Variational Bewley Preferences

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Abstract

This paper characterize preference relations over Anscombe and Aumann acts and give necessary and sufficient conditions that guarantee the existence of a utility function u on consequences and an *ambiguity index* η on the set of probabilities on the states of the nature such that, for all acts f and g,

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta(p) \geq \int u(g)dp, \forall p \in \Delta.$$

The function u represents the decision maker´s risk attitudes, while the ambiguity index $\eta(p)$ about the prior p captures its relative degree of plausibility. The axiomatic basis for this class of preference waiver completeness and transitivity, and an interesting property is that cycles are avoided.

The Bewley´s model of choice under uncertainty with transitive and incomplete preferences is included in this class of preferences as well the subjective expected utility model. As new examples, we can describe some special class of preferences, *e.g.*, the intransitive and incomplete entropic Bewley preferences obtained through the relative entropic ambiguity index.

1 Introduction

In the 80s two alternatives axiomatic approaches appeared as foundations for the distinction proposed by Frank Knight (1921) between risk and uncertainty. Bearing in mind that risk is characterized by randomness with well defined probabilities and uncertainty captures randomness with vague probabilities, both Gilboa and Schmeidler (1989) and Bewley (2002) proposed a set of axioms for preference relations on uncertainty acts endogenously getting a set of probabilities compatible with the decision maker's beliefs, which led to *multiple priors* models. On the other hand, previous theoretical developments as the famous

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axiomatizations of subjective expected utility theory (SEU), as proposed by Savage (1954) and Anscombe and Aumann (1963), had suggest that the Knight´s distinction is irrelevant because any uncertainty can be modeled through probabilities.

Multiple priors models were inspired also by the well known objection to the theory of subjective probability formulated by Ellsberg (1961), which formed the basis for the notion for *ambiguity*: an event is ambiguous if it has a unknown probability. In fact, Ellsberg showed that individuals may prefer gambles with precise probability to gambles with unknown odds, that is, ambiguity matters for choice[1].

As regards the axiomatic foundation, while the maxmin expected utility model (MEU) of Gilboa and Schmeidler (1989) remove the independence axiom from the Anscombe and Aumann list of axiom, the theory proposed by Bewley (2002) for Knightian uncertainty or ambiguity presents as main behavioral feature the lack of completeness in the decision maker´s preference relation[2]. In this model a set of priors C determines a preference relation via an unanimity rule: an act f is strictly better than an act g if and only if the expected utility of f is strictly higher than the expected utility of g for every prior p in the set C.

Ghirardato, Maccheroni, Marinacci and Siniscalchi (GMMS, 2003) provide a derivation of Bewley´s model in the purely subjective probability framework *a la* Savage, but such derivation differs from Bewley´s on some aspects, most important is that Bewley considers as primitive a strict preference relation while GMMS propose a representation using a reflexive relation as primitive which delivers an unanimity rule where f is at least as desirable as g if and only if the expected utility of f is at least as high as the expected utility of g for every prior p in the set C[3].

The main idea of Bewley´s model is that the presence of uncertainty might make the agent confused, which induces he stay with her status quo. In fact, the ambiguity aversion may reduces her confidence in her ability to compare some alternatives, as consequence, we observe the incompleteness of the preference relation as explained above. An important point is the inertia assumption proposed by Bewley: we view the agent in question as considering a set of priors beliefs in her decision on whether to abandon the status quo, and some degree of dominance using this set of priors is requiring before moving away from her status quo.

However, a natural point is that decision makers could presents a non uni-

[1] Although many arguments showed the that people often fail to behave in accordance with the subjective expected utility, Savage and Anscombe-Aumann provided as a legacy an important framework that serves as the basis for much of the recent developments in the theory of decision under ambiguity.

[2] Recently, Nascimento and Riella (2008) unified both approaches through an axiomatic foundation for a class of incomplete and ambiguity averse preferences in the sense of Ghirardato and Marinacci (2002).

[3] Ghirardato, Maccheroni and Marrinaci (GMM, 2004) obtained the same result in the Anscombe and Aumann´s set up for general state space. See also Giroto and Holzer (2005).

form opinion among the class of plausible models[4], and such factor should rule out the original Bewley´s unanimity principle that implicitly requires an uniform importance among plausible priors. Intuitively, assuming that the decision maker has a non uniform degree of confidence among such priors, then the full dominance as in the unanimity rule might be more coherent with a relative dominance rule, where less plausibility implies in some amount of acceptable loss in terms of expected utility. Bewley´s model is too extreme in the sense that a plausible priors is given by only probabilities in the set C and every plausible prior has an identical degree of confidence.

Aiming to get a model that captures the previous considerations, we characterized preference relations on the set of Anscombe and Aumann´s acts where a relative dominance rule is obtained. Our axiomatization generates a decision rule that generalizes the model proposed by Bewley (2002) via the notion of ambiguity index as proposed in Maccheroni, Marinacci and Rustichini (2006)´s foundation of variational preferences: the decision maker´s subjective ambiguity index is a special mapping η over the set of all probability measures Δ with values in $\mathbb{R}_+ \cup \{+\infty\}$ and the preference relation $\succsim$ satisfies,

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta\left(p\right) \geq \int u(g)dp,\ \forall p \in \Delta.$$

Where, u is the von Neumann-Morgenstern utility function over the set of lotteries X. Note that if $\eta\left(\cdot\right) = \delta_C\left(\cdot\right)$ for some (convex and closed) set of probability measures then we obtain the Bewley´s decision rule. We axiomatize preferences, called *variational Bewley preferences*, consistent with the decision rule above by showing how it rests on a simple set of axioms that generalizes the Bewley´s model as studied by Ghirardato, Maccheroni and Marinacci (2004), being more precisely, we do not impose transitivity.

In fact, in this paper both completeness as transitivity are not imposed for a preference relation. It is consistent with Aumann (1962) complains about the inaccurate description of actual behavior implied by completeness axiom and the normative viewpoint demanding that decision makers should make well comparison of every pair of alternatives. Mandler (2005) extended this criticism to incomplete and intransitive preferences by showing that even decision makers with such kind of preferences are not necessarily subject to money-pumps. Interestingly, we show that even intransitive any variational Bewley preference is acyclic.

The paper is organized as follows. After introducing the setup in Section 2 and the set of axioms in Section 3, we present the main representation result in Section 4. In Section 5, we derive conditions in order to obtain the countable additive case. In Section 6, we discuss the ambiguity revealing properties, in the sense of Ghirardato, Maccheroni and Marinacci (2004), featured by the class of preferences characterized in the main result. In Section 7, we study some special cases, namely the incomplete preferences of Bewley (1989) as well

[4]Following the MEU tradition, some work captures a similar idea, e.g., Maccheroni, Marinacci and Rustichini (2006) and Chateauenuf and Faro (2006).

its special case given by the SEU model of Anscombe and Aumann (1963). Also, we derive some special class of variational Bewley preferences, *e.g.*, the intransitive and incomplete entropic preferences obtained through the relative entropic ambiguity index. Proofs and related material are collected in the Appendix.

2 Framework

Consider a set S of *states of nature (world)*, endowed with an σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all the (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[5]. Moreover, we denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = sup_{s\in S} |a(s)|$ (called *sup norm*) and will denote by $B(\Sigma)$ the supnorm closure of $B_0(\Sigma)$.

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s)), \text{for all}\, s \in S$. We note that $u(f) \in B_0(S, \Sigma)$ whenever f belongs to $\mathcal{F}$.

Let x belong to X, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set $\mathcal{F}_c$ of the constant acts in $\mathcal{F}$.

Additionally, we assume that the set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of *outcomes* Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as re-started by Fisburn (1970).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act:

$$\begin{aligned}\alpha f + (1-\alpha)g \quad &: \quad S \to X \\ (\alpha f + (1-\alpha)g)(s) \quad &= \quad \alpha f(s) + (1-\alpha)g(s).\end{aligned}$$

Also, given two acts $f, g \in \mathcal{F}$ and an event $A \in \Sigma$ we denote by fAg the act h such that $h\,|_A = f$ and $h\,|_{A^c} = g$.

The decision maker's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose the usual symmetric and asymmetric components are denoted by $\sim$ and $\succ$.

We denote by $\Delta := \Delta(\Sigma)$ the set of all (finitely additive) probability measures $p : \Sigma \to [0, 1]$ endowed with the natural restriction of the well known weak* topology $\sigma(ba, B)$. We say that a mapping $\eta : \Delta \to [0, \infty]$ is grounded if $\{\eta = 0\} := \{p \in \Delta : \eta(p) = 0\} \neq \emptyset$ and its effective domain is defined by $dom(\eta) := \{\eta < \infty\}$. Also, η is weak* lower semicontinuous if $\{\eta \leq r\}$ is weak* closed for each $r \geq 0$. Moreover, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q \in \Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely contin-

[5] Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

uous w.r.t. q, *i.e.*, $\Delta^{\sigma}(q) = \{p \in \Delta^{\sigma} : p \ll q\}$, where $p \ll q$ means that $\forall A \in \Sigma,\ q(A) = 0 \Rightarrow p(A) = 0$.

Functions of the form $\eta : \Delta \to [0, \infty]$ will play a key role in the paper because it will capture the subjective degree of plausibility of the decision makers. We denote by $\mathcal{N}(\Delta)$ the class of these functions such that η is grounded, convex and weak* lower semicontinuous.

3 Axioms

Next we describe the axioms imposed in this paper on a preference relation $\succsim$ on the set of Anscombe and Aumann acts $\mathcal{F}$:

(**Axiom 1**) $\succsim$ is reflexive: For any $f \in \mathcal{F}$, $f \succsim f$.

(Axiom 2) The restriction on lotteries $\succsim|_{X \times X}$ is nontrivial, complete and transitive.

(**Axiom 3**) Archimedean Continuity. For all $f, g, h \in \mathcal{F}$ the sets:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\} \text{ and } \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\} \text{ are closed in } [0,1].$$

(**Axiom 4**) Monotonicity. For every $f, g \in \mathcal{F}$:

$$\text{if } f(s) \succsim (\succ)\, g(s) \text{ for any } s \in S \text{ then } f \succsim (\succ)\, g.$$

(**Axiom 5**) S-Independence: For every $f, g, h_1, h_2 \in \mathcal{F}$, and every $\alpha \in (0,1)$,

$$f \succsim g \text{ and } h_1 \succsim h_2 \text{ iff } \alpha f + (1-\alpha)h_1 \succsim \alpha g + (1-\alpha)h_2.$$

(**Axiom 6**) Unboundedness. There are $x, y \in X$ such that, for each $\alpha \in (0,1)$, there exist $z, \widehat{z} \in X$ such that

$$\alpha z + (1-\alpha)\, y \succ x \succ y \succ \alpha \widehat{z} + (1-\alpha)\, x.$$

Since we are following the standard notion of weak preference, *i.e.,* given two acts f and g the relation $f \succsim g$ means that "f is at least as good as g", Axiom 1 seems very natural because it says that any act is at least as good as the same. On the other hand, we relax the usual completeness and transitivity conditions about preferences over uncertainty acts.

Axiom 2 means that preferences over consequences satisfies standard assumptions concerning the *classical notion of rationality*, and also there is at least one par of consequences for which the decision maker is not indifferent between then. Axiom 3 and Axiom 6 are technical assumptions.

Axiom 4 is a state-independence condition for both weak and strict sense of preference, saying that decision makers always prefer acts delivering state-wise better payoffs, regardless of the state where the better payoffs occur.

Axiom 5 says that if a decision maker has two well defined preference between two pars of acts then for any two acts obtained through mixtures from the two best and worst acts of originals comparisons, respectively, then the preference between new acts obtained should respect the original ordering. We note that Axiom 6 is stronger than the usual Independence axiom, in fact, for the latter it is enough to consider $h_1 = h_2$. Also, recall that under transitivity assumption both Axiom 5 and the Independence axiom are equivalent conditions[6].

4 Main Theorem

We now derive our general representation that relies on Axioms A1-A6.

Theorem 1 *Let $\succsim$ be a preference relation on the set of Anscombe-Aumann acts $\mathcal{F}$. Then the following conditions are equivalent:*

(1) $\succsim$ satisfies assumptions A.1-A.6.

(2) There exists an affine utility index $u : X \to \mathbb{R}$, with $u(X) = \mathbb{R}$, and a function $\eta : \Delta \to [0, \infty]$ that belongs to $\mathcal{N}(\Delta)$ such that, for all f and g in $\mathcal{F}$,

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta(p) \geq \int u(g)dp,\ \forall p \in \Delta.$$

Moreover, u in (2) is unique up to positive linear transformation and for each u there is a (unique) minimal $\eta^* : \Delta \to [0, \infty]$ consistent with the decision rule above and is given by

$$\eta^*(p) = \sup_{(f,g) \in \succsim} \left(\int (u(g) - u(f))\, dp \right),\ \forall p \in \Delta.$$

The representation above involves a mapping η defined on $\Delta \subset B(\Sigma)^*$ which is a grounded, convex and weak* lower semicontinuous function, hence it can be viewed as a *Fenchel conjugate* of some functional on $B(\Sigma)$, which is one of the most classic tool in variational analysis[7]. This motivates the following definition:

Definition 2 *A preference $\succsim$ on $\mathcal{F}$ is called variational Bewley preference if it satisfies Axioms A.1–A.6.*

Following the Bewley inertia idea, an interesting interpretation for η says that $\eta(p)$ measure the maximal expected loss accepted by the decision maker if p is true model. Note that $\eta(p) < \eta(q)$ says that p is subjectively more plausible than q. So, the dominance reflects such difference on the decision maker´s confidence among priors:

$$f \succsim g \Leftrightarrow \int (u(f) - u(g))dp \geq -\eta(p),\ \forall p \in \Delta,$$

i.e., for priors the most plausible priors (*i.e.*, for priors $p \in \{\eta = 0\}$) we have the dominance *a la* Bewley

$$\int u(f)dp \geq \int u(g)dp,\ \forall p \in \{\eta = 0\},$$

[6] Technically, Axiom 5 says that the preference $\succsim$ is a convex subset of $\mathcal{F} \times \mathcal{F}$.

[7] See, for instance, Brézis (1984), page 8.

otherwise, the decision maker is willing to accept at most a loss (in terms of expected value) equal to $-\eta(p)$ for abandon the status quo.

An interesting aspect of our representation rule is concerns about the indifference relation $\sim$, in fact, since $f \sim g$ iff $f \succsim g$ and $g \succsim f$, the main theorem entails that

$$f \sim g \text{ iff } \eta(p) \geq \left| \int u(f)\, dp - \int u(g)\, dp \right|, \forall\, p \in \Delta.$$

Hence, indifference is equivalent to the fact that, for any prior, the module of the difference between the corresponding expected utilities is limited by the prior's plausibility. So, for priors with full plausibility the difference should be null; on the other hand, by considering priors with small plausibility degree, indifference in preference is consistent with the possibility of a significant difference between the corresponding expected values.

By Theorem 1, variational Bewley preferences can be represented by a pair (u, η^*). Hence, we will write u and η^* to denote our class of preferences. From now on, when we consider a variational Bewley preference, we will write u and η to denote the elements of such a pair. Next we give the uniqueness properties of this representation.

Corollary 3 *Two pairs (u, η^*) and (u_1, η_1^*) represent the same variational Bewley preference $\succsim$ if and only if there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u + \beta$ and $\eta_1^* = \alpha \eta^*$.*

An interesting consequence of this uniqueness result together with the Bewley´s unanimity rule, as characterized by Ghirardato Maccheroni and Marinacci (2004), is that our preference relation, in general, is not transitive. For instance, if we assume that "ambiguity index" η^* is given by the well known relative entropic index then the induced preference is not transitive because $\alpha\eta^*$, with $\alpha > 0$,is never an indicator function[8].

Fortunately, variational Bewley preferences are not subject to money-pumps and it is a consequence of next proposition saying that variational Bewley preferences are acyclic.

Proposition 4 *If a preference $\succsim$ on the set of Anscombe and Aumann act is a variational Bewley preference then the induced asymmetric component $\succ$ is acyclic. In fact,*

$$f \succ g \Rightarrow V(f) > V(g),$$

where V is a variational representation of a variational preference (MMR 2006) given by

$$V(f) = \min_{p \in \Delta} \left(\int u(f)\, dp + \eta^*(p) \right).$$

[8] For more details see Section 7.

5 Countable Additive Priors

In our previous analysis we considered the set Δ of all finitely additive probabilities. By its very convenient analytical properties in applications it is very useful to consider the case of countably additive probabilities. As we will see momentarily that this is the case for the construction of some interesting examples.

If we add the transitivity condition in order to recover the Bewley model as in Ghirardato, Maccheroni and Marinacci (2004), we have that the well know Monotone Continuity axiom due to Arrow (1970) is equivalent to the conditions saying that probabilities in the set of multiple priors C are all countably additive, provided is a σ-algebra[9].

Fortunately, the monotone continuity axiom also ensure in our main result that only countably additive probabilities matter. Formally, the monotone continuity axiom follows as:

(**Axiom 7**) Monotone Continuity: We say that a preference relation $\succsim$ on $\mathcal{F}$ is *monotone continuous* if for all consequences $x, y, z \in X$ such that $y \succ z$, and for all sequences of events $\{A_n\}_{n\geq 1}$ with $A_n \downarrow \emptyset$, there exists $k \geq 1$ such that $y \succsim xA_kz$.

Proposition 5 *Let $\succsim$ be a preference relation as in Theorem 1. The following statements are equivalents:*

(i) The preference relation also satisfies the monotone continuity axiom,

(ii) The set dom (η^*) consists of countably additive probabilities.

6 A Characterization of Ambiguity Levels

For the precise result concerning the characterization of η on the main result as a ambiguity level we need the following definition:

Definition 6 *(Ghirardato, Maccheroni and Marinacci, 2004) We say that the preference relation $\succsim_1$ reveals more ambiguity than $\succsim_2$ if for any acts f and g*

$$f \succsim_1 g \Rightarrow f \succsim_2 g$$

The decision maker 2 (with utility index u_2 and ambiguity index η_2^*) has a richer unambiguous preference than the decision maker 1 (with utility index u_1 and ambiguity index η_1^*) because the decision maker 2 behaves as if he is better informed about the decision problem.

Proposition 7 *The following statements are equivalents:*

a) The preference relation $\succsim_1$ reveals more ambiguity than $\succsim_2$

b) Both decision makers has the same attitudes towards risk (w.l.g, $u_1 = u_2$) and $\eta_1^* \leq \eta_2^*$.

[9] See, for instance, Proposition B.1 of Ghirardato, Maccheroni and Marinacci (2004).

Now, consider that the subjective expected utility is the benchmark for absence of ambiguity. We say that preference relation $\succsim$ reveals ambiguity when such preference reveals more ambiguity than some subjective expected utility preference $\succsim_{SEU}$. As consequence of the Proposition 7 and by $\{\eta = 0\} \neq \emptyset$, the class of preferences characterized in Theorem 1 reveals ambiguity.

7 Special Cases

In this section we study in some more detail special classes of variational Bewley preferences: the Knightian uncertainty model of Bewley (2002) and some preferences we just introduced, *e.g.*, the incomplete and intransitive entropic preferences.

7.1 Bewley Incomplete Preferences

Begin with the Knightian uncertainty model choice model axiomatized by Bewley (2002). As we mentioned in Introduction, the Bewley model is characterized by transitivity, an axiom that we dropped in our main result. Next we show in detail the relationship between transitivity and our main decision rule obtained in Theorem 1. In particular, when we add transitivity, the only probabilities in Δ that matter are those to which the decision maker attributes maximum plausibility that is, those in $\{\eta^* = 0\}$, otherwise probabilities presents null plausibility, *i.e.*, $\Delta = \{\eta^* = 0\} \cup \{\eta^* = \infty\}$. Also, note that transitivity implies that every probability that matter has the same degree of plausibility.

Proposition 8 *Let $\succsim$ be a variational Bewley preference. The following conditions are equivalent:*
(i) The preference $\succsim$ satisfies transitivity;
(ii) For all $f, g \in \mathcal{F}$

$$f \succsim g \text{ iff } \int u(f)dp \geq \int u(g)dp, \text{ for any } p \in \{\eta^* = 0\}\,;$$

(iii) The function η^* takes on only values 0 and ∞.

7.2 Divergence Bewley preferences

We now introduce a new class of variational Bewley preferences that play an important role in the rest of this section. Assume there is an underlying probability measure $q \in \Delta^\sigma$. Given a convex continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(1) = 0$ and $\lim_{t\to\infty} \phi(t)/t = 1$, the ϕ-divergence of $p \in \Delta$ with respect to q is given by

$$D_\phi(p \parallel q) = \begin{cases} \int \phi\left(\frac{dp}{dq}\right) dq, & \text{if } p \in \Delta^\sigma(q) \\ \infty, & \text{otherwise.} \end{cases}$$

The mappings $D_\phi(\cdot \parallel \cdot)$ are well known as standard divergences, which are a widely used notion of distance between distributions in statistics and information theory[10]. The two most important divergences are the *relative entropy* given by $\phi(t) = t\ln t - t + 1$, and the *relative Gini concentration index* given by $\phi(t) = (t-1)^2/2$.

The next lemma due to Maccheroni et. al. (2006) presents some important properties of divergences.

Lemma 9 *A divergence $D_\phi(\cdot \parallel q) : \Delta \to [0,\infty]$ is a grounded, convex, and lower semicontinuous function, and the sets $\{p \in \Delta : D_\phi(p \parallel q) \leq t\}$ are weakly compact subsets of $\Delta^\sigma(q)$ for all $t \in \mathbb{R}$.*

Thanks to the above properties, preferences $\succsim$ on $\mathcal{F}$ that satisfies the following rule

$$f \succsim g \Leftrightarrow \int \{u(f) - u(g)\}\, dp \geq -\theta D_\phi(p \parallel q), \forall p \in \Delta,$$

where $\theta > 0$ and $u : X \to \mathbb{R}$ is an affine function, belong to the class of variational Bewley preferences. In view of their interesting properties, we call them *divergence Bewley preferences.*

Theorem 10 *Divergence Bewley preferences are monotone continuous variational Bewley preferences with index of ambiguity aversion given by*

$$\eta^* : p \in \Delta \to \theta D_\phi(p \parallel q).$$

Concerning the analysis of comparative attitudes, the next simple consequence of Proposition 7 shows that they depend only on the parameter θ, which can therefore be interpreted as a coefficient of ambiguity level. In order to be more specific about ϕ, we speak of ϕ-divergence Bewley preferences.

Corollary 11 *Given two ϕ-divergence Bewley preferences $\succsim_1$ and $\succsim$, the following statements are equivalents:*

Proposition 12 *a) The preference relation $\succsim_1$ reveals more ambiguity than $\succsim_2$*
b) Both decision makers has the same attitudes towards risk (w.l.g, $u_1 = u_2$) and $\theta_1 \leq \theta_2$.

This result says that divergence Bewley preferences become revealing more and more (less and less, resp.) ambiguity as the parameter becomes closer and closer to 0 (closer and closer to ∞, resp.). In fact, since for any $p \in \Delta^\sigma(q)$

$$\lim_{\theta\to\infty} \theta D_\phi(p \parallel q) = \begin{cases} \infty, \text{ if } p \neq q \\ 0, \text{ if } p = q \end{cases},$$

[10] Csiszár (1963) introduced the notion of ϕ-divergences $D_\phi(\cdot \parallel \cdot)$ for probability measures and Liese and Vajda (1987) extended ϕ-divergences $D_\phi(\cdot \parallel \cdot)$ to finite or infinite measures.

we obtain that divergence Bewley preferences tend, more and more, as $\theta \to \infty$, to rank acts according to the SEU criterion with subjective probability q. On the other hand, since for any $p \in \Delta^{\sigma}(q)$

$$\lim_{\theta \to 0} \theta D_{\phi}(p \parallel q) = 0,$$

we obtain that divergence Bewley preferences tend more and more, $\theta \to 0$, to rank acts according to the very cautious criteria. For example, when q has a finite support supp(q) such cautious criteria says that[11]

$$f \succsim g \text{ iff } u(f(s)) \geq u(g(s)), \ \forall s \in \operatorname{supp}(q).$$

We commented that the two most important divergences are the relative entropy and the relative Gini concentration index given, which motivates the following examples:

Example 13 *If* $\eta = \theta R(\cdot \parallel q) : \Delta \to [0, \infty]$, *where* $q \in \Delta^{\sigma}$ *(σ-additive probability) and*

$$R(p \parallel q) = \begin{cases} \int \log\left(\frac{dp}{dq}\right) dp \text{ if } p \ll q \\ \infty, \text{ otherwise} \end{cases}$$

is the relative entropy index (w.r.t q), we obtain a preference relation in a similar spirit of Hansen and Sargent (2001)´s robustness model, but with a decision rule a la *Bewley, which we dub as entropic Bewley preferences.*

Example 14 *if* $\eta = \theta G(\cdot \parallel q) : \Delta \to [0, \infty]$, *where* $q \in \Delta^{\sigma}$ *and*

$$G(p \parallel q) = \begin{cases} \frac{1}{2} \int \left(\frac{dp}{dq} - 1\right)^2 dq \text{ if } p \ll q \\ \infty, \text{ otherwise} \end{cases}$$

is the classic concentration Gine index, which is related to the well known model proposed by Tobin (1958) and Markowitz (1952). In fact, Macherroni, Marinacci and Rustichini (2006) showed that such ambiguity index for variational preferences entails the Tobin and Markowitz preference. We say that $\succsim$ is a Gine Bewley preference if $\succsim$ is a divergence Bewley preference for which $\theta G(\cdot \parallel q)$ is the ambiguity index.

7.3 More examples

Completing the list of examples we proposed two cases not included as Bewley multiple prior preferences or divergence Bewley preferences:

[11] For the general case we need to assume some topological struture on the state spade because

$$\operatorname{supp}(q) := \cap \{E \subset S : E \text{ is closed and } q(E^c) = 0\}$$

Example 15 *If* $\eta = \theta R(\cdot \parallel C) : \Delta \to [0,\infty]$, *where* $q \in \Delta^{\sigma}$ *and*

$$R(p \parallel C) = \inf_{q \in C} R(p \parallel q)$$

is the relative entropy index w.r.t. C[12]*, we obtain an interesting generalization of entropic Bewley preferences. In fact, note that if* C *is not a singleton it means that the decision maker has a multiple set of* full *plausible priors and such decision maker reveals more ambiguity that any decision maker* a la *entropic Bewley preferences with same parameter* θ *and reference prior* q *belonging to* C*.*

Example 16 *Now we consider a example without any requirement of countable additivity. Consider the mapping* $\eta = \theta F o \xi_v(\cdot) : \Delta \to [0,\infty]$*, where* $\theta > 0$*,* $F : [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$ *is an increasing convex functions with* $F(0) = 0$*,* $v : \Sigma \to [0,1]$ *is a convex capacity*[13]*, and*

$$\xi_v(p) = \sup_{E \in \Sigma} \{v(E) - p(E)\}, \ \forall p \in \Delta$$

is the plausibility index w.r.t. the capacity v*. In this case, the set of full plausible priors is the core of the capacity* v*, in fact*

$$\xi_v(p) = 0 \Leftrightarrow p \in core(v).$$

8 Final Remark

We saw that variational Bewley preferences, in general, is not transitive. For a concrete example, consider the case of two states of nature and a decision rule given by:

$$a \succcurlyeq b \text{ iff } \alpha a_1 + (1-\alpha) a_2 + (-\ln \alpha) \geq \alpha a_1 + (1-\alpha) a_2, \forall \alpha \in [0,1]$$

i.e.,

$$a \succcurlyeq b \text{ iff } g_{a,b}(\alpha) \geq \ln \alpha, \ \forall \alpha \in [0,1],$$

where $g_{a,b}(\alpha) = \alpha(a_1 - a_2 + b_2 - b_1) + (a_2 - b_2)$. Note that $g_{a,b}(0) = a_2 - b_2$ and $g_{a,b}(1) = a_1 - b_1$, so $a \succcurlyeq b$ implies that $a_1 \geq b_1$. In this case, the decision maker views the state 2 as a miracle but he is prudent and does not accept large losses in the case of a future miracle. It is possible to find $a, b, c \in \mathbb{R}^2_+$ where $g_{a,b}(\cdot) \geq \ln(\cdot)$, $g_{b,c}(\cdot) \geq \ln(\cdot)$ but not $g_{a,c}(\cdot) \geq \ln(\cdot)$ (suitable parameters where $b_1 - c_1 < a_1 - b_1 < a_1 - c_1$ and $b_2 - c_2 > a_2 - b_2 > a_2 - c_2$).

[12] See Lemma 4 (page 41) of Strzalecki (2007).

[13] A capacity satisfies
a) $v(\emptyset) = 0, v(S) = 1$;
b) $A \subset B \Rightarrow v(A) \leq v(B)$;
Convexity means (note that c implies b)
c) For any $A, B \in \Sigma$,

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B).$$

Also, the core of v is given by

$$core(v) := \{p \in \Delta : p(E) \geq v(E), \forall E \in \Sigma\}.$$

9 Appendix

We recall that $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm. We denote by $ba(\Sigma)$ the Banach space of all finitely additive set functions on Σ endowed with the total variation norm, which is isometrically isomorphic to the norm dual of $B_0(\Sigma)$, so, in this case the weak* topology $\sigma(ba.B_0)$ of $ba(\Sigma)$ coincides with the event-wise convergence topology.

Given a binary relation $\trianglerighteq$ on $B_0(\Sigma)$, some properties follows as:

- $\trianglerighteq$ is *reflexive* if $a \trianglerighteq a$ for every $a \in B_0(\Sigma)$;
- $\trianglerighteq$ is *transitive* whenever $a, b, c \in B_0(\Sigma)$, if $a \trianglerighteq b$ and $b \trianglerighteq c$ then $a \trianglerighteq c$;
- $\trianglerighteq$ is *non-trivial* if the exists $a, b \in B_0(\Sigma)$ such that $a \trianglerighteq b$ but not $b \trianglerighteq a$ (in this case we wrote $a \triangleright b$);
- $\trianglerighteq$ is *continuous* if given $\{a_n\}_n, \{b_n\}_n$ sequences in $B_0(\Sigma)$ such that $a_n \trianglerighteq b_n$ for all $n \in N$, $a_n \to a$ and $b_n \to b$ then $a \trianglerighteq b$;
- $\trianglerighteq$ is *Archimedean* if for all $a, b, c \in B_0(\Sigma)$ the sets $\{\alpha \in [0,1] : \alpha a + (1-\alpha) b \trianglerighteq c\}$ and $\{\alpha \in [0,1] : c \trianglerighteq \alpha a + (1-\alpha) b\}$ are closed in $[0,1]$;
- $\trianglerighteq$ is *s-affine* if for all $a, b, c_1, c_2 \in B_0(\Sigma)$ and $\alpha \in (0,1)$ such that $c_1 \trianglerighteq c_2$,
$$a \trianglerighteq b \text{ iff } \alpha a + (1-\alpha) c_1 \trianglerighteq \alpha b + (1-\alpha) c_2;$$
- $\trianglerighteq$ is *affine* if for all $a, b, c \in B_0(\Sigma)$ and $\alpha \in (0,1)$,
$$a \trianglerighteq b \text{ iff } \alpha a + (1-\alpha) c \trianglerighteq \alpha b + (1-\alpha) c;$$
- $\trianglerighteq$ is *monotonic* if $a \geq b$ then $a \trianglerighteq b$;
- $\trianglerighteq$ is *monotonic continuous* if for any $r_1, r_2, t \in \mathbb{R}$ such that $r_1 1_S \triangleright r_2 1_S$ and for all sequences of events $\{A_n\}_{n\geq 1}$ with $A_n \downarrow \emptyset$, there exists $n_0 \geq 1$ such that $r_1 1_S \trianglerighteq x A_{n_0} z$.

Lemma 17 *A reflexive, affine and monotonic binary relation* $\succcurlyeq$ *on* $B_0(\Sigma)$ *is continuous if and only if it is Archimedian.*

Proof. The proof follows from Gilboa, Maccheroni, Marinacci and Schmeidler (2008): In fact, they showed that an affine and monotonic preorder is continuos if and only if it is Archimedean. First, we note that it is obvious that if $\trianglerighteq$ is convex then $\trianglerighteq$ affine because $\trianglerighteq$ is reflexive. So, we can mimic Lemma 1 (page 32), Lemma 2 (page 33)[14], and Lemma 3 (page 35) without transitivity. ■

Theorem 1:

Proof. (1) $\Rightarrow$ (2) :

[14] Concerning Lemma 2, note that in our case K is the whole set of real numbers, which is not an important assumption for this result.

By Axiom 2, Axiom 3 and Axiom 5 the restriction of $\succsim$ on $X \times X$ satisfies the set of the von Neumann-Morgenstern (1944)´s axioms and then there exist a non constant function $u : X \to \mathbb{R}$ such that $x \succsim y$ if and only if $u(x) \geq u(y)$ such that for any $x, y \in X$ and $\alpha \in (0,1)$,

$$u(\alpha x + (1-\alpha) y) = \alpha u(x) + (1-\alpha) u(y),$$

i.e., u is an affine function. Moreover, u is unique up to positive linear transformation[15]. Also, an important fact comes from the Axiom 6 about Unboundedness, in fact, we obtain that $u(X) = \mathbb{R}$ (see, for instance, MMR 2006).

Now we define the binary relation $\trianglerighteq$ over the set $B_0(\Sigma) = \{u(f) : f \in \mathcal{F}\}$ by:

$$a \trianglerighteq b \Leftrightarrow f \succsim g, \text{ for some } f, g \in \mathcal{F} \text{ such } a = u(f) \text{ and } b = u(g).$$

We note that $\trianglerighteq$ is well defined on $B_0(\Sigma)$ and

$$a \trianglerighteq b \Leftrightarrow f \succsim g, \text{ for any } f, g \in \mathcal{F} \text{ such } a = u(f) \text{ and } b = u(g).$$

We note that $\trianglerighteq$ is:

Reflexive: Given $a \in B_0(\Sigma)$ we have that $a = u(f)$ for some $f \in \mathcal{F}$, and since $\succsim$ is reflexive $f \succsim f$ which implies that $a \trianglerighteq a$;

Non-trivial: We know that $\succsim$ is non-trivial because there exists $x, y \in X$ such that $x \succsim y$ but not $y \succsim x$, so by considering the constant functions $a := u(x)1_S$ and $b := u(y)1_S$ on $B_0(\Sigma)$ we have that $a \trianglerighteq b$ but not $b \trianglerighteq a$, *i.e.*, $a \triangleright b$;

S-affine: Consider $a, b, c_1, c_2 \in B_0(\Sigma)$ and $\alpha \in (0,1)$ such that $c_1 \trianglerighteq c_2$. Hence there exist f, g, h_1, h_2 such that $a = u(f)$, $b = u(g)$, $c_1 = u(h_1)$, and $c_2 = u(h_2)$, in particular $h_1 \succsim h_2$. Since $\succsim$ satisfies the s-independence,

$$\begin{aligned} a &\trianglerighteq b \Leftrightarrow f \succsim g \Leftrightarrow \alpha f + (1-\alpha) h_1 \succsim \alpha g + (1-\alpha) h_2 \\ &\Leftrightarrow u(\alpha f + (1-\alpha) h_1) \trianglerighteq u(\alpha g + (1-\alpha) h_2) \\ &\Leftrightarrow \alpha a + (1-\alpha) c_1 \trianglerighteq \alpha b + (1-\alpha) c_2. \end{aligned}$$

Archimedean: Consider $a = u(f), b = u(g)$, and $c = u(h) \in B_0(\Sigma)$, then

$$\begin{aligned} \{\alpha \in [0,1] : \alpha a + (1-\alpha) b \trianglerighteq c\} &= \{\alpha \in [0,1] : \alpha u(f) + (1-\alpha) u(g) \trianglerighteq u(h)\} \\ &= \{\alpha \in [0,1] : u(\alpha f + (1-\alpha) g) \trianglerighteq u(h)\} \\ &= \{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}, \end{aligned}$$

is closed in $[0,1]$ because the Archimedean Continuity of $\succsim$, and a similar argument shows that $\{\alpha \in [0,1] : c \trianglerighteq \alpha a + (1-\alpha) b\}$ is closed too.

Monotonic: If $a = u(f) \geq b = u(g)$ then $f(s) \succsim g(s)$ for any $s \in S$ and the monotonicity of $\succsim$ implies that $f \succsim g$, hence $a \trianglerighteq b$.

Now we define a very useful mapping $\eta^* : \Delta \to \mathbb{R} \cup \{+\infty\}$ for our representation and it is given by the following rule: for any probability $p \in \Delta$,

$$\eta^*(p) = \underset{(f,g) \in \succsim}{sup} \left(\int (u(g) - u(f))\, dp \right) = \underset{(a,b) \in \trianglerighteq}{sup} \left(\int (b - a)\, dp \right).$$

[15] See, for instance, section 2.2 of Föllmer and Schied (2004).

Since $(a,a) \in \trianglerighteq$ for each $a \in B_0(\Sigma)$, it is true that $\eta^*(p) \geq \left(\int (a-a)\,dp\right) = 0$, i.e., η^* is a *non-negative* function.

Now, we define the mapping:

$$(\Delta \times \trianglerighteq) \ni (p,(a,b)) \mapsto \rho_{(a,b)}(p) = \int (b-a)\,dp.$$

Clearly, for each $(a,b) \in \trianglerighteq$ the function $\rho_{(a,b)}(\cdot) : \Delta \to \mathbb{R}$ is linear and weak* continuous. Also, since the supremum of continuous (lower semicontinuous function) is lower semicontinuous[16] we have that

$$\eta^*(\cdot) = \sup_{(a,b)\in\trianglerighteq} \rho_{(a,b)}(\cdot)$$

is weak* lower semicontinuous. Moreover, η^* is convex because the supremum of linear functions is a convex function[17].

Now we intent to show that $\{\eta = 0\} \neq \emptyset$. First we will show that $\inf_{p\in\Delta}\eta^*(p) = 0$ and for this part of the proof we need the following result:

von Neumann´s minimax theorem[18]*: Let M and N be convex subsets of vector spaces supplied with topologies If M is compact and $\phi : M \times N$ satisfy:*

i) for any $y \in N$, $M \ni x \mapsto \phi(x,y)$ is convex and lower semicontinuous;

ii) for any $x \in M$, $N \ni y \mapsto \phi(x,y)$ is concave.

Then

$$\inf_{x\in M}\sup_{y\in N}\phi(x,y) = \sup_{y\in N}\inf_{x\in M}\phi(x,y).$$

In our case $M = \Delta$ and $N = \trianglerighteq$. We note since $\trianglerighteq$ is s-affine, if $(a,b) \in \trianglerighteq$ and $(c_1,c_2) \in \trianglerighteq$ then for any $\beta \in [0,1]$, we have that

$$(\beta a + (1-\beta)c_1, \beta b + (1-\beta)c_2) \in \succcurlyeq,$$

i.e., $\trianglerighteq$ is a convex subset of $B_0(\Sigma)^2$ and, clearly, Δ is a convex subset of $ba(\Sigma)$. Also, by the Banach-Alaoglu-Bourbaki theorem[19], Δ is a weak* compact subset of $ba(\Sigma)$. By what we have observed η^* is convex and weak* lower semicontinuous. Moreover, it is easy to see that the function

$$(a,b) \mapsto \rho_{(a,b)}(p) = \int (b-a)\,dp$$

is affine (hence concave) for each $p \in \Delta$. Hence, by the minimax theorem and the fact that (by monotonicity) $(a,b) \in \trianglerighteq$ implies that $a(s_0) \geq b(s_0)$ for some

[16] See, for instance, Brézis (1984), page 8.

[17] See, for instance, Brézis (1984), page 9.

[18] The proof of this classical result can be found in Aubin and Ekeland (1984), chapter 6.

[19] See, for instance, Brézis (1984) page 42.

s_0 :

$$\begin{aligned}
\inf_{p\in\Delta} \underset{(a,b)\in\trianglerighteq}{sup} \left(\int (b-a)\,dp\right) &= \\
\underset{(a,b)\in\trianglerighteq}{sup} \inf_{p\in\Delta} \left(\int (b-a)\,dp\right) &= \\
& \underset{(a,b)\in\trianglerighteq}{sup} \underbrace{\inf_{s\in S} (b(s)-a(s))}_{\leq 0} \overset{(\succcurlyeq \text{ is reflexive})}{=} 0.
\end{aligned}$$

Now we will show that there exists some $q \in \Delta$ such that $\eta^*(q) = 0$. Since $\eta^*(q) \geq 0$, it is enough to show that there exists $q \in \Delta$ such that $\eta^*(q) \leq 0$, *i.e.*, it is possible to find $q \in \Delta$ such that

$$\int (a-b)\,dp \geq 0 \text{ for any } (a,b) \in\trianglerighteq .$$

Denoting $E = B_0(\Sigma)$ and E^* its dual, then our problem is to find some $x^* \in E^*$ such that

$$\begin{aligned}
\langle x^*, 1_S \rangle &\geq 1 \\
\langle x^*, -1_S \rangle &\geq -1 \\
\langle x^*, a-b \rangle &\geq 0, \text{ for any } (a,b) \in\trianglerighteq .
\end{aligned}$$

The mathematical tool for this kind of problem was given by Fan (1956), page 126:

Ky Fan´s theorem: Given an arbitrary set Λ, let the system

$$\langle x^*, x_i \rangle \geq \alpha_i, i \in \Lambda \qquad (\Xi)$$

of linear inequalities; where $\{x_i\}_{i\in\Lambda}$ be a family of elements, not all 0, in real normed linear space E, and $\{\alpha_i\}_{i\in\Lambda}$ be a corresponding family of real numbers. Let $\sigma := sup \sum_{j=1}^{n} \lambda_j \alpha_{i_j}$ when $n \in N$, and λ_j vary under conditions: $\lambda_j \geq 0$, $\forall j \in \{1, ..., n\}$ and $\left\| \sum_{j=1}^{n} \lambda_j x_{i_j} \right\|_E = 1$. Then the system (Ξ) has a solution $x^* \in E^*$ if and only if σ is finite.

Let us consider $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$ and $\mathbf{1}_S$, $-\mathbf{1}_S$, $(a_j, b_j) \in\trianglerighteq$, $3 \leq j \leq n$ such that:

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \sum_{j=3}^{n} \lambda_j \ (a_j - b_j) \right\|_\infty = 1,$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \ (a_j - b_j) \leq \mathbf{1}_S,$$

hence,

$$\lambda_1 - \lambda_2 + \sum_{j=3}^{n} \lambda_j \int (a_j - b_j)\, dp \leq 1 \text{ for any } p \in \Delta,$$

since

$$-\eta^*(p) = \inf_{(a,b)\in \trianglerighteq} \left(\int (a-b)\, dp \right),$$

we obtain that

$$\lambda_1 - \lambda_2 - \eta^*(p) \sum_{j=3}^{n} \lambda_j \leq 1 \text{ for any } p \in \Delta,$$

we saw that $\inf_{p\in\Delta} \eta^*(p) = 0$, hence $\sup_{p\in\Delta} \{-\eta^*(p)\} = 0$ and,

$$\lambda_1 - \lambda_2 = \lambda_1 - \lambda_2 + \sup_{p\in\Delta} (-\eta^*(p)) \sum_{j=3}^{n} \lambda_j \leq 1,$$

i.e., $\sum_{j=1}^{n} \lambda_j \alpha_j \leq 1$; where $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_j = 0$, $3 \leq j \leq n$. Hence σ is finite and by Ky Fan's theorem there exists $q \in \Delta$ such that $\eta^*(q) = 0$.

For the last statement in the theorem note that if $f_0 \succsim g_0$ then $\eta^*(p) \geq \int (u(g_0) - u(f_0))\, dp$ for any $p \in \Delta$, hence

$$\int u(f_0) dp + \eta^*(p) \geq \int u(g_0) dp \text{ for any } p \in \Delta.$$

Conversely, if $(f_0, g_0) \notin \succsim$ then $(a_0, b_0) \notin \trianglerighteq$, where $a_0 = u(f_0)$ and $b_0 = u(g_0)$. Since $\trianglerighteq$ is a nonempty, convex (by s-independence) and closed (by Lemma 17) subset of $B_0(\Sigma) \times B_0(\Sigma)$. Using the separation theorem[20] there exists $q \in \Delta$ that defines the linear functional $\Psi((a,b)) = \int (b-a)\, dq$ over $B_0(\Sigma) \times B_0(\Sigma)$, and[21]

$$\int (b_0 - a_0)\, dq > \sup_{(a,b)\in \trianglerighteq} \int (b-a)\, dq = \eta^*(q),$$

[20] See, for instance, the theorem I.7 at page 7 in Brézis (1984).

[21] In fact, since by Schatten (1950), $(B_0(\Sigma) \times B_0(\Sigma))^* = B_0(\Sigma)^* \times B_0(\Sigma)^*$ we obtain that $\Psi((a,b)) = \int b dq_1 - \int a dq_2$ with $q_1, q_2 \in ba(\Sigma)$. Since $(a,a) \in \trianglerighteq$ for all $a \in B_0(\Sigma)$ we obtain that $\Psi((a,a)) = 0$; in fact, if $\Psi((a,a)) \neq 0$ we obtain that

$$\sup_{k\in\mathbb{Z}} \Psi((ka,ka)) = \infty,$$

but

$$\Psi((a_0,b_0)) > \sup_{k\in\mathbb{Z}} \Psi((ka,ka)),$$

a contradiction. In particular, $q_1 = -q_2$. Also, we note that $q_1 \geq 0$; if $q_1(E) < 0$ for some $E \in \Sigma$ by monotonicity we obtain that $(nq_1(E) 1_S, 0) \in \trianglerighteq$ for any $n \geq 1$ and

$$\Psi((a_0,b_0)) > \sup_{n} \{-nq_1(E)\} = \infty,$$

a contradiction. Finally, w.l.g. we may suppose that $q_1(S) = 1$.

i.e., there exists $q \in \Delta$ such that,

$$\int u(f_0)dq + \eta^*(q) < \int u(g_0)dq.$$

$(2) \Rightarrow (1)$:

It is straightforward.

That η^* is minimal is easy: in fact, if there exists a grounded, convex and lower semicontinuous function $\gamma : \Delta \to [0, \infty]$ such that for any $f, g \in \mathcal{F}$,

$$f \succsim g \Leftrightarrow \int u(f)dp + \gamma(p) \geq \int u(g)dp, \, \forall p \in \Delta,$$

then

$$\gamma(p) \geq \int u(g)dp - \int u(f)dp, \, \forall p \in \Delta \text{ and } \forall (f,g) \in \succsim$$

so, for any $p \in \Delta$

$$\gamma(p) \geq \sup_{(f,g)\in\succsim} \left(\int u(g)dp - \int u(f)dp \right) = \eta^*(p).$$

■

Proof of Corollary 3:

Proof. Let (u, η^*) represent $\succsim$ as in Theorem 1. Taking another representation (u_1, η_1^*) of $\succsim$ as in Theorem 1, by its key equivalence u and u_1 are affine representations of the restriction of $\succsim$ to the set of consequence X. Hence, by a well known uniqueness results, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u + \beta$. By the characterization of η^* obtained in Theorem 1 for any probability p,

$$\begin{aligned}
\eta_1^*(p) &= \sup_{(f,g)\in\succsim} \left(\int (u_1(g) - u_1(f))\, dp \right) \\
&= \sup_{(f,g)\in\succsim} \left(\int \alpha u(g) + \beta - (\alpha u(g) + \beta)\, dp \right) \\
&= \sup_{(f,g)\in\succsim} \left(\int \alpha u(g) - \alpha u(g)\, dp \right) = \alpha \eta^*(p).
\end{aligned}$$

■

Proof of Proposition4:

Proof. Consider the asymmetric component $\succ \subset \mathcal{F} \times \mathcal{F}$ induced from a variational Bewley preference $\succsim$. Since it is well know a sufficient condition for $\succ$ to be acyclic is the existence of a real-valued function V on $\mathcal{F}$ such that

$$f \succ g \Rightarrow V(f) > V(g).$$

Now, consider

$$V(f) = \min_{p\in\Delta} \left(\int u(f)\, dp + \eta^*(p) \right),$$

if there exist acts f, g such that $f \succ g$ and $V(f) \leq V(g)$ then

$$\begin{aligned}
\int u(f)\,dp + \eta^*(p) &\geq \int u(g)\,dp,\ \forall p \in \Delta, \\
\exists p_0 &\in \Delta \text{ s.t. } \int u(g)\,dp_0 + \eta^*(p_0) < \int u(f)\,dp_0, \\
\text{and } \exists q_1, q_2 &\in \Delta \text{ s.t. } \int u(f)\,dq_1 + \eta^*(q_1) \leq \int u(g)\,dq_2 + \eta^*(q_2).
\end{aligned}$$

Hence, since η is convex, for any $n \in \mathbb{N}$,

$$\begin{aligned}
& \int u(f)\,d\left(\overbrace{n^{-1}q_1 + \left(1-n^{-1}\right)p_0}^{:=q_1np_0}\right) + \eta^*(q_1np_0) \\
\leq & \int u(f)\,d(q_1np_0) + n^{-1}\eta^*(q_1) + \left(1-n^{-1}\right)\eta^*(p_0) \\
\leq & \int u(g)\,d(q_2np_0) + n^{-1}\eta^*(q_2) + \left(1-n^{-1}\right)\eta^*(p_0)
\end{aligned}$$

which entails,

$$\begin{aligned}
& \int u(g)\,dp_0 + \eta^*(p_0) \\
= & \liminf_{n\to\infty}\left\{\int u(g)\,d(q_2np_0) + n^{-1}\eta^*(q_2) + \left(1-n^{-1}\right)\eta^*(p_0)\right\} \\
\geq & \liminf_{n\to\infty}\left\{\int u(f)\,d(q_1np_0) + \eta^*(q_1np_0)\right\} \\
\geq & \liminf_{n\to\infty}\int u(f)\,d(q_1np_0) + \liminf_{n\to\infty}\eta^*(q_1np_0)],
\end{aligned}$$

Since η^* is weak* lower semi-continuous and $(q_1np_0)(E) \to p_0(E)$ for any $E \in \Sigma$, we obtain

$$\int u(g)\,dp_0 + \eta^*(p_0) \geq \int u(f)\,dp_0 + \eta^*(p_0) > \int u(f)\,dp_0,$$

a contradiction. Hence, $f \succ g \Rightarrow V(f) > V(g)$ and $\succ$ is acyclic. ■

Lemma 18 *Consider a preference relation $\succsim$ as in Theorem 1 and some particular utility index $u : X \to \mathbb{R}$ consistent with $\succsim|_{X\times X}$. For any $f, g \in \mathcal{F}$ there exists a minimal $c_{(f,g)} \geq 0$ such that for any $c \geq c_{(f,g)}$*

$$f \succsim g \text{ iff } \int u(f)dp + \eta^*(p) \geq \int u(g)dp, \text{ for any } p \in \{\eta^* \leq c\}.$$

In fact, $c_{(f,g)} = \sup uog - \inf uof$.

Proof. The implication ($\Rightarrow$) is obvious. Now, suppose that

$$c \geq c_{(f,g)} = \sup uog - \inf uof.$$

Now consider $p \in \Delta$ such that $\eta^*(p) \geq c_{(f,g)}$. Since $uoh \in [\inf uoh, \sup uoh]$ for $h \in \{f, g\}$ we have that

$$uog - uof \in [\inf uog, \sup uog],$$

also,

$$\int (u(g) - u(f))\, dp \leq \|u(g) - u(f)\|_\infty \leq \sup uog - \inf uof \leq \eta^*(p).$$

Hence, if for some $c \geq c_{(f,g)}$

$$\int u(f)dp + \eta^*(p) \geq \int u(g)dp, \text{ for any } p \in \{\eta^* \leq c\}$$

then $f \succsim g$. ■

Proof of Proposition 5:

Proof. (i) implies (ii):

Let $p \in ba(\Sigma) \setminus ca(\Sigma)$ be a non-countably additive probability. Hence there exists a sequence of events $\{A_n\}_{n\geq 1}$ such that $A_n \downarrow \emptyset$ and $p(A_n) \downarrow \alpha > 0$. So, since $u(X) = \mathbb{R}$ for each $n \geq 1$ there exists some x_n such that $u(x_n) = n^{-1}$. Consider $z \in X$ such that $u(z) = 0$. Hence, monotonicity implies that $x_n \succ z$.

Now, by considering $x_m \in \left\{u^{-1}\left((\alpha n)^{-1} + m\right)\right\}$, $m \geq 1$, we obtain by the monotonic continuity axiom that there exist $k = k(n)$ such that

$$x_n \succsim x_m A_k z.$$

Hence,

$$\begin{aligned}
\eta^*(p) &\geq \int (u(x_m A_k z) - u(x_n))\, dp \\
&= \left((\alpha n)^{-1} + m\right) p(A_k) - n^{-1} \\
&= mp(A_k) + \frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right),
\end{aligned}$$

so, for any $m \geq 1$

$$\begin{aligned}
\eta^*(p) &\geq \lim_{n\to\infty}\left(mp(A_k) + \frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right)\right) \\
&= \lim_{n\to\infty} mp\left(A_{k(n)}\right) + \lim_{n\to\infty}\frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right) \\
&\geq m\alpha,
\end{aligned}$$

which implies that $\eta^*(p) = \infty$. Hence, if $\eta^*(p) < \infty$ then $p \in ca(\Sigma)$.

(ii) implies (i):

Let $x, y, z \in X$ such that $y \succ z$ and a sequences of events $\{A_n\}_{n\geq 1}$ with $A_n \downarrow \emptyset$. If $y \succsim x$ we have by monotonicity (y statewise dominates $xA_n z$) that $y \succsim xA_n z$ $\forall n \geq 1$. On the other hand, consider the case where $x \succ y$. We need to show that there exists some $n_0 \geq 1$ such that

$$y \succsim xA_{n_0} z.$$

By the previous Lemma, choosing $c = u(x) - u(y) + 1$ it is enough to show that for any $p \in \{\eta^* \leq c\}$,

$$u(y) + \eta^*(p) \geq \int u(xA_n z)\, dp.$$

Recalling that η^* is weak* lower semicontinous we have that $\{\eta^* \leq c\}$ is a weak* compact set of countably additive probabilities, so it is a weak compact subset of countably additive probabilities. By Theorem IV.9.1 of Dunford and Schwartz (1958) it follows that if $\varepsilon > 0$ and $A_n \downarrow \emptyset$ there exists n_o such that $p(A_n) < \varepsilon$ for any $n \geq n_0$ and all $p \in \{\eta^* \leq c\}$. Hence, putting $\varepsilon = [u(y) - u(z) + \eta^*(p)] / [u(x) - u(z)]$ we know that there exists n_0 such that

$$p(A_n) < [u(y) - u(z) + \eta^*(p)] / [u(x) - u(z)],$$

for any $n \geq n_0$ and for any $p \in \{\eta^* \leq u(x) - u(y)\}$. Hence, for any p such that $\eta^*(p) \leq c$

$$u(y) + \eta^*(p) > p(A_n)\, u(x) + u(z)\,(1 - p(A_n)) = \int u(xA_n z)\, dp,$$

and we conclude that $y \succsim xA_n z$ for any $n \geq n_0$. ■

Proposition 7

Proof. $a) \Rightarrow b)$ Concerning the same risk attitudes, it follows from Ghirardato et. al. (2004), Corollary B.3, i.e., we can take $u_1 = u_2 = u$.

By assumption $f \succsim_1 g \Rightarrow f \succsim_2 g$, i.e., $\succsim_1 \subset \succsim_2$. So, for any $p \in \Delta$:

$$\begin{aligned}
\eta_1^*(p) &= \underset{(f,g)\in\succsim_1}{sup} \left(\int (u(g) - u(f))\, dp \right) \\
&\leq \underset{(f,g)\in\succsim_2}{sup} \left(\int (u(g) - u(f))\, dp \right) = \eta_2^*(p).
\end{aligned}$$

$b) \Rightarrow a)$ Consider $(f, g) \in \succsim_1$, i.e., $\int u(f)dp + \eta_1^*(p) \geq \int u(g)dp$, $\forall p \in \Delta$. Since $\eta_2^* \geq \eta_1^*$, we obtain that for any $p \in \Delta$,

$$\int u(f)dp + \eta_2^*(p) \geq \int u(f)dp + \eta_1^*(p) \geq \int u(g)dp,$$

i.e., $f \succsim_2 g$. ■

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