Interim Outcomes and Bargaining Solutions

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Abstract

In this paper, we employ a unified approach to Nash's two-person bargaining problem by using a class of axioms, which we term Common Disagreement Point (CDP) axioms. These axioms describe under what circumstances parties that expect to face sometimes uncertain nested or non-nested bargaining sets can reach interim outcomes. By doing so, these axioms portray a bargaining process, and thereby bridge the gap between cooperative and non-cooperative bargaining; some of these axioms are also conducive to defining the relative bargaining power of parties via relative gains and concessions. We show that the bargaining process could lead to the Discrete Raiffa, Nash or Kalai/Smorodinsky solutions depending on *when* parties, who face uncertain bargaining compromises, are willing to reach interim outcomes.

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1 Introduction

Binmore (1994, p. 21) summarized the fundamental role that bargaining plays in our lives as follows: "much negotiation [and exchange] in real life" entail relationships which "create a surplus that would otherwise be unavailable" to the parties (e.g., the potential buyer and the seller of a house, employer employee, landowner tenant): "if you have a fancy house to sell that is worth \$2m to you and \$3m to me, then ... a surplus of \$1m is available for us to split." The significance of this very simple yet fruitful bargaining problem was recognized as early as 1881 by Edgeworth and for a very long period of time it was notoriously deemed to lack a clear solution; the only thing researchers had

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concluded during that period was that a specific bargaining outcome would depend on parties' bargaining power. Later, von Neumann and Morgenstern (1944) tried to provide a formal solution to this bargaining problem; it nevertheless coincided with Edgeworth's "contract curve" yielding the entire set of individually rational and Pareto optimal outcomes.

By 1950, Nash proposed a framework which allowed a unique feasible outcome to be selected as the solution of a given bargaining problem. He formalized the bargaining problem as a pair (S, d) where $S \subset \mathbb{R}^2$ is a convex and compact utility possibility set and d is the disagreement point; the latter is the utility allocation that results if no agreement is reached by both parties. The first solution to the problem was provided by Nash (1950). It was axiomatically characterized by four axioms, namely by Symmetry (SYM), Weak Pareto Optimality (WPO), Scale Invariance (SI), and Independence of Irrelevant Alternatives (IIA). By 1953, Raiffa criticized the Nash solution (and especially the IIA axiom) and proposed another solution which essentially described a discrete bargaining process but has never been characterized axiomatically. Kalai and Smorodinsky (1975), raising similar criticisms, were able to characterize a new solution concept which, as the Discrete Raiffa solution did, placed significant emphasis on the parties' ideal payoffs (i.e., parties' highest possible individually rational payoffs).[1]

So far, all known bargaining solutions were initially characterized axiomatically with the help of a crucial independence or monotonicity axiom (pioneered by Nash (1950) and Kalai and Smorodinsky (1975) respectively): When the solution outcome is irresponsive to the changes in the bargaining set, that axiom is coined as the independence axiom; when at least one of solution payoffs may be altered following a change in the bargaining set, it is dubbed as the monotonicity axiom.[2] The second generation characterizations of these solution concepts shifted the focus to changes in the disagreement payoffs as well as to the consideration of uncertain disagreement points (pioneered by Thomson (1987) and Chun and Thomson (1990), respectively). Both generations of characterizations were essential since a bargaining problem consists of a bargaining set and a disagreement point. Many valuable lessons were learned from both strands of these

[1] There have been several other solution concepts that have been characterized axiomatically since then (namely, the Egalitarian solution (Kalai, 1977; Roth, 1979), the Equal Sacrifice solution (Aumann and Maschler, 1985; Chun, 1988), the Perles/Maschler solution (Perles and Maschler, 1985), the Equal Area solution (Anbarci, 1993; Anbarci and Bigelow, 1994), the Average Payoff solution (Anbarci, 1995)), and the Dictatorial solutions (Bigelow and Anbarci, 1993).

[2] As a matter of fact, the Nash, Kalai/Smorodinsky, the Perles/Maschler solution, the Equal Area solution, the Average Payoff solution have been all initially characterized by SYM, WPO, SI and an independence or monotonicity axiom.

characterizations; they identified a wide variety of situations under which these solution concepts are appropriate.

The axiomatic bargaining theory typically lacks a description of the bargaining process. To rectify that, Nash (1953)'s Demand Game established a new research agenda, which has been commonly referred to as the Nash program (see Binmore (1998)). It utilizes the strategic (non-cooperative) approach to provide non-cooperative foundations for cooperative bargaining solution concepts by describing an explicit bargaining process. Another attempt was to try to insert the bargaining process into axiomatic bargaining. That way, for the solution concept using such an axiom, there would be less need for identifying non-cooperative foundations. Two notable examples are the Midpoint Domination (MD) axiom (Sobel, 1981) and the Step-by-Step Negotiation (SSN) axiom (Kalai, 1977); the former was used by Moulin (1983) in the characterization of the Nash solution, and the latter by Kalai (1977) in the characterization of the class of Proportional solutions, with the Egalitarian solution as the special case.

Both of these axioms have recognized the significance of interim outcomes that parties could reach between the initial disagreement point and potential solution outcomes at the Pareto frontier; such interim outcomes help eliminate the most lop-sided portions of a bargaining set as well as the most inefficient portions, which either one party or both parties would strongly dislike (in effect, the meta-bargaining attempts by van Damme (1986) and Anbarci and Yi (1992) too pertain to eliminations of such parts of the bargaining set deemed undesirable individually or jointly).

MD and SSN fulfilled an important role in pointing to the need of axioms entailing a bargaining process via reaching interim outcomes. However, they have not been generalized subsequently to give rise to a class of axioms which would be instrumental in characterizing some other existing solution concepts. This prevented the identification of other potentially important situations under which other solution concepts too would be appropriate. This paper aims to highlight the role of interim outcomes in a unified way by proposing a class of axioms. These axioms describe under what circumstances parties that expect to face sometimes uncertain nested or non-nested bargaining sets can reach interim outcomes. By describing circumstances of obtaining interim outcomes, these axioms portray a bargaining process. This attempt in a sense aims to achieve the Nash program within the confines of axiomatic bargaining, bridging the axiomatic (cooperative) and strategic (non-cooperative) approaches.

A major accomplishment of our framework is axiomatic characterization of the Discrete Raiffa solution, which has eluded researchers so far (the desirable feature of Raiffa's attempt, namely its description of a bargaining process, nevertheless led many researchers to seek and find characterizations of a "Continuous" version of the Raiffa solution; see Livne (1989), and Peters and van Damme (1991)). We also provide a variation of that axiomatic characterization. We then provide two main axiomatic characterizations of the Nash solution (as well as some variations of these characterizations) and an axiomatic characterization of the Kalai/Smorodinsky solution (as well as a couple of variations of it). The latter characterization of the Kalai/Smorodinsky solution and the second main characterization of the Nash solution entail certain properties - such as (1) parties' relative gains over d, and (2) parties' relative concessions in a particular bargaining problem, T, with respect to another bargaining problem S's solution outcome - that pertain to perceived relative bargaining powers of parties. Roughly speaking, we show that the bargaining process could lead to the Discrete Raiffa, Nash or Kalai/Smorodinsky solutions depending on *when* parties, who face uncertain bargaining compromises, are willing to reach interim outcomes.

The plan of the paper is as follows: In the next section, we motivate our Common Disagreement Point axioms. We then define some basic solutions and axioms. Following that we provide characterizations of the Discrete Raiffa, Nash and Kalai/Smorodinsky solutions. The final section concludes. All proofs are relegated to the Appendix.

2 Motivation of the Common Disagreement Point Axioms

In many situations, disagreement, or failure to reach at least an interim consensus, is costly for parties. Modifying an example from Eliaz, Ray and Razin (2007), consider two coalition member parties who may need to formulate a long-run response to, say, terrorism but may disagree very profoundly over the nature of an appropriate response. Both parties, however, might agree that complete inaction or disagreement is the worst of the options. Consider another example. If parties have to continuously encounter each other (such as two neighboring countries), then complete disagreement at any point may be very detrimental for their other endeavors in the future. Thus, they may feel strongly obliged to try improving upon their complete disagreement by jointly seeking some interim outcomes in the process. As mentioned above, the significant role of parties' attempts to reach interim outcomes has been recognized by the MD axiom and by the SSN axiom.

Here we develop a unified approach by using a class of axioms, which we term Common Disagreement Point (CDP) axioms. A crucial concept stringing the whole story together is the generic property, which we call the *common disagreement point*. But before we proceed further it would be useful to elaborate on the links between the domains of SSN (or MD) and CDP axioms.

In SSN, parties exactly know what time has in store for them. They know that they will face two nested bargaining sets in sequence, first the smaller one and then the augmented one. SSN requires that given two such bargaining sets, the solution outcome of the second augmented set should not be different regardless of whether the initial smaller bargaining set's solution outcome replaces its original disagreement point or its original disagreement point is used. Thus, the solution outcome of the initial bargaining set can perfectly function as an interim outcome. Kalai (1977) emphasized the advantage of this interim outcome in terms reducing the magnitude of the conflict at hand as follows:

> This principle is observed in actual negotiations (e.g., Kissinger's step-by-step) and it is attractive since it makes the implementation of a solution easier. It is also attractive because we can view every bargaining situation that we encounter in life as a first step in a sequence of predictable or unpredictable bargaining situations that may still arise. Thus the outcome of the current bargaining situation will be the threat point for the future ones.

Our CDP setup entails Kalai's setup as well as other cases, such as what time has in store for parties may not necessarily be two nested bargaining sets in sequence. It may also be the case that parties know certain salient features - such as the disagreement point, d, and the ideal point, b (i.e., the combination of parties' highest individually rational payoffs) - of their bargaining that lies ahead and they may learn about the rest of the bargaining set and the Pareto frontier in the future.[3] Suppose they will potentially

[3] The marketing literature (and recently the economics literature) provides well-established analysis and evidence that consumers do not consider all brands in a given market at once before making a purchase decision and that the set of brands changes in time as they learn more about that product; consequently, they start including some of the brands that they were initially unaware of (Chiang, Chib and Narasimhan, 1999; Goeree, 2008). A shopper in a typical supermarket faces 285 varieties of cookies, 230 different soups and 275 varieties of cereal (Schwartz, 2004).

Many financial decisions entail investing 401(k) plans among hundreds of available funds (Huberman and Regev, 2001). A similar example can also be found in university choice (see Dawes and Brown, 2004).

The setups in which individuals encounter alternatives sequentially have found themselves a place not only in the applied strands of marketing, finance and economics literatures but also in the theoretical economics literature - see, for instance, Rubinstein and Salant (2006).

Similarly, one can conceive that two bargaining partners may initially start considering a bargaining set where each party may initially only know, say, their maximum payoffs and their fallback positions; they may later learn more about other alternatives and start considering the alternatives they were initially unaware of as well.

face two bargaining sets with the same d and b. In that case, they can still agree on an interim outcome that can improve upon their current initial disagreement point without knowing which of the two bargaining sets they will face in the (d, b)-box.

As mentioned above, one other such attempt is by MD, which requires that the solution outcome should not yield to any party a payoff less than the midpoint of the (d, b)-box. Hence, if that midpoint is not Pareto optimal in a given (d, b)-box, MD implies that the midpoint should serve as an interim outcome for parties who adhere to a solution concept which satisfies it (both Nash and Discrete Raiffa solutions satisfy MD, for instance). Our common disagreement point axioms will identify such reasonable circumstances under which parties may expect to reach an interim outcome (or outcomes), and may provide hints as to what those interims outcomes may be.

The role of such interim outcomes (or improved disagreement points) is further magnified in environments where the negotiations may break down with some probability (such as the mechanism considered by Rubinstein, Safra and Thomson, 1991) and consequently the parties may receive their disagreement payoffs with that probability (i.e., the situation would then boil down to receiving a particular improved disagreement point rather than another one that is dominated by it with that termination probability).

3 The Model, Some Basic Axioms and Solution Concepts

A two-person bargaining problem is a couple (S, d) where $S \subset \mathbb{R}^2$ is the set of utility vectors that the players can achieve through cooperation and $d \in S$ is the utility that prevails in case of disagreement. We restrict S to be compact, convex and comprehensive.[4] Let Σ be the class of all two-person bargaining problems, $\Sigma^0 \subset \Sigma$ be the class of all bargaining problems (S, d) with the property that $x > d$ for some $x \in S$.[5] A bargaining problem (S, d) is *smooth* if S admits a unique supporting hyperplane at each utility vector on its boundary. Let Σ^s denote the class of all smooth problems. Unless stated otherwise, our results will consider bargaining problems in Σ. A bargaining problem (S, d) is *symmetric* if $d_1 = d_2$ and $(x_1, x_2) \in S$ implies $(x_2, x_1) \in S$. Define $IR(S, d) \equiv \{x \in S | x \geq d\}$ and $PO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2 \text{ and } x' \neq x, x' \geq x \Rightarrow x' \notin S\}$. Denote the ideal point of (S, d) as $b(S, d) = (b_1(S, d), b_2(S, d)),\ \text{ where } b_i(S, d) = \sup\{x_i | x \in IR(S, d)\}$. The midpoint of (S, d) is denoted by $m(S, d) = \frac{1}{2}(b(S, d) + d)$. A solution is a function $f : \Sigma \to \mathbb{R}^2$ such that for all $(S, d) \in \Sigma$, $f \in S$.

The disagreement point set of (S, d) with respect to f, $D(S, d, f) = \{x \in IR(S, d) | f(S, x) =$

[4] A set $S \subset \mathbb{R}^2$ is said to be comprehensive if $x, z \in S$ implies that $y \in S$ for all $x \leq y \leq z$.

[5] Given $x, y \in \mathbb{R}^2$, $x > y$ if $x_i > y_i$ for each i, and $x \geq y$ if $x_i \geq y_i$ for each i.

$f(S,d)\}$, is the set of all points x in S dominating d such that if we replace the initial disagreement point d with x and keep the utility feasibility set S unchanged, we can still reach the same bargaining solution outcome. $D(S,d,f)$ will be a key element of our analysis in this paper.

Next, we list some basic axioms that have been commonly used in the literature.

Pareto Optimality (PO) $f(S,d) \in PO(S)$.

Symmetry (SYM) If (S,d) is symmetric, then $f_1(S,d) = f_2(S,d)$.

Scale Invariance (SI) $T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a positive affine transformation if $T(x_1, x_2) = (a_1x_1 + b_1, a_2x_2 + b_2)$ for some positive constant a_i and constant b_i. We require that for such a transformation T, $f(T(S), T(d)) = T(f(S,d))$.

Individual Rationality (IR) $f(S,d) \geq d$.

Strong Individual Rationality (SIR) $f_i(S,d) \geq d_i$, with strict inequality whenever $x_i > d_i$ for some $x \in S$.

Independence of Non-individually Rational Alternatives (INIR) $f(S,d) = f(IR(S,d),d)$.

Disagreement Point Monotonicity (DM) If d and e are in S with $e_i = d_i$ and $e_j > d_j$, then $f_j(S,e) \geq f_j(S,d)$, for $i,j = 1,2$ with $i \neq j$.

Strong Disagreement Point Monotonicity (SDM) As DM with "$>$" instead of "$\geq$", but only if such a point $f(S,e)$ exists.

Disagreement Point Continuity (DCONT) For every bargaining set S and every sequence $d^1, d^2, ...$ in S, if $\lim_{n\to\infty} d^n = d \in S$ (in the Hausdorff topology), then $\lim_{n\to\infty} f(S,d^n) = f(S,d)$.

Pareto Continuity (PCONT) For all sequences $\{(S^n,d)\}$ in Σ, if $PO(S^n)$ converges to $PO(S)$ in the Hausdorff topology and $(S,d) \in \Sigma$, then $\lim_{n\to\infty} f(S^n,d) = f(S,d)$.

Midpoint Domination (MD) $f(S,d) \geq m(S,d)$.

MD requires that any reasonable agreement Pareto dominates the outcome of the random dictatorship. It is most notably satisfied by the Nash solution.

We introduce three solution concepts.

Definition 1 *The Nash solution N : For each $(S,d) \in \Sigma$, $N(S,d) = \arg\max\{\Pi_{j\in M}(x_j - d_j) | x \in IR(S,d)\}$, where $M = \{j |\ x_j > d_j$ for some $x \in S\}$*

Definition 2 *The Kalai/Smorodinsky solution KS : For each $(S,d) \in \Sigma$, $KS(S,d) = \max\{u \in S |$there exists $\alpha \in [0,1]$ such that $u = \alpha b(S,d) + (1-\alpha)d\}$.*

Definition 3 *The Discrete Raiffa solution DR : For each $(S,d) \in \Sigma$, consider a non-decreasing sequence $\{m_i\} \in S$ with $m_0 = m(S,d)$ and $m_i = m(S, m_{i-1})$, then $DR(S,d) =$*

$\lim_{i\to\infty} m_i$.

4 Characterization of the Discrete Raiffa Solution

To motivate our CDP properties, first we formally introduce the axiom of *Step-by-step Negotiations* (SSN) suggested by Kalai (1977). Let $\widetilde{\Sigma} = \{(S,d) \in \Sigma | d \notin PO(S)\}$ be the collection of all bargaining problems (S,d) in Σ in which the disagreement point d is not on the Pareto optimal boundary of S. A solution f satisfies (SSN) if whenever (S,d), $(T,d) \in \widetilde{\Sigma}$, $T \subset S$, and $S - f(T,d) \in \widetilde{\Sigma}$, then $f(S,d) = f(T,d) + f(S - f(T,d), 0)$. Think of bargaining as a dynamic negotiation process that involves multiple stages (or issues); this axiom then requires that the bargaining outcome is invariant under decomposition of the negotiation stages.

SSN has a strong requirement, and Kalai (1977) demonstrated that, combined with other mild conditions, it is able to uniquely characterize the *proportional* solutions.[6] Next, we will propose a novel axiom, which is a considerably weaker version of SSN. Recall that $D(S,d,f)$ represents the set of all points x in S dominating d such that, if we replace the initial disagreement point d with x, we can still reach the same bargaining outcome. Therefore, it can be treated as a collection of all points that are acceptable to *both* parties as *interim outcomes* during the negotiation process when the bargaining parties agree that the bargaining outcome will obey the allocation rule f. Before we introduce our first CDP axiom, we would like to restate SSN as follows:

Step-by-step Negotiations (SSN) Given two bargaining problems $(S,d), (T,d) \in \widetilde{\Sigma}$, $D(S,d,f) \cap D(T,d,f) \backslash \{d\} \supset \{f(T,d)\}$ whenever $T \subset S$ and $S - f(T,d) \in \widetilde{\Sigma}$.

For a given bargaining problem (S,d), this axiom requires that $D(S,d,f) \cap D(T,d,f) \backslash \{d\}$ is not only non-empty, but also contains $\{f(T,d)\}$ for *ALL* bargaining problems $(T,d) \in \widetilde{\Sigma}$ with $T \subset S$ and $S - f(T,d) \in \widetilde{\Sigma}$.

Consider two parties facing a bargaining situation with a disagreement outcome d. They only know the maximal utility each of them can receive (i.e., the ideal point) from bargaining, but are uncertain about the resulting Pareto optimal frontier from all possible underlying compromises. In this case, they may still be willing to reach an interim outcome instead of sticking in the status quo. Accordingly we must have $\cap_{(S,d)\in\Sigma^{b,d}} D(S,d,f) \backslash \{d\} \neq \emptyset$, where $\Sigma^{b,d}$ is the collection of all bargaining problems in Σ with ideal point b and disagreement point d. The common disagreement point axiom

[6] A solution f in two-person bargaining problem is *proportional* if there are strictly positive constants p^1 and p^2 such that for every $(S,d) \in \widetilde{\Sigma}$, $f(S,d) = d + \lambda(S,d)p$ where $p = (p^1, p^2)$ and $\lambda(S,d) = \max\{t | tp \in S - d\}$.

stated below is a weaker version of this requirement:

Common Disagreement Point in the (d,b)-Box (CDP-Box) Suppose $(S,d),(T,d) \in \Sigma$. If (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $b(S,d) = b(T,d)$, then $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$.

CDP-Box only requires that each pair of bargaining problems with the same disagreement point and ideal point have a non-empty intersection of their disagreement point sets. The intersection of the disagreement point sets for more than two bargaining problems, however, could be empty. It turns out that the above axiom is satisfied by the Egalitarian and Dictatorial solutions as well as by the Discrete Raiffa solution.

To characterize the Discrete Raiffa solution, the following new axiom is also required:

Independence of Alternatives Below Midpoint (IABM) Suppose $(S,d),(T,d) \in \Sigma$. If $IR(S,m(S,d)) = IR(T,m(T,d))$, then $f(S,d) = f(T,d)$.

Observe that, if the condition $IR(S,m(S,d)) = IR(T,m(T,d))$ holds, then $m(S,d) = m(T,d)$ and $b(S,d) = b(T,d)$. If MD holds, then parties know that the bargaining outcome will dominate the midpoint; thus it is reasonable for them to focus only on those alternatives dominating the midpoint in their negotiations, and those alternatives below the midpoint should not influence the bargaining outcome. It is also easy to verify that IABM is satisfied by the Nash, Kalai/Smorodinsky and Discrete Raiffa solutions. Hence IABM alone cannot distinguish the DR solution from the former two.

Proposition 1 *DR is the unique solution satisfying IABM, DCONT, MD and CDP-Box.*

Proposition 1 states that, with the help of IABM, DCONT and MD, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems, are willing to reach interim outcomes so long as these two problems share the same disagreement and ideal points, then the bargaining outcome must be DR.

Remark 1 *Consider a slightly modified version of CDP-Box as follows:*

$\alpha-$**Common Disagreement Point-Box ($\alpha-$CDP-Box)** Suppose $(S,d),(T,d) \in \Sigma$ and pick any $\alpha \in [0,1)$. If (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $b(S,d) = b(T,d)$, then there exists $x \in D(S,d,f) \cap D(T,d,f)$ with $x \geq \alpha d + (1-\alpha)(\min\{f_1(S,d), f_1(T,d)\}, \min\{f_2(S,d), f_2(T,d)\})$.

$\alpha-$CDP-Box strengthens CDP-Box by requiring there exists at least one common disagreement point which dominates $\alpha d+(1-\alpha)(\min\{f_1(S,d), f_1(T,d)\}, \min\{f_2(S,d), f_2(T,d)\})$. In terms of the negotiation process, it can be seen as a condition on the speed of convergence. This common disagreement point, however, can be arbitrarily close to d if we pick α sufficiently close to 1. It is straightforward to verify the following extension of Proposition 1.

Proposition 2 *DR is the unique solution satisfying MD, IABM and $\alpha-$CDP-Box.*

5 Characterizations of the Nash Solution

Nash (1950) used the following axiom, along with SYM, WPO, and SI, in characterizing the Nash solution, which we had not formally defined before (it is also called Contraction Independence in the bargaining literature):

Independence of Irrelevant Alternatives (IIA) For all (S, d) and (T, d) in Σ with $S \subset T$ and $f(T, d) \in S$, we have $f(S, d) = f(T, d)$.

Consider the axiom below:

Common Disagreement Point with Contraction (CDP-Contraction) Suppose (S, d) and (T, d) in Σ. If (i) $f(S, d) \neq d$ and $f(T, d) \neq d$, and (ii) $S \subset T$ and $f(T, d) \in S$, then $(D(S, d, f) \cup \{f(S, d)\}) \cap (D(T, d, f) \cup \{f(T, d)\}) \backslash \{d\} \neq \emptyset$.[7]

CDP-Contraction is weaker than IIA. While IIA requires that the bargaining solution outcome remains unchanged when the new bargaining set S is contained in the old bargaining set T but S still contain the solution of T, CDP-Contraction only requires that in such a case there is an interim outcome that both parties can agree upon for the time being.

Proposition 3 *N is the unique solution satisfying IR, PO, SYM, SI, DCONT and CDP-Contraction.*

It may or may not be very clear *ex-ante* what kind of economic and non-economic factors may determine a party's bargaining power relative to that of the other; nevertheless, it should be clear from an *ex-post* point of view that one party's gain from negotiation relative to the other must be monotone increasing with their bargaining power. This simple idea inspires our first definition of bargaining power in different contexts. It is as follows: For any $x, y \in \mathbb{R}^2$ and $x \neq y$, let $l[x, y]$ be the line segment connecting x and y, and $\theta(x, y)$ be the gradient (slope) of $l[x, y]$. Suppose the bargaining solution outcome is $f(S, d) \geq d$ for a given bargaining problem (S, d), then the gradient $\theta(d, f(S, d))$, which measures the relative gains in bargaining, could be a good index of bargaining power (See Figure 1).

[7] Note that $f(S, d)$ is not necessarily in $D(S, d, f)$. The terms $\{f(S, d)\}$ and $\{f(T, d)\}$ can be dropped if f also satisfies PO and IR.

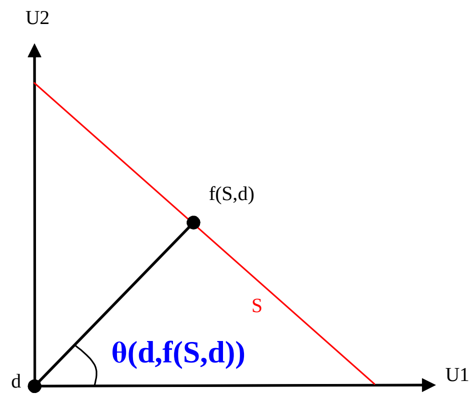

FIGURE 1

$\theta(d, f(S,d)) = 0$ implies Agent 1 has complete bargaining power, $\theta(d, f(S,d)) = \infty$ implies that Agent 2 has complete bargaining power, and Agent 1's bargaining power is monotone decreasing with θ. If $\theta(d, f(S,d)) = \theta(d, f(T,d))$, then parties receive the same relative gains over two bargaining problems (S,d) and (T,d) (See Figure 2):

Common Disagreement Point with Identical Relatives Gains (CDP-Gains) Suppose $(S,d),(T,d) \in \Sigma$. If (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(d, f(S,d)) = \theta(d, f(T,d))$, then $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$.

This axiom states that beginning with the same disagreement point d, if two parties perceive (correctly) that they are going to receive the same relative gains in two bargaining problems (S,d) and (T,d), then there exists at least one allocation in $S \cap T$ that is agreeable by both parties as a *common* interim outcome; from that point on they may continue their negotiation to split the remaining surplus in several particular ways. This axiom is also satisfied by the Egalitarian solution and the Dictatorial solutions as well as the Nash solution. It is closely related to the axiom of *disagreement point convexity* introduced by Peters and Van Damme (1991):

Disagreement Point Convexity (DPC) $f(S, \alpha d + (1-\alpha) f(S,d)) = f(S,d)$ for all $\alpha \in [0,1]$.

DPC requires that $D(S,d,f) \supset l(d, f(S,d))$. If the premises of CDP-Gains hold, then DPC implies that $D(S,d,f) \cap D(T,d,f) \supset l[d, \min\{f(S,d), f(T,d)\}]$.[8] Therefore DPC implies CDP-Gains, but not vice versa. Consider the $\epsilon-$egalitarian solution, $\epsilon - E$, such that (1) if $E_1(S,d) - d_1 = E_2(S,d) - d_2 \geq \epsilon$, it assigns $(E_1(S,d) - \epsilon, E_2(S,d) - \epsilon)$, where

[8]Note that $\min\{f(S,d), f(T,d)\}$ is well-defined when $\theta(d, f(S,d)) = \theta(d, f(T,d))$.

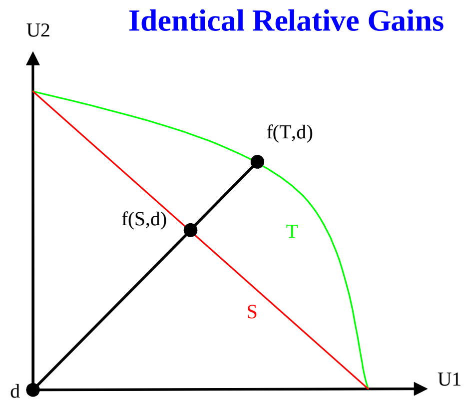

FIGURE 2

$\epsilon > 0$. (2) if $E_1(S,d) - d_1 = E_2(S,d) - d_2 < \epsilon$, it assigns d.[9] $\epsilon - E$ satisfies DCONT and CDP-Gains, but violates DPC.

Proposition 4 *N is the unique solution satisfying DCONT, MD and CDP-Gains.*

Proposition 4 can be interpreted as follows. With the help of DCONT and MD, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d, are willing to reach interim outcomes as long as they expect to receive the same relative gains over these two possible bargaining problems, then the bargaining outcome must be N, the compromise that maximizes the product of their bargaining gains.

Remark 2 *DCONT is merely a technical condition and can be dropped if we modify the axiom of CDP-Gains slightly as follows.*

$\alpha-$**Common Disagreement Point with Identical Relative Gains** ($\alpha-$**CDP-Gains)** Suppose $(S,d),(T,d) \in \Sigma$ and pick any $\alpha \in [0,1)$. A solution f satisfies $\alpha-$**CDP-Gains** if (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(d, f(S,d)) = \theta(d, f(T,d))$ implies that there exists $x \in D(S,d,f) \cap D(T,d,f)$ with $x \geq \alpha d + (1-\alpha)(\min\{f_1(S,d), f_1(T,d)\}, \min\{f_2(S,d), f_2(T,d)\})$.

$\alpha-$CDP-Gains strengthens CDP-Gains by requiring that there exists at least one common disagreement point which dominates $\alpha d + (1-\alpha)(\min\{f_1(S,d), f_1(T,d)\}, \min\{f_2(S,d), f_2(T,d)\})$. Again, from the negotiation process point of view, it can be seen as a condition on the speed of convergence. This common disagreement point can be arbitrarily close to d if

[9] $E(S,d)$ stands for the Egalitarian solution.

we pick α sufficiently close to 1. Note that DPC implies $\alpha-$CDP-Gains as well. Therefore, the following straightforward extension of Proposition 4 improves Theorem 1 of de Clippel (2007).

Proposition 5 *N is the unique solution satisfying MD and $\alpha-$CDP-Gains for all $\alpha \in (0, 1)$.*

Remark 3 *Peters and Van Damme (1991) demonstrates that N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and DPC. The following Proposition improves their result.*

Proposition 6 *N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and CDP-Gains.*

Proof. *It is straightforward to show that SIR, DCONT, SYM, SI and CDP-Gains imply DPC.* ■

6 Characterization of the Kalai/Smorodinsky Solution

Recall that $b(S, d)$ represents the ideal point of the bargaining problem (S, d), which may or may not be in S. If $b(S, d) \in S$, then we consider the bargaining situation *conflict-free* as both parties can reach their highest utility levels simultaneously. Thus $\theta(b(S, d), f(S, d))$ measures the relative concessions they made when there is a conflict of interest. A bargaining problem (S, d) can also be viewed as an abstract description of the situation between two parties negotiating over multiple issues. If there are more issues added to the negotiations, the utility possibility set should expand from S to a new set T accordingly ($S \subset T$) as argued in Section 2 by the quote from Kalai (1977). Let the bargaining outcomes be $f(S, d)$ and $f(T, d)$ respectively with $f(S, d) < f(T, d)$. Suppose the parties negotiate issues sequentially; then the set $IR(T, f(S, d))$ can be seen as the collection of all possible surpluses generated from adding new issues to the old ones, and $b(T, f(S, d)) - f(S, d)$ is the maximal surplus pair they can receive if the added issues are conflict-free.

Note that $\theta(b(T, f(S, d)), f(S, d))$ measures parties' relative concessions in $(T, f(S, d))$ with respect to $f(S, d)$. Therefore parties should *not* expect that their bargaining power will change from adding more issues if the concessions they have made are independent with those added issues; that is, when $\theta(b(T, f(S, d)), f(T, d)) = \theta(b(T, f(S, d)), f(S, d))$

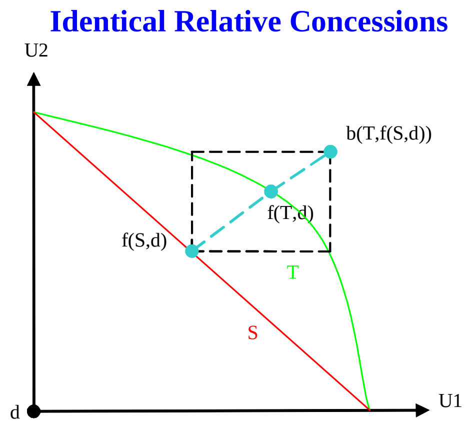

FIGURE 3

(See Figure 3). Accordingly the axiom of common disagreement point based on this concept can be stated as follows:[10]

Common Disagreement Point with Identical Relative Concessions (CDP-Concessions) Suppose $(S,d),(T,d) \in \Sigma$ with $S \subset T$. If (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(b(T,f(S,d)),f(T,d)) = \theta(b(T,f(S,d)),f(S,d))$, then $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$; moreover, $f_i(T,d) = b_i(T,d)$ for some i only if $b(T,d) \in T$.

Proposition 7 *KS is the unique solution satisfying SDM, DCONT, MD and CDP-Concessions.*[11]

With the help of SDM, DCONT and MD, Proposition 7 states that if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d, are willing to reach interim outcomes as long as they expect to take the same relative concessions over these two possible bargaining problems, then the bargaining outcome must be KS.

Remark 4 *It can readily be seen that the axiom of MD can be replaced by PO.*

[10] The last requirement "$f_i(T,d) = b_i(T,d)$ for some i only if $b(T,d) \in T$" is there to guarantee that $b(T,x) \neq f(T,d)$ will hold for all $x \in IR(T,d)\backslash\{f(T,d)\}$; otherwise $\theta(b(T,f(S,d)),f(T,d))$ may not be well-defined. This condition can be dropped if we restrict the domain of bargaining problems to be non-level or replace DCONT by PCONT in characterizing KS.

[11] Note that SDM, instead of its weaker version, DM, is required in the characterization of KS. However, even though N does not satisfy SDM in Σ, it does satisfy it in Σ^s nevertheless. Hence, clearly one cannot distinguish KS, N, and DR from each other - at least in Σ^s - solely on the basis of SDM.

Proposition 8 *KS is the unique solution satisfying SDM, DCONT, PO and CDP-Concessions.*

Remark 5 *Consider a revised version of CDP-Concessions below:*

$\alpha-$**Common Disagreement Point with Identical Relative Concessions ($\alpha-$CDP-Concessions)** Suppose $(S,d),(T,d) \in \Sigma$ with $S \subset T$ and $\alpha \in [0,1)$. If (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(b(T,f(S,d)),f(T,d)) = \theta(b(T,f(S,d)),f(S,d))$, then there exists $x \in D(S,d,f) \cap D(T,d,f)$ with $x \geq \alpha d+(1-\alpha)(\min\{f_1(S,d),f_1(T,d)\},\min\{f_2(S,d),f_2(T,d)\}$ moreover, $f_i(T,d) = b_i(T,d)$ for some i only if $b(T,d) \in T$.

It is straightforward to show the following:

Proposition 9 *KS is the unique solution satisfying SDM, MD and $\alpha-$CDP-Concessions.*

Proposition 10 *KS is the unique solution satisfying SDM, PO and $\alpha-$CDP-Concessions.*

7 Conclusion

Although there were previous non-unified attempts that tried to bring bargaining process into Nash's bargaining problem (via the Step-by-Step Negotiation axiom by Kalai (1977) and the Midpoint Domination (1981) by Sobel (1981), previous characterizations of bargaining solutions typically relied on crucial axioms entailing changes in the bargaining set and in the disagreement point, and did not describe the bargaining process. In this paper, we highlight the important role interim outcomes and bargaining process play in a unified way. The class of axioms we use, the Common Disagreement Point axioms, enable us to provide axiomatic characterizations of the Nash and Kalai/Smorodinsky solutions and most notably of the Discrete Raiffa solution which had not been characterized before (some of the characterizations of the former two solution concepts utilize the concept of bargaining power with respect to relative gains of parties over the disagreement point and their relative concessions in a bargaining set over another bargaining set's solution outcome). The central message delivered in this paper is that the bargaining outcome depends on under what circumstances parties facing uncertain underlying compromises are willing to reach interim outcomes. This attempt bridges the axiomatic and strategic approaches to bargaining within the confines of axiomatic bargaining. The use of interim outcomes is one way of bringing bargaining process into axiomatic bargaining. Future research may identify further fruitful ways in that direction.

8 Appendix

Proof of Proposition 1. It is obvious that DR satisfies these four axioms. Suppose f satisfies IABM, DCONT, MD and CDP-Box and we show that $f = DR$. Pick any $(S,d) \in \Sigma$. It is sufficient to show that $f(S,d) = f(S, m(S,d))$. Consider a bargaining problem (T,d) where $T = conv\{d, (d_1, b_2(S,d)), (b_1(S,d), d_2)\}$. MD implies that (i) $f(T,d) = m(S,d)$, and (ii) $D(T,d,f) = l[d, m(S,d)]$. By CDP-Box, there exists a common disagreement point $a \in l[d, m(S,d)]$ such that $f(S,d) = f(S,a)$. IABM excludes all points below $m(S,d)$ to be a common disagreement point. Hence, $a = m(S,d)$. ■

Proof of Proposition 3. It is straightforward to verify that N satisfies PO, SYM, SI, DCONT and CDP-Contraction. We will show that, if f satisfies these five axioms, then it must be $f = N$. It is sufficient to show that IR, PO, DCONT and CDP-Contraction imply IIA. Pick (S,d) and (T,d) in Σ with $S \subset T$ and $f(T,d) \in S$. Without loss of generality assume $IR(S,d)\backslash\{d\} \neq \varnothing$ and $IR(T,d)\backslash\{d\} \neq \varnothing$. $f(T,d) \neq d$ and $f(S,d) \neq d$ by PO. We show that $f(S,d) = f(T,d)$. Since f satisfies CDP-Contraction, there are three possible cases to be considered:

(i) If $f(S,d) = f(T,d)$, then we are done.

(ii) $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$. Pick $a^1 \in D(S,d,f) \cap D(T,d,f)\backslash\{d\}$. By definition, we have $f(S,a^1) = f(S,d)$ and $f(T,a^1) = f(T,d)$. If $a^1 \in PO(S) \cap PO(T)$, then $f(T,d) = f(T,a^1) = a^1 = f(S,d)$ by IR. $a^1 \notin PO(S)\backslash PO(T)$ by IR again. Suppose now $a^1 \notin PO(S)$. Starting at a^1 as a new disagreement point and repeatedly applying CDP-Contraction gives us a non-decreasing sequence $\{a^i\}$ such that $f(S,a^i) = f(S,d)$ and $f(T,a^i) = f(T,d)$ for all i. Define $a \equiv \lim_{i\to\infty} a^i$, then $f(S,a) = f(S,d)$ and $f(T,a) = f(T,d)$ by DCONT. It can be readily seen that $a = f(T,d)$. Hence $f(S,d) = f(T,d)$.

(iii) $f(S,d) \in D(T,d,f)$ or $f(T,d) \in D(S,d,f)$. $f(T,d) \in D(S,d,f)$ immediately implies $f(S,d) = f(T,d)$. If $f(S,d) \in D(T,d,f)$, then we claim $f(S,d) = f(T,d)$. Suppose to the contrary and $f(S,d) \neq f(T,d)$. Since $f(T,d) = f(T, f(S,d))$, $f(S,d)$ is Pareto dominated by $f(T,d) \in S$, a contradiction. ■

Proof of Proposition 4. It is obvious that N satisfies these three axioms. We will show that, if f satisfies these three axioms, then it must be $f = N$. First, we show that the statement is true in $\Sigma\backslash\Sigma^0$. Pick any $(S,d) \in \Sigma\backslash\Sigma^0$. By convexity of S, we have either $\sup\{x_1 | x \in IR(S,d)\} = d_1$ or $\sup\{x_2 | x \in IR(S,d)\} = d_2$. Without loss of generality, assume $\sup\{x_1 | x \in IR(S,d)\} = d_1$, then $IR(S,d) = l[d, (d_1, k)]$ by convexity again, where $k \equiv \sup\{x_2 | x \in IR(S,d)\}$. MD immediately implies $f_1(S,d) = d_1$. If $k = d_2$, then MD implies $f_2(S,d) = d_2$ and we have $f(S,d) = d = N(S,d)$ in this case. Suppose now $k > d_2$. Consider a new problem (T,d) with $T = l[d, (d_1, 2k - d_2)]$. MD implies

$f_2(T,d) \geq f_2(S,d) > d_2$, and $\theta(d, f(S,d)) = \theta(d, f(T,d))$. Consequently there exists a point $a^1 \in l[d,(d_1, f_2(S,d))]\backslash\{d\}$ such that $f(S,d) = f(S,a^1)$ and $f(T,d) = f(T,a^1)$ by CDP-Gains. Taking a^1 as a new common disagreement point and iteratively invoking the axiom of CDP-Gains, we get a strictly increasing sequence $\{a^i\}$ in $l[d,(d_1, f_2(S,d))]$ with $f(S,d) = f(S,a^i)$ and $f(T,d) = f(T,a^i)$ for all i. Let $a \equiv \lim_{i\to\infty} a^i$. $f(S,a) = f(S,d)$ and $f(T,a) = f(T,d)$ by DCONT. It can be readily seen that $a = f(s,d)$. We claim $f_2(S,d) = k$. Suppose to the contrary that $f_2(S,d) = \gamma < k$. Since $\lim_{i\to\infty} a^i = a = f(S,d) = (d_1, \gamma)$, there exists $a^j = (a_1^j, a_2^j)$ with $a_2^j > 2\gamma - k$ for some j. MD implies $f_2(S,d) \geq (a_2^j + k)/2 > (2\gamma - k + k)/2 = \gamma$, a contradiction. Therefore we have $f(S,d) = (d_1, \sup\{x_2 | x \in IR(S,d)\}) = N(S,d)$.

Next we show that if f satisfies DCONT, MD and CDP-Gains in Σ^0, then $f = N$. The proof is based on the following nice characterization of the Nash solution by de Clippel (2007).

Lemma 1 *(Theorem 1, de Clippel (2007)) N is the unique solution satisfying MD and DPC in Σ^0.*

With this Lemma in hand, it is sufficient to show that DCONT, MD and CDP-Gains imply DPC. Pick any (S,d) in Σ^0 and let $f(S,d)$ be its solution. MD implies $f(S,d) > d$. Consider a bargaining problem (T^ε, d) with $T^\varepsilon = conv\{d, (2f_1(S,d) - d_1 - \frac{\varepsilon}{f_2(S,d)-d_2}, d_2), (d_1, 2f_2(S,d) - d_2 - \frac{\varepsilon}{f_1(S,d)-d_1})\}$.[12] MD implies that (i) $f(T^\varepsilon, d) = (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})$, which in turn implies that $\theta(d, f(S,d)) = \theta(d, f(T^\varepsilon, d))$, and (ii) $D(T^\varepsilon, d, f) = l[d, (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})]$. CDP-Gains tells us that at least one point $a^1 \in l[d, (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})]$ is in the disagreement point set of (S,d) with respect to f. Starting at a^1 as a new disagreement point and repeating the argument above gives us a strictly increasing sequence $\{a^n\}$ such that $a^n \in D(S,d,f)$ $\forall n$. $\lim_{n\to\infty} a^n = (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})$ by CDP-Gains and DCONT. Invoking DCONT and CDP-Gains again gives us $(f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)}) \in D(S,d,f)$. Ranging ε from 0 to $2(f_1(S,d) - d_1)(f_2(S,d) - d_2)$ gives us DPC. ■

Proof of Proposition 7. It is straightforward to see that KS satisfies these four axioms. Suppose f satisfies SDM, DCONT, MD and CDP-Concessions; then we will show that $f = KS$ must hold. The proof is constructive and consists of four steps:

(I) $f(l[d,(d_1+b,d_2)],d) = b(l[d,(d_1+b,d_2)],d) = (d_1+b,d_2)$, where $b > 0$. Let $H \equiv l[d,(d_1+2b,d_2)]$. By MD, $f_1(H,d) \geq f_1(l[d,(d_1+b,d_2)],d) > d_1$.CDP-Concessions

[12] "conv" denotes "the convex hull of."

implies that there exists $a^1 \in l[d,(d_1+b,d_2)]$ such that $f(l[d,(d_1+b,d_2)],d) = f(l[d,(d_1+b,d_2)],a^1)$ and $f(H,d) = f(H,a^1)$. Iteratively applying CDP-Concessions gives us a strictly increasing sequence $\{a^i\}$ with $f(l[d,(d_1+b,d_2)],d) = f(l[d,(d_1+b,d_2)],a^i)$ and $f(H,d) = f(H,a^i)$ for all i. It can be shown that $\lim a^i = f(l[d,(d_1+b,d_2)],d) = (d_1+b,d_2)$ by DCONT, MD and CDP-Concessions.

(II) Similarly we have $f(l[d,(d_1,d_2+c)],d) = b(l[d,(d_1,d_2+c)],d) = (d_1,d_2+c)$.

(III) If $T = conv\{d,(d_1+b,d_2),(d_1,d_2+c),(d_1+b,d_2+c)\}$ for some $b,c>0$, then $f(T,d) = b(T,d) = (d_1+b,d_2+c)$. Suppose to the contrary that $f(T,d) = z \neq (d_1+b,d_2+c)$. MD implies $z \geq m(T,d) > d$. Denote $L(z,(d_1+b,d_2+c))$ to be the straight line going through z and (d_1+b,d_2+c), and define $\eta \equiv \inf\{x |\ x \in L(z,(d_1+b,d_2+c)) \cap IR(S,d)\}$. η is well-defined as the partial order $\geq$ in $\mathbb{R}^2$ induces a linear order in $L(z,(d_1+b,d_2+c))$. Apparently $\eta \geq d$. There are two possible cases:

(i) $\eta = d$. Consider a new bargaining problem (W,d) with $W = conv\{d,(d_1+b,d_2),(d_1,d_2+c)\}$. Notice that $b(T,d) = b(W,d) = (d_1+b,d_2+c)$. MD implies (a) $f(W,d) = (d_1+\frac{b}{2},d_2+\frac{c}{2})$, and (b) $D(W,d,f) = l[d,(d_1+\frac{b}{2},d_2+\frac{c}{2})]$. Accordingly we have $\theta(b(T,f(W,d)),f(T,d)) = \theta(b(T,f(W,d)),f(W,d))$, and there exists $a^1 \in l[d,(d_1+\frac{b}{2},d_2+\frac{c}{2})]$ such that $f(T,d) = f(T,a^1)$ and $f(W,d) = f(W,a^1)$ by CDP-Concessions. Again by repeatedly applying CDP-Concessions we get a strictly increasing sequence $\{a^i\}$ with $a^i \in l[d,(d_1+\frac{b}{2},d_2+\frac{c}{2})]$ such that $f(T,d) = f(T,a^i)$ and $f(W,d) = f(W,a^i)$ for all i. It can be shown that $\lim a^i = (d_1+\frac{b}{2},d_2+\frac{c}{2}) = m(T,d)$, and $f(T,d) = f(T,m(T,d))$ by DCONT. Taking $m(T,d)$ as a new disagreement point and iteratively applying the equation $f(T,d) = f(T,m(T,d))$ shows that $f(T,d) = (d_1+b,d_2+c)$, contradicting our premise that $f(T,d) = z \neq (d_1+b,d_2+c)$.

(ii) If $\eta \neq d$, then either $\eta = (\alpha,d_2)$ for some $\alpha \in (d_1,d_1+b)$ or $\eta = (d_1,\beta)$ for some $\beta \in (d_2,d_2+c)$. Without loss of generality, assume $\eta = (\alpha,d_2)$ for some $\alpha \in (d_1,d_1+b)$. From (I) we have $f(l[d,\eta],d) = \eta$. Consequently $\theta(b(T,f(l[d,\eta],d)),f(T,d)) = \theta(b(T,f(l[d,\eta],d)),f(l[d,\eta],d))$, and DCONT, MD and CDP-Concessions imply $f(T,d) = f(T,\eta)$. Taking η as a new disagreement point and following the same steps in (i) we get $f(T,d) = (d_1+b,d_2+c)$, contradicting our premise that $f(T,d) = z \neq (d_1+b,d_2+c)$.

In sum, we have shown that $f(T,d) = b(T,d) = (d_1+b,d_2+c)$.

(IV) Pick any (S,d) in Σ. If $b(S,d) \in S$, then $IR(S,d)$ has one of three forms defined in (I)-(III). Therefore we have $f(S,d) = b(S,d) = KS(S,d)$. Assume now $b(S,d) \notin S$. $f(S,d) \geq m(S,d)$ by MD; moreover, we show that $f(S,d) \in PO(S)$. Consider a bargaining problem (W,d) with $W = conv\{d,f(S,d),(f_1(S,d),d_2),(d_1,f_2(S,d))\} \subset S$. We know $f(W,d) = f(S,d)$ from (III). Hence $\theta(b(S,f(W,d)),f(S,d)) = \theta(b(S,f(W,d)),f(W,d))$. Repeatedly invoking CDP-Concessions and DCONT concludes that $f(S,d) \in D(S,d,f)$.

Consequently, $f(S,d)$ must be in $PO(S)$ by MD.

Define $\Gamma \equiv \{x \in IR(S,d) |\ \theta(b(S,x), f(S,d)) = \theta(b(S,x), x)$ and $x \leq f(S,d)\}$. Since $f_i(S,d) \neq b_i(S,d)$, $\Gamma \backslash \{f(S,d)\}$ is non-empty. It can be shown that either $\Gamma \cap \{x \in IR(S,d)|x_1 = d_1\} \neq \varnothing$ or $\Gamma \cap \{x \in IR(S,d)|x_2 = d_2\} \neq \varnothing$. There are two cases to be considered:

(i) If $d \in \Gamma$, then $f(S,d) = l[d, b(S,d)] \cap PO(S) = KS$.

(ii) If $d \notin \Gamma$, then either $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$ or (d_1, β) for some $\beta \in (d_2, f_2(S,d))$. Without loss of generality assume $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$. $f(l[d,(\alpha,d_2)],d) = (\alpha, d_2)$ by (I). Since $\theta(b(S, f(l[d,(\alpha,d_2)],d), f(S,d)) = \theta(b(S, f(l[d,(\alpha,d_2)],d)), f(l[d,(\alpha,d_2)],d))$, there exists $\xi \in l[d,(\alpha,d_2)] \backslash \{d\}$ such that $f(S,d) = f(S,\xi)$, which violates SDM. Therefore d must be in Γ. ■

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