A Strategic Analysis of the *War against Terror*

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Motivation

- The study of terrorism has been an active field of research in international relations since the early 1970s, with a considerable increase in interest after 9/11.
- An extensive review of the literature can be found in the book "The Political Economy of Terrorism" by Enders and Sandler (2006).
- The primary contributors have been political scientists, but they have typically not applied economic or strategic modelling.
- Recently, a growing number of researchers have applied game theory to study terrorism, but important features are not modelled, such as:
    - the military efficiency and the political power of countries fighting terror
    - the benefit that countries obtain from cooperating against terror
    - the change over time of terrorist resources, etc

This Paper

- We analyze a two stage game where a transnational terrorist organization interacts with an arbitrary number of countries.
- We distinguish between proactive measures (aggressively fighting terrorism) and defensive measures (protecting against attacks).
- Countries differ in the following parameters:
  - The political and economic power.
  - The effectiveness of the proactive measures.
  - The benefit that each obtains from cooperating against terrorism.
  - The value which they assign to the damage that can be inflicted on them upon a successful terrorist attack.

This Paper

- Both the terrorist organization and the countries act strategically:
  - Countries can use both proactive and defensive measures against terrorism.
  - All countries use defensive measures but the group of countries proactively fighting terror is determined endogenously.
  - The terrorist organization allocates resources among all the countries simultaneously.
- We study the static game analytically and its dynamic version both analytically and numerically.
- In the multi-period game, the resources of the terrorist change over time depending on its success and the group of countries proactively fighting terror changes accordingly.

The Static Model

- There are n countries. The set of countries is denoted by $N = \{1, 2, ..., n\}$.
- There is a terror organization T with initial resources R_0.
- There is a subset $N_0 \subseteq N$ which represents the group of countries taking proactive measures against T.
- The countries and T are engaged in a two stage game $G(N_0, R_0)$.

At the first stage of the game $G(N_0, R_0)$:

- Each country $i \in N_0$ chooses a proactive effort level x_i to fight T, where $x_i = 0$ if $i \notin N_0$.
- The $\{x_i\}_{i \in N_0}$ are chosen simultaneously and independently. As a result, the total resources of T are reduced from R_0 to R, where

$$R \equiv R\left(R_0, \sum_{i \in N_0} x_i\right) < R_0.$$

- The function R increases in R_0 and decreases in $\sum_{i \in N_0} x_i$.

At the second stage of the game $G(N_0, R_0)$:

- R becomes commonly known.
- Each country $i \in N$ chooses a defensive effort level y_i, which is the monetary investment to protect the country against a terror attack.
- Simultaneously, T allocates his total resources R among the N countries by choosing $\{R_i\}_{i \in N}$ such that:

$$R = \sum_{i=1}^{n} R_i, \text{ with } R_i \geq 0.$$

- Note that T may attack any country in N and not only the ones in N_0.

- The damage that T can inflict on a country $i \in N$ is random with mean
$$\lambda_i \equiv \lambda\left(P_i, R_i, y_i\right).$$
- P_i measures the political and economic power of country $i \in N$.
- The function λ is increasing in R_i and P_i and decreasing in y_i.
- The monetary value that country $i \in N$ assigns to a unit of damage inflicted by T is denoted by v_i.

- The countries in N_0 obtain a political/economic benefit from their cooperation against T.
- The benefit of i depends on the contribution x_i to the total proactive effort and is denoted by $b_i(x_i)$. If $N_0 = \{i\}$, then $b_i(x_i) \equiv 0$.
- The monetary cost for country $i \in N_0$ of providing a proactive effort level of x_i is denoted by $c_i(x_i)$.

- The expected payoff of country $i \in N$ is:

$$\pi_i(N_0) = \left\{ \begin{array}{ll} b_i(x_i) - c_i(x_i) - y_i - v_i \lambda(P_i, R_i, y_i) & \text{if } i \in N_0 \\ -y_i - v_i \lambda(P_i, R_i, y_i) & \text{if } i \notin N_0 \end{array} \right\}$$

- The expected payoff of T is the total expected damage it inflicts:

$$\pi_T = \sum_{i=1}^{n} \lambda(P_i, R_i, y_i)$$

where $R_i \geq 0$, $\sum_{i=1}^{n} R_i = R$ and

$$R \equiv R\left(R_0, \sum_{i \in N_0} x_i\right)$$

- To be able to explicitly characterize the subgame perfect equilibrium (SPE) of $G(N_0, R_0)$, we choose the following functions:

$$R\left(R_0, \sum_{i \in N_0} x_i\right) = R_0 \exp\left(-\epsilon \sum_{i \in N_0} x_i\right), \ \epsilon \geq 0$$

$$\lambda(P_i, R_i, y_i) = \frac{P_i R_i}{y_i}$$

$$c_i(x_i) = \frac{1}{2}\gamma_i x_i^2$$

$$b_i(x_i) = \left\{ \begin{array}{ll} b_i x_i & \text{if } N_0 \neq \{i\} \\ 0 & \text{if } N_0 = \{i\} \end{array} \right\}$$

- We start with the assumption that $b_i \geq 0$ for all $i \in N$. Later on, we extend the analysis to allow for $b_i < 0$.
- With the above functional forms, the expected payoff functions are:

$$\begin{aligned} \pi_T &= \sum_{i=1}^{n} \frac{P_i R_i}{y_i} \\ \pi_i (N_0) &= \left\{ \begin{array}{cc} b_i x_i - \frac{1}{2}\gamma_i x_i^2 - y_i - \frac{v_i P_i R_i}{y_i} & \text{if } i \in N_0 \\ -y_i - \frac{v_i P_i R_i}{y_i} & \text{if } i \notin N_0 \end{array} \right\} \end{aligned}$$

Proposition 1. The game $G\left(N_0, R_0\right)$ has a unique pure strategy SPE. It is characterized by:

1. $x_i^* = \frac{b_i}{\gamma_i} + \frac{\frac{P_i}{\gamma_i}\left[x^* - \sum_{k\in N_0}\frac{b_k}{\gamma_k}\right]}{\sum_{k\in N_0}\frac{P_k}{\gamma_k}}$ for all $i \in N_0$, where x^* uniquely solves:
2. $x^* = \sum_{k\in N_0} x_k^* = \sum_{k\in N_0}\frac{b_k}{\gamma_k} + \frac{\epsilon\left(\sum_{k\in N_0}\frac{P_k}{\gamma_k}\right)R_0^{0.5}\exp\left(-\frac{\epsilon}{2}x^*\right)}{\left(\sum_{k=1}^{n}\frac{P_k}{v_k}\right)^{0.5}}$
3. $y_i^* = \frac{P_i R_0^{0.5}\exp\left(-\frac{\epsilon}{2}x^*\right)}{\left(\sum_{k=1}^{n}\frac{P_k}{v_k}\right)^{0.5}}$, $i \in N$. Therefore $\frac{y_i^*}{y_j^*} = \frac{P_i}{P_j}$.
4. $R_i^* = \frac{P_i R_0 \exp\left(-\epsilon x^*\right)}{v_i\left(\sum_{k=1}^{n}\frac{P_k}{v_k}\right)}$, $i \in N$. Therefore $\frac{R_i^*}{R_j^*} = \frac{\frac{P_i}{v_i}}{\frac{P_j}{v_j}}$.

In equilibrium,

5. $R_i^* = \frac{(y_i^*)^2}{P_i v_i}$, $i \in N$
6. $\lambda_i^* = \frac{P_i}{v_i} \frac{R_0^{0.5}}{\left(\sum_{k=1}^n \frac{P_k}{v_k}\right)^{0.5}} \exp\left(-\frac{\epsilon}{2} x^*\right)$, $i \in N$
7. $y_i^* = \frac{\gamma_i x_i^* - b_i}{\epsilon}$, $i \in N_0$ and $\epsilon > 0$
8. $$\pi_i(N_0) = \left\{ \begin{array}{cc} \left(b_i - \frac{2\gamma_i}{\epsilon}\right) x_i^* - \frac{1}{2}\gamma_i (x_i^*)^2 + \frac{2b_i}{\epsilon} & \text{if } i \in N_0 \\ -\frac{2P_i R_0^{0.5} \exp\left(-\frac{\epsilon}{2} x^*\right)}{\left(\sum_{k=1}^n \frac{P_k}{v_k}\right)^{0.5}} & \text{if } i \in N \backslash N_0 \end{array} \right\}$$
9. $\pi_T^* = \left(\sum_{k=1}^n \frac{P_k}{v_k}\right)^{0.5} R_0^{0.5} \exp\left(-\frac{\epsilon}{2} x^*\right)$

- In what follows, we let $B_i = \frac{P_i}{\gamma_i}$ and $B = \sum_{k \in N_0} \frac{P_k}{\gamma_k}$.

Proposition 2.

1. $\frac{\partial x^*}{\partial v_i} > 0; \frac{\partial x^*}{\partial b_i} > 0; \frac{\partial x^*}{\partial \gamma_i} < 0; \frac{\partial x^*}{\partial R_0} > 0; \frac{\partial x^*}{\partial P_i} < 0$ if $i \notin N_0$ and $\frac{\partial x^*}{\partial P_i} > 0$ if $B_i > \frac{1}{2}B$ or $\frac{v_i}{v_j} \leq 2\frac{\gamma_i}{\gamma_j}, i, j \in N_0$.
2. For all $i, j \in N_0$, $k \in N$, $j, k \neq i$, $\frac{\partial x_i^*}{\partial v_i} > 0; \frac{\partial x_i^*}{\partial v_k} > 0; \frac{\partial x_i^*}{\partial P_i} > 0; \frac{\partial x_i^*}{\partial P_k} < 0;$ $\frac{\partial x_i^*}{\partial b_i} > 0; \frac{\partial x_i^*}{\partial b_j} < 0; \frac{\partial x_i^*}{\partial \gamma_i} < 0; \frac{\partial x_i^*}{\partial \gamma_j} > 0; \frac{\partial x_i^*}{\partial R_0} > 0.$
3. For all $i, j \in N$, $j \neq i$, $\frac{\partial y_i^*}{\partial v_i} > 0; \frac{\partial y_i^*}{\partial v_j} > 0; \frac{\partial y_i^*}{\partial P_i} > 0; \frac{\partial y_i^*}{\partial P_j} < 0; \frac{\partial y_i^*}{\partial R_0} > 0;$ $\frac{\partial \lambda_i^*}{\partial v_i} < 0; \frac{\partial \lambda_i^*}{\partial v_j} > 0; \frac{\partial \lambda_i^*}{\partial P_i} > 0; \frac{\partial \lambda_i^*}{\partial P_j} < 0; \frac{\partial \lambda_i^*}{\partial R_0} > 0.$
4. For all $i \in N_0$, $\frac{\partial y_i^*}{\partial b_i} < 0;\ \frac{\partial y_i^*}{\partial \gamma_i} > 0;\ \frac{\partial \lambda_i^*}{\partial b_i} < 0;\ \frac{\partial \lambda_i^*}{\partial \gamma_i} > 0.$
5. For all $i \in N$ and $j \in N_0$, $\frac{\partial y_i^*}{\partial b_j} < 0; \frac{\partial y_i^*}{\partial \gamma_j} > 0; \frac{\partial \lambda_i^*}{\partial b_j} < 0; \frac{\partial \lambda_i^*}{\partial \gamma_j} > 0.$

Proposition 3.

1. For all $i, j \in N$, $j \neq i$, $\frac{\partial R_i^*}{\partial v_i} < 0$ and $\frac{\partial R_i^*}{\partial v_j} > 0$.
2. For all $i, j \in N$, $j \neq i$, $\frac{\partial R_i^*}{\partial P_i} > 0$ if $B_i \leq \frac{1}{2}B$ and $\frac{\partial R_i^*}{\partial P_j} < 0$.
3. For all $i \in N_0$, $\frac{\partial R_i^*}{\partial b_i} < 0$ and for all $i \in N$ and $j \in N_0$, $\frac{\partial R_i^*}{\partial b_j} < 0$.
4. For all $i \in N_0$, $\frac{\partial R_i^*}{\partial \gamma_i} > 0$ and for all $i \in N$ and $j \in N_0$, $\frac{\partial R_i^*}{\partial \gamma_j} > 0$.

In what follows, we let $\pi_i^* = \pi_i^*(N_0)$.

Proposition 4.

1. For all $i, j \in N$ and $j \neq i$, $\frac{\partial \pi_i^*}{\partial v_i} < 0$ and $\frac{\partial \pi_i^*}{\partial v_j} < 0$.
2. For all $i, j \in N$ and $j \neq i$, $\frac{\partial \pi_i^*}{\partial P_i} < 0$ and $\frac{\partial \pi_i^*}{\partial P_j} > 0$.
3. For all $i \in N_0$ $\frac{\partial \pi_i^*}{\partial b_i} > 0$; For all $i \in N$, $j \in N_0$, $i \neq j$, $\frac{\partial \pi_i^*}{\partial b_j} > 0$.
4. The payoff of $i \in N_0$ depends on γ_i as follows:
    - $\frac{\partial \pi_i^*}{\partial \gamma_i} < 0$ if $B_i \geq \frac{1}{2}B$.
    - If $B_i < \frac{1}{2}B$, $\exists\ \widehat{R}_0$ s.t. $\frac{\partial \pi_i^*}{\partial \gamma_i} < 0$ if $R_0 < \widehat{R}_0$ and $\frac{\partial \pi_i^*}{\partial \gamma_i} > 0$ if $R_0 > \widehat{R}_0$.
    - For every $i \in N$, $j \in N_0$, $j \neq i$, $\frac{\partial \pi_i^*}{\partial \gamma_j} < 0$.

$\uparrow$	x^*	x_i^*	x_j^*	y_i^*	y_j^*	R_i^*	R_j^*	λ_i^*	λ_j^*	π_i^*	π_j^*	$\frac{R_i^*}{R^*}$	$\frac{R_j^*}{R^*}$	$\frac{x_i^*}{x^*}$
v_i	+	+	+	+	+	−	+	−	+	−	−	−	+	$*_1$
P_i	$*_3$	+	−	+	−	$*_2$	−	+	−	−	+	+	−	+
b_i	+	+	−	−	−	−	−	−	−	+	+	0	0	+
γ_i	−	−	+	+	+	+	+	+	+	$*_4$	−	0	0	−
R_0	+	+	+	+	+	+	+	+	+	−	−	0	0	0

$*_1$ + if $\frac{B_i}{B} > \frac{A_i}{A}$.

$*_2$ + if $B_i \leq \frac{1}{2}B$.

$*_3$ If $i \notin N_0$, $\frac{\partial x^*}{\partial P_i} < 0$. If $i \in N_0$, $\frac{\partial x^*}{\partial P_i} > 0$ if $B_i > \frac{1}{2}B$ or $\frac{v_i}{v_j} \leq 2\frac{\gamma_i}{\gamma_j}$ $\forall\ i, j \in N_0$.

$*_4$ + iff $B_i < \frac{1}{2}B$ and R_0 is sufficiently large. Otherwise $\frac{\partial \pi_i^*}{\partial \gamma_i} < 0$.

- Next, we explain the results of the propositions and we provide numerical examples with $N = 2$.

Results for the Static Game

A Change in the Initial Resources of the Terrorist

- In the numerical examples, $i \in N_0$ whenever we refer to x_i, b_i and γ_i. Also,

$$P_1 = 1, P_2 = 2, v_i = 1, \gamma_i = 1, b_i = 0.1, \epsilon = 0.1,\ R_0 = 1$$

- An increase in R_0:
  - Increases the proactive $(x_i \uparrow, x_j \uparrow)$ and defensive $(y_i \uparrow, y_j \uparrow)$ efforts of all countries .
  - In spite of this counterterrorist reaction, the expected damage increases and the expected payoff decreases for every country.
  - The available resources of T go up and so does his payoff.
- Hence, a more powerful terrorist decreases the benefit of all countries.

R_0	R	R_1	R_2	y_1	y_2	x_1	x_2	x	π_T
1	0.963	0.321	0.642	0.566	1.133	0.156	0.213	0.37	1.70
5	4.720	1.573	3.146	1.254	2.508	0.225	0.350	0.57	3.76

Results for the Static Game

A Change in the Damage Valuation

- An increase in v_i:
  - Increases the efforts of i, $(x_i \uparrow, y_i \uparrow)$, causing a significant shift of terrorist resources from i to j $\left(R_i \downarrow, R_j \uparrow\right)$. In turn, j considerably increases the defensive effort $\left(y_j \uparrow\right)$ and also the proactive effort $\left(x_j \uparrow\right)$ to reduce the power of T.
  - The shift in resources increases the expected damage for j and decreases it for i, while the payoff decreases for all countries $(v_i\lambda_i \uparrow)$.
  - Since the total proactive effort x is higher, the resources of T go down. Further, the terrorist payoff π_T decreases.
- Consistent with evidence that higher defensive efforts by a country divert attacks to softer targets: Enders and Sandler (93, 04, 05).

v_1	v_2	R	R_1	R_2	y_1	y_2	x_1	x_2	x	π_T
1	1	0.963	0.321	0.642	0.566	1.133	0.156	0.213	0.37	1.70
1	5	0.956	0.683	0.273	0.826	1.652	0.182	0.265	0.44	1.15

- An increase in P_i:
  - Increases the proactive and the defensive effort of i, $(x_i \uparrow, y_i \uparrow)$ as well as the relative resources that T allocates to i $\left(\frac{R_i}{R} \uparrow, \frac{R_j}{R} \downarrow\right)$.
  - In anticipation, j free rides on i by decreasing his proactive effort $(x_j \downarrow)$ and he also decreases his defensive effort $(y_j \downarrow)$.
  - The shift in resources increases the expected damage of i and decreases it for j, while the opposite happens to the payoffs.
- This is consistent with evidence that more powerful countries exert bigger proactive efforts and other countries free ride.

P_1	P_2	R	R_1	R_2	y_1	y_2	x_1	x_2	x	π_T
1	2	0.963	0.321	0.642	0.566	1.133	0.156	0.213	0.37	1.70
1	5	0.957	0.159	0.797	0.399	1.996	0.139	0.299	0.43	2.39

Results for the Static Game

A Change in the Efficiency of the Proactive Measures

- An increase in γ_i (i is less efficient externally):
  - Decreases the proactive effort of i $(x_i \downarrow)$, which is substituted with a higher defensive effort $(y_i \uparrow)$. To compensate, the other countries increase their proactive effort $\left(x_j \uparrow\right)$ so that i free rides on them.
  - The total proactive effort x decreases, and this increases the available resources of T, who allocates more to each country $\left(R_i \uparrow, R_j \uparrow\right)$. This induces an increase in the defensive effort of j $\left(y_j \uparrow\right)$.
  - The expected damage increases for all countries. The payoff of j decreases and the payoff of T increases.
  - If $B_i < \frac{1}{2}B$ and R_0 is sufficiently large, the total proactive cost of i decreases and **the payoff of i increases even though he becomes less efficient**. Otherwise, the payoff of i decreases.

γ_1	γ_2	R	R_1	R_2	y_1	y_2	x_1	x_2	x	π_T
1	1	0.963	0.321	0.642	0.566	1.133	0.156	0.213	0.37	1.70
1	5	0.980	0.326	0.653	0.571	1.143	0.157	0.042	0.20	1.71

Results for the Static Game

Why can a country benefit from being less efficient?

- A per unit increase in γ_i:
  - Has a direct effect on the cost $\frac{1}{2}\gamma_i x_i^2$ that is equal to $\frac{1}{2}x_i^2$.
  - Has an indirect effect on the cost of $\gamma_i x_i \frac{\partial x_i}{\partial \gamma_i}$. If x_i is sufficiently large, $\frac{\partial x_i}{\partial \gamma_i} = -\frac{B-B_i}{B}\frac{x_i}{\gamma_i}$ and the indirect cost effect is $-\frac{B-B_i}{B}x_i^2$. If $B_i < \frac{1}{2}B$, this effect is proportional to x_i^2.
  - Has a direct effect on the benefit $b_i x_i$ that is equal to $b_i \frac{\partial x_i}{\partial \gamma_i}$. If x_i is sufficiently large, this is of the magnitude of a term that is linear in x_i.
  - Decreases the total proactive effort and increases the resources of T, but the decrease in x approaches a constant when x_i increases indefinitely.
- If x_i is very large, the quadratic cost saving effect $(x_i \downarrow)$ outweights the two other negative effects and $\frac{\partial \pi_i}{\partial \gamma_i} > 0$.
- Since x_i increases indefinitely with R_0, if $B_i < \frac{1}{2}B$ and R_0 is sufficiently large, $\frac{\partial \pi_i}{\partial \gamma_i} > 0$.

- An increase in b_i:
  - Increases the proactive effort of i and decreases the proactive efforts of the other countries, who free ride on i $\left(x_i \uparrow, x_j \downarrow\right)$.
  - The total proactive effort x increases (first order effect) and the available resources of T decrease. Hence, all countries decrease their defensive efforts $\left(y_i \downarrow, y_j \downarrow\right)$.
  - The expected damage decreases and the payoff increases for every country, while the payoff of T declines.
- Hence, an increase in the benefit from cooperation of a country benefits all countries.

b_1	b_2	R	R_1	R_2	y_1	y_2	x_1	x_2	x	π_T
1	1	0.963	0.321	0.642	0.566	1.133	0.156	0.213	0.37	1.70
1	5	0.926	0.308	0.617	0.555	1.111	0.155	0.611	0.76	1.66

Proposition 5. *The expected damage that T inflicts on $i \in N$*

1. *Decreases to zero if one of the following is true: (i) v_i increases indefinitely; (ii) γ_j for any $j \in N_0$ decreases to zero; (iii) b_j for any $j \in N_0$ increases indefinitely.*
2. *Increases indefinitely if one of the following is true: (v) R_0 increases indefinitely; (vi) P_i increases indefinitely.*

- In what follows, we allow for $b_i < 0$ for some or all $i \in N$. Note that a country i with $b_i < 0$ may still have an incentive to proactively fight T if either P_i or v_i are sufficiently large or γ_i is sufficiently small.
- To deal with this case, we denote by $\widetilde{x}_i\left(N_0\right)$ the solution of the first order equilibrium conditions, namely,

$$\widetilde{x}_i\left(N_0\right) = A_i + \frac{B_i\left[\widetilde{x}\left(N_0\right) - A\left(N_0\right)\right]}{B\left(N_0\right)} \tag{1}$$

where $\widetilde{x}\left(N_0\right)$ is the unique solution x of

$$x = A\left(N_0\right) + \frac{\epsilon B\left(N_0\right) R_0^{0.5} \exp\left(-\frac{\epsilon}{2}x\right)}{C\left(N\right)^{0.5}} \tag{2}$$

- Clearly, if $\widetilde{x}_i\left(N_0\right) \geq 0$ for all $i \in N_0$, then $x_i^*\left(N_0\right) = \widetilde{x}_i\left(N_0\right)$ for all $i \in N_0$, but $\widetilde{x}_i\left(N_0\right)$ may be negative.

Proposition 6. *Let $N_0 \neq \phi$ be a subset of N and let $k \in N \backslash N_0$ and $N_0' = N_0 + k$. Suppose that $\widetilde{x}_k(N_0') \geq 0$. Then, for all $i \in N_0$,*

1. $\widetilde{x}(N_0') \geq \widetilde{x}(N_0)$.
2. $\widetilde{x}_i(N_0') \leq \widetilde{x}_i(N_0)$ *and* $\widetilde{y}_i(N_0') \leq \widetilde{y}_i(N_0)$.
3. $\widetilde{\lambda}_i(N_0') \leq \widetilde{\lambda}_i(N_0)$, $\widetilde{\pi}_i(N_0') \geq \widetilde{\pi}_i(N_0)$ and $\widetilde{\pi}_T(N_0') \leq \widetilde{\pi}_T(N_0)$.
4. All the inequalities above are strict if $\widetilde{x}_k(N_0') > 0$.
5. All the inequalities in (1)-(3) are reversed if $\widetilde{x}_k(N_0') \leq 0$ and they are strict if $\widetilde{x}_k(N_0') < 0$.

Definition (Sustainability): A non-empty subset N_0 of N is sustainable iff

1. $\pi_i^*(N_0) \geq \pi_i^*(N_0 \backslash i)$ for all $i \in N_0$
2. $\pi_k^*(N_0) \geq \pi_k^*(N_0 + k)$ for all $k \notin N_0$

- This implies that N_0 is sustainable if no country in N_0 benefits from leaving and no country outside N_0 benefits from joining N_0.

Proposition 7.

1. There exists a sustainable set N_0 where for all $i \in N_0$, $x_i^*(N_0) = \widetilde{x}_i(N_0) > 0$.
2. Suppose that $b_i \geq 0$ for all $i \in N$. Then, $N_0 = N$ is the only sustainable set.
3. If N_0 is sustainable, then Proposition 6 applies to the equilibrium outcome, namely it holds with * replacing $\widetilde{}$.

- The proof of Proposition 7 is constructive and uses Lemmas A and B.
- **Lemma A.** Let $N_0 \subset N$, $i \in N_0$ and $k \notin N_0$. Then, (1) if $\widetilde{x}_k\left(N_0+k\right) \geq 0$ and $x_i\left(N_0+k\right) \leq 0$ and at least one inequality is strict, then $\frac{b_k}{P_k} > \frac{b_i}{P_i}$; (2) if $\widetilde{x}_k\left(N_0+k\right) \leq 0$ and $x_i\left(N_0+k\right) \geq 0$ and at least one inequality is strict, then $\frac{b_k}{P_k} < \frac{b_i}{P_i}$.
- **Proof of Lemma A.** (1) By the equilibrium first order conditions,

$$\widetilde{x}_i\left(N_0+k\right) = A_i + \frac{B_i\left[\widetilde{x}\left(N_0+k\right) - A\left(N_0+k\right)\right]}{B\left(N_0+k\right)} \leq 0$$

$$\widetilde{x}_k\left(N_0+k\right) = A_k + \frac{B_k\left[\widetilde{x}\left(N_0+k\right) - A\left(N_0+k\right)\right]}{B\left(N_0+k\right)} > 0$$

Therefore, $-\frac{A_i}{B_i} = -\frac{b_i}{P_i} \geq \frac{\left[\widetilde{x}\left(N_0+k\right)-A\left(N_0+k\right)\right]}{B\left(N_0+k\right)} > -\frac{A_k}{B_k} = -\frac{b_k}{P_k}$. The proof of part (2) is similar.■

- **Lemma B.** Suppose that $\frac{b_1}{P_1} \geq \frac{b_2}{P_2} \geq ... \geq \frac{b_n}{P_n}$. Let $1 < m < k \leq n$. If $\widetilde{x}_m(1, 2, ..., m) \leq 0$, then $\widetilde{x}_k(1, 2, ..., m-1, k) \leq 0$.
- **Proof of Lemma B.**
  - Claim. $\widetilde{x}_k(1, 2, ..., m, k) \leq 0$. Otherwise, if $\widetilde{x}_k(1, 2, ..., m, k) > 0$, then by part (2) of Proposition 6, $\widetilde{x}_m(1, 2, ..., m, k) < \widetilde{x}_k(1, 2, ..., m) \leq 0$. By Lemma A, $\frac{b_k}{P_k} > \frac{b_m}{P_m}$, a contradiction.
  - Suppose to the contrary that $\widetilde{x}_k(1, 2, ..., m-1, k) > 0$. Then, $\widetilde{x}_m(1, 2, ..., m, k) > 0$. Otherwise, by part (5) of Proposition 6, $0 \geq \widetilde{x}_k(1, 2, ..., m, k) \geq \widetilde{x}_k(1, 2, ..., m-1, k)$, a contradiction.
  - Applying again part (2) of Proposition 6, we have that $\widetilde{x}_k(1, 2, ..., m, k) \geq \widetilde{x}_k(1, 2, ..., m-1, k) > 0$ and this contradicts the claim.■

Results for the Static Game

Proof of Proposition 7

- Without loss of generality assume that $\frac{b_1}{P_1} \geq \frac{b_2}{P_2} \geq ... \geq \frac{b_n}{P_n}$.
- **First**. Let $N_{0,1} = \{1\}$.
  - If $\widetilde{x}_2(1,2) \leq 0$, then by Lemma B $\widetilde{x}_k(1,k) \leq 0$ for all $1 < k \leq n$ and $N_{0,1}$ is sustainable (recall that $b_1 = 0$ if $|N_0| = 1$ and hence $\widetilde{x}_1(1) = x_1^*(1) > 0$).
  - Otherwise, let $N_{0,2} = \{1,2\}$. By Lemma A, $\widetilde{x}_1(1,2) > 0$. If $\widetilde{x}_3(1,2,3) \leq 0$, then by Lemma B $N_{0,2}$ is sustainable. Otherwise, if $\widetilde{x}_3(1,2,3) > 0$, then by Lemma A, $\widetilde{x}_2(1,2,3) > 0$ and $\widetilde{x}_1(1,2,3) > 0$.
  - There is a unique $1 \leq m \leq n$ such that $\widetilde{x}_m(1,2,...,m) > 0$ and $\widetilde{x}_{m+1}(1,2,...,m+1) \leq 0$. The sustainable set is $\{1,2,...,m\}$.
- **Second.** Suppose that $b_i \geq 0$ for all $i \in N$. In this case, for every $N_0 \subset N$ and $k \notin N_0$, $\widetilde{x}_k(N_0 + k) > 0$, implying that $\pi_k^*(N_0 + k) > \pi_k(N_0)$ and N_0 is therefore not sustainable.■

Numerical Example with N=5

The Sustainable Cooperating Group

Proactive Effort with $N = 5$, $\rho = 0.01$ and different R_0								
Country	v_i	γ_i	b_i	P_i	$R_0 = 1$	$R_0 = 10$	$R_0 = 100$	$R_0 = 900$
1	1	1	0.1	1	0.14	0.23	0.52	1.20
2	1	1	-0.1	1	0	0.03	0.32	1.00
3	1	1	-0.2	1	0	0	0.22	0.90
4	1	1	-0.5	1	0	0	0	0.60
5	1	1	-1	1	0	0	0	0.10

- Countries with a negative b_i might find it beneficial to join the cooperating group if the terrorist is sufficiently powerful.

Numerical Example with N=5

The Sustainable Cooperating Group

Proactive Effort with $N = 5$, $R_0 = 5$, $\rho = 0.01$ and different ϵ								
Country	v_i	γ_i	b_i	P_i	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.11$	$\epsilon = 0.2$
1	1	1	0.1	1	0.14	0.19	0.20	0.28
2	1	1	-0.01	1	0.03	0.08	0.09	0.17
3	1	1	-0.05	1	0	0.04	0.05	0.13
4	1	1	-0.1	1	0	0	0.007	0.08
5	1	1	-0.15	1	0	0	0	0.03

- Countries with a negative b_i might find it beneficial to join the cooperating group if their proactive effort reduces the resources of the terrorist effectively.

External Effort with $N = 5$, $\rho = 0.01$							
Country	v_i	γ_i	b_i	P_i	x_i	P_i	x_i
1	1	1	0.1	1	0.14	1	0.10
2	1	1	-0.1	1	0	1	0
3	1	1	-0.2	1	0	100	0.19
4	1	1	-0.5	1	0	200	0.28
5	1	1	-1	1	0	300	0.17

- A change in the power (P_i) of a country affects the sustainable cooperating group.
- For example, a country i that has a lower benefit from cooperation than country j $(b_i < b_j < 0)$ might join N_0 (while j stays out) if it is more powerful than j $(P_i > P_j)$.

- In the M period game, the resources of T evolve as follows. At the first stage of period 1, the resources are equal to $R_{0,1}$. Let $N_{0,1}$ be a sustainable set of countries.
- The countries in $N_{0,1}$ attack T with efforts $\{x_{i,1}\}_{i \in N_{0,1}}$, chosen simultaneously and independently, and the resources of T reduce to:

$$r_1 = R_{0,1} \exp(-\epsilon \sum_{i \in N_{0,1}} x_{i,1})$$

- At the second stage of period 1, T allocates r_1 so that $r_1 = \sum_{i=1}^{n+1} R_{i,1}$, where $R_{i,1}$ are the resources allocated to country $i \in N$ and $R_{i,n+1}$ are the resources kept for future use.
- Simultaneously with T, the countries choose defensive levels $\{y_{i,1}\}_{i \in N}$ in an attempt to avoid successful terrorist attacks.

- We let $D_{i,1}$ be the random variable that measures the degree of damage to country $i \in N$ and we let $d_{i,1}$ be the realization of $D_{i,1}$.
- We assume that
    - $D_{i,1}$ takes integer values
    - $(D_{i,1})_{i=1}^{n}$ are mutually independent
    - $D_{i,1} \sim Poisson\,(\lambda_{i,1})$, where

$$\lambda_{i,1} = E\,(D_{i,1}) = \frac{P_i R_{i,1}}{y_{i,1}}$$

- $d_{i,1} = 0$ means that T fails to successfully attack country $i \in N$.

- At the beginning of the second period, the resources of T are:

$$R_{0,2} = r_1 + \rho \sum_{i=1}^{n} d_{i,1}$$

  where $\rho > 0$ is a constant fraction of the actual damage. Let $N_{0,2}$ be a sustainable set of countries with respect to $R_{0,2}$.

- The countries in $N_{0,2}$ attack T with proactive efforts $\{x_{i,2}\}_{i \in N_{0,2}}$ at the first stage of period 2, reducing his resources to

$$r_2 = R_{0,2} \exp(-\epsilon \sum_{i \in N_{0,2}} x_{i,2})$$

- At the second stage of period 2, T allocates $r_2 = \sum_{i=1}^{n+1} R_{i,2}$ and the countries choose $\{y_{i,2}\}_{i \in N}$ simultaneously.

- The previous process continues every period. The resources of T at the beginning of period $m+1$, for $1 \leq m \leq M-1$, are thus equal to:

$$R_{0,m+1} = r_m + \rho \sum_{i=1}^{n} d_{i,m}$$

and a sustainable set $N_{0,m}$ is generated.

- In addition, we have that

$$r_m = R_{0,m} \exp(-\epsilon \sum_{i \in N_{0,m}} x_{i,m})$$

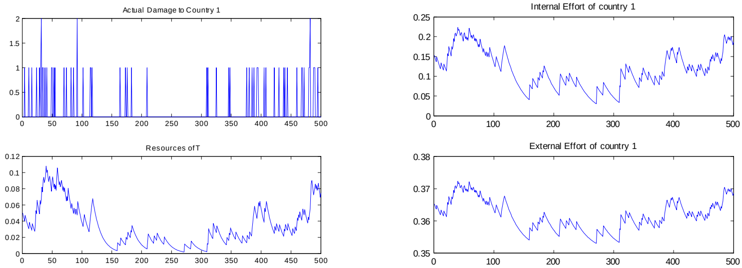
where $r_m = \sum_{i=1}^{n+1} R_{i,m}$ and $\sum_{i=1}^{n} R_{i,m}$ is the sum of resources allocated by T to the countries in period m.

The Dynamic Game with M Periods

- We let $G(N_0, R_0)$ be the static game in which R_0 is the initial resource of T. Further, we let $G_m(N_{0,m}, R_{0,m})$ be the interaction of T and the countries in N in period m.
- The dynamic game G_M is defined to be the game where the players interact as in $G_m(N_{0,m}, R_{0,m})$ in every period m, $1 \leq m \leq M$.
- In such a game, the strategies may be quite sophisticated:
  - T could find it optimal in some periods to leave part of the resources for future use.
  - T or the countries may not find it optimal to play every period the equilibrium strategies of the static game $G(N_{0,m}, R_{0,m})$.
- However, if the damages to the countries follow a Poisson distribution, the equilibrium dynamic strategies are the equilibrium static strategies.

- **Proposition 8:** The dynamic game G_M has a unique subgame perfect equilibrium. In every period $1 \leq m \leq M$,
  - T behaves myopically and it allocates all its resources r_m to maximize its expected payoff in $G\left(N, R_{0,m}\right)$.
  - Similarly, each country chooses its proactive and defensive efforts according to the equilibrium actions in $G\left(N, R_{0,m}\right)$.
- In what follows, we report numerical results of the dynamic game. The graphs are for a given simulation but the results in the Tables report averages over many simulations.

$$R_0 = 0.05, \epsilon = 0.1, v_i = 1, \gamma_i = 1, b_i = 0.35, P_i = 1, \rho = 0.01$$

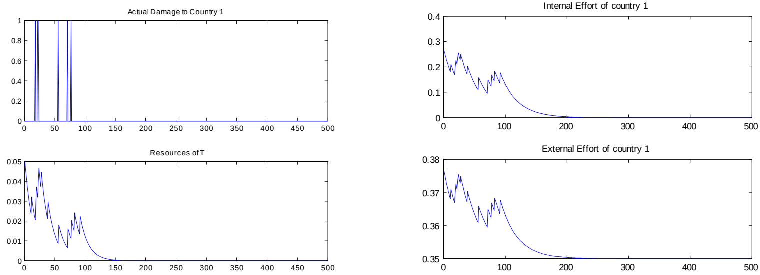


- In this example, the countries and the terrorist coexist.

Numerical Example with N=2

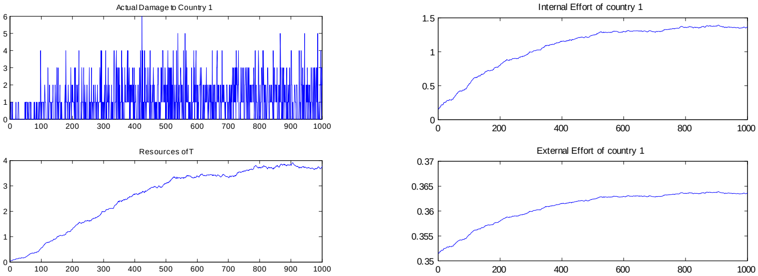
Symmetric Case-Example 2

$$R_0 = 0.05, \epsilon = 0.1, v_i = 3, \gamma_i = 1, b_i = 0.35, P_i = 1, \rho = 0.01$$



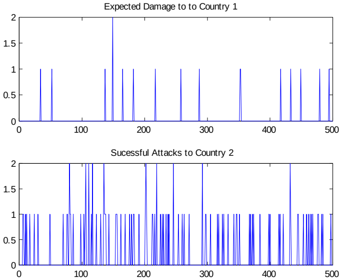
- The damage valuation is higher and the terrorist is defeated.

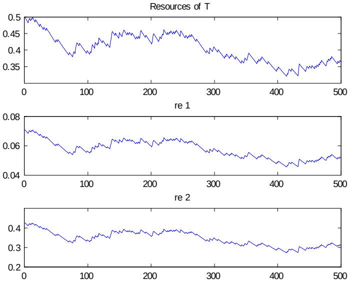
$$R_0 = 0.05, \epsilon = 0.01, v_i = 1, \gamma_i = 1, b_i = 0.35, P_i = 1, \rho = 0.01$$



- The proactive effort is not effective in reducing the resources of the terrorist, who defeats the countries.

$$v_i = 5, \gamma_i = (10, 4), b_i = (0.15, 0.1), P_i = (0.1, 0.6), R_0 = 0.5$$



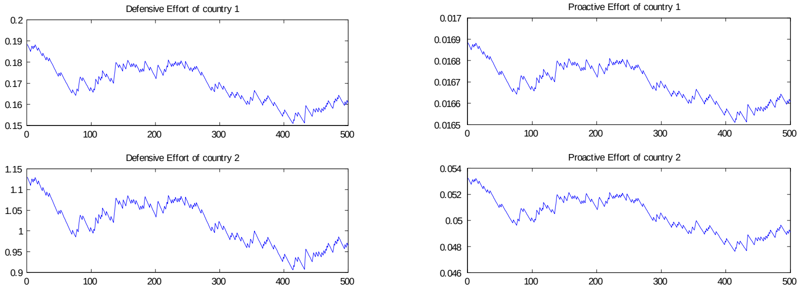


- Country 1 (US) benefits less from cooperation and it is 6 times more powerful and 2.5 times militarly more efficient than 2 (Spain). We see that 1 is allocated 6 times more resources by the terrorist than 2.

Numerical Example with N=2

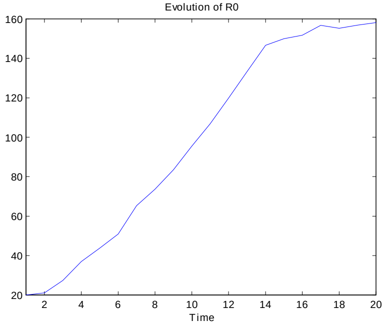
Asymmetric Case-Example 4

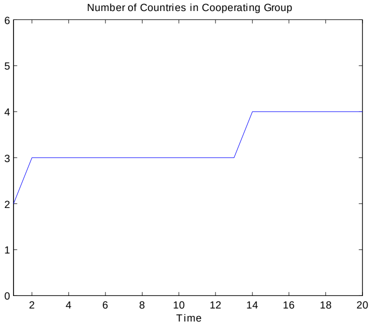
$$v_i = 5, \gamma_i = (10, 4), b_i = (0.15, 0.1), P_i = (0.1, 0.6), R_0 = 0.5$$



- The US exerts six times more defensive effort and about three times more proactive effort than Spain.

$$v_i = 1,\ P_i = 1,\ \gamma_i = 1,\ b_i = \{0.1, -0.1, -0.2, -0.5, -1\}$$





- At $t = 1$, $N_{0,1} = \{1, 2\}$ but as the resources of T increase over time, more countries join the cooperating group, $N_{0,20} = \{1, 2, 3, 4\}$.

v_1	v_2	R_1	R_2	y_1	y_2	x_1	x_2	d_1	d_2
1	0.5	0.194	0.777	0.417	0.834	0.141	0.183	0.42	1.67
1	1	0.236	0.473	0.443	0.886	0.144	0.188	0.46	0.89
1	5	0.327	0.131	0.489	0.979	0.149	0.198	0.53	0.19
1	10	0.368	0.073	0.520	1.040	0.152	0.204	0.51	0.09
1	100	0.392	0.007	0.520	1.041	0.152	0.204	0.53	0.01

$P_1 = 1,\ P_2 = 2,\ \gamma_i = 1,\ b_i = 0.1,\ \epsilon = 0.1,\ \rho = 0.01$

- As v_i increases, all countries increase their defensive effort y_i and their proactive effort x_i, decreasing the total resources of T.
- As v_i increases indefinitely, the resources that T allocates to i go to zero and so does the realized damage d_i.
- If v_i is sufficiently high for every country T is eventually defeated. Otherwise, T coexists with the countries.

P_1	P_2	R_1	R_2	y_1	y_2	x_1	x_2	d_1	d_2
1	0.1	0.652	0.065	0.716	0.071	0.171	0.107	0.70	0.06
1	1	0.236	0.473	0.443	0.886	0.144	0.188	0.46	0.89
1	5	0.122	0.613	0.330	1.651	0.133	0.265	0.32	1.63
1	10	0.068	0.680	0.250	2.502	0.125	0.350	0.22	2.52
1	20	0.037	0.758	0.189	3.792	0.119	0.479	0.19	3.81

$v_i = 1,\ b_i = 0.1,\ \gamma_i = 1,\ \epsilon = 0.1,\ \rho = 0.01$

- As P_i increases, country i increases x_i and y_i and the terrorist allocates more resources to i and less to j. In turn, country j decreases both x_j and y_j.
- The realized damage and the resources of T increase, since T assigns more resources to powerful countries and a successful attack on these countries attracts more followers.

Numerical Examples

Effects of a Change in the Efficiency of the Proactive Effort

γ_1	γ_2	R_1	R_2	y_1	y_2	x_1	x_2	d_1	d_2
1	0.5	0.142	0.285	0.329	0.659	0.133	0.331	0.35	0.70
1	1	0.236	0.473	0.443	0.886	0.144	0.188	0.46	0.89
1	5	0.494	0.988	0.680	1.360	0.168	0.047	0.71	1.36
1	10	0.554	1.108	0.723	1.446	0.172	0.024	0.71	1.45
1	50	0.601	1.203	0.756	1.512	0.175	0.005	0.74	1.49

$P_1 = 1,\ P_2 = 2,\ v_i = 1,\ b_i = 0.1,\ \epsilon = 0.1,\ \rho = 0.01$

- As γ_i increases, country i uses less x_i and more y_i. To compensate, country j uses more x_j.
- The expected damage increases for both countries and this increases the resources of T. Thus, j also uses more y_j.
- If γ_i is sufficiently low for at least one country, T is eventually defeated. Otherwise, T coexists with the countries.

b_1	b_2	R_1	R_2	y_1	y_2	x_1	x_2	d_1	d_2
0.1	0.01	0.333	0.667	0.550	1.099	0.155	0.120	0.56	1.04
0.1	0.1	0.236	0.473	0.443	0.886	0.144	0.188	0.46	0.89
0.1	0.2	0.173	0.346	0.361	0.722	0.136	0.272	0.34	0.73
0.1	0.5	0.096	0.193	0.245	0.491	0.124	0.549	0.22	0.46

$P_1 = 1,\ P_2 = 2,\ \gamma_i = 1, v_i = 1,\ \epsilon = 0.1,\ \rho = 0.01$

- As b_i increases, country i uses more x_i and country j uses less x_j.
- In turn, this decreases the damage and the resources of T and all countries decrease their defensive efforts y_i and y_j.
- If b_i is sufficiently high for at least one country, T is defeated. Otherwise, T will coexist with the countries.

- In the first version of the Colonel Blotto game by Borel (1921):
  - Each player divides one unit among several positions.
  - Whoever assigns a higher quantity to a majority of positions wins.
- In more popular versions by Read (1957, 1961) or Dresher (1961):
  - There are n targets A_1, ..., A_n with values $a_1 > a_2 >, ..., > a_n$.
  - Blue has one attacking unit and Red has one defending unit which they both allocate to the targets.
  - If a target A_i is undefended, it is destroyed and Blue gains a_i. If a defended target is attacked, Blue gains pa_i, with $p \in (0, 1)$.
- Blotto games are zero sum games and the solution is in general a mixed strategy equilibrium.

- The literature on terrorism and game theory has studied issues as:
  - Government concessions (mostly regarding hostages).
  - The terrorists's choice of a target,.
  - The governments' counterterrorist responses.
- We mostly relate to the literature on counterterrorism.
- The literature on counterterrorism:
  - Mostly deals with 2 by 2 games with two countries (the terrorist is not a strategic player): Lee (1988), Arce and Sandler (2003).
  - Only recently analyzes extensive form games: Sandler and Siqueira (2006) and Rosendorff and Sandler (2004).

- Sandler and Siqueira (2006): Study two versions where the terrorist is not a strategic player and where two targets choose independently defensive measures (version 1) or proactive measures (version 2):
  - The measures determine the probability of successful attacks.
  - The level of defensive and proactive measures is not optimal and it depends on the magnitude of the externalities.
- Rosendorff and Sandler (2004): Two player game where the government chooses first the proactive effort and the terrorist chooses then the type of attack (normal or a spectacular). They:
  - Assume that proactive policies increase terrorist recruitment.
  - Show that a country which either (i) values damage more or (ii) is more powerful will exert more proactive efforts.

- For a given country, the proactive effort against terror:
    - Increases with its effectiveness, the valuation of the damage, the political/economic power, the benefit from cooperation against terror and the initial resources of the terrorist.
- For a given country, the defensive effort:
    - Increases with the valuation of the damage, the political/economic power and the initial resources of the terrorist.
    - Decreases with the benefit from cooperation against terror and the effectiveness of the proactive effort.

- The payoff of a country:
  - Increases with the benefit from cooperating against terror.
  - Decreases with the valuation of the damage and the political/economic power.
  - Increases with the effectiveness of the proactive effort if the initial resources of the terrorist are not too large. Surprisingly, it decreases with the effectiveness of the proactive effort provided that the initial resources of T are large and the relative power or the country is less than 50% of the total power.

- The expected damage on a country:
  - Increases with the initial resources of the terrorist and the political/economic power.
  - Decreases with the valuation of the damage, the benefit from cooperation and the effectiveness of the proactive effort.
  - Decreases to zero if one of the following increases indefinitely: the valuation of the damage an attack can cause, the benefit of cooperating and the effectiveness of the proactive effort against terror.
  - Increases indefinitely if the initial resources of the terrorist increase indefinitely.

- A non-empty sustainable cooperating group exists wether or not some or all the benefits from cooperation are negative.
- If the benefit from cooperation is positive, then the only sustainable cooperating group is the set of all countries.

- In the dynamic game, the terrorist is defeated if any one of the following is sufficiently large:
  - The monetary valuation of the damage that T can cause for all countries.
  - The effectiveness of the proactive effort of some country.
  - The benefit from cooperation of some country.
- If any of the above does not hold, the terrorist will coexist with the countries.
- The results of the static game regarding changes in power, military efficiency, benefit from cooperation and damage valuation of the countries also hold in the dynamic simulations.

Future Work

- Extend the analysis to the case where the benefit from cooperation depends on the specific cooperating group.
- Analyze the case in which the cooperating group of countries act as one entity to achieve the first best and analyze if countries are over or under investing in proactive and defensive efforts.