y\in Y} = \frac{\alpha_2(C_2)}{\beta}(\pi_y^{\omega_2}(a_1,C_2))_{y\in Y} + \frac{\beta-\alpha_2(C_2)}{\beta}(\pi_y^{\omega_2}(a_1,D_2))_{y\in Y},$$

we have $\mathrm{rank}\Pi_{(1,\omega_1,A_1^*)(2,\omega_2)}(\alpha) = 2 < 3 = \mathrm{rank}\Pi_{(1,\omega_1)(2,\omega_2)}(\alpha)$. Thus, the second clause of the definition does not hold. If $\mathrm{supp}\alpha_1 = \{a_1\}$ and $A_1^* = \{C_1, D_1\}$, then, $\mathrm{rank}\Pi_{(1,\omega_1,A_1^*)}(\alpha) + \mathrm{rank}\Pi_{(2,\omega_2)}(\alpha) = 2+2 > 3 = \mathrm{rank}\Pi_{(1,\omega_1,A_1^*)(2,\omega_2)}(\alpha)$, as before, so that the second clause is not satisfied. Hence, no profile α can distinguish $(1,\omega_1)$ from $(2,\omega_2)$.

In what follows, we prove that the folk theorem fails in this example due to the failure of statewise distinguishability. Specifically, we show that the maximal score $k^*(\lambda)$ in direction $\lambda' = ((1,0),(0,1))$ is strictly less than the value $\lambda' \cdot g(C_1,C_2) < \max_{v\in V^*} \lambda' \cdot v$. To do this, we use the fact that the monitoring technology has an additive form, so that it suffices to consider only the pure action profiles, as in Lemma 4.1 of FL.[18]

[18] FL used a more restricitve definition of "additive monitoring structure," but the proof of their Lemma 4.1 applies to any case where the effect of one player's action on the distribution of signals is independent of the action of the other player.

Claim 2. *For* $\alpha = (C_1, C_2)$,

$$k^*(\alpha, \lambda') \leq \lambda' \cdot g(C_1, C_2) - \frac{1-\beta}{\beta}(g_2^{\omega_2}(C_1, D_2) - g_2^{\omega_2}(C_1, C_2)).$$

Proof. See Appendix. The intuition is that we need to deter player 2's deviation to D_2 for state ω_2, and this punishment is somewhat costly for $\beta < 1$. To see this, note first that when $\beta = 1$ so that player 2's contribution is not discounted, then the score $k^*((C_1, C_2), \lambda')$ attains the efficient value $\lambda' \cdot g(C_1, C_2)$.[19] However, for $\beta < 1$, the distribution of public signals by (C_1, C_2) for state ω_2 is a linear combination of that for the case of $\beta = 1$ and the distribution by (C_1, D_2) for state ω_2. This implies that player 2's continuation payoff after (C_1, C_2) for state ω_2 is given by an average of that for $\beta = 1$ and the continuation payoff after (C_1, D_2) for state ω_2. Recall that the continuation payoff after (C_1, D_2) must be low, in order to deter the deviation to D_2. Hence, player 2's continuation payoff after (C_1, C_2) is lower than that for $\beta = 1$, which gives $k^*((C_1, C_2), \lambda')$ less than $\lambda' \cdot g(C_1, C_2)$. Note that this inefficiency vanishes as $\beta \to 1$, since the coefficient on the continuation payoff after (C_1, D_2) is $(1-\beta)$. *Q.E.D.*

Claim 3. *For* $\alpha = (D_1, C_2)$, $k^*(\alpha, \lambda') \leq \lambda' \cdot g(D_1, C_2) - \frac{1-\beta}{\beta}(g_2^{\omega_2}(D_1, D_2) - g_2^{\omega_2}(D_1, C_2))$.

Proof. The same as in the previous claim. *Q.E.D.*

Claim 4. *For* $\alpha = (C_1, D_2)$, $k^*(\alpha, \lambda') \leq \lambda' \cdot g(C_1, D_2)$.

Proof. Since $\pi_y^{\omega_1}(C_1, D_2) = \pi_y^{\omega_2}(C_1, D_2)$ and $\pi_y^{\omega_1}(D_1, D_2) = \pi_y^{\omega_2}(D_1, D_2)$ for all $y \in Y$, the set of the constraints in the LP-Average problem for λ' is isomorphic with that for $\lambda'' = ((0,0), (1,1))$. Then the maximal score for λ' equals that for λ'', and the statement follows from Lemma 1 (b). *Q.E.D.*

Claim 5. *For* $\alpha = (D_1, D_2)$, $k^*(\alpha, \lambda') \leq \lambda' \cdot g(D_1, D_2)$.

Proof. The same as in the last lemma. *Q.E.D.*

[19] The reason is as follows. For $\beta = 1$, the payoffs and the distribution of public signals do not depend on state ω. Then the LP-Average problem for λ' is identical with the LP problem for a single state ω, e.g., the LP-Average problem for $\lambda = ((1,1), (0,0))$ Since the matrix $\Pi_{(1,\omega_1)(2,\omega_1)}(C_1, C_2)$ has pairwise full rank, we obtain $k^* = \lambda' \cdot g(C_1, C_2)$.

Now we combine these claims to show that $k^*(\lambda') < \lambda' \cdot g(C_1,C_2)$. Since $g_1^{\omega_1}(C_1,D_2) = g_1^{\omega_2}(C_1,D_2)$, we have

$$\begin{aligned}\lambda' \cdot g(C_1,D_2) &= g_1^{\omega_1}(C_1,D_2) + g_2^{\omega_2}(C_1,D_2) = g_1^{\omega_2}(C_1,D_2) + g_2^{\omega_2}(C_1,D_2)\\ &< g_1^{\omega_2}(C_1,C_2) + g_2^{\omega_2}(C_1,C_2) \leq g_1^{\omega_1}(C_1,C_2) + g_2^{\omega_2}(C_1,C_2) = \lambda' \cdot g(C_1,C_2).\end{aligned}$$

Also,

$$\begin{aligned}\lambda' \cdot g(D_1,C_2) =& g_1^{\omega_1}(D_1,C_2) + g_2^{\omega_2}(D_1,C_2)\\ =& g_1^{\omega_1}(C_1,C_2) + g_2^{\omega_2}(C_1,C_2)\\ &+ (g_1^{\omega_1}(D_1,C_2) + g_2^{\omega_1}(D_1,C_2) - g_1^{\omega_1}(C_1,C_2) - g_2^{\omega_1}(C_1,C_2))\\ <& g_1^{\omega_1}(C_1,C_2) + g_2^{\omega_2}(C_1,C_2)\\ =& \lambda' \cdot g(C_1,C_2).\end{aligned}$$

Here, the second equality comes from the additive structure, which implies that $g_2^{\omega_1}(D_1,C_2) - g_2^{\omega_1}(C_1,C_2)) = g_2^{\omega_2}(D_1,C_2) - g_2^{\omega_2}(C_1,C_2)$. Combined with the previous claims, it follows from the above claims that $k^*(\lambda') < \lambda' \cdot g(C_1,C_2)$. Thus PPXE cannot approximate the efficient payoff vector $g(C_1,C_2)$.

7 Incomplete Information and Belief-Free Equilibria

7.1 PPXE of Incomplete-Information Games

So far we have assumed that the players have symmetric information about the state. Now suppose that each player i observes a private signal $\theta_i \in \Theta_i$ at the beginning of the game, where Θ_i is a partition of Ω. Any public strategy s_i of the game where player i has a trivial partition, $\Theta_i = \{(\Omega)\}$ induces a public strategy for any non-trivial partition Θ_i: player i simply ignores the private information and sets $s_i'(h,\theta_i) = s_i(h)$ for all h and all θ_i. Since by definition play in a PPXE is optimal regardless of the state, any PPXE for the symmetric-information game (where all players have the trivial partition) induces a PPXE for any incomplete-information game (any partitions Θ_i) with the same payoff functions and prior. That is, if strategy profile s is a PPXE of the symmetric-information game, then the profile s' where $s_i'(h,\theta_i) = s_i(h)$ for all players i, types θ_i, and histories h is a PPXE of the incomplete-information game. Moreover, since θ_i is private information, any strategy that conditions on θ_i will not be a function of only the public information. Thus the PPXE of the incomplete-information games are isomorphic to the PPXE of the associated symmetric-information game, so the limit PPXE payoffs can be computed using LP-average, and our sufficient conditions for the folk theorem still apply.

However, we would expect the folk theorem to hold under weaker conditions if players are allowed to condition their play on their private information. Player *can* condition on their private information in the "belief-free equilibrium" studied by Hörner and Lovo (2008) and Hörner, Lovo, and Tomala (2008). These papers define a belief-free equilibrium for games with observable actions and incomplete information to be a strategy profile s such that for each state ω, profile s is a subgame-perfect equilibrium of the game where all players know the state is ω.[20]

When the information partitions are trivial (and actions are perfectly observed) belief-free equilibrium is equivalent to PPXE. In this case the game is one of complete information, and players have no way to learn the state, so one way to study the game is to replace the payoff functions in each state with their expected value, and apply subgame-perfect equilibrium to the resulting standard game. It may be that the folk theorem holds in this game, but the set of PPXE is empty, which might raise some questions about the strength of the robustness argument for ex-post equilibria; we are agnostic on the status of PPXE when the folk theorem fails but efficient payoffs can be supported by other sorts of equilibria.

When the information partitions are non-trivial, belief-free equilibrium allows a larger set of strategies than does symmetric-information PPXE, so the limit PPXE payoffs must be a weak or strict subset of the limit payoffs of belief-free equilibria. In the next subsection we study games with the monitoring structure of Hörner and Lovo (2008) and Hörner, Lovo, and Tomala (2008), and show that the inclusion is strict: some limit payoffs of belief-free equilibria are not limit payoffs of PPXE. In ongoing work Fudenberg and Yamamoto (2009), we define the

[20] Hörner and Lovo (2008) study two-player games where the information partition has a product structure; Hörner, Lovo, and Tomala (2008) extends the analysis to general partitions and N-player games. These papers assume that players do not observe their realized payoffs as the game is played: The players' only information is their initial private signal θ_i and the sequence of realized actions.

notion of a "Bayesian perfect public ex-post equilibrium," which allows players to condition on their initial private information in addition to the public history. This equilibrium concept reduces to the belief-free equilibrium of Hörner and Lovo (2008) and Hörner, Lovo, and Tomala (2008) when actions are perfectly observed. We then develop the appropriate linear programming characterization of limit equilibrium payoffs, which we hope to use to extend the results of Hörner, Lovo, and Tomala (2008) to games where actions are imperfectly observed and the monitoring structure is unknown.

7.2 Incomplete Information and Perfectly Observed Actions

Consider the following example from Hörner and Lovo (2008). There are two players, $\boldsymbol{I} = \{1,2\}$, and two states, $\Omega = \{\omega_1, \omega_2\}$. Player 1 knows the state, but player 2 does not: $\Theta_1 = \{(\omega_1),(\omega_2)\}$ and $\Theta_2 = \{(\Omega)\}$. Player $i \in \boldsymbol{I}$ chooses actions $a_i \in A_i = \{T,B\}$, and observes a public signal $y \in A$. Assume that $\pi_y^{\omega}(a) = 1$ for $y = a$, so that actions are perfectly observable, and players cannot learn the state from the signals. The payoff matrix conditional on ω_1 is

	T	B
T	1,1	$-L$, $1+G$
B	$1+G$,$-L$	0, 0

where $0 < L - G < 1$. This game can be regarded as prisoner's dilemma where T is cooperation and B is defection. On the other hand, the payoff matrix conditional on an ω_2 is

	L	R
U	0,0	$-L$, $1+G$
D	$-L$,$1+G$	1, 1

Note that this game is also prisoner's dilemma, but now the role of each action is reversed; B is cooperation and T is defection.

Hörner and Lovo (2008) show that player 1's best limit payoff in belief-free equilibrium is $1+\frac{G}{1+L}$ in each state, which is the highest payoff consistent with individual rationality for player 2 in the games where the state is known. We will show that PPXE cannot attain this high limit payoff. Intuitively, this is because (a) the public signals do not directly reveal the state, so with trivial partitions ($\Theta_i =$

$\{(\Omega)\}$) there is no way players can learn the state, and (b) the same conclusion obtains if player 1 does start out knowing the state but we restrict attention to equilibria in which player 1's play doesn't depend on his prior information.

Because the PPXE payoff set for games with asymmetric information is identical with that for the corresponding symmetric-information game, we can compute the limit set of PPXE payoffs for asymmetric-information games by using LP-Average.

Lemma 14. *Suppose that $Y = A$ and $\pi_y^\omega(a) = 1$ for $y = a$. Then $k^*(\alpha,\lambda) \leq \lambda \cdot g(\alpha)$ for all α and λ.*

Proof. Let (v,w) be a solution to LP-Average associated with (λ,α,δ). By definition, (v,w) satisfies all the constraints in LP-Average, and since $Y = A$ we can treat the continuation payoffs as a function of the realized actions. Then,

$$\begin{aligned}
k^*(\alpha,\lambda) &= \sum_{i\in \boldsymbol{I}} \sum_{\omega\in\Omega} \lambda_i(\omega)\cdot v_i(\omega) \\
&= \sum_{i\in \boldsymbol{I}} \sum_{\omega\in\Omega} \lambda_i(\omega) \left((1-\delta) g_i^\omega(\alpha) + \delta \sum_{a\in A} \alpha(a) w_i(a,\omega) \right) \\
&= (1-\delta)\lambda\cdot g(\alpha) + \delta \sum_{a\in A} \alpha(a)\lambda\cdot w(a) \\
&\leq (1-\delta)\lambda\cdot g(\alpha) + \delta \sum_{a\in A} \alpha(a) k^*(\alpha,\lambda) \\
&= (1-\delta)\lambda\cdot g(\alpha) + \delta k^*(\alpha,\lambda).
\end{aligned}$$

(The inequality follows from constraint (iii) in LP-Average.) Subtracting $\delta k^*(\alpha,\lambda)$ from both sides and dividing by $(1-\delta)$, we obtain $k^*(\alpha,\lambda) \leq \lambda\cdot g(\alpha)$, as desired. *Q.E.D.*

Consider λ such that $\lambda_1(\omega_1) = \lambda_1(\omega_2) = 1$ and $\lambda_2(\omega_1) = \lambda_2(\omega_2) = 0$. It follows from the above lemma that for any α,

$$k^*(\alpha,\lambda) \leq \lambda\cdot g(\alpha) = g_1^{\omega_1}(\alpha) + g_1^{\omega_2}(\alpha).$$

Note that the value $g_1^{\omega_1}(\alpha) + g_1^{\omega_2}(\alpha)$ is maximized by $\alpha = (T,T)$ or $\alpha = (B,B)$, and its value is 1. Hence,

$$k^*(\lambda) = \sup_\alpha k^*(\alpha,\lambda) = 1.$$

This result shows that Q is contained in the hyperplane $H(\lambda,1)=\{v\in \boldsymbol{R}^{\boldsymbol{I}\times|\Omega|}|v_1(\omega_1)+v_1(\omega_2)\leq 1\}$, so that

$$\forall v\in \lim_{\delta\to 1}E(\delta), \quad v_1(\omega_1)+v_1(\omega_2)\leq 1.$$

In words, the sum of player 1's equilibrium payoffs for state ω_1 and for ω_2 cannot exceed 1. On the other hand, since the equilibrium payoff must be above the minimax payoff for each state, we have

$$\forall v\in \lim_{\delta\to 1}E(\delta), \quad v_1(\omega_1)\geq 0 \quad \text{and} \quad v_1(\omega_2)\geq 0.$$

Therefore, we obtain

$$\forall v\in \lim_{\delta\to 1}E(\delta), \quad 0\leq v_1(\omega_1)\leq 1 \quad \text{and} \quad 0\leq v_1(\omega_2)\leq 1$$

Obviously, this value is less than the best belief-free equilibrium payoff, $1+\frac{G}{1+L}$.

8 Concluding Remarks

This paper has restricted attention to the set of PPXE, and analyzed them with extensions of the techniques used to analyze PPE in games where the monitoring structure is known. When the statewise full rank conditions hold, along with the standard individual and pairwise full rank conditions, the set of PPXE satisfies an ex-post folk theorem, even if the set of static ex-post equilibria is empty. When a static ex-post equilibrium does exist, there is an ex-post PPXE folk theorem under even milder informational conditions.

Of course for a given discount factor the full set of sequential equilibria of these games is larger than the set of PPXE, and can permit a larger set of payoffs. In particular, because the game has finitely many actions and signals per period and is continuous at infinity, sequential equilibria exist for any discount factor, even if the set of PPXE is empty,[21] so PPXE is not well-adapted to the study of games with uncertain monitoring structures and very impatient players. Conversely, when players are patient and mostly concerned with their long-run payoff,

[21] This follows from the facts that sequential equilibria exist in the finite-horizon truncations (Kreps and Wilson (1982)) and that the set of equilibrium strategies is compact in the product topology (Fudenberg and Levine (1983)).

our informational conditions imply that there are PPXE where players eventually learn what the state is, and obtain the same payoffs as if the state was publicly observed.

Appendix

A.1 Proof of Proposition 1

Proposition 1. *If a subset W of $\boldsymbol{R}^{I\times|\Omega|}$ is bounded and ex-post self-generating with respect to δ, then $W \subseteq E(\delta)$.*

Proof. Let $v \in W$. We will construct a PPXE that yields v. Since $v \in B(\delta, W)$, there exist a profile α and a function $w: Y \to W$ such that (α, v) is ex-post enforced by w. Set the action profile in period one to be $s|_{h^0} = \alpha$ and for each $h^1 = y^1 \in Y$, set $v|_{h^1} = w(h^1) \in W$. The play in later periods is determined recursively, using $v|_{h^t}$ as a state variable. Specifically, for each $t \geq 2$ and for each $h^{t-1} = (y^\tau)_{\tau=1}^{t-1} \in H^{t-1}$, given a $v|_{h^{t-1}} \in W$, let $\alpha|_{h^{t-1}}$ and $w|_{h^{t-1}} : Y \to W$ be such that $(\alpha|_{h^{t-1}}, v|_{h^{t-1}})$ is ex-post enforced by $w|_{h^{t-1}}$. Then, set the action profile after history h^{t-1} to be $s|_{h^{t-1}} = \alpha|_{h^{t-1}}$, and for each $y^t \in Y$, set $v|_{h^t=(h^{t-1},y^t)} = w|_{h^{t-1}}(y^t) \in W$.

Because W is bounded and $\delta \in (0,1)$, payoffs are continuous at infinity so finite approximations show that the specified strategy profile $s \in S$ generates v as an average payoff, and its continuation strategy $s|_{h^t}$ yields $v|_{h^t}$ for each $h^t \in H^t$. Also, by construction, nobody wants to deviate at any moment of time, given any state $\omega \in \Omega$. Because payoffs are continuous at infinity, the one-shot deviation principle applies, and we conclude that s is a PPXE, as desired. *Q.E.D.*

A.2 Proof of Proposition 2

Proposition 2. *If a subset W of $\boldsymbol{R}^{I\times|\Omega|}$ is compact, convex, and locally ex-post generating, then there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for all $\delta \in (\overline{\delta}, 1)$.*

Proof. Suppose that W is locally ex-post generating. Since $\{U_v\}_{v\in W}$ is an open cover of the compact set W, there is a subcover $\{U_{v^m}\}_m$ of W. Let $\overline{\delta} = \max_m \delta_{v^m}$. Choose $u \in W$ arbitrarily, and let U_{v^m} be such that $u \in U_{v^m}$. Since $W \cap U_{v^m} \subseteq B(\delta_{v^m}, W)$, there exist α_u and $w_u : Y \to W$ such that (α_u, u) is ex-post enforced by

w_u for δ_{v^m}. Given a $\delta \in (\overline{\delta}, 1)$, let

$$w(y) = \frac{\delta - \delta_u}{\delta(1-\delta_u)}u + \frac{\delta_u(1-\delta)}{\delta(1-\delta_u)}w_u(y)$$

for all $y \in Y$. Then, it is straightforward that (α_u, u) is enforced by $(w(y))_{y\in Y}$ for δ. Also, $w(y) \in W$ for all $y \in Y$, since u and $w(y)$ are in W and W is convex. Therefore, $u \in B(\delta, W)$, meaning that $W \subseteq B(\delta, W)$ for all $\delta \in (\overline{\delta}, 1)$. (Recall that u and δ are arbitrarily chosen from W and $(\overline{\delta}, 1)$.) Then, from Proposition 1, $W \subseteq E(\delta)$ for $\delta \in (\overline{\delta}, 1)$, as desired. *Q.E.D.*

A.3 Proof of Lemma 12

Lemma 12. *Suppose that Condition 6 holds. Then, $k^*(\lambda) = \infty$ for direction λ such that there exist $i \in \boldsymbol{I}$, $j \in \boldsymbol{I}$, $\omega \in \Omega$, and $\tilde{\omega} \neq \omega$ such that $\lambda_i(\omega) \neq 0$ and $\lambda_j(\tilde{\omega}) \neq 0$.*

Proof. Let (i, ω) and $(j, \tilde{\omega})$ be such that $\lambda_i(\omega) \neq 0$, $\lambda_j(\tilde{\omega}) \neq 0$, and $\tilde{\omega} \neq \omega$. Let α be a profile that is ex-post enforceable and statewise identifiable for (i, ω) and $(j, \tilde{\omega})$. In what follows, we show that $k^*(\alpha, \lambda) = \infty$.

First, we claim that for every $K > 0$, there exist $(z_i(y, \omega), z_j(y, \tilde{\omega}))_{y\in Y}$ such that (1) holds for all $a_i \in A_i$, (2) holds for all $a_j \in A_j$, and (3) holds for all $y \in Y$. To prove that this system of equations indeed has a solution, let $A_i' \subseteq A_i$ provide a basis for the space spanned by $(\pi_y^\omega(a_i', \alpha_{-i}))_{y\in Y}$, meaning that the set $\{(\pi_y^\omega(a_i', \alpha_{-i}))_{y\in Y}\}_{a_i'\in A_i'}$ is a basis for the space, so that $\text{rank}\Pi'_{(i,\omega)}(\alpha) = \text{rank}\Pi_{(i,\omega)}(\alpha) = |A_i'|$. Then if (1) holds for all $a_i' \in A_i'$, then (1) for $a_i'' \notin A_i'$ is satisfied as well. Likewise, let $A_j' \subseteq A_j$ provide a basis for the space spanned by $(\pi_y^{\tilde{\omega}}(a_j', \alpha_{-j}))_{y\in Y}$ for all $a_j' \in A_j'$; if (2) holds for all $a_j' \in A_j'$, then (2) for $a_j'' \notin A_j'$ is satisfied. Thus, for the above system to have a solution, it suffices to show that there exist $(z_i(y, \omega), z_j(y, \tilde{\omega}))_{y\in Y}$ such that (1) holds for all $a_i \in A_i'$, (2) holds for all $a_j \in A_j'$, and (3) holds for all $y \in Y$. Eliminate (3) by solving for $z_j(y, \tilde{\omega})$. Then, there remain $|A_i'| + |A_j'|$ linear equations, and its coefficient matrix is $\Pi'_{(i,\omega)(j,\tilde{\omega})}(\alpha)$, which is constructed by stacking two matrices $\Pi'_{(i,\omega)}(\alpha)$ and $\Pi'_{(j,\tilde{\omega})}(\alpha)$. It follows from statewise identifiability that

$$\text{rank}\Pi'_{(i,\omega)(j,\tilde{\omega})}(\alpha) = \text{rank}\Pi'_{(i,\omega)}(\alpha) + \text{rank}\Pi'_{(j,\tilde{\omega})}(\alpha) = |A_i'| + |A_j'|.$$

Therefore, we can indeed solve the system.

Let $(\tilde{v},\tilde{w})$ be a pair of a payoff vector and a function such that $\tilde{w}$ enforces $(\tilde{v},\alpha)$. Let $K > \max_{y\in Y}\lambda\cdot\tilde{w}(y)-\lambda\cdot\tilde{v}$, and choose $(z_i(y,\omega),z_j(y,\tilde{\omega}))_{y\in Y}$ to satisfy (1) through (3). Then, let

$$w_l(y,\overline{\omega}) = \begin{cases} \tilde{w}_i(y,\omega)+z_i(y,\omega) & \text{if} \quad (l,\overline{\omega})=(i,\omega) \\ \tilde{w}_j(y,\tilde{\omega})+z_j(y,\tilde{\omega}) & \text{if} \quad (l,\overline{\omega})=(j,\tilde{\omega}) \\ \tilde{w}_l(y,\overline{\omega}) & \text{otherwise} \end{cases}$$

for each $y\in Y$. Also, let

$$v_l(\overline{\omega}) = \begin{cases} \tilde{v}_i(\omega)+\frac{K}{\lambda_i(\omega)} & \text{if} \quad (l,\overline{\omega})=(i,\omega) \\ \tilde{v}_j(\tilde{\omega}) & \text{if} \quad (l,\overline{\omega})=(j,\tilde{\omega}) \\ \tilde{v}_l(\overline{\omega}) & \text{otherwise} \end{cases} .$$

Then, as in the proof of Lemma 6, this (v,w) satisfies constraints (i) through (iii) in LP-Average. Therefore, $k^*(\alpha,\lambda)\geq\lambda\cdot v=\lambda\cdot\tilde{v}+K$. Since K can be arbitrarily large, we conclude $k^*(\alpha,\lambda)=\infty$. *Q.E.D.*

A.4 Proof of Lemma 13

Lemma 13. *Suppose that a profile α is ex-post enforceable and statewise distinguishes (i,ω) from $(j,\tilde{\omega})$. Then, $k^*(\alpha,\lambda)=\infty$ for direction λ such that $\lambda_i(\omega)>0$ and $\lambda_j(\tilde{\omega})\neq 0$.*

Proof. Fix an A_i^* consistent with the statewise distinguishability condition. Let $A_i'\subseteq \text{supp}\alpha_i$ provide a basis for the space spanned by

$$\{(\pi_y^{\omega}(a_i',\alpha_{-i}))_{y\in Y}|a_i'\in\text{supp}\alpha_i\},$$

and $A_j'\subseteq A_j$ provide a basis for the space spanned by

$$\{(\pi_y^{\tilde{\omega}}(a_j',\alpha_{-j}))_{y\in Y}|a_j'\in A_j\}.$$

Then, let $A_{ij}'\subseteq A_i^*\bigcup A_j$ provide a basis for the space spanned by

$$\{(\pi_y^{\omega}(a_i',\alpha_{-i}))_{y\in Y}|a_i'\in A_i\}\bigcup\{(\pi_y^{\tilde{\omega}}(a_j',\alpha_{-j}))_{y\in Y}|a_j'\in A_j\}$$

such that $(A_i' \bigcup A_j') \subseteq A_{ij}'$. Note that the second equality of clause (ii) of statewise distinguishability guarantees the existence of a basis with elements in $A_i^* \bigcup A$, and that $a_i \in A_{ij}' \bigcap (A_i \setminus \text{supp}\alpha_i)$ if and only if $a_i \in A_i^* \setminus \text{supp}\alpha_i$.

First, we claim that for every $K > 0$, there exist $(z_i(y,\omega), z_j(y,\tilde{\omega}))_{y\in Y}$ such that

$$\sum_{y\in Y} \pi_y^\omega(a_i,\alpha_{-i}) z_i(y,\omega) = \frac{K}{\delta\lambda_i(\omega)} \tag{5}$$

for all $a_i \in \text{supp}\alpha_i$,

$$\sum_{y\in Y} \pi_y^\omega(a_i,\alpha_{-i}) z_i(y,\omega) \leq \frac{K}{\delta\lambda_i(\omega)} \tag{6}$$

for all $a_i \notin \text{supp}\alpha_i$,

$$\sum_{y\in Y} \pi_y^{\tilde{\omega}}(a_j,\alpha_{-j}) z_j(y,\tilde{\omega}) = 0 \tag{7}$$

for all $a_j \in A_j$, and

$$\lambda_i(\omega) z_i(\omega) + \lambda_j(\tilde{\omega}) z_j(y,\tilde{\omega}) = 0 \tag{8}$$

for all $y \in Y$. To prove this claim, eliminate (8) by solving for $z_j(y,\tilde{\omega})$. Then, instead of (7), we have

$$\sum_{y\in Y} \pi_y^{\tilde{\omega}}(a_j,\alpha_{-j}) z_i(y,\omega) = 0 \tag{9}$$

for all $a_j \in A_j$. By construction, for each $a_i \in \text{supp}\alpha_i \setminus A_{ij}'$, a vector $(\pi_y^\omega(a_i,\alpha_{-i}))_{y\in Y}$ is represented by a linear combination of $(\pi_y^\omega(a_i',\alpha_{-i}))_{y\in Y}$ for $a_i' \in A_{ij}' \bigcap \text{supp}\alpha_i$. Hence, if (5) holds for all $a_i' \in A_{ij}' \bigcap \text{supp}\alpha_i$ then (5) for $a_i \in \text{supp}\alpha_i \setminus A_{ij}'$ is satisfied as well. Likewise, if (9) holds for all $a_j' \in A_{ij}' \bigcap A_j$, then (9) for $a_j \in A_j \setminus A_{ij}'$ is satisfied. Moreover, if (5) holds for all $a_i' \in A_{ij}' \bigcap \text{supp}\alpha_i$, (9) holds for all $a_j' \in A_{ij}' \bigcap A_j$, and

$$\sum_{y\in Y} \pi_y^\omega(a_i'',\alpha_{-i}) z_i(y,\omega) = 0 \tag{10}$$

for all $a_i'' \in A_{ij}' \bigcap (A_i \setminus \text{supp}\alpha_i)$, then (6) for $a_i \in A_i \setminus (\text{supp}\alpha_i \bigcup A_{ij}')$ is satisfied, as a vector $(\pi_y^\omega(a_i,\alpha_{-i}))_{y\in Y}$ is represented by a linear combination of $(\pi_y^\omega(a_i',\alpha_{-i}))_{y\in Y}$ for $a_i' \in A_{ij}' \bigcap A_i$ and $(\pi_y^\omega(a_j',\alpha_{-j}))_{y\in Y}$ for $a_j' \in A_{ij}' \bigcap A_j$ with weights such that $\sum_{a_i' \in \text{supp}\alpha_i} \kappa^\omega(a_i') \leq 1$. Therefore, to establish the above claim, it suffices to show that there exist $(z_i(y,\omega))_{y\in Y}$ such that (5) holds for all $a_i \in A_{ij}' \bigcap \text{supp}\alpha_i$, (10) holds for all $a_i \in A_{ij}' \bigcap (A_i \setminus \text{supp}\alpha_i)$, and (9) holds for all $a_j \in A_{ij}' \bigcap A_j$. Note that this system consists of $|A_{ij}'|$ linear equations, and its coefficient matrix consists of rows

$(\pi_y^\omega(a_i', \alpha_{-i}))_{y\in Y}$ for $a_i' \in A_{ij}' \bigcap A_i$ and $(\pi_y^\omega(a_j', \alpha_{-j}))_{y\in Y}$ for $a_j' \in A_{ij}' \bigcap A_j$. Since this matrix has rank $|A_{ij}'|$, we can indeed solve the system.

Let $(\tilde{v}, \tilde{w})$ be a pair of a payoff vector and a function such that $\tilde{w}$ enforces $(\tilde{v}, \alpha)$. Let $K > \max_{y\in Y} \lambda \cdot \tilde{w}(y) - \lambda \cdot \tilde{v}$, and choose $(z_i(y,\omega), z_j(y,\tilde{\omega}))_{y\in Y}$ to satisfy (5) through (8). Then, let

$$w_l(y,\overline{\omega}) = \begin{cases} \tilde{w}_i(y,\omega) + z_i(y,\omega) & \text{if} \quad (l,\overline{\omega}) = (i,\omega) \\ \tilde{w}_j(y,\tilde{\omega}) + z_j(y,\tilde{\omega}) & \text{if} \quad (l,\overline{\omega}) = (j,\tilde{\omega}) \\ \tilde{w}_l(y,\overline{\omega}) & \text{otherwise} \end{cases}$$

for each $y \in Y$. Also, let

$$v_l(\overline{\omega}) = \begin{cases} \tilde{v}_i(\omega) + \frac{K}{\lambda_i(\omega)} & \text{if} \quad (l,\overline{\omega}) = (i,\omega) \\ \tilde{v}_j(\tilde{\omega}) & \text{if} \quad (l,\overline{\omega}) = (j,\tilde{\omega}) \\ \tilde{v}_l(\overline{\omega}) & \text{otherwise} \end{cases}.$$

We claim that this (v,w) satisfies all the constraints in LP-Average. Obviously, constraints (i) and (ii) are satisfied for all $(l,\overline{\omega}) \in (\boldsymbol{I}\times\Omega)\setminus\{(i,\omega),(j,\tilde{\omega})\}$, as $v_l(\overline{\omega}) = \tilde{v}_i(\overline{\omega})$ and $w_l(y,\overline{\omega}) = \tilde{w}_l(y,\overline{\omega})$. Also, since (5) and (6) hold and $\tilde{w}$ enforces $(\alpha, \tilde{v})$, we obtain

$$\begin{aligned} &(1-\delta)g_i^\omega(a_i,\alpha_{-i}) + \delta\sum_{y\in Y}\pi_y^\omega(a_i,\alpha_{-i})w_i(y,\omega) \\ =&(1-\delta)g_i^\omega(a_i,\alpha_{-i}) + \delta\sum_{y\in Y}\pi_y^\omega(a_i,\alpha_{-i})\left(\tilde{w}_i(y,\omega) + z_i(y,\omega)\right) \\ =&\left((1-\delta)g_i^\omega(a_i,\alpha_{-i}) + \delta\sum_{y\in Y}\pi_y^\omega(a_i,\alpha_{-i})\tilde{w}_i(y,\omega)\right) + \frac{K}{\lambda_i(\omega)} \\ =&\tilde{v}_i(\omega) + \frac{K}{\lambda_i(\omega)} \end{aligned}$$

for all $a_i \in \text{supp}\alpha_i$, and

$$\begin{aligned} &(1-\delta)g_i^\omega(a_i,\alpha_{-i}) + \delta\sum_{y\in Y}\pi_y^\omega(a_i,\alpha_{-i})w_i(y,\omega) \\ =&(1-\delta)g_i^\omega(a_i,\alpha_{-i}) + \delta\sum_{y\in Y}\pi_y^\omega(a_i,\alpha_{-i})\left(\tilde{w}_i(y,\omega) + z_i(y,\omega)\right) \\ \leq&\tilde{v}_i(\omega) + \frac{K}{\lambda_i(\omega)} \\ =&v_i(\omega) \end{aligned}$$

for all $a_i \notin \text{supp}\alpha_i$. Hence, (v,w) satisfies constraints (i) and (ii) for (i,ω). Likewise, it follows from (7) that (v,w) satisfies constraints (i) and (ii) for $(j,\tilde{\omega})$. Furthermore, using (8) and $K > \max_{y\in Y} \lambda \cdot \tilde{w}(y) - \lambda \cdot \tilde{v}$,

$$\begin{aligned}
\lambda \cdot w(y) &= \lambda \cdot \tilde{w}(y) + \lambda_i(\omega) z_i(y,\omega) + \lambda_j(\tilde{\omega}) z_j(y,\tilde{\omega}) \\
&= \lambda \cdot \tilde{w}(y) \\
&< \lambda \cdot \tilde{v} + K \\
&= \lambda \cdot v
\end{aligned}$$

for all $y \in Y$, and hence constraint (iii) holds.

Therefore, $k^*(\alpha,\lambda) \geq \lambda \cdot v = \lambda \cdot \tilde{v} + K$. Since K can be arbitrarily large, we conclude $k^*(\alpha,\lambda) = \infty$. *Q.E.D.*

A.5 Proof of Claim 2

Claim 2. *For $\alpha = (C_1,C_2)$ and $\lambda = ((1,0),(0,1))$,*

$$k^*(\alpha,\lambda) \leq \lambda \cdot g(C_1,C_2) - \frac{1-\beta}{\beta}(g_2^{\omega_2}(C_1,D_2) - g_2^{\omega_2}(C_1,C_2)).$$

Proof. Consider the associated LP-Average problem, and choose (v,w) to satisfy constraints (i) through (iii) of this problem. From player 2's IC constraint for state ω_2, we have

$$\begin{aligned}
&\beta(q_H(w_2(H,\omega_2) - w_2(L,\omega_2)) + q_M(w_2(M,\omega_2) - w_2(L,\omega_2))) \\
&\qquad\qquad \geq \frac{1-\delta}{\delta}(g_2^{\omega_2}(C_1,D_2) - g_2^{\omega_2}(C_1,C_2)).
\end{aligned}$$

Then,

$$\begin{aligned}
v_1(\omega_1)+v_2(\omega_2) =&(1-\delta)g_1^{\omega_1}(C_1,C_2)+\delta\sum_{y\in Y}\pi_y^{\omega_1}(C_1,C_2)w_1(y,\omega_1)\\
&+(1-\delta)g_2^{\omega_2}(C_1,C_2)+\delta\sum_{y\in Y}\pi_y^{\omega_2}(C_1,C_2)w_2(y,\omega_2)\\
=&(1-\delta)(g_1^{\omega_1}(C_1,C_2)+g_2^{\omega_2}(C_1,C_2))\\
&+\delta\sum_{y\in Y}\pi_y^{\omega_1}(C_1,C_2)(w_1(y,\omega_1)+w_2(y,\omega_2))\\
&-\delta(1-\beta)(q_H(w_2(H,\omega_2)-w_2(L,\omega_2))+q_M(w_2(M,\omega_2)-w_2(L,\omega_2)))\\
\leq&(1-\delta)(g_1^{\omega_1}(C_1,C_2)+g_2^{\omega_2}(C_1,C_2))+\delta(v_1(\omega_1)+v_2(\omega_2))\\
&-\frac{(1-\delta)(1-\beta)}{\beta}(g_2^{\omega_2}(C_1,D_2)-g_2^{\omega_2}(C_1,C_2))
\end{aligned}$$

Arranging,

$$v_1(\omega_1)+v_2(\omega_2)\leq g_1^{\omega_1}(C_1,C_2)+g_2^{\omega_2}(C_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(C_1,D_2)-g_2^{\omega_2}(C_1,C_2)).$$

So we have

$$\lambda\cdot v\leq\lambda\cdot g(C_1,C_2)-\frac{1-\beta}{\beta}(g_2^{\omega_2}(C_1,D_2)-g_2^{\omega_2}(C_1,C_2)).$$

This proves the desired result. *Q.E.D.*

References

Abreu, D., D. Pearce, and E. Stacchetti (1986): "Optimal Cartel Equilibria with Imperfect Monitoring," *Journal of Economic Theory* 39, 251-269.

Abreu, D., D. Pearce, and E. Stacchetti (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica* 58, 1041-1063.

Athey, S., and K. Bagwell (2001): "Optimal Collusion with Private Information," *RAND Journal of Economics* 32, 428-465.

Aumann, R., and S. Hart (1992): *Handbook of Game Theory with Economic Applications, Volume 1*. North Holland, New York, NY.

Aumann, R., and M. Maschler (1995): *Repeated Games with Incomplete Information*. MIT Press, Cambridge, MA. With the collaboration of R.E. Stearns.

Bhaskar, V., G.J. Mailath, and S. Morris (2008): "Purification in the Infinitely Repeated Prisoner's Dilemma," *Review of Economic Dynamics* 11, 515-528.

Bergemann, D., and S. Morris (2007): "Belief-Free Incomplete Information Games," mimeo.

Cripps, M., and J. Thomas (2003): "Some Asymptotic Results in Discounted Repeated Games of One-Side Incomplete Information," *Mathematics of Operations Research* 28, 433-462.

Deb, J. (2008): "Cooperation and Community Responsibility: A Folk Theorem for Repeated Random Matching Games," mimeo.

Dekel, E., D. Fudenberg, and D.K. Levine (2004): "Learning to Play Bayesian Games," *Games and Economic Behavior* 46, 282-303.

Ely, J., J. Hörner, and W. Olszewski (2005): "Belief-Free Equilibria in Repeated Games," *Econometrica* 73, 377-415.

Ely, J., and J. Välimäki (2002): "A Robust Folk Thorem for the Prisoner's Dilemma," *Journal of Economic Theory* 102, 84-105.

Fudenberg, D., and D.K. Levine (1983): "Subgame-Perfect Equilibria of Finite and Infinite Horizon Games," *Journal of Economic Theory* 31, 251-268

Fudenberg, D., and D.K. Levine (1994): "Efficiency and Observability in Games with Long-Run and Short-Run Players," *Journal of Economic Theory* 62, 103-135.

Fudenberg, D., D.K. Levine, and E. Maskin (1994): "The Folk Theorem with imperfect public information," *Econometrica* 62, 997-1040.

Fudenberg, D., D.K. Levine, and S. Takahashi (2007): "Perfect Public Equilibrium when Players are Patient," *Games and Economic Behavior* 61, 27-49.

Fudenberg, D., and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting and with Incomplete Information," *Econometrica* 54, 533-554.

Fudenberg, D., and Y. Yamamoto (2009): "Bayesian Perfect Public Ex-Post Equilibria," mimeo.

Gossner, O., and N. Vieille (2003): "Strategic Learning in Games with Symmetric Information," *Games and Economic Behavior* 42, 25-47.

Green, E.J., and R.H. Porter (1984): "Noncooperative Collusion under Imperfect Price Information," *Econometrica* 52, 87-100.

Hart, S. (1985): "Nonzero-Sum Two-Person Repeated Games with Incomplete Information," *Mathematics of Operations Research* 10, 117-153.

Hörner, J., and S. Lovo (2008): "Belief-Free Equilibria in Games with Incomplete Information," forthcoming in *Econometrica*.

Hörner, J., S. Lovo, and T. Tomala (2008): "Belief-Free Equilibria in Games with Incomplete Information: the N-Player Case," mimeo.

Kandori, M. (1992): "The Use of Information in Repeated Games with Imperfect Monitoring," *Review of Economic Studies* 59, 581-593.

Kandori, M. (2008): "Weakly Belief-Free Equilibria in Repeated Games with Private Monitoring," mimeo.

Kreps, D., and R. Wilson (1982): "Sequential Equilibria," *Econometrica* 50, 863-894.

Levin, J. (2003): "Relational Incentive Contracts," *American Economic Review* 93, 835-857.

Miller, D. (2007): "The Dynamic Cost of Ex Post Incentive Compatibility in Repeated Games of Private Information," mimeo.

Piccione, M. (2002): "The Repeated Prisoner's Dilemma with Imperfect Private Monitoring," *Journal of Economic Theory* 102, 70-83.

Radner, R., R. Myerson, and E. Maskin (1986): "An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria," *Review of Economic Studies* 53, 59-70.

Takahashi, S. (2008): "Community Enforcement when Players Observe Partners' Past Play," mimeo.

Wiseman, T. (2005): "A Partial Folk Theorem for Games with Unknown Payoff Distributions," *Econometrica* 73, 629-645.

Wiseman, T. (2008) "A Partial Folk Theorem for Games with Private Learning," mimeo.

Yamamoto, Y. (2007): "Efficiency Results in N Player Games with Imperfect Private Monitoring," *Journal of Economic Theory* 135, 382-413.

Yamamoto, Y. (2009): "A Limit Characterization of Belief-Free Equilibrium Payoffs in Repeated Games," *Journal of Economic Theory* 144, 802-824.