Context Effects: A Representation of Choices from Categories*

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Abstract

This paper axiomatizes a utility representation for reference-dependent preferences over menus of lotteries. Most papers in the relevant literature use as the reference point some status quo element and characterize the complete set of preferences by using various assumptions on how the conditional preferences depend on this reference point. In our paper, we will analyze instead the case in which the perceived value of a menu depends on some subjectively selected reference point contained in that set. More specifically, we will axiomatize preferences over menus of lotteries exhibiting the following context-effects bias. An inferior option in a menu, even though never chosen, may make the better bundles of that menu appear to the decision maker to be more attractive by comparison and thus it may distort the objecive ranking of the various available options. An informal way of referring to this behavioral bias is that of 'making a good deal'. In addition, since menus are the standard objects of choice that reveal an individual's preference for flexibility, we will also study the case in which we allow for the presence of some underlying uncertainty between the moment of the choice of the menu and the moment of the choice of the most preferred option within the menu.

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1 Introduction

For considerations of simplicity and analytical tractability, the standard economic model of decision-making posits that the individual is able to rank alternatives independently of the context and in particular without taking any specific alternative as a reference point. This works in certain contexts, at least as an approximation, however, we now know from psychology that there are interesting settings in which systematic deviations occur. Simonson and Tversky (1992) present the following seemingly anomalous choice pattern. Participants in an experiment were asked to choose between a roll of towels and a box of tissues. Two separate versions of the experiment were designed. In one version, the participants were presented with two brands of towels and one brand of tissues, whereas in the second version, they were presented with one brand of towels and two brands of tissues. In the category with two brands, the quality of one of the options was clearly inferior to the other option from that category. The two superior brands of towels and tissues were included in both versions of the experiment. The results of the experiment showed that the market shares of both high quality brands were significantly greater when the inferior brand introduced belonged to the same category. A second similar example cited in Simonson and Tversky (1992) is that of a tactic frequently used by sales people. Thus, a common approach used to convince potential buyers to purchase a certain product is to present it along with another product and then argue that the former product is a much better deal in comparison with the later. Both examples show how the introduction of inferior options in a menu can potentially increase the attractiveness of the options in that menu relative to the elements in the other menus.

To illustrate these ideas formally, let x and y denote the high quality brands from the experiment of Simonson and Tversky (1992) and let x_0 and y_0 denote the corresponding lower quality brands. In order to capture the fact that some options belong to the same category, we will describe the choice made by the participants when presented for instance with the options x, x_0 and y as $c(\{x, x_0\}, \{y\})$. Then, the results of the experiment suggest that a significant share of the consumers exhibited the following pattern of choice: $c(\{x, x_0\}, \{y\}) = \{x\}$ and $c(\{x\}, \{y, y_0\}) = \{y\}$. But note that according to the standard model of rationality, the distribution of products into categories and the presence of the inferior products x_0 or y_0 should not affect the consumers' preferences in any way and thus, we should have $c(\{x, x_0\}, \{y\}) = c(\{x\}, \{y, y_0\})$.[1] This is the departure from the pattern of choice of a standard decision maker that we will study in this paper.

As suggested in the previous paragraph, the primitive object considered in our model *without uncertainty* is a choice correspondence that maps a set of options grouped into some possibly nondisjoint categories into a nonempty subset. More formally, for any sets $A_1, ..., A_n$, denote by

[1] For future research we intend to study in more detail a model of rational choice from sets grouped into categories.

$c(A_1, ..., A_n)$ to be the nonempty subset of $A = A_1 \cup ... \cup A_n$ that are the chosen options from A when these options are presented grouped into the categories $A_1, ..., A_n$. In general, by a *category* we will understand a *menu* or a *context,* that is a subset of the set of available options that are perceived by the decision maker as being tied together.[2] As a real world example, consider an individual who contemplates buying a car and note that while two types of cars of the same brand can be compared based on some objective characteristics such as price, featured options, warranty etc., this comparison becomes more difficult when it is made across brands. Thus, while we may consider that the individual sees the available options as the set of objects of choice as is standard in the literature, it does make sense to consider that in fact he perceives these options as being grouped into menus, where a menu is for instance a brand or a county of origin.[3]

By identifying the choice of an option from a category with a preference for the corresponding category we will employ throughout our analysis an *induced* preference relation over categories. In other words, we will define the induced preference relation $\succeq$ over sets by saying that $A \succeq B$ if and only if an individual selects some option from menu A when he has both A and B available. Slightly more formally, if A and B are disjoint, then we say that $A \succeq B$ if and only if $c(A, B) \cap A \neq \varnothing$. For nondisjoint categories, we will extend the preference by continuity by considering sequences of mutually disjoint sets converging to the relevant sets. The corresponding details are presented formally in Section 3.1.1. Now, note that the standard rationality axioms on the preferences over singletons, that is, transitivity, independence and continuity, have intuitive counterparts for the induced preference over the categories when the corresponding sets are disjoint.[4] In addition, whenever it is the case that sets which are close enough in the relevant metric are also close enough in preference, the above axioms can be extended by standard continuity arguments to nondisjoint sets.

[2] We will use the terms *category* and *menu* interchangeably throughout the rest of the paper when refering to the model without uncertainty. In the case of the model with uncertainty that we will present below, since the primitive object will be a revealed, rather than induced, preference relation over these sets, we will employ the term *menu* to conform to the standard practice in the related literature.

[3] We mention that we will not attempt here to derive a subjective construction of the menus of choice based for instance on considerations of similarity. Instead, we will restrict attention to those applications in which products are objectively grouped into categories that the outside observer can infer based either on some objective information or on some prior analysis of the decision maker's behavior. However, we mention that the model we study here has clear behavioral implications that would allow identification of the relevant categories as long as the application at hand satisfies the main modelling assumptions we use in our paper, that is, the existence of well defined objective categories and the presence of the context effects bias. For instance, in the car buying example, while it is not immediately clear whether a menu is for instance a brand or a country of origin, this can be infered, at least in theory, in an obvious way when the behavioral bias studied here is present.

[4] As an example, transitivity of the preferences over categories would suggest patterns of choice of the following flavor. Assume x is preferred over y, when x is presented alongside x_0 and y is presented alongside y_0. Also, assume that the decision maker chooses y when he has available the two categories $\{y, y_0\}$ and $\{z, z_0\}$. Then, transitivity of the preferences over menus would require that x be chosen, when the available categories are $\{x, x_0\}$ and $\{z, z_0\}$.

As an example for the way in which we will employ this induced preference relation, consider again the experiment from Simonson and Tversky (1992). Note firstly that from $c(\{x, x_0\}, \{y\}) = \{x\}$ and $c(\{x\}, \{y, y_0\}) = \{y\}$ we can infer the following rankings: $\{x, x_0\} \succ \{y\}$ and $\{y, y_0\} \succ \{x\}$. Using transitivity of the preferences over categories, it follows that $\{x\} \succeq \{y\}$ would imply $\{y, y_0\} \succ \{y\}$, while $\{y\} \succeq \{x\}$ would imply $\{x, x_0\} \succ \{x\}$. In addition, although not explicitly tested in the experiment, the fact that x is a superior brand to x_0 suggests that $\{x\} \succ \{x_0\}$. Therefore, we basically obtained the following pattern of choice $\{x, x_0\} \succ \{x\} \succ \{x_0\}$. Note now again that the presence of x_0 should not affect the preferences in any way so with the above mentioned interpretation of the preference over categories, standard rationality would imply that $\{x, x_0\} \sim \{x\} \succ \{x_0\}$.

The specific assumption used in this paper is that in the process of contemplating each of the available options, the decision maker exhibits *a reference-point bias* or *context-effects bias*. More exactly, the presence of a relatively inferior option in the menu makes the rest of the elements of that menu appear more attractive *by comparison* with that inferior element.[5] Bhargava, Kim and Srivastava(2000) designed and tested a model of *comparative judgement* in which they analyzed how the market shares of two superior products varied with the change in the *relative* values of the two superior products when a third inferior product was introduced with characteristics that were meant to make it highlight only one of the two superior products. The results suggested a statistically significant negative correlation between the valuation of the reference point and the market share of the targeted superior product. While the design of their experiment did not include an objective grouping of options into categories and the valuations of the products were computed based on their objective characteristics rather than subjectively, the relevant conclusion in our setting is that the main determinant of the context effects bias is the relative valuation of the products with respect to the reference point. Thus, a decrease in the quality of the reference point of a menu would increase the overall attractiveness of the rest of the options from that menu. Clearly, we will not consider those cases in which the presence of the inferior element may affect the perceived value of the menu as a whole because of issues such as brand image. For instance, it is clear that a car manufacturer offering a very low quality model might affect the image of the brand as a whole and thus we are not analyzing this type of applications. Instead, for this particular case we may consider for instance an average quality model coming with full options but at an extremely high price as being the reference point. Thus, by an inferior product we may understand either a highly priced product, a low quality one or a combination of the two.

Before moving on to presenting the theoretical model, we mention a couple of important facts.

[5] We note here that we restrict attention to the simple case in which the decision maker uses a unique reference point in assessing the perceived value of a given menu. Naturally, the identifying assumption for this unique point is that it is the least valuable option from each menu when the choice of the menu is made.

Firstly, in this paper we are analyzing the impact of the context-effects bias as observed in the Simonson and Tversky (1992) example and we abstract away from the potential presence of any other behavioral biases by practically imposing the standard rationality axioms on the preferences over singleton sets.[6] By assuming away any other deviations from the standard model, the departure from the choices implied by the standard definition of rationality can be attributed solely to the presence of the behavioral bias that we study here. Secondly, we emphasize that the model we study here applies mainly to preferences over objects whose assessment of the characteristics and trade-offs between these characteristics are mostly *subjective*. When the available options can be ranked objectively across menus, it is less likely that decision makers would exhibit the behavioral bias.[7] Many applications do satisfy this requirement. For instance, in the car buying example, it is clear that two cars of different brands are difficult to rank objectively even when there is a clear ranking based purely on the objective characteristics.

We define a *reference-dependent representation* of the induced preference relation over categories as follows. The utility of any compact set of lotteries A is:

$$V(A) = \max_{x \in A} u(x) - \theta \min_{y \in A} u(y) \tag{1}$$

where $\theta \in (0, 1)$ is a parameter measuring the strength of the reference point bias, $u(\cdot)$ is the utility function that represents the individual's preferences over singletons and $\arg\min_{y \in A} u(y)$ is the endogenous reference point. We consider here the case of a constant value of the parameter θ. As an extension of this model in another direction, we may also study in future research the case in which θ decreases as the gap between the most preferred choice and the reference point increases. This would allow for the marginal effect of the bias to vanish as additional inferior products are introduced in the menu.

To see how the above representation captures the type of behavior presented in Simonson and Tversky (1992) consider the following numerical example. Assume $\theta = 0.5$ and $u(x) = 10$, $u(x_0) = 5$ so that as required we have $\{x\} \succ \{x_0\}$. Then $V(\{x\}) = u(x)(1 - \theta) = 5$ while $V(\{x, x_0\}) = u(x) - \theta u(x_0) = 7.5$ so indeed $\{x, x_0\} \succ \{x\}$. Then, since $V(\{y\}) = u(y)(1 - \theta)$ it

[6] Thus, the choices within menus are in line with the standard model of rationality. A formal way of expressing this assumption that is that $x \in C(A)$ if and only if $x \in c(\{x\}, \{y\})$ for all $y \in A$. See Section 3.1.2 for more details.

[7] In fact, many other behavioral biases are unlikely to be present when the decisions are made under complete objective information. For instance, consider the 'hot hand fallacy bias', which is the tendency for people to overestimate the likelihood of a certain event when a sequence of observations consistent with that event is observed. Now, consider a person who exhibits this bias in a strong manner when the probabilities are subjective. If the same person is given an urn with a known number of balls of different colors and then asked to extract a certain number of balls which all come out of the same color, it is unlikely that he would change his beliefs about the composition of the balls in the urn in any other way than the one consistent with the Bayesian updating.

follows that for all values of $u(y) \in (10, 15)$ we will have $\{x, x_0\} \succ \{y\} \succ \{x\}$ so the preferences over menus given by our representation will be consistent with the preferences identified in the Simonson and Tversky (1992) example. Finally, to see that the parameter θ measures the strength of the reference point bias take $u(y) = 12$ and consider a different $\theta_1 = 0.2$. Then, $V_1(\{y\}) = 9.6$ while $V_1(\{x, x_0\}) = 9$ so $\{y\} \succ_1 \{x, x_0\} \succ_1 \{x\}$. Therefore, in this case the parameter θ is too small and the strength of the reference point bias is insufficient to overcome the fact that option y is better than x.

Then, the primitive choice correspondence has the following representation:

$$
\begin{aligned}
c(A_1, ..., A_n) &= \bigcup_{A_i \in \mathcal{A}} \arg\max_{x \in A_i} u(x) \text{ where} && (2)\\
\mathcal{A} &= \arg\max_{A_i \in \{A_1, ..., A_n\}} V(A_i) && (3)
\end{aligned}
$$

We will present the necessary and sufficient conditions on the choice correspondence that guarantee that it has a representation as suggested by (2), (3) and (1). Throughout the rest of the paper, we will refer to the preceeding model as the *basic model,* in order to distinguish it from the model with uncertainty that we will introduce immediately.

Since menus are the objects of choice that reveal a decision maker's preference for flexibility, we will also extend the analysis to allow for the presence of uncertainty between the moment of the choice of the menu and that of the choice within the menu. In other words, following Kreps(1979), we will identify a menu with an ex ante *observable* action that after some uncertainty is resolved will make a certain set of outcomes available ex post. Thus, unlike the case considered in the basic model, the observability of the ex ante actions renders the preference over menus a *revealed* preference. We allow for this uncertainty to be *completely subjective* meaning that besides assuming a subjective distribution over the ex post contingencies as in standard Savage type models, we also assume that the actual space of ex post contingencies is subjective. Allowing for uncertainty between the two stages makes sense especially in those cases in which there is some cost of switching between menus and the choice of the specific element from the menu is made either significantly later or repeatedly over a long period of time. In these cases, when choosing the menu the decision maker has to contemplate various potential realizations of his future tastes and thus introducing uncertainty is necessary.[8]

[8] As an example, consider an individual choosing between various online movie rental services such as Netflix and Blockbuster. Each of these services comes in a variety of options combining benefits and prices, thus basically making the initial choice of the type of program a choice over menus. Assume now that after some careful consideration, a consumer decides to subscribe to Netflix and then selects one of the available options regarding the number of DVDs he can rent at a time. Now, while it may happen that he is sure of the number of movies that he would desire to watch in a given period, it may as well happen that he needs to test the various options to see which fits him best.

For the model in which we allow for the presence of uncertainty, the *reference-dependent representation under uncertainty* that we will axiomatize is the following. The utility of any compact set of lotteries A is:

$$V(A) = \int_S \left[\max_{z \in A} U(z,s) \right] \mu(ds) - \theta \min_{x \in A} \left[\int_S U(x,s)\mu(ds) \right] \tag{4}$$

where S is a state space capturing the subjective uncertainty with μ a positive measure over S, $U(z,s)$ is the ex post state utility of lottery z in state s. Note that this time the reference point is the *ex ante* least preferred option within the menu, that is $\arg\min_{x \in A} \left[\int_S U(x,s)\mu(ds) \right]$. The set of ex post utilities will satisfy an additional condition presented formally in Section 2, that will allow us to identify the behavioral bias modeled by our representation as being a reference point bias. More specifically, we will assume that in the second period after the uncertainty is resolved the decision maker cannot reverse *or almost* reverse his ex ante tastes. Thus, while we allow for the presence of *some* uncertainty in the model, we do assume the existence of some underlying phenomenon that makes the ex ante preferences relevant for the ex post stage. We will study the behavioral implications of both a finite and an infinite state space S.[9] Finally, note that the representation in (1) is a particular case of the representation in (4) with S being any singleton set.

Most papers in the reference-dependent preferences literature use as the reference point the actual or expected value of the endowment or of the status quo.[10] Then they usually assume a standard set of preferences conditional on each possible value of the reference point and characterize the complete set of preferences by using various assumptions on how the conditional preferences depend on the reference point. The paper which is the closest to our model is Neilson (2006) which axiomatizes an additive representation over bundles whose first component acts as a reference point against which *all* the rest of the components of the bundle are evaluated. In contrast, in our model the reference point is endogenous and the value of the bundle is determined only by those options that are the most preferred. A recent paper by Ok, Ortoleva and Riella (2007) identifies the departures from standard rationality that we study with the existence of a subset of highlighted alternatives out of which the decision maker chooses his most preferred one. This set of highlighted alternatives is basically the counterpart of the most preferred menu in our paper so the two papers are in some sense complementary. More exactly, they analyze conditions on a standard choice correspondence from sets that guarantee that the departure from rationality as suggested by the weak axiom of revealed preference is due to the existence of a most preferred category, when it is

In this second case it is clear that at the moment when he makes the initial decision to choose Netflix, he would consider all of his possible ex post preferences and thus introducing uncertainty in a model that attempts to capture his behavior is necessary. Moreover, it is only natural to assume that the uncertainty is subjective in this example.

[9] An infinite state space appears for instance in models in which the individual has a continuous distribuition of the ex post tastes over the characteristics of the available options.

[10] See for instance Koszegi and Rabin (2004), Masatlioglu and Ok (2005) or Sagi (2006).

a priori known that this most preferred category and in particular the relevant reference point is always unique. Our paper considers the case of objectively given categories by using a non standard primitive choice correspondence and presents a theory of *how* those possibly multiple most preferred categories are selected.

Preferences over menus were considered for the first time by Kreps(1979). He identified an act with the choice of a set of future options out of which at a later stage the decision maker chooses his most preferred element. He interpreted the agent's preference for the flexibility offered by the menu as being generated by some underlying *subjective* uncertainty that will be resolved between the moment when the choice of the menu is made and the moment when the choice from the menu is made. This allowed him to show that under sufficiently weak conditions, the decision maker behaves as if the uncertainty were described by a subjective state space, where each state is identified with an ex post subjective utility. The preferences that we model in this paper belong to the class of preferences modeled by Dekel, Lipman and Rustichini (2001), henceforth DLR(2001), in which they considered menus of lotteries instead of deterministic bundles so that they could restrict the ex post state utilities to be of the expected utility form. This allowed them to address the main problem that the Kreps representation had, that is the nonuniqueness of the subjective state space. Also, unlike Kreps(1979) they allowed for subjective states of negative measure to capture not only a preference for flexibility but also a preference for commitment. In addition, following Kreps(1992) they interpreted the resulting model as being also a model of unforeseen objective contingencies. There is a large body of literature that built on the class of preferences introduced by DLR(2001). Gul and Pesendorfer(2001) were the first to give meaning to the abstract subjective states derived in earlier papers. Thus, they imposed conditions on preferences such that the resulting state space consists of one state of negative measure representing a temptation preference and one state of positive measure representing the second period preferences which combine a normative preference and the temptation preference. This combination of normative and temptation preferences has been implemented in the meantime in other papers to model various behavioral biases, such as non-bayesian updating, cognitive dissonance, etc. For an example, see for instance Epstein and Kopylov (2006). In another direction, the preference for commitment has been interpreted in Sarver(2005) not as being driven by the presence of temptation but by the anticipation of regret.

The rest of the paper is organized as follows. In section 2 we present the assumptions that are common to most of the papers using a Dekel, Lipman and Rustichini (2001) framework for the representation over menus. This includes the standard axioms that deliver their representation. Also, in Section 2 we compare our representation with other representations that built on that framework to underline the differences they impose on the subjective state delivered by Dekel, Lipman and Rustichini (2001). In section 3 we present our additional axioms and the two main results which state the equivalence between the axioms and the two representations. We break this section into two parts, corresponding to the cases in which we do not assume or we do assume the presence of uncertainty. For the basic model, we will also present the definition of the induced preference relation in terms of the choice correspondence and the axiomatic foundations of the representation of the choice correspondence. In addition, we will also study here the uniqueness of the representations and analyze comparative strengths of the context effects bias. In section 4 we present some applications of the preferences studied in this paper to principal-agent models and auctions, while section 5 concludes the paper. Most proofs are relegated to the Appendix.

2 The Framework and the Representations

Let Z be a possibly infinite compact metric space of *outcomes* or prizes and let $\Delta(Z)$ denote the set of probability measures on Z endowed with the topology of convergence in distribution. Let $K(\Delta(Z))$ denote the collection of all nonempty closed subsets of $\Delta(Z)$. Endowing $K(\Delta(Z))$ with the Hausdorff topology[11] we make it a compact metric space. Elements of $\Delta(Z)$ will be called *lotteries* and will be denoted by x, y, z, etc., while the typical elements $K(\Delta(Z))$ will be called *menus* and will be denoted by A, B, C, etc. Also, when convenient we will refer to a certain menu by explicitly mentioning the elements of that menu, such as $\{x\}$ or $\{x, y, z\}$. The decision maker is assumed to have a *induced or revealed* preference relation $\succeq$ over the elements in $K(\Delta(Z))$.[12] For any two menus $A, B \in K(\Delta(Z))$ and any $\alpha \in [0,1]$, define their convex combination as $\alpha A + (1-\alpha)B \equiv \{z \in \Delta(Z) : z = \alpha x + (1-\alpha)y$, for some $x \in A$ and $y \in B\}$. Finally, denote by $B(\Delta(Z))$, the set of binary menus of elements from $\Delta(Z)$.

We will impose throughout the paper a subset of the following standard axioms on the preference. The weaker version of the Independence axiom called Binary Independence is sufficient for the reference-dependent representation without uncertainty. This was noted for the first time by Kopylov(2005) for a representation of temptation-driven preferences over menus.

Axiom 1 ***(Weak Order).*** *$\succeq$ is a complete and transitive binary relation.*

Axiom 2 ***(Continuity).*** *For any $A \in K(\Delta(Z))$, the upper and lower contour sets, $\{B \in$*

[11] In this case, this is the topology generated by the Hausdorff metric. For a given compact metric space (Z, d), the Haussdorff metric is defined by the distance $d_h(A, B) = \inf\{\varepsilon > 0 : B \subset O_\varepsilon(A) \text{ and } A \subset O_\varepsilon(B)\}$ for all $A, B \in K(\Delta(Z))$, where $O_\varepsilon(A) = \{z \in Z : d(z, A) < \varepsilon\}$ denotes the $\varepsilon-$neighbourhood of the set A in the metric d. When Z is finite the metric d used on $\Delta(Z)$ is the usual Euclidean metric and the Hausdorff metric becomes: $d_h(A, B) = \max\left\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{x\in B}\min_{y\in A} d(x, y)\right\}$.

[12] For the sake of maintaining a coherent exposition for the two models we study, we will defer the presentation of the definition of the induced preference relation in terms of the primitive choice correspondence to Section 3.1.1.

$K(\Delta(Z)) : B \succeq A\}$ and $\{B \in K(\Delta(Z)) : B \preceq A\}$, are closed in the Hausdorff metric topology.

Axiom 3 ***(Independence).*** *For all $A, B, C \in K(\Delta(Z))$ and any $\alpha \in (0,1)$, $A \succ B$ implies $\alpha A + (1-\alpha)C \succ \alpha B + (1-\alpha)C$.*

Axiom 4 ***(Binary Independence).*** *For all $A, B, C \in B(\Delta(Z))$ and any $\alpha \in (0,1)$, $A \succ B$ implies $\alpha A + (1-\alpha)C \succ \alpha B + (1-\alpha)C$.*

For a detailed interpretation and motivation of the first three axioms for the case of a revealed preference relation over menus, see DLR(2001). For the basic model, the interpretation of the Weak Order and Continuity axioms is clear. The motivation for Binary Independence that we will provide here is just a very slight adaptation of the argument offered by Gul and Pesendorfer (2001). If we denote by $\alpha \circ A + (1-\alpha)C$ the lottery that would yield the menu A with probability α and the menu C with probability $1-\alpha$, then the standard motivation for the independence axiom suggests that $A \succ B$ should imply $\alpha \circ A + (1-\alpha)C \succ \alpha \circ B + (1-\alpha)C$. Note now that if the uncertainty regarding A and C is resolved *before* the choice is made, then the reference point bias affects the choice within the menus A and C. On the other hand, if this uncertainty is resolved *after*, then the bias affects the choice within the abstract menu $\alpha A + (1-\alpha)C$. [13] If the decision maker is indifferent as to the timing of the resolution of this uncertainty, $\alpha \circ A + (1-\alpha)C \succ \alpha \circ B + (1-\alpha)C$ should imply that $\alpha A + (1-\alpha)C \succ \alpha B + (1-\alpha)C$ as stated by the axiom. Thus, the Binary Independence is a combination of the usual independence axiom applied to binary menus and the indifference with respect to the timing of the resolution of uncertainty.[14]

The representations in (1) and (4) that we will axiomatize in this paper are particular cases of the *additive expected utility representation* as defined and axiomatized for the first time by DLR(2001). We will also frequently refer to it throughout as the *DLR representation*.

Definition 5 *An* ***additive expected utility representation*** *of $\succeq$ is a nonempty possibly infinite set S, a state dependent utility function $U : \Delta(Z) \times S \to \mathbb{R}$ and a finitely additive signed Borel measure μ on S, such that $V : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$ by*

$$V(A) = \int_S \left[\max_{z \in A} U(z, s) \right] \mu(ds) \tag{5}$$

[13]Note that if $A = \{x, x_0\}$, $C = \{z, z_0\}$ with $\{x\} \succ \{x_0\}$ and $\{z\} \succ \{z_0\}$, then the best element in the menu $\alpha A + (1-\alpha)C$ is $\alpha x + (1-\alpha)z$ whereas the reference point of that menu is $\alpha x_0 + (1-\alpha)z_0$. This follows from the restriction of our independence axiom to singleton sets, which is the standard independence axiom.

[14]Note that this axiom is one of the behavioral implications of a constant strength of the reference point bias so relaxing this axiom appropriately would be the approach that would result in a reference dependent representation with a variable value of the parameter θ.

is continuous and represents $\succeq$ and each $U(\cdot, s)$ is an expected utility function in that for each $s \in S$ there exists $u_s : Z \to \mathbb{R}$ such that $U(z, s) = z \cdot u_s$.

The *state* in the above representation can be uniquely identified by the corresponding ex post state utility so we will occasionally refer to a state s by naming the corresponding expected utility function $U(\cdot, s)$.

For the case when the state space S is finite the representation can be equivalently written as:

$$V(A) = \sum\nolimits_{s \in S} \left[\max_{z \in A} U(z, s) \right] \mu(s) = \sum\nolimits_{s \in S^+} \left[\max_{z \in A} u_s(z) \right] - \sum\nolimits_{s \in S^-} \left[\max_{z \in A} u_s(z) \right] \tag{6}$$

where $S^+ \equiv \{s \in S : \mu(s) > 0\}$ and $S^- \equiv \{s \in S : \mu(s) < 0\}$ and $u_s(\cdot) \equiv |\mu(s)| U(\cdot, s)$. In writing the above we used the fact that the measure over states and the state utility are not separately identified in models of state-dependent utility, so they can be combined together.

Note that the above definition allows the measure μ over the states to be signed. DLR(2001) call *positive states* and *negative states*, the states in the support of the positively signed and respectively negatively signed components of μ. Intuitively, as stated in DLR(2001), the positive states would reveal the agent's desire for flexibility, while the negative states would reveal his desire for commitment.

In our paper, unlike the other papers using a DLR(2001) framework, the agent will be assumed to not have any kind of commitment issues. Therefore we impose throughout an additional axiom on preferences called Monotonicity, which will basically impose that weakly larger sets in the partial order given by inclusion be weakly preferred by the decision maker.[15] This is a condition consistent with the assumption of the agent not experiencing commitment problems.

Axiom 6 ***(Monotonicity).*** *For all $A, B \in K(\Delta(Z))$ with $A \subseteq B$, we have $B \succeq A$.*

DLR(2001) and Dekel, Lipman, Rustichini and Sarver (2005) prove the following result.

Theorem 7 *When the set Z is finite, the preference $\succeq$ has an additive expected utility representation with a measure μ which is always positive if and only if it satisfies Weak Order, Independence, Continuity, and Monotonicity.*

The effects of imposing Monotonicity on preferences are the following. Firstly, as mentioned above the axiom insures that the measure over the states from the representation is everywhere

[15] The Monotonicity axiom is part of the axiomatization of the preference for flexibility in Kreps(1979).

positive. Secondly, it will allow us to obtain a stronger property of the measure, that is σ-additivity instead of finite additivity as in DLR(2001). Finally, Dekel, Lipman, Rustichini and Sarver (2005) also show that if Monotonicity is not imposed, the Continuity axiom as presented above needs to be strenghtened to an axiom which they call Strong Continuity in order to get the additive expected utility representation with a signed measure. The additional condition on preferences that is needed, called L-Continuity in their paper, basically delivers the Lipschitz continuity of the representation. Here, since we do assume Monotonicity, we may impose the weaker continuity condition given by the Continuity axiom presented above.

We will also consider the case when the state space from the DLR representation is finite. A necessary and sufficient condition to obtain a finite state space with an everywhere positive measure under the axioms from Theorem 2 was found in Dekel, Lipman and Rustichini (2005). The authors call this additional axiom Finiteness.

Axiom 8 ***(Finiteness)*** *Every menu $A \in K(\Delta(Z))$ has a finite critical set, where a critical set of a menu A is a any set A' such that for all B with $A' \subseteq hull(B) \subseteq hull(A)$ we have $B \sim A$.*[16]

Dekel, Lipman and Rustichini (2005) prove the following result.

Theorem 9 *When the set Z is finite, the preference $\succeq$ has an additive expected utility representation with a measure μ which is always positive and with a finite state space S if and only if it satisfies Weak Order, Independence, Continuity, Monotonicity and Finiteness.*

For the basic model without uncertainty we will use the following definition:

Definition 10 *Let Z be a compact metric space.*[17] *A **reference-dependent representation** of the induced preference relation $\succeq$ consists of a constant $\theta \in (0,1)$ and a utility function $u(\cdot)$ of the expected utility form representing the preferences over lotteries in $\Delta(Z)$, such that $V : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$ by*

$$V(A) = \max_{x \in A} u(x) - \theta \min_{y \in A} u(y) \tag{7}$$

represents the preference $\succeq$.

[16] $hull(A) = \{z \in Z : z = \sum_{i=1}^{k} \lambda_i z_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$ and $z_i \in A\}$ denotes the convex hull of a set A.

[17] The proof of the representation result for the model without uncertaity employs the strategy developed by Kopylov(2005) for his model of temptation. Therefore, as he suggests, the representation result can be extended from $\Delta(Z)$ to any compact metric space X endowed with a continuous mixture operation. Thus, for some compact metric space $\Delta(Z)$, the space X can be the product space $(\Delta(Z))^N$ with N finite, or the space of all convex and compacts subsets of $\Delta(Z)$. Also, X can be any convex and compact subset of a metric linear space.

For the representation under uncertainty, the specifics of the representation from DLR(2001) require a number of normalizations. Firstly, as explained in DLR(2001) the state space is just an index set that allows reference to different ex post preferences over the elements of $\Delta(Z)$. Moreover, the ex post state utilities which are assumed to be of the expected utility form are identified only up to affine transformations, so we follow the approach in DLR(2001) and restrict the state space S to the set of normalized utilities

$$S^N \equiv \left\{ s \in \mathbb{R}^N : \sum\nolimits_{k=1}^{N} s^k = 0 \text{ and} \sum\nolimits_{k=1}^{N} (s^k)^2 = 1 \right\}. \tag{8}$$

Throughout the rest of the paper we will use $s \in S^N$ to refer both to a second period contingency as well as to the normalized expected utility function representing the preferences in that state. Thus, the utility of a lottery $x \in \Delta(Z)$ in state s will be $U(x,s) = x \cdot s = \sum\nolimits_{k=1}^{N} x^k s^k$ where $s = (s^1, ..., s^N) \in S^N$ is the normalized expected utility function that represents the state s preferences. Note now that the restrictions of the Weak Order, Continuity and Independence axioms to $\Delta(Z)$ imply by standard results the existence of an expected utility function $v(\cdot)$ that represents the restriction of $\succeq$ to $\Delta(Z)$. Sarver(2005) shows that since S^N contains the normalization of any affine function on $\Delta(Z)$, there exists $s_* \in S^N$ and $\lambda \geq 0$ such that $v(x) = \lambda \sum\nolimits_{k=1}^{N} x^k s_*^k$ for all $x \in \Delta(Z)$.

We will define now formally a normalized representation of the preferences exhibiting the reference point bias. To that end, we firstly denote the ball of radius ε around s where $\varepsilon > 0$ and $s \in S$ by $N_\varepsilon(s) \equiv \{s' \in S : d(s', s) < \varepsilon\}$ where $d(\cdot, \cdot)$ is the usual Euclidean metric in $\mathbb{R}^N$.

Definition 11 *Let Z be any finite set. A* ***normalized reference-dependent representation under uncertainty*** *of $\succeq$ consists of a nonempty possibly infinite measurable set $S \subset S^N$, a Borel measure μ on S^N, with S being the unique support*[18] *of μ and a constant $\theta \in (0,1)$, such that*

(i) $V : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$ by

$$V(A) = \int_S \left[\max_{z \in A} (z \cdot s) \right] \mu(ds) - \theta \min_{x \in A} \left[\int_S (x \cdot s) \mu(ds) \right] \tag{9}$$

represents the preference $\succeq$;

(ii) the utility of a lottery $x \in \Delta(Z)$ in state $s \in S$ is $x \cdot s$;

[18] The support of a Borel σ-additive measure μ, if it exists, is a closed set, denoted S, satisfying: *(1)* $\mu(S^c) = 0$; and *(2)* If G is open and $G \cap S \neq \varnothing$, then $\mu(G \cap S) > 0$. Theorem 10.13 in Aliprantis and Border (1999) shows that if the underlying topological space on which μ is defined is second countable or if μ is tight, then μ has a (unique) support. In our case, S^N is clearly second countable, so the definition is correct.

(iii) if $s_* \in S^N$ is the normalized utility that represents the restriction of $\succeq$ to $\Delta(Z)$ then there exists $\varepsilon > 0$ such that $S \subset S^N \backslash N_\varepsilon(-s_*)$.[19]

We emphasize here that besides S and μ which are usual elements in a normalized DLR representation, the parameter θ and the restriction *(iii)* on the set of ex post utility functions will also will be deduced from preference as a part of the representation.

Note that the functional form in (9) for $V(\cdot)$ can be rewritten as:

$$V(A) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu(ds) + \max_{x\in A}\theta \left[-\int_S (x \cdot s)\mu(ds)\right], \tag{10}$$

and thus our representation is indeed a particular form of an additive expected utility representation with all states having associated a positive measure.

As mentioned in the Introduction, the condition *(iii)* on the set of ex post utilities will allow us to identify the behavioral bias modeled by our representation as being a reference point bias. Note that (9) implies that the ex ante preferences over singletons are represented by the utility function $v(x) \equiv (1-\theta)\int_S (x \cdot s)\mu(ds)$. Thus, by inspecting (10) it is clear that the preferences represented by $v^-(x) \equiv -\int_S (x \cdot s)\mu(ds)$ could constitute just another ex post state in a DLR(2001) framework with the property that these ex post preferences are exactly the reverse of the ex ante preferences over singletons. We rule out this possibility by making the arguably reasonable assumption that in the second period after the uncertainty is resolved the decision maker cannot reverse *or almost* reverse his ex ante tastes. Thus, as already mentioned in the Introduction, while we allow for the existence of *some* uncertainty in the model, we still assume *some* consistency between the ex ante and the ex post preferences over the singletons in $\Delta(Z)$. This consistency is imposed by the identification of the term $\max_{x\in A} \theta \left[-\int_S (x \cdot s)\mu(ds)\right]$ from the equivalent representation in (10) with the impact of a reference-point bias and by the condition *(iii)* from Definition 11 which excludes ex post preferences exactly or almost opposite to the ex ante tastes.

In the remaining of this Section we will present what differentiates this representation from other DLR type representations, more specifically the representations from Gul and Pesendorfer (2001), Sarver (2005) or Dekel, Lipman and Rustichini (2005). This part can be skipped without any loss of understanding of the model.

The thing that differentiates various models constructed in a DLR(2001) framework is the particular structure imposed on the ex post states. For instance, in our model there exist no negative states and one positive state of strictly positive measure having the corresponding utility a negative affine transformation of the utilities of the rest of the states.

[19] When the state space is finite, condition *(iii)* can be written as $-s_* \notin S$. Also, note that $-s_* \in S^N$.

In Gul and Pesendorfer (2001) the equivalent DLR representation is the following:

$$V(A) = \max_{x \in A} u_1(x) - \max_{y \in A} u_2(y) \tag{11}$$

where u_1 is the utility that represents the second period preference relation and u_2 is the temptation component of these second period preferences. Therefore, in this representation there is one positive state and one negative state with no particular mathematical relation between them.

In Sarver (2005) the equivalent DLR representation of his *regret representation* is:

$$V(A) = \max_{z \in A} \left[(1+K) \int_S U(z,s)\mu(ds) \right] - \int_S \left[\max_{x \in A} KU(x,s) \right] \mu(ds) \tag{12}$$

where $K \geq 0$. Thus, in this case there is a number, possibly infinite of negative states and one positive state whose corresponding state utility is a positive affine transformation of the utilities corresponding to the negative states.[20]

The equivalent representation from Dekel, Lipman and Rustichini (2005) of what they call the *temptation representation* is:

$$V(A) = \sum\nolimits_{s \in S} \left[\max_{z \in A} U(z,s) \right] \mu(s) - \sum\nolimits_{s \in S} \left\{ \sum\nolimits_{j \in J_s} \left[\max_{y \in A} \overline{U}(y,j) \right] \right\} \mu(s) \tag{13}$$

which is a generalization of the one from Gul and Pesendorfer(2001) in the sense that it assumes multiple ex post states and for each ex post state multiple ex post temptations. This representation has a number of positive states and for each positive state a number of corresponding negative states with some underlying structure among them. Unlike the other representations, in Dekel, Lipman and Rustichini (2005) the state space is assumed to be finite.

As a last remark for this section, we emphasize that the actual representation from each of the papers mentioned above is different from that presented here as being its equivalent DLR representation in order to capture the corresponding behavioral trait that is analyzed in each paper.

3 The Axioms and the Main Results

As noted before, the representations in (7) and (9) are special types of an additive expected utility representation with an everywhere positive measure. Thus, it will be necessary that the preference

[20] The regret representation as defined in Sarver (2005) has the Borel measure μ positive. However, as mentioned in that paper as well, the equivalent DLR representation is signed and has a negative component.

satisfy Weak Order, Continuity, (Binary) Independence[21] and Monotonicity. Two additional axioms will be sufficient for the preference $\succeq$ to have a reference-dependent representation when combined with the above four axioms. The first additional axiom is common to both the case without and with uncertainty. This is basically the axiom that captures the departure from the standard model of rationality that we study in this paper.

Axiom 12 ***(CEB: Context-Effects Bias) :*** *For any pair $(A, x) \in K(\Delta(Z)) \times \Delta(Z)$, such that $\{y\} \succ \{x\}$ for all $y \in A$, we have $A \cup \{x\} \succ A$.*[22]

What Axiom CEB says is that if the decision maker has the set of possible choices A expanded by adding an option, say a singleton $\{x\}$, which from an ex ante point of view is strictly worse that the rest of the elements in the menu, then the agent will strictly prefer the new expanded set $A \cup \{x\}$ to the initial one A. The reason for which the agent will strictly prefer the menu containing the additional option is because the inferior lottery x will be chosen as the reference point and thus the overall attractiveness of the menu will increase. Note that because the preferences in the text of the axiom are strict, Axiom CEB imposes that the agent has a *strict* preference for having *additional* strictly inferior outcomes in the menu of choice. This corresponds to the restriction that $\theta > 0$ in the representation in (7) and (9).

3.1 The basic model

Since as mentioned earlier, menus are the natural objects of choice that reveal the decision maker's preference for flexibility, in order to ensure that preferences over menus do *not* in fact capture also some preference for flexibility we will impose the following axiom:

Axiom 13 ***(NDF: No Desire for Ex Post Flexibility) :*** *For any finite set $A \in K(\Delta(Z))$ and any $x \in \Delta(Z)$ such that $\{x\} \succ \{y\}$ for some $y \in A$, we have that $A \cup \{x\} \succ A$ if and only if $\{x\} \succ \{y\}$ for all $y \in A$.*

Thus, when an additional option x improves the menu A, since $\{x\} \succ \{y\}$ for *some* $y \in A$ implies that x does *not* act as the reference point for the menu $A \cup \{x\}$, it must be that x will be the new option selected from this set. Therefore it must be that $\{x\} \succ \{y\}$ for *all* $y \in A$.

[21] As mentioned above, for the representation without uncertainty the Binary Independence axiom is sufficient.

[22] For the model without uncertainty a sufficient version of Axiom CEB is the folowing. For any $x, y, z \in \Delta(Z)$ with $\{x\} \succeq \{y\} \succ \{z\}$ we have $\{x, y, z\} \succ \{x, y\}$. Also, under Monotonicity a suficient version of Axiom CEB is: $\{x\} \succeq \{y\} \succ \{z\}$ implies $\{x, z\} \succ \{x, y\}$.

Conversely, when an option that is strictly preferred to all options in the menu is added, then the new menu will be strictly preferred. Note that when the individual exhibits some preference for flexibility, this restriction on preferences does not necessarily hold. This is because option x, even though ex ante inferior to some option in A, might turn out to be strictly preferred to all options in A in some ex post contingency so that the preference $A \cup \{x\} \succ A$ would follow.

The following is the main representation result for the basic model without uncertainty:

Theorem 14 *The induced preference $\succeq$ has a representation as in (7) if and only if it satisfies Weak Order, Continuity, Binary Independence, Monotonicity, Axiom CEB and Axiom NDF.*

Proof. See Appendix A1.

Corollary 15 *Suppose that (u_1, θ_1) and is a representation of some preferences $\succeq$ satisfying Weak Order, Continuity, Binary Independence, Monotonicity, Axiom CEB and Axiom NDF. Then, if (u_2, θ_2) is also a representation of $\succeq$ and there exists $x_0, y_0 \in \Delta(Z)$ such that $x_0 \succ y_0$, we must have $\theta_1 = \theta_2$ and $u_2 = \alpha u_1 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.*

Proof. See Appendix A2.

In order to analyze the comparative strength of the bias we will restrict attention to preferences that have the same ranking over the lotteries.

Definition 16 *We say that the preference $\succeq_1$ exhibits a stronger context effects bias than $\succeq_2$ if whenever $A \succ_2 B$ for some $A, B \in K(\Delta(Z))$ it follows that $A \succ_1 B$.*

To motivate the above definition, note that $A \succ_2 B$ implies one of the following two cases. Firstly, it may be that the most preferred option in A is better than all elements in B, or that the reference point of A is superior to the reference point of B or a combination of the two. Secondly, it may happen that the most preferred option in A is worse than some option in B from a normative point of view, but that B has a reference point sufficiently better than that of A that the context effects bias reverses this normative ranking and makes the some option in A appear more attractive than all elements in B. Now, note that in the first case since $\succeq_1$ has the same ranking over lotteries as $\succeq_2$ it should follow that $A \succ_1 B$. In the second case, since $\succeq_1$ is supposed to have a stronger context effects bias than $\succeq_2$, the compensation should occur in this case as well and thus it should again be that $A \succ_1 B$.

Theorem 17 *Suppose that* (u_1, θ_1) *and* (u_2, θ_2) *are the reference-dependent representations of two preferences* $\succeq_1$ *and* $\succeq_2$ *such that there exist* $x_0, y_0 \in \Delta(Z)$ *with* $x_0 \succ_2 y_0$*. Then,* $\succeq_1$ *exhibits a stronger context effects bias than* $\succeq_2$ *if and only if* $\theta_1 \geq \theta_2$ *and* $u_2 = \alpha u_1 + \beta$ *for some* $\alpha > 0$ *and* $\beta \in \mathbb{R}$.[23] *Moreover, if there exist some sets* $A, B \in K(\Delta(Z))$ *with* $A \sim_1 B$ *and* $A \nsim_2 B$*, then* $\theta_1 > \theta_2$*.*

Proof. See Appendix A3.

3.1.1 Formal definition of the induced preference relation for the basic model

As stated in the Introduction, in the case without uncertainty the observed behavior does not consist of choices of menus but of options from the various menus available. Therefore, we will define here formally the preference relation over categories using as a primitive only the following choice correspondence. As a reminder, for any compact sets $A_1, ..., A_n$, we denoted the choice correspondence $c(A_1, ..., A_n)$ as being the *nonempty* subset of $A = A_1 \cup ... \cup A_n$, which are the chosen elements when the decision maker has available the options in A, grouped into menus $A_1, ..., A_n$.

Lemma 18 *For any sets* $A, B \in K(\Delta(Z))$*, there exist two sequences* A_n, B_n *in* $K(\Delta(Z))$ *with* $A_n \cap B_n = \varnothing$ *for each* n*, such that* $A_n \to A$ *and* $B_n \to B$ *in the Hausdorff metric.*

Proof. See Appendix A4.

For any $A, B \in K(\Delta(Z))$, define $\mathcal{M}_{A,B} \equiv \{(A_n, B_n) \in K(\Delta(Z)) \times K(\Delta(Z)) : A_n \to A$ and $B_n \to B$ in the Hausdorff metric and $A_n \cap B_n = \varnothing$ for all $n\}$ and note that by Lemma 18, $\mathcal{M}_{A,B}$ is nonempty. We will define now the preference relation over categories for the basic model without uncertainty. Part *(i)* is the natural definition for disjoint sets, while part *(ii)* extends the binary relation to the rest of $K(\Delta(Z)) \times K(\Delta(Z))$ by continuity.

Definition 19 *(i) When* $A \cap B = \varnothing$*, we say that* $A \succeq B$ *if and only if* $c(A, B) \cap A \neq \varnothing$*.*

(ii) When $A \cap B \neq \varnothing$*, we say that* $A \succ B$ *if and only if for all sequences* $A_n, B_n \in \mathcal{M}_{A,B}$*, we have* $A_n \succ B_n$ *for all sufficiently high* n*. When* $A \cap B \neq \varnothing$ *and the previous does not imply either* $A \succ B$ *or* $B \succ A$*, we say that* $A \sim B$*.*

[23] As it is the case with all the other results for the model without uncertainty, it is sufficient to restrict the requirement in Definition 16 to hold only for binary menus for the result of this Theorem to hold.

The definition above ensures that the induced preference relation is complete. Secondly, the specification of $\succeq$ in Definition 19 has the following important merit. Thus, note firstly that, as presented in (2), the story underlying our representation suggests that the choice correspondence $c(A_1, ..., A_n)$ is the union of the sets of maximizers of the utility function u over those sets $A_i \in \{A_1, ..., A_n\}$ that maximize the function V. On the other hand, the preference $\succeq$ over sets in $K(\Delta(Z))$ is represented by the function V as defined in (7). Now, with the help of the Lemma 18 presented above, it is straightforward to show that the choice correspondence and the preference relation as suggested by our representation satisfy the *consistency conditions* required by Definition 19.[24] Finally, we mention that while all of our axioms could also be defined in terms of the primitive choice correspondence, the definition of the axioms that we employed throughout, that is the one that uses the induced preference relation over menus, has the advantage of being more transparent by avoiding the various technicalities brought about especially by comparisons of nondisjoint sets.

3.1.2 Axiomatic Foundation for the Representation of the Choice Correspondence

We close the presentation of the basic model with two conditions on the primitive choice correspondence that together with some relatively straightforward arguments, would justify the interpretation of the choices within menus and across menus as suggested by the representation of the choice correspondence in (2) and (3). The first of these two conditions is the formal definition of our underlying assumption that the choices within menus are in line with the standard models of rationality.

Axiom 20 ***(SRWM: Standard Rationality Within Menus)*** *For any $A \in K(\Delta(Z))$, $x \in c(A)$ if and only if $x \in c(\{x\}, \{y\})$ for all $y \in A$.*

To interpret the axiom, note that since all options in A are evaluated with respect to the same reference point and we assume away any other behavioral bias affecting choices within menus, $x \in c(A)$ if and only if x is weakly preferred to all the other options in A. But whenever x is weakly preferred to y, we should have $x \in c(\{x\}, \{y\})$ because comparison between singleton menus is not affected by the reference point bias. Therefore, we require the equivalence from the text of Axiom SRWM. Next, we will define the set $c^*(A, u)$ as the subset of A consisting of the most preferred options according to utility function u and then we will show that under Axiom SRWM, this set coincides with $c(A)$, when u is chosen to be the utility function from our reference dependent representation of preference relation induced by the choice correspondence c.

Definition 21 *For any utility function u, define $c^*(A, u) \equiv \{x \in A : u(x) \geq u(y)$ for all $y \in A\}$.*

[24] It is clear that there exist other specifications that would formally define the preference relation over categories starting from the choice correspondenc in a manner that is consistent with our representation.

Remark 22 *Suppose that the choice correspondence* $c(\cdot)$ *satisfies Axiom SRWM and that the induced preference relation according to Definition 19 has a representation as in (7) with utility function* $u(\cdot)$*. Then, for any* $A \in K(\Delta(Z))$*, we have* $c(A) = c^*(A, u)$*.*

Proof. By Axiom SRWM, $x \in c(A)$ if and only if $x \in c(\{x\}, \{y\})$ for all $y \in A$. Employing the Definition 19, $x \in c(\{x\}, \{y\})$ is written in terms of the induced preference relation as $\{x\} \succeq \{y\}$. In turn, by the representation in (7), we have $\{x\} \succeq \{y\}$ if and only if $u(x) \geq u(y)$. Thus, given the definition of $c^*(A, u)$, we conclude as desired that $x \in c(A)$ if and only if $x \in c^*(A, u)$. ■

The second condition relates the choices across menus with the choices within menus:

Axiom 23 ***(CAM: Choices Across Menus)*** *For any sets* $A_1, ..., A_n$ *in* $K(\Delta(Z))$*, we have:*

$$c(A_1, ..., A_n) = \bigcup_{\{A_i : A_i \succeq A_j \text{ for all } j \in \{1,...,n\}\}} c(A_i)$$

To see the motivation for this axiom, assume for simplicity that $A_1, ..., A_n$ are disjoint. Note then that if $x \in c(A_1, ..., A_n)$, there must exist some $i \in \{1, ..., n\}$ with $x \in A_i$, such that x appears when presented in the context given by the menu A_i, at least as attractive as any other element in $A_1 \cup ... \cup A_n$, presented in their respective contexts. But then, this comparison between x and the rest of the elements in $A_1 \cup ... \cup A_n$ should also work when considering pairwise menus, so given the definition of the preference relation, it must be that $A_i \succeq A_j$ for all $j \in \{1, ..., n\}$. In addition, since the preferences within menus are not affected by any behavioral bias, x should belong to $c(A_i)$. The converse argument is very similar. We mention that while we use the induced preference relation $\succeq$ in the definition of the axiom, this is ultimately a condition on the choice correspondence. Now, by combining the results of Theorem 14 and Remark 22, a straightforward argument would show that Axiom CAM and Axiom SRWM together with the other axioms that guarantee the representation in (7), constitute the axiomatic foundation of the representation of the choice correspondence as presented in (1), (2) and (3). This result is presented without proof in the Theorem 24 below.

Theorem 24 *The choice correspondence* $c(\cdot)$ *has a representation as suggested by (1), (2) and (3) if and only if it satisfies Axiom CAM, Axiom SRWM, Weak Order, Continuity, Binary Independence, Monotonicity, Axiom CEB and Axiom NDF.*

3.2 The model with uncertainty

Axiom CEB presented above, will provide the departure from the standard rational preferences as suggested by the presence of a behavioral bias. However, when allowing for an *infinite* state space this departure is provided by Axiom CEB only when combined with the Axiom CEB-2 presented below. This is because, as suggested earlier, when allowing for the presence of uncertainty, it may happen that an ex ante inferior option, may still provide some ex post flexibility to the elements of a set and thus the pattern of choice suggested by Axiom CEB would be valid even without assuming any reference point bias. In order to have a departure from the standard rational preferences, we need to assume the existence of at least one set A and at least one lottery y strictly worse from an ex ante point of view to all elements of A such that in any ex post state there exists an element in A that is at least as preferred as y. Then, imposing Axiom CEB to the sets A and $A \cup \{y\}$ would provide the departure. Now, in the case of a finite state space, the pair (A, y) with the desired properties clearly exists when we maintain the assumption that the second period preferences cannot be exactly the reversed ex ante preferences.[25] In the case of an infinite state space, the existence of such a pair (A, y) will be imposed by Axiom CEB-2 below. But before presenting Axiom CEB-2 we will make a remark that suggests that imposing axiomatically the existence of such a pair is correct when the preferences that we are studying are represented by a utility function as in (9).

Remark 25 *When the preferences $\succeq$ admit a normalized reference-dependent representation as in (9), there exist a set $A \in K(\Delta(Z))$ with $A \subset int(\Delta(Z))$ and a lottery $y \in \Delta(Z)$ such that: (i) for any $x \in A$ we have $x \cdot s_* > y \cdot s_*$ and (ii) for any $s \in S$ there exists $x \in A$ such that $x \cdot s > y \cdot s$.*

Proof. See Appendix B1 for some notation on support functionals and then Lemma 48.

The second non standard axiom for the case under uncertainty with an infinite subjective state space is the following.

Axiom 26 ***(CEB-2):*** *There exists a set $A \in K(\Delta(Z))$ with $A \subset int(\Delta(Z))$ and a set $B \in K(\Delta(Z))$ with $A \subset B$ and $\{x\} \succ \{y\}$ for all $x \in A$ and some $y \in B$, such that for all lotteries $z \in \Delta(Z)$ with $\{y\} \succeq \{z\}$ for all $y \in B$, we have $B \cup \{z\} \sim A \cup \{z\}$.*

The condition on preferences that is imposed by Axiom CEB-2 is motivated by the fact that each set has a unique reference point whose choice is restricted to be made ex ante. To see this, note that if A and B are such that $B = A \cup \{y\}$ and y does not provide any ex post flexibility to A, then the indifference $B \cup \{z\} \sim A \cup \{z\}$ is required to hold for *all* $z \in \Delta(Z)$ with $\{y\} \succeq \{z\}$. The motivation for this requirement is the following. On the one hand, by choice of y, it provides no ex post flexibility to the set $A \cup \{z\}$. On the other, since y is weakly preferred to z from an ex ante point of view, y will not be the reference point chosen from $A \cup \{y, z\}$. Since under no

[25] See the necessity part of the proof of Theorem 28 below for a formal argument.

circumstances the agent would choose y over the elements in $A\cup\{z\}$, he is as well off having at hand the menu $A\cup\{z\}$ as he is having the larger menu $A\cup\{y,z\}$. Therefore, we impose on preferences the required indifference.

We mention here that while the condition from the text of the axiom is valid for an infinite number of sets in $K(\Delta(Z))$, we do not impose this condition to hold for *all* these sets simply because there may exist $A\in K(\Delta(Z))$ such that there exists no lottery $y\notin A$ that does not provide any ex post flexibility to A. Also, we do not impose the indifference of the sets constructed as in the text of Axiom CEB-2, but for *all* B that contain a lottery y such that y is strictly less preferred to all elements of A from the ex ante point of view. This is because a lottery which is ex ante inferior to all $x\in A$ could turn out ex post to be better to all elements of A and then we should have $B\cup\{z\}\succ A\cup\{z\}$. The weak restriction imposed in the text of the Axiom that the condition is valid for at least one pair (A,B) is sufficient to obtain the desired representation for *all* sets due to the additional structure provided by the EU form of the ex post utilities.[26]

Now we are ready to state our main result for the model with uncertainty, which gives us the axiomatization of the representation in (9) for the set of preferences characterized by the above axioms:

Theorem 27 *The preference $\succeq$ has a representation as in (9) if and only if it satisfies Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2.*

Proof. See Appendix B1 and Appendix B2.

While the complete proof of Theorem 27 can be found in the Appendix, we will present here for intuition a sketch of the proof under the assumption that the state space S is *finite*. This will show how Axiom CEB and Axiom CEB-2 work to give us the desired representation and thus, the argument will constitute a rough roadmap for the proof of the representation for the general infinite state spaces.

Weak Order, Continuity, Independence and Monotonicity imply that the preference over menus has the following representation, with μ a positive measure:

$$V(A)=\sum\nolimits_{s\in S}\left[\max_{z\in A}(z\cdot s)\right]\mu(s),\text{ for all }A\in K(\Delta(Z))\tag{14}$$

[26]Note also that we impose that an element of B be strictly worse than all elements of A. Without this condition, it is clear that the axiom would not have any bite since we could always let B be exactly the set A. The additional condition that B is a superset of A is meant only to simplify the notation in the proof of the main theorem when we characterize a menu by the corresponding support functional.

Let us denote by $v(\cdot)$ the restriction of $V(\cdot)$ to singletons. Thus, v represents the ex ante preference over lotteries and using the representation in (14) we have: $v(z) = \sum_{s \in S}(z \cdot s)\mu(s)$, for all $z \in \Delta(Z)$. Moreover, as claimed in Section 2 since v is an affine function, there exists $s_* \in S^N$, such that $v(z) = \lambda(z \cdot s_*)$ for all $z \in \Delta(Z)$.

Now, note that as we showed above, the representation in (9) is a particular case of a DLR representation in which the utility associated with one of the states is a negative affine transformation of the utilities associated with the rest of the states. We will prove here that under Axiom CEB and Axiom CEB-2, the representation in (14) must have exactly that structure on the ex post states, which basically comes down to showing that $-s_* \in S$. The rest of the proof consists of showing that given that structure, the representation can be written as in (9). This second part of the proof is just simple algebra manipulations and its presentation is relegated to the Appendix where the proof is presented for the general case of infinite state spaces.

Let $A \in int(K(\Delta(Z)))$ and $B \in K(\Delta(Z))$ be as in the definition of Axiom CEB-2, that is $A \subset B$ and $\{x\} \succ \{y\}$ for all $x \in A$ and some $y \in B$. Then by Axiom CEB we must have $B \succ A$ so using the representation in (14) it follows that

$$\sum_{s \in S} \left[\max_{x \in B}(x \cdot s)\right]\mu(s) > \sum_{s \in S} \left[\max_{x \in A}(x \cdot s)\right]\mu(s).$$

Since by Monotonicity the measure μ is positive this implies that there must exist $s' \in S$ such that $\max_{x \in B}(x \cdot s') > \max_{x \in A}(x \cdot s')$. Denote the strict lower contour sets associated with an expected utility function s and a lottery $y \in \Delta(Z)$ by $L_s(y) \equiv \{x \in \Delta(Z) : x \cdot s < y \cdot s\}$. Then, a state utility s will be a negative affine transformation of s_* if and only if $L_s(y) \cap L_{s_*}(y) = \varnothing$ for all $y \in \Delta(Z)$. Assume by contradiction that there is no such state utility as the one that we are looking for, that is $L_s(y) \cap L_{s_*}(y) \neq \varnothing$ for all $s \in S$. We will show that in this case if Axiom CEB holds then Axiom CEB-2 must be violated.

Take $z \in L_{s'}(y) \cap L_{s_*}(y)$ which is nonempty by the contradiction assumption. Then, since $y \in B$ we will have $\max_{x \in B}(x \cdot s') > z \cdot s'$ and then immediately $\max_{x \in B \cup \{z\}}(x \cdot s') = \max_{x \in B}(x \cdot s') > \max\left(\max_{x \in A}(x \cdot s'), z \cdot s'\right) = \max_{x \in A \cup \{z\}}(x \cdot s')$. Therefore:

$$\begin{gathered} V(B \cup \{z\}) = \sum_{s \in S \setminus \{s'\}} \left[\max_{x \in B \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s')\max_{x \in B \cup \{z\}}(x \cdot s') \; > \\ \sum_{s \in S \setminus \{s'\}} \left[\max_{x \in B \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s')\max_{x \in A \cup \{z\}}(x \cdot s') \quad \geq \\ \sum_{s \in S \setminus \{s'\}} \left[\max_{x \in A \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s')\max_{x \in A \cup \{z\}}(x \cdot s') \; = V(A \cup \{z\}). \end{gathered}$$

Therefore, $V(B \cup \{z\}) > V(A \cup \{z\})$ so there exists $z \in L_{s_*}(y)$ such that $B \cup \{z\} \succ A \cup \{z\}$ which violates Axiom CEB-2 as claimed. In conclusion, there must exist a state $s \in S$ that is a negative affine transformation of s_*.

Proving the necessity of Axiom CEB is straightforward. To see that Axiom CEB-2 must also be satisfied when the preferences can be represented by a utility function as in (9), take some lottery $y \in int(\Delta(Z))$ and for each $s \in S$, take $x_s \in H_s(y) \cap L_{-s_*}(y) \cap \Delta(Z)$, where $H_s(y) \equiv \{x \in R^N : x \cdot s = y \cdot s\}$. Let $A \equiv \cup_{s \in S} x_s$ and $B \equiv A \cup \{y\}$. Then, by the choice of the set A, we will have $\{x\} \succ \{y\}$ for all $x \in A$. On the other hand, for any $z \in L_{s_*}(y) \cup H_{s_*}(y)$we will have

$$V(A \cup \{z\}) = \sum_{s \in S} \left[\max_{w \in A \cup \{z\}} (w \cdot s) \right] \mu(ds) - \theta \min_{w \in A \cup \{z\}} \left[\sum_{s \in S} (w \cdot s) \mu(ds) \right] =$$

$$\sum_{s \in S} \left[\max \left(\max_{w \in A} (w \cdot s), z \cdot s \right) \right] \mu(ds) - \theta \left[\sum_{s \in S} (z \cdot s) \mu(ds) \right] =$$

$$\sum_{s \in S} \left[\max \left(\max_{w \in B} (w \cdot s), z \cdot s \right) \right] \mu(ds) - \theta \min_{w \in B \cup \{z\}} \left[\sum_{s \in S} (w \cdot s) \mu(ds) \right] = V(B \cup \{z\}).$$

In the above we used the fact that the restriction of the representation to singletons implies $\{x\} \succ \{y\} \Leftrightarrow (1-\theta) \sum_{s \in S} (x \cdot s) > (1-\theta) \sum_{s \in S} (x \cdot s)$ and the fact that for each $s \in S$, there exists $x_s \in A \cap H_s(y)$ implies $\max_{w \in A \cup \{y\}} (w \cdot s) = \max_{w \in A} (w \cdot s)$.

While in the case of a finite state space, it is sufficient to show that the state $-s_*$ must be one of the states from the DLR representation, for the case of an infinite state space, this is not enough. This is because the state $-s_*$ can always be added to the state space and assign a measure zero, but that would not give us the representation in (9) because we could not exclude the case $\theta = 0$ where the axioms do not necessarily hold because there is no reference-point bias. Thus, in the proof of the sufficiency of the axioms for the infinite state space case, the main challenges are to show that the DLR measure of $-s_*$ is strictly positive and to show the existence of the empty neighborhood of $-s_*$. In addition, the necessity part of the proof of Theorem 27 also needs a rather elaborate approach. This is because the infinite set $\cup_{s \in S} x_s$ with x_s chosen as above is not necessarily closed and thus not necessarily compact, so we need to take $A = cl(\cup_{s \in S} x_s)$. But then the fact that we select $x_s \in L_{-s_*}(y)$ for each $s \in S$ does not necessarily imply $\{x\} \succ \{y\}$ for all $x \in A$ and this invalidates the required conclusion. Part *(iii)* of the Definition 11 will help overcome this problem but the construction is still not straightforward.

As argued above, when the state space is finite the restriction on preferences given by Axiom CEB-2 is not necessary. The following representation theorem deals with this case. To prove the theorem, it is enough to show that the set of axioms that we assume imply Axiom CEB-2. Then the argument from the sketch of the proof of Theorem 27 would complete the argument. Showing that Axiom CEB-2 must be satisfied can be done by following an argument close to the one used above.

Theorem 28 *The preference $\succeq$ has a representation as in (9) with a finite state space if and only if it satisfies Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Finiteness.*

3.2.1 Uniqueness of the representation for the model with uncertainty

Proving uniqueness of the representation is essential because it allows the interpretation the objects of the representation as intended. Most importantly in our case, the fact that the parameters of the representation are identified ensures that when observing choice it is feasible to disentangle the impact on behavior of the context effects from the impact of the presence of subjective uncertainty.

Our representation in (9) is identified by the elements of the set (μ, θ) where μ is a probability over the S^N and θ measures the strength of the behavioral bias. The following theorem shows that both μ and θ are identified from preferences.

Theorem 29 *Suppose that (μ_1, θ_1) is a normalized representation of some preferences $\succeq$ satisfying Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2. Then, if (μ_2, θ_2) is also a normalized representation of $\succeq$ we must have $\theta_1 = \theta_2$ and $\mu_1 = \mu_2$.*

Proof. See Appendix B3.

While the formal proof can be found in the Appendix, we will present here an intuitive argument for why the identification is possible in this model. The strategy is to find a way to separate the effect of the reference point bias in preference from the response to the underlying uncertainty. This is done by constructing a menu for which the existence of an ex post stage and thus of the uncertainty is practically inconsequential. This would show identification of the parameter θ and then the identification of the measure μ would follow.

Thus, Remark 25 shows that if a preference relation $\succeq$ has a normalized reference-dependent representation then, there exists a compact set $A \subset int(\Delta(Z))$ and a lottery $y \in int(\Delta(Z))$ such that $\{x\} \succ \{y\}$ for all $x \in A$ and $\max_{x \in A} (x \cdot s) > y \cdot s$ for all $s \in S$, where S is the set of ex post states from the representation in (9). By continuity of the preference and of the scalar product it follows immediately that there also exists $y' \in int(\Delta(Z))$ with $\{x\} \succ \{y'\}$ and $\max_{x \in A} (x \cdot s) > y' \cdot s$ for all $s \in S$.

Now, since $V_i(A) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu_i(ds) - \theta_i \min_{x\in A} \left[\int_S (x \cdot s)\mu_i(ds)\right]$ for $i \in \{1, 2\}$ are two representations of the same preference $\succeq$, standard arguments imply that $V_1 = \alpha V_2 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Moreover, $v_i(z) \equiv V_i(\{z\})$ the corresponding restrictions to the singletons also satisfy $v_1 = \alpha v_2 + \beta$. But $v_i(z) = (1-\theta_i)\int_S (x \cdot s)\mu_i(ds)$, so $V_i(A) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu_i(ds) - \theta_i \left[\min_{x\in A} v_i(x)\right]$.

Now, using the particular properties of the sets $A \cup \{y\}$ and $A \cup \{y'\}$ presented above, it follows that

$$V_i(A \cup \{y\}) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu_i(ds) - \theta_i \left[v_i(y)\right] \text{ and}$$

$$V_i(A \cup \{y'\}) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu_i(ds) - \theta_i \left[v_i(y')\right].$$

Therefore, $V_i(A \cup \{y'\}) - V_i(A \cup \{y\}) = \theta_i \left[v_i(y) - v_i(y')\right]$ and then

$$\theta_1 = \frac{V_1(A\cup\{y'\})-V_1(A\cup\{y\})}{v_1(y)-v_1(y')} = \frac{[\alpha V_2(A\cup\{y'\})+\beta]-[\alpha V_2(A\cup\{y\})+\beta]}{[\alpha v_2(y)+\beta]-[\alpha v_2(y')+\beta]} = \frac{V_2(A\cup\{y'\})-V_2(A\cup\{y\})}{v_2(y)-v_2(y')} = \theta_2.$$

Therefore, the parameter θ is indeed identified. The rest of the proof consists of showing that this implies $\int_S \left[\max_{z\in A}(z \cdot s)\right] \mu_1(ds) = \int_S \left[\max_{z\in A}(z \cdot s)\right] (\alpha\mu_2)(ds)$ for any compact set $A \subset \Delta(Z)$ and then using Lemma 14 from Sarver(2005) to conclude that $\mu_1 = \alpha\mu_2$. This immediately implies that $\mu_1 = \mu_2$, because the measures are normalized to be probabilities. The remaining details are presented formally in the Appendix.[27]

4 Applications

4.1 Principal-Agent Models

We will present here a simple example of a second degree price discrimination model, which will show how sellers facing consumers who exhibit a reference point bias of the type studied in our paper could exploit this irrational behavior from the part of the consumers to increase their profits.

Consider a seller who faces a demand composed of two types of consumers. The seller has the ability to design a set of menus,[28] each menu containing products who are characterized by two

[27] The above argument does not consitute a formal proof of the fact that the parameter θ is identified. This is because in order to apply Lemma 48 we would need first to show that the two states s_* are the same for both representations. This is done formally in the Appendix.

[28] For instance, many firms specializing in retail sale open outlets under different brands, each type of outlet containing in turn a variety of options. Similarly, most car manufacturers produce cars under different brands, each

variables: the quality q and the price p. The consumers have different tastes so they have different valuations for the quality of the product. Thus, a share $\lambda \in (0,1)$ of them called the low types have valuation given by the function $v_l(\cdot)$, while the rest of the consumers, called the high types, have valuation given $v_h(\cdot)$ such that the single crossing property $v_h'(q) > v_l'(q)$ for all q is satisfied. The utility of a consumer is quasilinear in the price of the good so the utility derived by a consumer of type v_i from a product (q,p) is:

$$u_i(q,p) = v_i(q) - p \tag{15}$$

Without loss of generality, we assume that the outside option for the consumers has value zero so that they would consider buying a product only when $u_i(q,p) \geq 0$. Finally, the cost for the seller of manufacturing a product *that is sold in the market* of quality q is $c(q)$, where $c(\cdot)$ is strictly increasing, differentiable and convex. Thus, for the sake of simplicity we assume here that the cost for the seller of expanding the menu with options that are not meant to be sold is zero.

In addition to the above standard specification of a principal-agent model, which we will call from now on the benchmark model, we assume that the seller knows that consumers' preferences exhibit a context effects bias of the type that we study in our paper. However, for the sake of simplicity, we assume away uncertainty from the part of the consumers regarding their future tastes. Thus, a consumer v_i evaluates a menu $A \subset \mathbb{R}^2_+$ with the following utility function:

$$U_i(A) = \max_{\{(q,p)\in A | u_i(q,p)\geq 0\}} u_i(q,p) - \theta \min_{\{(q',p')\in A | u_i(q',p')\geq 0\}} u_i(q',p') \tag{16}$$

Note in the above definition of $U_i(\cdot)$ we assumed that the consumer uses as reference points only one of those options that he could potentially end up buying. In other words, products that are too bad to satisfy the consumer's incentive rationality constraint do not affect his choices.[29]

It is clear that with this simple specification of the model, the seller cannot do better than by restricting his strategy to offer only two menus, one for each type of consumer. In turn, each menu would consist only of two options, one being the option targeted to be purchased by the consumer, while the second option being designed to be used as the reference point. Therefore, the seller's

brand coming again in different forms. The aim of these marketing strategies is to screen the consumers according to their valuations for the quality of the product sold. They are therefore examples of second degree price discrimination with the particular feature that the sellers group their products into menus, each menu being designed to appeal to a certain class of consumers. Within each menu the seller may devise an additional price discrimination mechanism.

[29]The effect of this assumption is that it puts an upper bound on the amount of additional profit that the seller can extract from the higher type relative to the benchmark model.

problem is:[30]

$$\max_{\{((q_i,p_i),(q_i',p_i'))\in\mathbb{R}_+^2\times\mathbb{R}_+^2\}_{i\in\{h,l\}}} \lambda\left[p_l - c(q_l)\right] + (1-\lambda)\left[p_h - c(q_h)\right] \text{ s.t., for } i \in \{h,l\} \tag{17}$$

$$u_i(q_i', p_i') \geq 0 \tag{IRR_i}$$

$$u_i(q_i, p_i) \geq u_i(q_i', p_i') \tag{ICP_i}$$

$$U_i(\{(q_i,p_i),(q_i',p_i')\}) \geq U_i(\{(q_{-i},p_{-i}),(q_{-i}',p_{-i}')\}) \tag{ICM_i}$$

Proposition 30 *The solution to the seller's problem (17) has a value of the objective function higher than the one corresponding to the seller's problem in the benchmark model. As in the benchmark model, the solution to the problem (17) implies no distortion for the high type and zero surplus for the low type. The quality offered to the low type is higher than in the benchmark model. The profits extracted by the principal are increasing in the strength of the reference point bias.*

Proof. See Appendix C.

Therefore, as claimed above, the sellers who face consumers exhibiting a reference point bias of the type studied in our paper can increase their profits by designing menus as in problem (17) when this action is feasible. This is achieved by constructing a menu for the higher type that would relax his incentive compatibility constraint so that the seller can extract more of his surplus than he does in the benchmark model.[31]

4.2 Qualifying Auctions

Consider the following auction format which in practice is frequently used when a public or private entity sells a valuable asset or awards rights to a complex contract. The auction consists of two stages. In the first stage, all firms make some non-binding offers that are used as expressions of interest. The auctioneer then ranks the offers and asks the companies who submitted the top offers to resubmit other, this time binding offers which are ranked to select the winner. Ye(2007) showed

[30] The first restriction IRR_i ensures that the reference product is a viable product for the consumer of type i. The restriction ICP_i ensures that consumers will eventually purchase the target product while the restriction ICM_i ensures that the consumers select the menus that are designed for their types. Finally, note that IRR_i and ICP_i together also ensure that the targeted product satisfies the incentive rationality constraint.

[31] The result of Proposition 30 can be extended from the simple second degree price discrimination model employed here to any principal-agent model in which it is reasonable to assume that the agents may exhibit a reference point bias and the principals have the ability to tie options together in such a way that the resulting objects are perceived by the agents as menus in the sense used in this paper.

that unless the entry cost is fully subsidized by the auctioneer *and* there is no information updating about the value of the asset in the second round, there is no symmetric increasing equilibrium exhibiting efficient entry.[32]

We will present in this section a situation in which an equilibrium with the desired features might exist under the current specification of the auction procedure. Thus, note that in the second stage of the process if there is no objective and clear economic indicator that would automatically rank the offers, the auctioneer may exhibit a reference-point bias by taking into account not only the absolute value of each firms' offer, but also the improvement perceived in each firms' second stage offer relative to the first stage.[33] Alternatively, the reference point bias could also be an expression of the auctioneer's desire to discipline the bidders toward making fair offers. Therefore, when firms have reason to believe that the auctioneer may exhibit a reference point bias, they would not make unbounded offers in the first stage as the Ye's paper concluded, but instead would need to solve a trade-off between offering an initial good enough offer to remain in competition and leaving room for perceived improvement. Therefore, in a qualifying auction, if the auctioneer exhibits a sufficiently strong reference point bias, then a symmetric equilibrium exhibiting efficient entry *may* exist. We will show now formally why the presence of a reference point bias might solve the issue of the nonexistence of an equilibrium in Ye's model.

A number N of potential buyers observe private signals X_i regarding the valuation of the asset. The signals are independent draws from the same distribution with cdf $F_X(\cdot)$. The bidders make non-binding first stage offers a_i out of which the auctioneer selects the n highest ones and the corresponding bidders move on to the second round. In the second stage, the remaining bidders incur an entry cost c and acquire a second private signal Y_i. Again, the signals are independent draws from a distribution with cdf $F_Y(\cdot)$. The final value of the asset for bidder i is $v(X_i, Y_i)$ with the function v being strictly increasing in both arguments. Each bidder makes a second binding offer b_i and the seller selects the winner as the bidder with the highest value of $b_i - \theta a_i$, where θ is the strength of the reference point bias. The winner pays his own bid b_i, so his utility will be $v(X_i, Y_i) - b_i - c$, while the remaining second round participants end up with utility $-c$.

[32] Efficient entry is defined as the situation in which the bidders with the highest first stage valuations of the asset enter the second stage. Therefore, since in the absence of a symmetric increasing equilibrium efficient entry cannot be guaranteed, in order to avoid the ensuing efficiency loss, Ye(2007) suggests implementing a new procedure that at least in theory would guarantee efficient bidding. The procedure would make the first stage offers binding and require the bidders entering the second stage to pay an entry fee equal to the highest rejected first stage bid. This is hardly implementable in practice because the equilibrium first stage bids are the true expected valuations.

[33] Since the procedure is used in auctioning of complex assets or contracts, it is likely that the offers are ranked based on more coordinates than just the final price. The evaluation of some of these coordinates, such as the bidder's past history of abiding to the terms of the contract or of dealing with the labor force may be highly subjective. This leaves room for the presence of a reference point bias in assessing the second round offers since the comparison of the first and second round offers for the same bidder is significantly easier than accross bidders.

Let $b_i(X_i, Y_i, a_i)$ be the optimal second stage bid for player i and denote by $d_i(X_i, Y_i, a_i) \equiv b_i(X_i, Y_i, a_i) - \theta a_i$. Then, the second stage problem for player i is to choose z_i to maximize: $\pi_2(X_i, Y_i, a_i, z_i) \equiv F_{\widehat{d}_{1:-i}(X_i)}(z_i)\left[v\left(X_i, Y_i\right) - z_i - \theta a_i\right] - c$, where $F_{\widehat{d}_{1:-i}(X_i)}(\cdot)$ is the equilibrium ex post cdf of the highest value of d among the $n-1$ remaining opponents of player i. The interim expected second period profits are then: $E\pi_2(X_i, a_i) \equiv \int \pi_2(X_i, y_i, a_i, d_i(X_i, y_i, a_i)) dF_Y(y_i)$. Therefore, the first stage problem is to choose w_i to maximize: $\pi_1(X_i, w_i) \equiv F_{\widehat{a}_{n:-i}}(w_i) E\pi_2(X_i, w_i)$, where $F_{\widehat{a}_{n:-i}}(\cdot)$ is the equilibrium cdf of the n^{th} highest first stage bid among the $N-1$ opponents of player i. Then, assuming differentiability and concavity of the relevant functions, the first stage bidding function $a_i(X_i)$ is the solution to the equation in w_i:

$$f_{\widehat{a}_{n:-i}}(w_i) E\pi_2(X_i, w_i) = F_{\widehat{a}_{n:-i}}(w_i)\left[-\frac{\partial}{\partial w_i} E\pi_2(X_i, w_i)\right] \tag{18}$$

The left hand side of (18) is the probability of being the marginal entrant to the second stage with a first stage bid of w_i multiplied by the corresponding expected profits conditional on being the marginal entrant. The right hand side is the probability of entering the second stage with that bid multiplied by the loss in expected second stage profits given due to an increased reference point when the first stage bid is increased. When these two effects of adjusting the value of w_i compensate each other we have the optimal bid of player i. Now, note that by the Envelope Theorem we have $-\frac{\partial}{\partial w_i} E\pi_2(X_i, w_i) = \theta \int \left[F_{\widehat{d}_{1:-i}(X_i)}(d_i(X_i, Y_i, w_i))\right] dF_Y(y_i)$ so when $\theta = 0$, which corresponds to the case of no reference point bias, no solution to this equation exists because the right hand side in (18) is zero, while the left hand side is not zero unless $E\pi_2(X_i, w_i) = 0$, which at least generically cannot happen. Thus, without a reference point bias, no equilibrium of this type exists which is basically the result from Ye(2007). However, when $\theta > 0$, equation (18) may have a solution and thus an equilibrium may exist under the current the current design of qualifying auctions.[34]

5 Conclusion

In this paper we proposed a model of choices affected by a context effects bias, that is a model of preferences over menus with the particular feature that certain options in a given menu, even though never selected for consumption, may influence the overall attractiveness of the elements in that menu by acting as a reference point against which the rest of the elements in the menu are compared. The testable implications of our model are different from those of other models studying

[34] Note that efficient entry is ensured if equilibrium first stage bidding function is increasing in the signal. In addition, in order to have efficienct auctions in the second stage, the term $b_i - \theta a_i$ should be increasing in $v(X_i, Y_i)$. While there are functional forms of the primitives for which these conditions are satisfied, this does not always happen.

preferences over menus affected by various behavioral biases, such as temptation or regret, in that in our model adding inferior options to a menu never decreases the perceived value of that menu, but can potentially increase it. In addition, we analyze the case in which we allow for the presence of some uncertainty between the moment when the menu is chosen and the moment when the element from the menu is selected. While in this case choices over menus also reveal a preference for flexibility, our model can distinguish between the effect of the context effects bias and that of the preference for flexibility under the relatively weak and, at least in theory, testable assumption, that the uncertainty is weak enough that there exist menus which can be decorated with inferior options that are never chosen ex post.

Finally, we mention here that depending on the empirical evidence relevant to various applications, the present model can be extended to capture preferences over objects of choice that do not necessarily satisfy our assumptions. This can be done by using the additional empirical evidence to construct a mapping from the larger set of objects of choice to the set that we use. For instance, consider the case when the empirical evidence suggests that options whose utility do not exceed a certain threshold do not play any role in the decision making. Choice data can identify the threshold given by the participation constraint of the specific application. Then preferences over menus that contain such inferior options can be modelled by employing a map that incorporates this additional information and removes from any menu the inferior alternatives and by then using the representation from this paper over the reduced menus.

For future reasearch, we intend to study the behavioral implications of a variable strength of the reference point bias so that as additional inferior options are introduced in a menu, themarginal benefit in terms of increasing the attractiveness of the superior products decreases and eventually vanishes. Also, we intend to study a model of rationality in which the available options are grouped into categories.

Appendix

A1. Proof of Theorem 14

Since the necessity of all axioms is straightforward, we will prove here the sufficiency of the axioms for the representation in (7). For the rest of the proof of this Theorem, whenever it is convenient, instead of a singleton set $\{z\}$, we will write the corresponding element z. Firstly, note that if $A \sim B$ for all $A, B \in K(\Delta(Z))$, with $A \subset B$, then we will have immediately that $A \sim A \cup B \sim B$ so $A \sim B$ for all $A, B \in K(\Delta(Z))$. Then, any constant utility function coupled with some $\theta \in (0,1)$ would represent the preferences. Therefore, assume that there exists $A, B \in K(\Delta(Z))$, with $A \subset B$ and $A \nsim B$. By Monotonicity we must have $B \succ A$ and using Axiom NDF it follows that we must have $x_0, y_0 \in B$ such that $x_0 \succ y_0$. Let $N_\delta(A)$ denote the $\delta-$neighborhood of any set A in the Hausdorff metric topology. Then, the Continuity axiom implies that there exists $\delta > 0$ such that

$$A \succ B, \text{ for all } A \in N_\delta(\{x_0\}) \text{ and } B \in N_\delta(\{y_0\}). \tag{19}$$

Since $\Delta(Z)$ endowed with the topology of convergence in distribution, is a compact metric space, there exists $\lambda_1 > 0$ such that $\lambda_1\Delta(Z) + (1-\lambda_1)x_0 \equiv \{\lambda_1 z + (1-\lambda_1)x_0 : z \in \Delta(Z)\} \subset N_\delta(\{x_0\})$. Similarly, there exists $\lambda_2 > 0$ such that $\lambda_2\Delta(Z) + (1-\lambda_2)y_0 \subset N_\delta(\{y_0\})$. Let $\lambda \equiv \min\{\lambda_1, \lambda_2\}$ and for any $z_1, z_2 \in \Delta(Z)$ denote by:

$$\lambda(z_1, z_2) \equiv \lambda z_1 + (1-\lambda)z_2 \tag{20}$$

Note that by (19) it follows that for any $x, y \in \Delta(Z)$, we have $\lambda(x, x_0) \succ \lambda(y, y_0)$.

Lemma 31 *For any x, y, x', y' and any $\alpha \in (0,1)$, we have*

$$\alpha\{\lambda(x,x_0), \lambda(y,y_0)\} + (1-\alpha)\{\lambda(x',x_0), \lambda(y',y_0)\} \sim \{\lambda\left(\left(\alpha x + (1-\alpha)x'\right), x_0\right), \lambda\left(\left(\alpha y + (1-\alpha)y'\right), y_0\right)\} \tag{21}$$

Proof. Using the definition of the mixtures of menus from Section 2, the set in the right hand side of (21) is $A = \{(\alpha\lambda(x,x_0) + (1-\alpha)\lambda(x',x_0), \alpha\lambda(x,x_0) + (1-\alpha)\lambda(y',y_0), \alpha\lambda(y,y_0) + (1-\alpha)\lambda(x',x_0), \alpha\lambda(y,y_0) + (1-\alpha)\lambda(y',y_0)\}$. Firstly, note that (19) and Binary Independence imply the following:

$$\begin{aligned}
\alpha\lambda(x,x_0) + (1-\alpha)\lambda(x',x_0) &\succ \alpha\lambda(x,x_0) + (1-\alpha)\lambda(y',y_0) && (22)\\
\alpha\lambda(x,x_0) + (1-\alpha)\lambda(x',x_0) &\succ \alpha\lambda(y,y_0) + (1-\alpha)\lambda(x',x_0) && (23)\\
\alpha\lambda(x,x_0) + (1-\alpha)\lambda(y',y_0) &\succ \alpha\lambda(y,y_0) + (1-\alpha)\lambda(y',y_0) && (24)\\
\alpha\lambda(y,y_0) + (1-\alpha)\lambda(x',x_0) &\succ \alpha\lambda(y,y_0) + (1-\alpha)\lambda(y',y_0) && (25)
\end{aligned}$$

Let $A' \equiv A \backslash \{\alpha\lambda(y, y_0) + (1-\alpha)\lambda(x', x_0)\}$ and note that by Monotonicity we must have $A \succeq A'$. On the other hand, by Axiom NDF, (22) and (24) it follows that $A' \succeq A$ so it must be that $A \sim A'$. Let $A'' \equiv A' \backslash \{\alpha\lambda(x, x_0) + (1-\alpha)\lambda(y', y_0)\}$ so again by Monotonicity, Axiom NDF, (23) and (25) we have $A'' \sim A'$. Then immediately $A \sim A''$ by transitivity which completes the proof. ■

Consider now the product space $\Delta(Z) \times \Delta(Z)$, and define on it the binary relation $\succeq_0$ as follows. For any $(x, y), (x', y') \in \Delta(Z) \times \Delta(Z)$, let:

$$(x, y) \succeq_0 (x', y') \text{ if and only if } \{\lambda(x, x_0), \lambda(y, y_0)\} \succeq \{\lambda(x', x_0), \lambda(y', y_0)\} \tag{26}$$

and note that the Weak Order and Continuity properties of the preference $\succeq$ imply Continuity and Weak Order of the relation $\succeq_0$. To prove the Independence property of $\succeq_0$, consider any pairs (x, y) and (x', y') such that $(x, y) \succ_0 (x', y')$. We want to show that for any pair (x'', y'') and any $\alpha \in (0, 1)$ we must have $\alpha(x, y) + (1-\alpha)(x'', y'') \succ_0 \alpha(x', y') + (1-\alpha)(x'', y'')$. Using the definition in (26) this can be rewritten as: $\{\lambda(\alpha x + (1-\alpha)x'', x_0), \lambda(\alpha y + (1-\alpha)y'', y_0)\} \succ \{\lambda(\alpha x' + (1-\alpha)x'', x_0), \lambda(\alpha y' + (1-\alpha)y'', y_0)\}$. Now, using Lemma 31, the above is equivalent to: $\alpha\{\lambda(x, x_0), \lambda(y, y_0)\} + (1-\alpha)\{\lambda(x'', x_0), \lambda(y'', y_0)\} \succ \alpha\{\lambda(x', x_0), \lambda(y', y_0)\} + (1-\alpha)\{\lambda(x'', x_0), \lambda(y'', y_0)\}$ for which, given the Binary Independence it is sufficient to have $\{\lambda(x, x_0), \lambda(y, y_0)\} \succ \{\lambda(x', x_0), \lambda(y', y_0)\}$. This last condition holds because of the assumption $(x, y) \succ_0 (x', y')$.

Therefore, using the Herstein-Milnor Theorem (see Herstein and Milnor (1953)), we conclude that there exists a linear and continuous function $U_0 : \Delta(Z) \times \Delta(Z) \to \mathbb{R}$ that represents the order relation $\succeq_0$. Moreover, since U_0 is identified only up to affine transformations, it is without loss of generality to take $U_0(x_0, y_0) = 0$. Then, using linearity of U_0, we have that for any $(x, y) \in \Delta(Z) \times \Delta(Z)$, $U_0(\frac{1}{2}(x, y) + \frac{1}{2}(x_0, y_0)) = \frac{1}{2}U_0(x, y)$, while on the other hand $U_0(\frac{1}{2}(x, y) + \frac{1}{2}(x_0, y_0)) = U_0(\frac{1}{2}(y, x_0) + \frac{1}{2}(x_0, y)) = \frac{1}{2}U_0(x, y_0) + \frac{1}{2}U_0(x_0, y)$. Therefore, $U_0(x, y) = U_0(x, y_0) + U_0(x_0, y)$.

Lemma 32 *$U_0(x_1, y_0) \geq U_0(x_2, y_0)$ if and only if $x_1 \succeq x_2$. $U_0(x_0, y_1) \geq U_0(x_0, y_2)$ if and only if $y_2 \succeq y_1$.*

Proof. Assume that $U_0(x_1, y_0) \geq U_0(x_2, y_0)$ and note that the fact that U_0 represents the preference $\succeq_0$ and the definition in (26) this implies $\{\lambda(x_1, x_0), y_0\} \succeq \{\lambda(x_2, x_0), y_0\}$. Assume by contradiction that $x_2 \succ x_1$, and note that by Binary Independence it follows $\lambda(x_2, x_0) \succ \lambda(x_1, x_0)$. Then, (19), $\lambda(x_2, x_0) \succ \lambda(x_1, x_0)$ and Axiom NDF imply $\{\lambda(x_2, x_0), y_0\} \succeq \{\lambda(x_1, x_0), \lambda(x_2, x_0), y_0\}$, so given the Monotonicity we must have $\{\lambda(x_1, x_0), \lambda(x_2, x_0), y_0\} \sim \{\lambda(x_2, x_0), y_0\}$. On the other hand, (19), $\lambda(x_2, x_0) \succ \lambda(x_1, x_0)$ and Axiom NDF also imply $\{\lambda(x_1, x_0), \lambda(x_2, x_0), y_0\} \succ \{\lambda(x_1, x_0), y_0\}$. Therefore, by transitivity $\{\lambda(x_2, x_0), y_0\} \succ \{\lambda(x_1, x_0), y_0\}$ which is a contradiction, so as claimed, it must be that $x_1 \succeq x_2$. Conversely, assume $x_1 \succeq x_2$ which implies $\lambda(x_1, x_0) \succeq \lambda(x_2, x_0)$. Now, since $\lambda(x_2, x_0) \succ y_0$, by Axiom NDF and Monotonicity we have $\{\lambda(x_1, x_0), y_0\} \sim \{\lambda(x_1, x_0), \lambda(x_2, x_0), y_0\} \succeq$

$\{\lambda(x_2,x_0),y_0\}$. Therefore, $U_0(x_1,y_0) \geq U_0(x_2,y_0)$. Take now $U_0(x_0,y_1) \geq U_0(x_0,y_2)$ so that $\{x_0,\lambda(y_1,y_0)\} \succeq \{x_0,\lambda(y_2,y_0)\}$ and assume by contradiction that $y_1 \succ y_2$. Then, $y_1 \succ y_2$, Binary Independence and (19) imply by Axiom CEB that $\{x_0,\lambda(y_1,y_0),\lambda(y_2,y_0)\} \succ \{x_0,\lambda(y_1,y_0)\}$ and by Axiom NDF and Monotonicity that $\{x_0,\lambda(y_1,y_0),\lambda(y_2,y_0)\} \sim \{x_0,\lambda(y_2,y_0)\}$. Therefore, by transitivity $\{x_0,\lambda(y_2,y_0)\} \succ \{x_0,\lambda(y_1,y_0)\}$, which is again a contradiction. Conversely, if $y_2 \succeq y_1$, then Binary Independence and (19) imply $x_0 \succ \lambda(y_2,y_0) \succeq \lambda(y_1,y_0)$, so by Axiom NDF and Monotonicity $\{x_0,\lambda(y_1,y_0)\} \succeq \{x_0,\lambda(y_1,y_0),\lambda(y_2,y_0)\} \succeq \{x_0,\lambda(y_2,y_0)\}$. This implies immediately $U_0(x_0,y_1) \geq U_0(x_0,y_2)$ so the proof of the Lemma is complete. ■

Lemma 33 *Consider any arbitrary finite set $A \in K(\Delta(Z))$ and let $x_m \in \underset{x\in A}{\arg\max} U_0(x,y_0)$ and $y_m \in \underset{y\in A}{\arg\max} U_0(x_0,y)$. Then $\lambda A + (1-\lambda)\{x_0,y_0\} \sim \{\lambda(x_m,x_0),\lambda(y_m,y_0)\}$.*

Proof. Take some arbitrary $a \in A$, and note that $U_0(x_m,y_0) \geq U_0(a,y_0)$ implies by Lemma 32, $\lambda(x_m,x_0) \succeq \lambda(a,x_0)$. Denote now by $A_0 \equiv \{x_m,y_m\}$ and take some $a \in A \backslash A_0$. Let $A_1 = A_0 \cup \{a\}$ and note that Monotonicity implies $\lambda A_1 + (1-\lambda)\{x_0,y_0\} \succeq \lambda A_0 + (1-\lambda)\{x_0,y_0\}$. On the other hand, since $\{\lambda(x_m,x_0),\lambda(y_m,y_0)\} \subset \lambda A_0 + (1-\lambda)\{x_0,y_0\}$, $\lambda(x_m,x_0) \succeq \lambda(a,x_0)$, $\lambda(x_m,x_0) \succeq \lambda(a,y_0)$, $\lambda(a,x_0) \succeq \lambda(y_m,y_0)$, $\lambda(a,y_0) \succeq \lambda(y_m,y_0)$, Axiom NDF implies iteratively that $\lambda A_0+(1-\lambda)\{x_0,y_0\} \succeq (\lambda A_0 + (1-\lambda)\{x_0,y_0\}) \cup \{\lambda(a,x_0)\} \succeq \lambda A_1+(1-\lambda)\{x_0,y_0\}$. Therefore, $\lambda A_0+(1-\lambda)\{x_0,y_0\} \sim \lambda A_1+(1-\lambda)\{x_0,y_0\}$. Repeating this argument inductively for each element in $A\backslash\{x_m,y_m\}$, we obtain in the end that $\lambda A_0 + (1-\lambda)\{x_0,y_0\} \sim \lambda A + (1-\lambda)\{x_0,y_0\}$. By the same argument it can be shown that $\lambda A_0 + (1-\lambda)\{x_0,y_0\} \sim \{\lambda(x_m,x_0),\lambda(y_m,y_0)\}$ so the proof of the Lemma is complete. ■

Lemma 34 *The function $U : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$, by*

$$U(A) \equiv \max_{x\in A} U_0(x,y_0) + \max_{y\in A} U_0(x_0,y) \tag{27}$$

represents the preference $\succeq$.

Proof. The steps in the proof of this lemma are straightforward adaptations of various steps from the proof of the main representation theorem in Kopylov (2005). Firstly, we show that U as defined above represents the preference $\succeq$ over $B(\Delta(Z))$. Take B, $B' \in B(\Delta(Z))$, such that $B \succ B'$. Then Binary Independence implies $\lambda B + (1-\lambda)\{x_0,y_0\} \succ \lambda B' + (1-\lambda)\{x_0,y_0\}$. Take $x_m \in \underset{x\in B}{\arg\max} U_0(x,y_0)$, $x'_m \in \underset{x\in B'}{\arg\max} U_0(x,y_0)$, $y_m \in \underset{y\in B}{\arg\max} U_0(x_0,y)$, $y'_m \in \underset{y\in B'}{\arg\max} U_0(x_0,y)$ and note that Lemma 33 implies $\{\lambda(x_m,x_0),\lambda(y_m,y_0)\} \succ \{\lambda(x'_m,x_0),\lambda(y'_m,y_0)\}$. Therefore, we will have $(\lambda(x_m,x_0),\lambda(y_m,y_0)) \succ_0 (\lambda(x'_m,x_0),\lambda(y'_m,y_0))$ and then $U_0(x_m,y_m) > U_0(x'_m,y'_m)$ and

$U(B) > U(B')$. Therefore, $B \succ B'$ implies $U(B) > U(B')$. Conversely, assume $U(B) > U(B')$ and we will show that $B \succ B'$. By the first part of the proof, $B' \succ B$ would imply $U(B') > U(B)$, so it must be that $B \succeq B'$. The continuity of U_0 implies the continuity of U. Therefore, since $U(B) > U(B')$, there must exist some $\alpha > 0$ such that $U(\alpha y_0 + (1-\alpha)B) > U(B')$ and $U(B) > U(\alpha x_0 + (1-\alpha)B')$. Note that using the first part of the proof, these imply $\alpha y_0 + (1-\alpha)B \succeq B'$ and $B \succeq \alpha x_0 + (1-\alpha)B'$. Now, since $B \succeq B'$ and $x_0 \succ y_0$ transitivity implies that we must have either $B \succ y_0$ or $x_0 \succ B'$. By Binary Independence, in the first case we will have $B \succ \alpha y_0 + (1-\alpha)B$ while in the second $\alpha x_0 + (1-\alpha)B' \succ B'$. Therefore, using the previous result it follows again by transitivity in either case that $B \succ B'$.

Take now A and A' two finite sets in $K(\Delta(Z))$. Then, using an argument similar to the one from the proof of Lemma 33, it follows that $A \sim \{x_m, y_m\}$ and $A' \sim \{x'_m, y'_m\}$ for some $x_m \in \arg\max_{x \in A} U_0(x, y_0)$, $x'_m \in \arg\max_{x \in A'} U_0(x, y_0)$, $y_m \in \arg\max_{y \in A} U_0(x_0, y)$ and $y'_m \in \arg\max_{y \in A'} U_0(x_0, y)$. Therefore, $A \succ A' \Leftrightarrow \{x_m, y_m\} \succ \{x'_m, y'_m\} \Leftrightarrow U(\{x_m, y_m\}) > U(\{x'_m, y'_m\}) \Leftrightarrow U(A) > U(A')$. For the second equivalence we used the fact that U represents $\succeq$ over $B(\Delta(Z))$, while for the last equivalence we used (27). Lemma 0 in Gul and Pesendorfer (2001) shows that the set of finite sets of $K(\Delta(Z))$ is dense in $K(\Delta(Z))$ under the Hausdorff metric. Therefore the continuity of the function U as defined by (27) and Continuity imply that U represents $\succeq$ over $K(\Delta(Z))$. ■

Now, note that Lemma 32 implies that $U(x_0, x)$ is an affine transformation of $-U(x, y_0)$, because they both represent the same preferences $\succeq$ over singletons. Moreover, since $\{x_0\} \succ \{y_0\}$, the two functions are not constant everywhere. Therefore, $U_0(x_0, x) = -\theta U_0(x, y_0) + \beta$ for some $\theta > 0$ and $\beta \in \mathbb{R}$. Take now some $z \in \Delta(Z) \backslash \{x_0, y_0\}$ and note that by (27), it follows that $U(\{\lambda(z, x_0)\}) = (1-\theta) U_0(\lambda(z, x_0), y_0) + \beta$ and $U(\{\lambda(z, y_0)\}) = (1-\theta) U_0(\lambda(z, y_0), y_0) + \beta$. Since $\lambda(z, x_0) \succ \lambda(z, y_0)$, we must have $U(\{\lambda(z, x_0)\}) > U(\{\lambda(z, y_0)\})$ because the restriction of U to $\Delta(Z)$ should represent the restriction of the preference $\succeq$ to singleton sets. On the other hand, by Lemma 32, we have $U_0(\lambda(z, x_0), y_0) > U_0(\lambda(z, y_0), y_0)$. Therefore, it must be that $\theta < 1$. Letting $u : \Delta(Z) \to \mathbb{R}$ be such that $u(x) = U_0(x, y_0)$ for all $x \in \Delta(Z)$, we conclude that

$$U(A) = \max_{x \in A} u(x) - \theta \min_{y \in A} u(x) \tag{28}$$

with $\theta \in (0,1)$ and u a linear utility function, represents the preference $\succeq$ over $K(\Delta(Z))$. ■

A2. Proof of Corollary 15

Since (u_1, θ_1) and (u_2, θ_2) both represent the preference $\succeq$, we will have for $i \in \{1,2\}$ that $\{\lambda(x, x_0), \lambda(y, y_0)\} \succ \{\lambda(x', x_0), \lambda(y', y_0)\}$ if and only if $u_i(\lambda(x, x_0)) - \theta_i u_i(\lambda(y, y_0)) > u_i(\lambda(x', x_0)) - \theta_i u_i(\lambda(y', y_0))$ where λ is chosen as in the proof of the Theorem 14 presented above. Let $U_0(x, y) \equiv$

$u_1(\lambda(x,x_0)) - \theta_1 u_1(\lambda(y,y_0))$ and $U_0'(x,y) \equiv u_2(\lambda(x,x_0)) - \theta_2 u_2(\lambda(y,y_0))$ for any $x,y \in \Delta(Z)$ and note that both U_0 and U_0' represent the preference $\succeq_0$ as defined by (26). Therefore, by the Herstein-Milnor Theorem we must have $U_0' = \alpha U_0 + \gamma$ for some $\alpha > 0$ and $\gamma \in \mathbb{R}$. Thus, $u_2(\lambda(x,x_0)) - \theta_2 u_2(\lambda(y,y_0)) = \alpha u_1(\lambda(x,x_0)) - \alpha\theta_1 u_1(\lambda(y,y_0)) + \gamma$ for all $x,y \in \Delta(Z)$. From this, it follows immediately using also the linearity of u_1 and u_2 that $u_2(\cdot) = \alpha u_1(\cdot) + \gamma_1$ and $\theta_2 u_2(\cdot) = \alpha\theta_1 u_1(\cdot) + \gamma_2$ for some $\gamma_1, \gamma_2 \in \mathbb{R}$. Then $(\theta_1 - \theta_2)u_2(\cdot) = \theta_1\gamma_1 - \gamma_2$ so since $x_0 \succ y_0$ implies that $u_2(\cdot)$ is not constant everywhere, it must be that $\theta_1 = \theta_2$. ■

A3. Proof of Theorem 17

Note firstly that the fact that $\succeq_1$ has a stronger context effects bias than $\succeq_2$ implies by the restriction of Definition 16 to singletons that $u_1(x) \geq u_1(y) \Rightarrow u_2(x) \geq u_2(y)$ for all $x,y \in \Delta(Z)$. It is a standard result in the literature that this implies in turn that $u_2 = \alpha u_1 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Take now some arbitrary $k \in \mathbb{R}_+$ and note that it is straightforward to show that the continuity of u_2 and the non-degeneracy assumption from Theorem 17 implies that there exist $x_a, x_b, y_a, y_b \in \Delta(Z)$ such that $u_2(x_b) > u_2(x_a) > u_2(y_b) > u_2(y_a)$ and $\frac{u_2(x_b)-u_2(x_a)}{u_2(y_b)-u_2(y_a)} = k$. Now note that $\theta_2 > k$ implies by the representation result that $\{x_a, y_a\} \succ_2 \{x_b, y_b\}$. On the other hand, whenever $\{x_a, y_a\} \succ_2 \{x_b, y_b\}$ it follows $\{x_a, y_a\} \succ_1 \{x_b, y_b\}$ so $\theta_1 > \frac{u_1(x_b)-u_1(x_a)}{u_1(y_b)-u_1(y_a)} = k$. Therefore, whenever $\theta_2 > k$ for some $k \in \mathbb{R}_+$ it follows $\theta_1 > k$ which implies that it must be that $\theta_1 \geq \theta_2$. Finally, it is clear that if there exist some sets $A, B \in K(\Delta(Z))$ with $A \sim_1 B$ and $A \nsim_2 B$, then it must be that $\theta_1 > \theta_2$. ■

A4. Proof of Lemma 18

By Lemma 0 in Gul and Pesendorfer (2001), for any $A, B \in K(\Delta(Z))$, there exist two sequences of subsets A_n, B_n of A and B respectively, such that each A_n and each B_n have exactly n elements and $A_n \to A$, $B_n \to B$ in the Hausdorff metric. Fix n, let $A_n = \{a_1, ..., a_n\}$ and $B_n = \{b_1, ..., b_n\}$ and choose some $x_n, y_n \in \Delta(Z)$ such that $x_n \neq y_n$ and $\frac{1}{2}(x_n + b_j) \neq \frac{1}{2}(y_n + a_i)$, for all $i,j \in \{1,...,n\}$. Now, for each $i,j \in \{1,...,n\}$, let $\alpha_{i,j}$ be such that $\frac{n-\alpha_{i,j}}{n}a_i + \frac{\alpha_{i,j}}{n}x_n = \frac{n-\alpha_{i,j}}{n}b_j + \frac{\alpha_{i,j}}{n}y_n$ if such a value of $\alpha_{i,j}$ exists in $[0,1]$. Note that by the choice of x_n and y_n, at most one such value of $\alpha_{i,j}$ can exist for each pair (i,j). Take some $\alpha_n \in [0,1] \setminus (\cup_{i,j}\{\alpha_{i,j}\})$ which we ensured to be nonempty and define $A_n' = \frac{n-\alpha_n}{n}A_n + \frac{\alpha_n}{n}\{x_n\}$ and $B_n' = \frac{n-\alpha_n}{n}B_n + \frac{\alpha_n}{n}\{y_n\}$. Then, by the choice of α_n, we have $A_n' \cap B_n' = \varnothing$ for all n, while on the other hand, it is clear that $\lim\limits_{n\to\infty} A_n' = \lim\limits_{n\to\infty} A_n = A$ and $\lim\limits_{n\to\infty} B_n' = \lim\limits_{n\to\infty} B_n = B$ in the Hausdorff metric. Therefore, the sequences A_n' and B_n' satisfy the required conditions, which completes the proof of the Lemma. ■

B1. Construction of the state space for the model with uncertainty

We will present here briefly the construction of the state space from Dekel, Lipman and Rustichini (2001) as we will utilize the concepts introduced there extensively in the rest of the proof.

Firstly, as shown in DLR(2001) under Weak Order, Continuity and Independence any set of lotteries in $\Delta(Z)$ is indifferent to its convex hull. Thus we can restrict attention to the set of *convex* sets[35] in $K(\Delta(Z))$, which we will denote from now on with $\widetilde{K}(\Delta(Z))$. Recall that the number of outcomes in Z is denoted by N and that S^N is the set of normalized expected utility functions on $\Delta(Z)$. Define by $C(S^N)$ the set of real-valued continuous functions on S^N and endow it with the topology given by the sup-norm metric. Embed $\widetilde{K}(\Delta(Z))$ into $C(S^N)$ by identifying each menu with its support function: $A \to \sigma_A$, with $\sigma_A(s) = \max\limits_{x \in A} \sum_{k=1}^{N} x^k s^k$. It is a standard result that the above mapping is an embedding, one-to-one and monotonic. Thus, for all $A, B \in \widetilde{K}(\Delta(Z))$, $\sigma_A(\cdot) = \sigma_B(\cdot)$ implies $A = B$ and $A \subset B$ implies $\sigma_A \leq \sigma_B$. The order used on $C(S^N)$ is the usual pointwise partial order. Also the support functional is affine, that is: $\sigma_{\beta A+(1-\beta)B} = \beta\sigma_A + (1-\beta)\sigma_B$.

Let C denote the subset of $C(S^N)$ that σ maps $\widetilde{K}(\Delta(Z))$ into, that is $C \equiv \{\sigma_A \in C(S^N) : A \in \widetilde{K}(\Delta(Z))\}$. Using this mapping and the Weak Order and Continuity axioms, DLR(2001) construct the continuous linear functional $W : C \to \mathbb{R}$ that basically represents the preference $\succeq$ over $\widetilde{K}(\Delta(Z))$:

$$W(\sigma_A) \geq W(\sigma_B) \text{ if and only if } A \succeq B. \tag{29}$$

As in the main text, define $v : \Delta(Z) \to \mathbb{R}$ to be the restriction of W to the set of support functions of the singleton sets: $v(x) \equiv W(\sigma_{\{x\}})$. It can be shown using the linearity of the support functions that v is affine, that is $v(\beta x + (1-\beta)y) = \beta v(x) + (1-\beta)v(y)$. In addition, as mentioned in Section 2, there exists $s_* \in S^N$ and $\lambda \geq 0$ such that $v(x) = \lambda \sum_{k=1}^{N} x^k s_*^k$ for all $x \in \Delta(Z)$.

Dekel, Lipman, Rustichini and Sarver(2005) show in the proof of their Theorem 2 that under Monotonicity, the functional W is increasing on the space $H^* = \{r_1\sigma_1 - r_2\sigma_2 : \sigma_1, \sigma_2 \in C$ and $\sigma_1, \sigma_2 \geq 0\}$ which is dense in $C(S^N)$. Since $f \leq ||f|| \cdot \mathbf{1}$ for any $f \in H^*$, where $\mathbf{1}$ is the function identically equal to 1, by the monotonicity of W we will have $W(f) \leq ||f||W(1)$ so W is bounded on H^*. Therefore, as in DLR (2001), W can be extended uniquely from C to the whole $C(S^N)$ preserving continuity and linearity. Also, since H^* is dense in $C(S^N)$, it follows immediately that W will be monotone on the whole $C(S^N)$. As in Royden (1988, page 355), W can be decomposed as $W = W^+ - W^-$ where W^+ and W^- are two positive linear functional forms. Using again the

[35] Note that $\int_S U(x,s)\mu(ds)$ is a linear function in x so even when A is not convex, the minimum of $\int_S U(x,s)\mu(ds)$ over A will be attained at an element of A. Thus, the reference point will always belong to A.

monotonicity of W and the definition of W^+ from Royden(1988) it is straightforward to show that $W(\cdot) = W^+(\cdot)$ and $W^-(\cdot) = 0$ on $C(S^N)$.

Then, W is a positive linear functional on $C(S^N)$ and S^N being compact, the functions in $C(S^N)$ have compact support since closed subsets of compact spaces are compact, so the Riesz-Markov Theorem from Royden(1988, page 352) can be used to write $W(f)$ as an integral of f against a σ-additive positive measure μ over S^N for any $f \in C(S^N)$. In particular,

$$W(\sigma_A) = \int_{S^N} \sigma_A(s)\mu(ds) \text{ for any } A \in \widetilde{K}(\Delta(Z)) \tag{30}$$

This last step delivers the DLR representation of the preference $\succeq$. However, note that we use here a different version of the Riesz Representation Theorem than the one used in DLR(2001). This is because the Monotonicity Axiom makes the functional W positive and thus we can obtain a σ-additive and positive Borel measure as opposed to a finitely additive and signed measure as in DLR(2001). As it will be seen below, the σ-additivity of the measure is necessary both for obtaining our reference-dependent representation as well as for proving the uniqueness of this representation. Next, we will impose the additional restrictions on preferences given by Axiom CEB and Axiom CEB-2 to obtain our specific representation from (9).

B2. Proof of Theorem 27

As a first step in the proof, we will rewrite Axiom CEB and Axiom CEB-2 by using the support functionals and the functional W instead of the preference relation.

Note that since $v(x)$ represents the preference over lotteries in $\Delta(Z)$, using the results from Appendix B1 we have $\{x\} \succ \{y\} \Leftrightarrow \lambda \sum_{k=1}^{N} y^k(-s_*^k) \geq \lambda \sum_{k=1}^{N} x^k(-s_*^k) \Leftrightarrow \lambda\sigma_{\{y\}}(-s_*) \geq \lambda\sigma_{\{x\}}(-s_*)$. Given two sets $A, B \in \widetilde{K}(\Delta(Z))$, if there exists $y \in B$ such that $\{x\} \succ \{y\}$ for all $x \in A$ we will have that $\lambda\sigma_{\{y\}}(-s_*) > \lambda\sigma_{\{x\}}(-s_*)$ for all $x \in A$ so $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$. Thus, in general if there exists a lottery in B that is strictly worse than all lotteries in A we can write this in a compact way as $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$. Similarly, if y is *weakly* worse than all elements in A, we have $\lambda\sigma_B(-s_*) \geq \lambda\sigma_A(-s_*)$. Also note that in order for Axiom CEB-2 to hold, more exactly for a lottery $y \in \Delta(Z)$ to exist such that $\{x\} \succ \{y\}$ for some other $x \in \Delta(Z)$, we need $\lambda > 0$ since otherwise all elements in $\Delta(Z)$ are indifferent to each other. Therefore, under Axiom CEB-2 we have $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$ if and only if $\sigma_B(-s_*) > \sigma_A(-s_*)$. Finally, for any two support functionals $\sigma_A, \sigma_B \in C$, denote their join by $\sigma_A \vee \sigma_B$, that is $(\sigma_A \vee \sigma_B)(\cdot) \equiv \max(\sigma_A(\cdot), \sigma_B(\cdot))$ and note that $\sigma_{A\cup B} = \sigma_A \vee \sigma_B$.

Using these results, the fact that $A \sim hull(A)$ for any $A \in K(\Delta(Z))$ and the fact that $A \subset B$ iff $\sigma_A \leq \sigma_B$, we can write Axiom CEB and Axiom CEB-2 in the following equivalent forms:

Axiom CEB: *For any* $A, B \in \widetilde{K}(\Delta(Z))$, *with* $\sigma_A \leq \sigma_B$ *such that* $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$, *we have* $W(\sigma_B) > W(\sigma_A)$.

Axiom CEB-2: *For any set* $A \in \widetilde{K}(\Delta(Z))$ *such that* $A \subset int(\Delta(Z))$, *there exists a set* $B \in \widetilde{K}(\Delta(Z))$ *with* $\sigma_A \leq \sigma_B$ *and* $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$, *such that for all lotteries* $z \in \Delta(Z)$ *with* $\lambda\sigma_{\{z\}}(-s_*) \geq \lambda\sigma_B(-s_*)$, *we have* $W(\sigma_{hull(B\cup\{z\})}) = W(\sigma_{hull(A\cup\{z\})})$.

We will introduce now some new notation. Denote by $\Omega \equiv \{z \in \mathbb{R}^N : \sum_{i=1}^N z_i = 1\}$ and note that Ω is the smallest affine set that contains $\Delta(Z)$, so $\Delta(Z)$ has a nonempty algebraic interior in Ω. For any set $B \in \widetilde{K}(\Delta(Z))$ we denote its *expanded* strict lower contour set corresponding to the utility $s_i \in \mathbb{R}^N\backslash\{0\}$ by $L_{s_i}(B) \equiv \{y \in \Omega : y \cdot s_i < z \cdot s_i$ for all $z \in B\}$ and by $\widetilde{L}_{s_i}(B)$ its expanded weak lower contour set. Also, $U_{s_i}(B) = \{y \in \Omega : y \cdot s_i > z \cdot s_i$ for all $z \in B\}$ denotes the corresponding expanded strict upper contour set. For $q \in \Delta(Z)$ denote the hyperplane generated by s_i as being the set $H_{s_i}(q) = \{z \in \Omega : z \cdot s_i = q \cdot s_i\}$. Note that while the sets denoted above by B are subsets of lotteries from $\Delta(Z)$, the sets $L_{s_i}(B)$ contain elements from the *whole* space Ω. The reason for using the expansion from $\Delta(Z)$ to Ω is to temporary avoid some boundary issues. For a set S of points in $\mathbb{R}^N\backslash\{0\}$, denote the convex cone generated by S with $cone(S) \equiv \{s \in \mathbb{R}^N : s = \sum_{i=1}^k \lambda_i s_i$ with $\lambda_i \geq 0$ and $s_i \in S$ for all $i \in \{1, ..., k\}$ and $k \in \mathbb{N}\}$. Also denote the convex hull generated by S with $hull(S) \equiv \{s \in \mathbb{R}^N : s = \sum_{i=1}^k \lambda_i s_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$ and $s_i \in S$ for all $i \in \{1, ..., k\}$ and $k \in \mathbb{N}\}$. Again, we mention that we defined the convex cone and hull in $\mathbb{R}^N\backslash\{0\}$, not only in the subset of normalized utilities S^N. The need for the expansion of S^N is slightly more subtle and can be understood by following the rest of the proof.[36]

Proof of Sufficiency in Theorem 27:

We will prove two results, Proposition 35 and Proposition 47, which together will bring us one step away from obtaining the structure on the state space from the DLR representation necessary for writing the representation of $W(\cdot)$ as in (9).

Proposition 35 *Under Axiom CEB and Axiom CEB-2 we must have* $\mu(\{-s_*\}) > 0$.

Proof. Assume by contradiction that $\mu(\{-s_*\}) = 0$ and take any set $A \in \widetilde{K}(\Delta(Z))$ with $A \subset int(\Delta(Z))$ and any superset $B' \in \widetilde{K}(\Delta(Z))$ of A such that there exists $x \in B'\backslash A$ with $\lambda\sigma_{\{x\}}(-s_*) > \lambda\sigma_A(-s_*)$. Note that in order for such a set B' to exist it must be that $\lambda > 0$ so we must also have $\sigma_{\{x\}}(-s_*) > \sigma_A(-s_*)$. We will break up most of the rest of the proof of Proposition 35 into a series of lemmas.

[36] See the footnote from Lemma 42 in the proof of Proposition 35 for some motivation for this expansion.

Lemma 36 *There exists a set $B \in \widetilde{K}(\Delta(Z))$ with $A \subset B \subset B' \cap int(\Delta(Z))$ and $\sigma_B(-s_*) > \sigma_A(-s_*)$. Moreover, under Axiom CEB we have $\int_{S^N} \sigma_B(s)\mu(ds) > \int_{S^N} \sigma_A(s)\mu(ds)$.*

Proof. If $x \in int(\Delta(Z))$, then let $B \equiv hull(A \cup \{x\})$. Since B' and $int(\Delta(Z))$ are convex and $A \cup \{x\} \subset B' \cap int(\Delta(Z))$, we will have $hull(A \cup \{x\}) \subset B' \cap int(\Delta(Z))$. On the other hand, $\sigma_B(-s_*) = \sigma_{A\cup\{x\}}(-s_*) = \max(\sigma_{\{x\}}(-s_*), \sigma_A(-s_*)) = \sigma_{\{x\}}(-s_*) > \sigma_A(-s_*)$. If $x \notin int(\Delta(Z))$ we will find some $x' \in int(\Delta(Z)) \cap (B' \backslash A)$ with $\sigma_{\{x'\}}(-s_*) > \sigma_A(-s_*)$ and then define $B \equiv hull(A \cup \{x'\})$ and repeat the argument above to prove the first part of the claim. Note that since A is closed it must be that $\sigma_A(-s_*) = \sigma_{\{y\}}(-s_*)$ for some $y \in A$. Also, since $-s_* \in S^N$, we have $-s_* \neq 0$ so $y \notin int(A)$. Take $x' = \frac{1}{2}(x+y)$ and note that $x' \in int(\Delta(Z)) \cap (B' \backslash A)$. On the other hand, by the affine property of $\sigma_{(\cdot)}$ we have $\sigma_{\{x'\}}(-s_*) = \frac{1}{2}\sigma_{\{x\}}(-s_*) + \frac{1}{2}\sigma_{\{y\}}(-s_*) > \sigma_A(-s_*)$. For the second part of the claim, note that $A \subset B$ implies $\sigma_A \leq \sigma_B$ and since $\sigma_B(-s_*) > \sigma_A(-s_*)$ we can appeal to Axiom CEB to conclude that $W(\sigma_B) > W(\sigma_A)$. Using the DLR(2001) representation from (30), this can be rewritten as $\int_{S^N} \sigma_B(s)\mu(ds) > \int_{S^N} \sigma_A(s)\mu(ds)$. ■

Lemma 37 *There exists an open set $\widehat{S}_1 \subset S^N$ with $\mu(\widehat{S}_1) > 0$ and $-s_* \in \widehat{S}_1$ such that $\sigma_B(s) > \sigma_A(s)$ for any $s \in \widehat{S}_1$.*

Proof. We have $\sigma_B(s) \geq \sigma_A(s)$ over S^N so assume by contradiction that $\sigma_B(\cdot) = \sigma_A(\cdot)$ everywhere but on a set of measure zero. Since $\sigma_B(\cdot)$ and $\sigma_A(\cdot)$ are both continuos functions on S^N which is compact, they are bounded and measurable. So by Proposition 5 from Royden(1988, pp. 82) we have $\int_{S^N} \sigma_B(s)\mu(ds) = \int_{S^N} \sigma_A(s)\mu(ds)$ which constitutes a contradiction to the result of Lemma 36. Also, since $\sigma_B(-s_*) > \sigma_A(-s_*)$ we may assume without loss of generality that $-s_* \in \widehat{S}_1$. ■

Lemma 38 *There exists $\varepsilon > 0$ such that $\mu(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))) > 0$ where $\overline{N}_\varepsilon(-s_*)$ is the closed ball of radius ε around $-s_*$ in $\mathbb{R}^N$.*

Proof. If this were not true we would then have $\mu\left(\widehat{S}_1 \backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$ for all $n \geq 1$ so $\mu\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) = \mu\left(\widehat{S}_1\right) - \mu\left(\widehat{S}_1 \backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) > 0$ for any $n \geq 1$. Note that $\left\{cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right\}$ is a decreasing sequence of sets with $\cap_{n=1}^{\infty}\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) = cone(\{-s_*\}) \cap \widehat{S}_1$. But $cone(\{-s_*\}) = \{\lambda(-s_*) : \lambda \geq 0\}$ and since $\widehat{S}_1 \subset S^N$ in which the utilities are normalized so that $\sum_{k=1}^{N}(s^k)^2 = 1$, we have $cone(\{-s_*\}) \cap \widehat{S}_1 = \{-s_*\}$. So, since $\mu\left(cone\left(\overline{N}_1(-s_*)\right) \cap \widehat{S}_1\right) \leq \mu\left(\widehat{S}_1\right) < \infty$ and μ is σ-additive we can use for instance Theorem 9.8(ii) in Aliprantis and Border(1999, pp. 337) to conclude that $\mu(\{-s_*\}) = \lim_{n\to\infty} \mu\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) =$

$\lim_{n\to\infty} \mu\left(\widehat{S}_1\right) > 0$ which contradicts the assumption that $\mu(\{-s_*\}) = 0$.[37] Thus, the set $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$ will be of strictly positive measure. ■

Lemma 39 *There exists a set $\widehat{S}_2 \subset \widehat{S}_1$ such that $\mu(\widehat{S}_2) > 0$ and $-s_* \notin cone\left(\widehat{S}_2\right)$.*

Proof. Even though $-s_* \notin \widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$, we cannot yet claim that $-s_* \notin cone(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*)))$. To obtain a set with this property, we will partition $\widehat{S}_2$ into 2^{N-1} elements constructed as follows. Firstly, note that by the normalization $\sum_{k=1}^N s^k = 0$ for all $s \in S^N$, we must have $-s_* \cdot v_1 = 0$, where $v_1 \equiv (1, ..., 1) \in \mathbb{R}^N$. Select next some other $N-2$ vectors such that $\{v_1, v_2, ..., v_{N-1}\}$ is a linearly independent set and $-s_* \cdot v_i = 0$, for all $i \in \{1, ..., N-1\}$. Note on the one hand that choosing $N-1$ such vectors is possible because the dimension of the underlying space is N. On the other hand, since $\{v_1, v_2, ..., v_{N-1}\}$ are linearly independent, the dimension of the set $R \equiv \{s \in \mathbb{R}^N : s \cdot v_i = 0, \text{ for } i \in \{1, ..., N-1\}\}$ is 1. Thus, since $-s_* \in R$, we have that $s \in R$ implies $s = \kappa(-s_*)$ for some $\kappa \in \mathbb{R}$. Let $H_1 \equiv \{s \in \mathbb{R}^N : s \cdot v_1 \geq 0\}$ and $H_2 \equiv \{s \in \mathbb{R}^N : s \cdot v_1 \leq 0\}$ and then construct iteratively the following sets. $H_{i_1,...,i_n,1} \equiv \{s \in H_{i_1,...,i_n} : s \cdot v_{n+1} \geq 0\}$, $H_{i_1,...,i_n,2} \equiv \{s \in H_{i_1,...,i_n} : s \cdot v_{n+1} \leq 0\}$ for $n = \{1, ..., N-2\}$. Let $S_{i_1,...,i_{N-1}} \equiv H_{i_1,...,i_{N-1}} \cap \left(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))\right)$ for all $\{i_1, ..., i_{N-1}\} \in \{1, 2\}^{N-1}$. Note that the 2^{N-1} elements $S_{i_1,...,i_{N-1}}$ thus constructed form a finite partition of $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$ so since $\mu(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))) > 0$, one of the elements of the partition which we denote $\widehat{S}_2$ must be of strict positive measure. Without loss of generality we may assume that $s \cdot v_i \geq 0$ for all $s \in \widehat{S}_2$ and $i \in \{1, ..., N-1\}$ because when $s \cdot v_i \leq 0$ for some i we may take $v_i' = -v_i$ instead of v_i and then except for some notation the elements of the partition of $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$ will be the same.

We will show now that $-s_* \notin cone(\widehat{S}_2)$. Assume by contradiction that this is not true, that is there exist $\{s_1, ..., s_m\} \subset \widehat{S}_2$ and $\{\phi_1, ..., \phi_m\} \in \mathbb{R}_+^m$ such that $-s_* = \sum_{i=1}^m \phi_j s_j$. We may assume without loss of generality that $s_j \neq s_*$ for any j, because when this is not true we must still be able to write $-s_*$ as a positive combination of the remaining elements from $\{s_1, ..., s_m\}$. Now, for any $i \in \{1, ..., N-1\}$ we have $-s_* \cdot v_i = 0$, $s_j \cdot v_i \geq 0$ and $-s_* = \sum_{i=1}^m \phi_j s_j$ imply $s_j \cdot v_i = 0$ for all $j \in \{1, ..., m\}$. Therefore, for any $j \in \{1, ..., m\}$, we have $s_j \cdot v_i = 0$ for all $i \in \{1, ..., N-1\}$ which implies $s_j \in R \cap S^N$. But $R \cap S^N = \{-s_*, s_*\}$ because of the normalization $\sum_{k=1}^N \left(s^k\right)^2 = 1$ for the elements in S^N and of the fact that $s \in R$ implies $s = \kappa(-s_*)$ for some $\kappa \in \mathbb{R}$. Since $s_j \neq s_*$ we must therefore have $s_j = -s_*$ for all j which is impossible because $-s_* \notin \widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$. This completes the proof of Lemma 39. ■

We denote by $diam(S) \equiv \sup\{d(s, s') : s, s' \in S\}$ the diameter of a nonempty set.

[37] It is important to emphasize that this is the point where the initial assumption that $\mu(\{-s^*\}) = 0$ plays its role in the argument.

Lemma 40 *There exists a closed set $\widehat{S}_3 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_3) > 0$ and diam$(\widehat{S}_3) \leq \delta$ for some $\delta < \frac{1}{2}$.*

Proof. We will use Proposition 15 from Royden(1988, pp. 63) which states that if E is a measurable set and $\varepsilon > 0$, then there exists a closed set $F \subset E$ such that $\mu(E \backslash F) < \varepsilon$. Since by Lemma 39 we have $\mu(\widehat{S}_2) > 0$, there exists ε such that $\mu(\widehat{S}_2) > \varepsilon > 0$. Applying the result from Royden we conclude that there exists a closed set $\widehat{S}_4 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_2 \backslash \widehat{S}_4) < \varepsilon$. But since $\widehat{S}_4 \subset \widehat{S}_2$ we have $\mu(\widehat{S}_2 \backslash \widehat{S}_4) = \mu(\widehat{S}_2) - \mu(\widehat{S}_4) > \varepsilon - \mu(\widehat{S}_4)$ from which it follows that $\mu(\widehat{S}_4) > 0$. Now, take some $\delta < \frac{1}{2}$ and consider the open cover of $\widehat{S}_4$ consisting of the sets $\{N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4\}_{s \in \widehat{S}_4}$. Since $\widehat{S}_4$ is a closed subset of the compact set S^N, it is compact so there exists a finite subcover of $\widehat{S}_4$. Since $\mu(\widehat{S}_4) > 0$, one of the elements of the subcover, let's say $N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4$ must be of strict positive measure. Applying again the result from Royden(1988) to the set $N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4$, we conclude that there exists a closed set $\widehat{S}_3 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_3) > 0$ and $\widehat{S}_3 \subset N_{\frac{\delta}{2}}(s)$ so that $diam(\widehat{S}_3) \leq diam\left(N_{\frac{\delta}{2}}(s)\right) \leq \delta$. ■

We call the dimension of a set, the dimension of the smallest affine set that includes that set. In our case, we may assume without loss of generality that the dimension of $\widehat{S}_3$ is N, so $\widehat{S}_3$ is of full dimension in $\mathbb{R}^N$. If this is not true, we can continue the argument below replacing everywhere N with the actual dimension of the set $\widehat{S}_3$.

Lemma 41 *There exists $\epsilon > 0$ such that $-s_* \notin cone(\cup_{s \in \widehat{S}_3} \overline{N}_\epsilon(s))$.*

Proof. Assume by contradiction that the claim is not true so that for any $n > 1$, we have $-s_* \in cone(\cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s))$. Thus, for any $n \geq 2$, there exist $\{\lambda_i^n, r_i^n\}_{i \in \{1,\ldots,p(n)\}}$ with $\lambda_i^n > 0$ and $r_i^n \in \cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)$ such that $-s_* = \sum_{i=1}^{p(n)} \lambda_i^n r_i^n$. We firstly claim that it is without loss of generality to take $p(n) = N+1$ for all n. To see this, note that $-s_* = \beta\left(\sum_{i=1}^{p(n)} \alpha_i r_i^n\right)$, where $\beta \equiv \left(\sum_{i=1}^{p(n)} \lambda_i^n\right)$ and $\alpha_i \equiv \frac{\lambda_i^n}{\beta}$. Since $\sum_{i=1}^{p(n)} \alpha_i = 1$, we have $r^n \equiv \sum_{i=1}^{p(n)} \alpha_i r_i^n \in hull\left(\cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)\right)$. By Carathéodory's Convexity Theorem (see for instance Theorem 5.17 from Aliprantis and Border(1999, pp. 173)) in an N-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination of at most $N+1$ vectors from that set. Thus, in our case there exist $\{\alpha_i^n, \overline{r}_i^n\}_{i \in \{1,\ldots,\, N+1\}}$ with $\alpha_i^n > 0$ and $\overline{r}_i^n \in \cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)$ such that $r^n = \sum_{i=1}^{N+1} \alpha_i^n \overline{r}_i^n$. Therefore, $-s_* = \sum_{i=1}^{N+1} (\beta \alpha_i^n) \overline{r}_i^n$ as desired.

Now, since $\widehat{S}_3$ is closed it follows that $\cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{n}}(s)$ is also closed. Moreover, for any $r_i^n \in \cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s) \subset \cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{4}}(s)$ we have $||r_i^n|| \leq ||r_i^n - s|| + ||s|| \leq \frac{5}{4}$, because $||s|| = 1$ when $s \in S^N$. Therefore, $\cup_{s \in \widehat{S}_3} \overline{N}_{\frac{1}{2}}(s)$ is a closed subset of the compact set $\left\{s \in \mathbb{R}^N : ||s|| \leq \frac{5}{4}\right\}$ so it is compact. Since $\{r_1^n\}$ is a sequence in a compact set, it has a convergent subsequence $r_1^{n_i} \to r_1^0$. Thus, it is without loss of generality to assume that $r_1^n \to r_1^0$ and then repeating the argument iteratively we can take $r_i^n \to r_i^0$ for all $i \in \{1,...,N+1\}$. We claim that $r_i^0 \in \widehat{S}_3$ for all i. To see this, note that if $r_i^0 \notin \widehat{S}_3$ for some i, since $\widehat{S}_3$ is closed we will have $d(r_i^0, \widehat{S}_3) = \chi > 0$. But then, take M' such that for any $n \geq M'$ we have $r_i^n \in N_{\frac{\chi}{2}}(r_i^0)$ and let $M \equiv \max(M', \frac{1}{\chi}) + 1$. Then, $\chi = d(r_i^0, \widehat{S}_3) \leq d(r_i^0, r_i^M) + d(r_i^M, \widehat{S}_3) < \frac{\chi}{2} + \frac{\chi}{2}$ which is impossible so it must be that $r_i^0 \in \widehat{S}_3$ for all i. Next, we show that for any i the real sequence $\{\lambda_i^n\}$ is bounded so that we can extract some convergent subsequence. Thus, we have $-s_* = \sum_{i=1}^{N+1} \lambda_i^n r_i^n = \left(\sum_{i=1}^{N+1} \lambda_i^n\right) r^n$, for some $r^n \in hull\left(\cup_{s\in\widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)\right)$. Now, note that since $diam(\widehat{S}_3) \leq \delta$ we will have $diam\left(hull\left(\cup_{s\in\widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)\right)\right) = diam\left(\cup_{s\in\widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)\right) \leq \delta + \frac{1}{n}$. Thus, for any $r \in hull\left(\cup_{s\in\widehat{S}_3} \overline{N}_{\frac{1}{2n}}(s)\right)$ we have $||r|| = d(r,0) \geq d(s,0) - d(r,s) \geq ||s|| - (\delta + \frac{1}{n}) = (1 - \frac{1}{n} - \delta)$ for any $s \in \widehat{S}_3$. Thus, $||-s_*|| = \left(\sum_{i=1}^{N+1} \lambda_i^n\right) ||r^n|| \geq \left(\sum_{i=1}^{N+1} \lambda_i^n\right)\left(1 - \frac{1}{n} - \delta\right)$, which since $||-s_*|| = 1$ and $\lambda_i^n \geq 0$ implies $\lambda_i^n \leq \frac{1}{\frac{1}{2} - \delta}$. Therefore, repeating the argument from above, we may assume without loss of generality that $\lambda_i^n \to \lambda_i^0 \geq 0$ for each $i \in \{1,...,N+1\}$. But then, the sequence $\sum_{i=1}^{N+1} \lambda_i^n r_i^n \to \sum_{i=1}^{N+1} \lambda_i^0 r_i^0$ as $n \to \infty$. Therefore, $-s_* = \sum_{i=1}^{N+1} \lambda_i^0 r_i^0$ with $r_i^0 \in \widehat{S}_3$ and $\lambda_i^0 \geq 0$ so $-s_* \in cone(\widehat{S}_3)$ which is a contradiction and thus the proof of Lemma 41 is complete.[38] ■

Lemma 42 *There exists a set of N linearly independent utilities*[39] $\{s_1,...,s_N\} \subset \mathbb{R}^N \backslash \{0\}$ *such that* $\mu(int(cone(\{s_1,...,s_N\})) \cap \widehat{S}_3) > 0$ *and* $-s_* \notin cone(\{s_1,...,s_N\})$.

Proof. For each $i \in \{1,...,N\}$, let $f_i \equiv (0,...,1,...,0) \in \mathbb{R}^N$ with 1 on the i^{th} position and $e_i \equiv f_i - \frac{1}{N}(1,...,1) \in \mathbb{R}^N$. It is straightforward to show that $\{e_1,...,e_N\}$ is a linearly independent set in $\mathbb{R}^N$ so it constitutes a basis for $\mathbb{R}^N$. For any $s \in \widehat{S}_3$ and $i \in \{1,...,N\}$, let $s_i^s \equiv s + \eta^s e_i$ for some $0 < \eta^s < \min(\epsilon, 1)$, where ϵ is given by Lemma 41 and note that $s = \sum_{i=1}^N \frac{1}{N} s_i^s$. We claim that for any $s \in \widehat{S}_3$ we can choose η^s such that the set $\{s_1^s,...,s_N^s\}$ is linearly independent. For this, we will show that $\sum_{i=1}^N \lambda_i s_i^s = 0$ must imply $\lambda_i = 0$ for all i. Since $\{e_1,...,e_N\}$ is a basis in $\mathbb{R}^N$, $s = \sum_{i=1}^N \gamma_i^s e_i$ for some $\gamma_i \in \mathbb{R}$. Let $\lambda \equiv \sum_{i=1}^N \lambda_i$ and $\gamma^s \equiv \sum_{i=1}^N \gamma_i^s$ and note that $\sum_{i=1}^N \lambda_i s_i^s = \sum_{i=1}^N (\lambda\gamma_i^s + \eta^s\lambda_i)e_i = \left(\lambda\gamma_1^s + \eta^s\lambda_1 - \frac{\lambda\gamma^s + \eta^s\lambda}{N}, ..., \lambda\gamma_N^s + \eta^s\lambda_N - \frac{\lambda\gamma^s+\eta^s\lambda}{N}\right)$. Setting this equal to 0, we obtain a system of N equations with N unknowns $\{\lambda_1,...,\lambda_N\}$, where the i^{th} equation is $\lambda_1(\gamma_i^s - \frac{\gamma^s+\eta^s}{N}) + ... + \lambda_i(\gamma_i^s + \eta^s - \frac{\gamma^s+\eta^s}{N}) + ... + \lambda_N(\gamma_i^s - \frac{\gamma+\eta^s}{N}) = 0$. We will show now

[38] Note here that unless we bound $p(n)$ above with $N+1$, the argument as presented here does not go through because it may well be that $p(n) \to \infty$ as $n \to \infty$.

[39] We emphasize here that the set $\{s_1,...,s_N\}$ is not required to belong to S^N, but to $\mathbb{R}^N \backslash \{0\}$. While we could adapt Lemma 43 below to conclude that $s_{n+1} \in hull(\{-s_1,...,-s_n\})$ and then also adapt the rest of the proof of Proposition 35 to avoid using cones and work only with states in S^N we cannot insure in the proof of this Lemma that it is possible to choose $\{s_1,...,s_n\}$ in S^N to satisfy the desired properties. Therefore, the need to work in the extended state space $\mathbb{R}^N \backslash \{0\}$ and use cones instead of convex hulls.

that the $N \times N$ coefficient matrix of this system has a non-zero determinant D. Thus:

$$\begin{aligned}
D &= \begin{vmatrix} \gamma_1^s + \eta^s - \frac{\gamma^s+\eta^s}{N} & \gamma_1 - \frac{\gamma^s+\eta^s}{N} & ... & \gamma_1^s - \frac{\gamma^s+\eta^s}{N} \\ \gamma_2^s - \frac{\gamma^s+\eta^s}{N} & \gamma_2^s + \eta^s - \frac{\gamma^s+\eta^s}{N} & ... & \gamma_2^s - \frac{\gamma^s+\eta^s}{N} \\ ... & ... & ... & ... \\ \gamma_N^s - \frac{\gamma^s+\eta^s}{N} & \gamma_N^s - \frac{\gamma^s+\eta^s}{N} & ... & \gamma_N^s + \eta^s - \frac{\gamma^s+\eta^s}{N} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & ... & 1 \\ \gamma_2^s - \frac{\gamma^s+\eta^s}{N} & \gamma_2^s + \eta^s - \frac{\gamma^s+\eta^s}{N} & ... & \gamma_2^s - \frac{\gamma^s+\eta^s}{N} \\ ... & ... & ... & ... \\ \gamma_N^s - \frac{\gamma^s+\eta^s}{N} & \gamma_N^s - \frac{\gamma^s+\eta^s}{N} & ... & \gamma_N^s + \eta^s - \frac{\gamma^s+\eta^s}{N} \end{vmatrix} \frac{N-1}{N}(\gamma^s + \eta^s) = \\
&= \frac{N-1}{N}(\gamma^s + \eta^s)\,(\eta^s)^{N-1}
\end{aligned}$$

For the first equation, we added rows 2 through N to the first row and then factored out the term $\frac{N-1}{N}(\gamma^s + \eta^s)$. For the second equation, we subtracted from each row $i \in \{2, ..., N\}$, the first row multiplied with $\gamma_i^s - \frac{\gamma^s+\eta^s}{N}$. Now note that since the only restriction on η^s is $0 < \eta^s < \min(\epsilon, 1)$ we can always select η^s such that $\eta^s \neq -\gamma^s$ so $D \neq 0$. Therefore, the system has a unique solution and since $\lambda_i = 0$ for $i \in \{1, ..., N\}$ solves the system, we obtain the desired conclusion that $\{s_1^s, ..., s_N^s\}$ is a linearly independent set.

We will show now that $s \in int(cone(\{s_1^s, ..., s_N^s\}))$, for which since $cone(\{s_1^s, ..., s_N^s\})$ is a convex set in an Euclidean space it suffices to show that $s \in al-int(cone(\{s_1^s, ..., s_N^s\}))$, the algebraic interior of the set $cone(\{s_1^s, ..., s_N^s\})$. Thus, we will show that for any $p \in \mathbb{R}^N$, there exists some $\overline{\alpha}^s > 0$ such that for all $\alpha \in [0, \overline{\alpha}^s)$, we have $(1-\alpha)s + \alpha p \in cone(\{s_1^s, ..., s_N^s\})$. Since $\{s_1^s, ..., s_N^s\}$ are linearly independent, they form a basis in $\mathbb{R}^N$ so $p = \sum_{i=1}^N \delta_i s_i^s$ with $\delta_i \in \mathbb{R}$. On the other hand, by construction $s = \sum_{i=1}^N \frac{1}{N} s_i^s$ so $(1-\alpha)s + \alpha p = \sum_{i=1}^N \left((1-\alpha)\frac{1}{N} + \alpha\delta_i\right) s_i^s$. Now, by denoting $\beta_i^\alpha \equiv (1-\alpha)\frac{1}{N} + \alpha\delta_i$ we will have $(1-\alpha)s + \alpha p = \sum_{i=1}^N \beta_i^\alpha s_i^s$ and noting that for α sufficiently small $\beta_i^\alpha \geq 0$ for all i, the argument is complete.

Employing the procedure presented above and using the Axiom of Choice construct the family of sets $\mathcal{F} = \{int(cone(\{s_1^s, ..., s_N^s\})) \cap \widehat{S}_3 : s \in \widehat{S}_3\}$. Since $s \subset int(cone(\{s_1^s, ..., s_N^s\})) \cap \widehat{S}_3$ for any s, the elements of $\mathcal{F}$ are nonempty and open relative to $\widehat{S}_3$. Thus, $\mathcal{F}$ is an open cover of $\widehat{S}_3$ which is compact as a closed subset of the compact set S^N so there exists a finite family $\mathcal{F}' \subset \mathcal{F}$ such that $\widehat{S}_3 \subset \underset{F \in \mathcal{F}'}{\cup} F$. Since $\mu(\widehat{S}_3) > 0$, one of the elements of $\mathcal{F}'$ must be of strictly positive measure so $\mu(int(cone(\{s_1^s, ..., s_N^s\})) \cap \widehat{S}_3) > 0$ for some $s \in \widehat{S}_3$. Now, since $d(s_1^s, s) = ||\eta^s e_i|| = \eta^s \frac{N-1}{N} < \epsilon$ we have $int(cone(\{s_1^s, ..., s_N^s\})) \subset int(cone(\overline{N}_\epsilon(s)))$. Therefore by Lemma 41 it follows that $-s_* \notin cone(\{s_1^s, ..., s_N^s\})$. Finally, for any i, $||s_i^s|| = ||s + \eta^s e_i|| \geq ||s|| - ||\eta^s e_i|| = 1 - \eta^s \frac{N-1}{N} > \frac{1}{N}$ implies $s_i^s \neq 0$. ■

Lemma 43 *Let $\{s_1, ..., s_n\} \subset \mathbb{R}^N \backslash \{0\}$ with $n \geq 1$ be such that $\cap_{i=1}^n L_{s_i}(B) \neq \varnothing$. Then, if $s_{n+1} \in \mathbb{R}^N \backslash \{0\}$ is such that $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) = \varnothing$, we must have $s_{n+1} \in cone(\{-s_1, ..., -s_n\})$.*

Proof. Note that since B is compact, $L_{s_i}(B) = L_{s_i}(z_{s_i})$ for some lottery $z_{s_i} \in B$ for all $i \in \{1, ..., n+1\}$. Moreover, $\cap_{i=1}^n L_{s_i}(B) \supset \cap_{i=1}^n L_{s_i}(q) \neq \varnothing$ for some $q \in \cap_{i=1}^n L_{s_i}(B)$ because for any $x \in \cap_{i=1}^n L_{s_i}(q)$ and any $i \in \{1, ..., n\}$ we will have $x \cdot s_i < q \cdot s_i \leq z_{s_i} \cdot s_i$. Therefore, the condition that $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) = \varnothing$ implies that $(\cap_{i=1}^n L_{s_i}(q)) \cap L_{s_{n+1}}(z_{n+1}) = \varnothing$. Also, $q \cdot s_{n+1} > z_{n+1} \cdot s_{n+1}$ because otherwise $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) \neq \varnothing$ since all elements in $\cap_{i=1}^n L_{s_i}(q)$ would be also in $L_{s_{n+1}}(B)$. We will show now that $L_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q)) = \varnothing$ and to this end, assume by contradiction that there exists some $y \in L_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q))$. Consider the set $V \equiv \{x \in \Omega : q + \tau(y - q) \text{ for some } \tau > 0\}$ and note for any $x \in V$ and $i \in \{1, ..., n\}$, we have $x \cdot s_i < q \cdot s_i$ because $y \cdot s_i < q \cdot s_i$. Therefore, $V \subset \cap_{i=1}^n L_{s_i}(q)$ so to prove our claim it is enough to show that $V \cap L_{s_{n+1}}(z_{n+1}) \neq \varnothing$. For this we need to find some $\tau > 0$ such that $(q + \tau(y - q)) \cdot s_{n+1} < z_{n+1} \cdot s_{n+1}$. Since $q \cdot s_{n+1} > z_{n+1} \cdot s_{n+1}$ as stated above and $y \in L_{s_{n+1}}(q)$ by the contradiction assumption, any $\tau > \frac{(q-z) \cdot s_{n+1}}{(q-y) \cdot s_{n+1}}$ would satisfy this requirement.

Consider now the following sets $H_{s_{n+1}}(q) = \{z \in \Omega : z \cdot s_{n+1} = q \cdot s_{n+1}\}$, $Y = \{w \in \mathbb{R}^n : w = ((z - q) \cdot s_1, ..., (z - q) \cdot s_n)$ or $w = (-(z - q) \cdot s_1, ..., -(z - q) \cdot s_n)$ for some $z \in H_{s_{n+1}}(q)\}$ and $Y' = \{w \in \mathbb{R}^n : w \leq 0\}$. Clearly, Y and Y' are closed and convex. We will show next that $Y \cap int(Y') = \varnothing$. Thus, we want to show that if $z \cdot s_{n+1} = q \cdot s_{n+1}$ then it cannot be that $z \cdot s_i < q \cdot s_i$ for all $i \in \{1, ..., n\}$ or $z \cdot s_i > q \cdot s_i$ for all $i \in \{1, ..., n\}$. We can assume that $s_i \neq -s_{n+1}$ because otherwise we would be done with the proof of the lemma, so what remains to prove is that $H_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q)) = \varnothing$ and $H_{s_{n+1}}(q) \cap (\cap_{i=1}^n U_{s_i}(q)) = \varnothing$. The first claim follows from the results we obtained above. Thus, note that if this were not true, that is if there exists $x \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q))$, since $\cap_{i=1}^n L_{s_i}(q)$ is open, we could take a sufficiently $\delta > 0$ such that $N_\delta(x) \subset \cap_{i=1}^n L_{s_i}(q)$. Since $x \in H_{s_{n+1}}(q)$, we have that $\beta x + (1 - \beta)y \in N_\delta(x) \cap L_{s_{n+1}}(q)$ for some $y \in L_{s_{n+1}}(q)$ and some β sufficiently small and we would thus obtain a contradiction with the fact that $L_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q)) = \varnothing$. As for the second part of the claim, note that if there exists $x \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^n U_{s_i}(q))$ we would have $x \cdot s_i > q \cdot s_i$ for all $i \in \{1, ..., n\}$ and $x \cdot s_{n+1} = q \cdot s_{n+1}$. Consider then the element $x' = q + \alpha(q - x)$ for some $\alpha > 0$. We will then have $x' \cdot s_i < q \cdot s_i$ for all $i \leq n$ and $x' \cdot s_{n+1} = q \cdot s_{n+1}$ so $x' \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q))$ which we know that cannot hold by the first part of the claim and thus we are done.

Given that Y and Y' are closed and convex and $Y \cap int(Y') = \varnothing$ we can use the Separating Hyperplane Theorem to obtain that there exists a vector $\phi \in \mathbb{R}^n \backslash \{0\}$ and a number $k \in \mathbb{R}$ such that such that $\phi \cdot w \geq k$ for all $w \in Y$ and $\phi \cdot w \leq k$ for all $w \in Y'$. But since $((q - q) \cdot s_1, ..., (q - q) \cdot s_n) \in Y \cap Y'$ we must have $k = \phi \cdot 0 = 0$. Also, note that for any $w \in Y$ we have $-w \in Y$ so $\phi \cdot w \geq 0$ and $\phi \cdot (-w) \geq 0$ so $\phi \cdot w = 0$. Moreover, note that since $\phi \cdot w \leq k = 0$ for all $w \in Y'$ we must have $\phi \geq 0$.

Therefore, we obtained that for any $z \in H_{s_{n+1}}(q)$, that is for any $z \in \Omega$ with $(z-q)\cdot s_{n+1} = 0$ we must have $(z-q)\cdot(\phi_1 s_1 + ... + \phi_n s_n) = 0$. Then, using for instance Theorem 5.81 from Aliprantis and Border (1999, pp. 207) we have that $s_{n+1} = \psi(\phi_1 s_1 + ... + \phi_n s_n)$ for some $\psi \in \mathbb{R}$. From the fact that $s_{n+1} \in \mathbb{R}^N \backslash \{0\}$ it follows that $\psi \neq 0$. Since $(\cap_{i=1}^{n} L_{s_i}(q)) \cap L_{s_{n+1}}(q) = \varnothing$ we must also have $\psi < 0$ so $s_{n+1} = \sum_{i=1}^{n} \alpha_i s_i$ with $\alpha_i \equiv \psi\phi_i \leq 0$ and the proof of the Lemma 43 is complete. ■

Lemma 44 $\cap_{i=1}^{N} L_{s_i}(B) \neq \varnothing$.

Proof. We will prove the lemma by induction. Clearly, we have $L_{s_1}(B) \neq \varnothing$ so assume that $\cap_{i=1}^{n} L_{s_i}(B) \neq \varnothing$ and by contradiction that $\cap_{i=1}^{n+1} L_{s_i}(B) = \varnothing$. By Lemma 43, it would follow that $s_{n+1} \in cone(\{-s_1, ..., -s_n\})$ so $s_{n+1} = \sum_{i=1}^{n} \alpha_i(-s_i)$ with $\alpha_i \geq 0$. But then, $s_{n+1} + \sum_{i=1}^{n} \alpha_i s_i = 0$ which contradicts the fact that $\{s_1, ..., s_{n+1}\}$ are linearly independent. Therefore, we must have $\cap_{i=1}^{n+1} L_{s_i}(B) \neq \varnothing$ and this completes the induction proof. ■

Lemma 45 *There exists a set* $\widehat{S}_5 \subset \widehat{S}_3$ *with* $\mu(\widehat{S}_5) > 0$ *and* $z' \in \Omega$ *such that* $z' \in U_{-s_*}(B) \cap \left(\cap_{s\in\widehat{S}_5} L_s(B)\right)$.

Proof. Since $\cap_{i=1}^{N} L_{s_i}(B) \neq \varnothing$ by Lemma 44 and $-s_* \notin cone(\{s_1, ..., s_N\})$ which implies immediately that $s_* \notin cone(\{-s_1, ..., -s_N\})$ we can use Lemma 43 to conclude that $L_{s_*}(B) \cap \left(\cap_{i=1}^{N} L_{s_i}(B)\right) \neq \varnothing$. But since $L_{s_*}(B) = \{y \in \Omega : y \cdot s_* < z \cdot s_* \text{ for all } z \in B\} = \{y \in \Omega : y \cdot (-s_*) > z \cdot (-s_*) \text{ for all } z \in B\} = U_{-s_*}(B)$ it follows that $U_{-s_*}(B) \cap (\cap_{i=1}^{N} L_{s_i}(B)) \neq \varnothing$. So we can take some $z' \in U_{-s_*}(B) \cap (\cap_{i=1}^{N} L_{s_i}(B))$. Moreover, since $z' \in \cap_{i=1}^{N} L_{s_i}(B)$ it follows that $z' \in \cap_{s\in cone(\{s_1,...,s_N\})} L_s(B)$. To see this, note firstly that $z' \cdot s_i < x \cdot s_i$ for all $x \in B$ and for each i. Take some $s = \sum_{i=1}^{N} \alpha_i s_i$ with $\alpha_i \geq 0$ for all i. Then for any $x \in B$ we will have $z' \cdot s = \sum_{i=1}^{N} \alpha_i (z' \cdot s_i) < \sum_{i=1}^{N} \alpha_i (x \cdot s_i) = x \cdot s$ so that $z' \in L_s(B)$. Now, denote by $\widehat{S}_5 \equiv cone(\{s_1, ..., s_N\}) \cap \widehat{S}_3$. Since $z' \in U_{-s_*}(B) \cap \left(\cap_{s\in cone(\{s_1,...,s_N\})} L_s(B)\right) \subset U_{-s_*}(B) \cap \left(\cap_{s\in\widehat{S}_5} L_s(B)\right)$ and $\widehat{S}_5 \subset \widehat{S}_3$ with $\mu(\widehat{S}_5) > 0$ by Lemma 42, the proof of the Lemma 45 is complete. ■

Lemma 46 *There exists a lottery* $z \in \Delta(Z)$ *such that for all* $s \in \widehat{S}_5$ *we have* $\sigma_B(s) > \sigma_{\{z\}}(s)$ *and* $\sigma_B(-s_*) < \sigma_{\{z\}}(-s_*)$.

Proof. Since B is compact we have $U_{-s_*}(B) = U_{-s_*}(\{z''\})$ for some $z'' \in B$. Since $B \subset int(\Delta(Z))$ we have $z'' \in int(\Delta(Z))$ so there exists $z \equiv \alpha z'' + (1-\alpha)z' \in int(\Delta(Z)$ for some sufficiently high $\alpha < 1$. Since $z' \in U_{-s_*}(B)$ we will have $\sigma_B(-s_*) < z' \cdot (-s_*)$. On the other hand, by choice of z'' we have $z'' \cdot (-s_*) \geq x \cdot (-s_*)$ for all $x \in B$ so $z'' \cdot (-s_*) \geq \sigma_B(-s_*)$. Therefore $\sigma_{\{z\}}(-s_*) = \alpha z'' \cdot (-s_*) + (1-\alpha) z' \cdot (-s_*) > \sigma_B(-s_*)$. Finally, for any $s \in \widehat{S}_5$ we have $\sigma_B(s) \geq z'' \cdot s$

while $z' \in L_s(B)$ implies $\sigma_B(s) > z' \cdot s$. Thus, $\sigma_B(s) > \alpha z'' \cdot s + (1-\alpha) z' \cdot s = z \cdot s = \sigma_{\{z\}}(s)$ and the proof of the lemma is complete. ■

We will complete now the proof of Proposition 35. Thus, consider the sets $A \cup \{z\}$ and $B \cup \{z\}$ and we want to show that we must have $W(\sigma_{hull(B\cup\{z\})}) > W(\sigma_{hull(A\cup\{z\})})$ which would be sufficient to exclude the case when $\mu(\{-s_*\}) = 0$. To see this, note that $B \subset B'$ implies $W(\sigma_{hull(B'\cup\{z\})}) \geq W(\sigma_{hull(B\cup\{z\})})$ so we found $z \in \Delta(Z)$ with $\lambda\sigma_{\{z\}}(-s_*) \geq \lambda\sigma_B(-s_*)$ and $W(\sigma_{hull(B'\cup\{z\})}) > W(\sigma_{hull(A\cup\{z\})})$. This contradicts Axiom CEB-2 because B' was chosen arbitrarily from those sets satisfying the requirements of the axiom. Thus, we have:

$$W(\sigma_{hull(B\cup\{z\})}) = \int_{S^N\setminus\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) \geq$$

$$\geq \int_{S^N\setminus\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) >$$

$$> \int_{S^N\setminus\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) = W(\sigma_{hull(A\cup\{z\})})$$

where the weak inequality comes from the fact that $A \subset B$ so $\sigma_B(s) \geq \sigma_A(s)$ for all s. The strict inequality comes from the fact that for any $s \in \widehat{S}_5 \subset \widehat{S}_3$ with $\mu(\widehat{S}_5) > 0$ we have $\sigma_B(s) > \sigma_{\{z\}}(s)$ and $\sigma_B(s) > \sigma_A(s)$ so that $(\sigma_B \vee \sigma_{\{z\}})(s) > (\sigma_A \vee \sigma_{\{z\}})(s)$. Therefore, we must have $\mu(\{-s_*\}) > 0$ and thus the proof of Proposition 35 is complete. ■

Proposition 47 *Under Axiom CEB and Axiom CEB-2 there exists $\varepsilon > 0$ such that $\mu(N_\varepsilon(-s_*)\setminus\{-s_*\}) = 0$.*

Proof. Most steps in the proof of this proposition are identical to steps from the proof of the previous proposition so we will present in detail only the step at which the two proofs differ. Assume Axiom CEB is satisfied and by contradiction that the statement of Proposition 47 is false. Thus, for any $\varepsilon > 0$ we have $\mu(N_\varepsilon(-s_*)\setminus\{-s_*\}) > 0$. Repeat the steps from Lemmas 36-37 in the proof of Proposition 35 to construct the open set $\widehat{S}_1 \subset S^N$ with $\mu(\widehat{S}_1) > 0$ and $-s_* \in \widehat{S}_1$ such that $\sigma_B(s) > \sigma_A(s)$ for any $s \in \widehat{S}_1$. We will next show that the result from Lemma 38 is true in this case as well.[40] Then, the rest of the proof will go through as above and thus we would conclude that Axiom CEB-2 is violated which would constitute the contradiction.

We have $-s_* \in \widehat{S}_1$ and we claim that there exists $\varepsilon > 0$ such that $\mu\left(\widehat{S}_1\setminus cone(\overline{N}_\varepsilon(-s_*))\right) > 0$ where $\overline{N}_\varepsilon(-s_*)$ is the *closed* ball of radius ε around $-s_*$ in $\mathbb{R}^N$. If this were not true we would then have $\mu\left(\widehat{S}_1\setminus cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$ for all $n \geq 1$. Note that $\left\{\widehat{S}_1\setminus cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right\}$ is an increasing

[40] Note that it is the proof of Lemma 38 where we used the contradiction assumption that $\mu(\{-s_*\}) = 0$ in the proof of Proposition 35.

sequence of sets with $\cup_{n=1}^{\infty}\left(\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = \widehat{S}_1\backslash cone(\{-s_*\})$. But $\widehat{S}_1\backslash cone(\{-s_*\}) = \widehat{S}_1\backslash\ \{-s_*\}$ because $cone(\{-s_*\})\cap\widehat{S}_1 = \{-s_*\}$ as argued in the proof of the Lemma 38 from Proposition 35. So, we can use Theorem 9.8(i) in Aliprantis and Border(1999, pp. 337) to conclude that $\mu(\widehat{S}_1\backslash\{-s_*\}) = \lim_{n\to\infty}\mu\left(\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$. Since $-s_*\in\widehat{S}_1$ and $\widehat{S}_1$ is open, there must exist an open neighborhood $N_\delta(-s_*)$ of $-s_*$ included in $\widehat{S}_1$ such that $\mu(N_\delta(-s_*)\backslash\{-s_*\}) = 0$ which would contradict our assumption. Therefore, there must exist some $\varepsilon > 0$ such that $\mu\left(\widehat{S}_1\backslash cone(\overline{N}_\varepsilon(-s_*))\right) > 0$ which completes the proof of our claim and thus of the Proposition 47. ■

We will complete now the proof of the sufficiency of the Axioms. Using (30) for any $A\in\widetilde{K}(\Delta(Z))$, we can write:

$$W(\sigma_A) = \int_{S^N\backslash\{-s_*\}}\left[\max_{x\in A}(x\cdot s)\right]\mu(ds) + \max_{x\in A}(x\cdot(-s_*))\mu(\{-s_*\}). \tag{31}$$

In particular, for $A = \{z\}$ we will have:

$$W(\sigma_{\{z\}}) = \int_{S^N\backslash\{-s_*\}}(z\cdot s)\mu(ds) + (z\cdot(-s_*))\mu(\{-s_*\}).$$

But as shown above, $W(\sigma_{\{z\}}) = v(z) = \lambda(z\cdot s_*)$ so we have:

$$z\cdot s_* = \frac{1}{\lambda+\mu(\{-s_*\})}\int_{S^N\backslash\{-s_*\}}(z\cdot s)\mu(ds)$$

so using (31) we get:

$$W(\sigma_A) = \int_{S^N\backslash\{-s_*\}}\left[\max_{x\in A}(x\cdot s)\right]\mu(ds) + \max_{z\in A}\left[-\frac{\mu(\{-s_*\})}{\lambda+\mu(\{-s_*\})}\int_{S^N\backslash\{-s_*\}}(z\cdot s)\mu(ds)\right]$$

In conclusion, since $W(\sigma_A) = V(A)$ and $A\sim hull(A)$, for any $A\in K(\Delta(Z))$ we get the desired normalized reference-dependent representation:

$$V(A) = \int_S\left[\max_{x\in A}(x\cdot s)\right]\widetilde{\mu}(ds) - \theta\min_{z\in A}\left[\int_S(z\cdot s)\widetilde{\mu}(ds)\right] \tag{32}$$

where $\theta\equiv\frac{\mu(\{-s_*\})}{\lambda+\mu(\{-s_*\})}$, $S\equiv S^N\backslash N_\varepsilon(-s_*)$ and $\widetilde{\mu}(ds)\equiv\frac{\mu(ds)}{\mu(S^N)-\mu(\{-s_*\})}$ for $s\neq -s_*$ and $\widetilde{\mu}(\{-s_*\})\equiv 0$. Note that since $\mu(\{-s_*\}) > 0$ by Proposition 35 and $\lambda > 0$ we will have $\theta\in(0,1)$. Also, since we have $\mu(N_\varepsilon(-s_*)\backslash\{-s_*\}) = 0$ by Lemma 47 it follows that $\widetilde{\mu}(N_\varepsilon(-s_*)) = \widetilde{\mu}(N_\varepsilon(-s_*)\backslash\{-s_*\}) + \widetilde{\mu}(\{-s_*\}) = \frac{\mu(N_\varepsilon(-s_*)\backslash\{-s_*\})}{\mu(S^N)-\mu(\{-s_*\})} = 0$ and thus condition *(iii)* from Definition 11 is also satisfied. This completes the sufficiency part of the proof of Theorem 27. ■

Proof of Necessity in Theorem 27:

We will show next that the a preference relation which can be represented by a utility function as in (9) must satisfy Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2. The fact that the preference will satisfy the first three of the axioms is true because the representation in (9) is just a particular form of a DLR representation which implies those axioms. Also, given the equivalent representation in **(2)** it is clear that the preference must also satisfy Monotonicity so it remains to show that also Axiom CEB and Axiom CEB-2 must be satisfied.

Firstly, note that if $V(A) = \int_S \left[\max_{x\in A}(x\cdot s)\right]\mu(ds) - \theta \min_{z\in A}\left[\int_S (z\cdot s)\mu(ds)\right]$ represents the preference over menus in $K(\Delta(Z))$, then for any $x, y \in \Delta(Z)$ we have $\{x\} \succ \{y\}$ if and only if $\int_S (x\cdot s)\mu(ds) > \int_S (y\cdot s)\mu(ds)$.

For Axiom CEB, take any $A, B \in K(\Delta(Z))$, with $A \subset B$ such that there exists $y \in B$ with $\{x\} \succ \{y\}$ for all $x \in A$. Then, using the representation in (9) we will have $V(B) = \int_S \left[\max_{x\in B}(x\cdot s)\right]\mu(ds) - \theta \min_{z\in B}\left[\int_S (z\cdot s)\mu(ds)\right] \geq \int_S \left[\max_{x\in A}(x\cdot s)\right]\mu(ds) - \theta \min_{z\in B}\left[\int_S (z\cdot s)\mu(ds)\right] > \int_S \left[\max_{x\in A}(x\cdot s)\right]\mu(ds) - \theta \min_{z\in A}\left[\int_S (z\cdot s)\mu(ds)\right] = V(A)$. The second inequality comes from the fact that $y \in B$ and $\{y\} \prec \{x\}$ for all $x \in A$ implies $\min_{z\in B}\left[\int_S (z\cdot s)\mu(ds)\right] < \min_{z\in A}\left[\int_S (z\cdot s)\mu(ds)\right]$.

To prove that Axiom CEB-2 must be satisfied, the following lemma will constitute the main step of the argument. Note that by part *(iii)* of the representation in (9) there exists $\varepsilon > 0$ such that $\mu(N_\varepsilon(-s_*)) = 0$.

Lemma 48 *When the preferences $\succeq$ admit a normalized reference-dependent representation as in (9), there exist a compact set $A' \subset int(\Delta(Z))$ and a lottery $y \in int(\Delta(Z))$ such that $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$ and $\sigma_{A'}(s) > \sigma_{\{y\}}(s)$ for all $s \in S^N \backslash N_\varepsilon(-s_*)$.*

Proof. Again the proof will share some steps which are similar to steps from the proof of Proposition 35 so we will present them in less detail here. However, we emphasize that while in both proofs the initial steps consist of partitioning some compact set of ex post states into a finite number of subsets such that the states contained in each subset share some common properties, there is an important difference in terms of the ultimate goal of these arguments. Thus, in the proof of Proposition 35 the partitioning was done so that we could then claim that one of these subsets must be of strictly positive measure since the measure of the initial set was strictly positive. In the proof of this Lemma, the goal is to partition the set $S^N \backslash N_\varepsilon(-s_*)$ into a finite number of subsets so that we can resolve the problems raised by the infiniteness of the state space.

Firstly, since $-s_* \notin S^N \backslash cone(N_\varepsilon(-s_*))$ we can use an argument similar to the one from the proof of Lemma 39 from Proposition 35 to cover $S^N \backslash N_\varepsilon(-s_*)$ with 2^{N-1} elements $\{S_1, ..., S_{2^{N-1}}\}$ such that $-s_* \notin cone(S_j)$ for any j. By taking their closures, we can assume that the elements are all closed sets. Then, using the approach from Lemma 40, we can partition each S_j to obtain a cover of $S^N \backslash N_\varepsilon(-s_*)$ with elements indexed by a finite set J, such that $diam(S_j) \leq \delta$ for some $\delta < \frac{1}{2}$ and all $j \in J$. Again, by taking closures we can assume that S_j are closed for all j. Next, as in Lemma 41 we can show that for each $j \in J$ there exists $\epsilon_j > 0$ such that for each j we have $-s_* \notin cone(\cup_{s \in S_j} \overline{N}_{\epsilon_j}(s))$. Thus, as in Lemma 42 we can find a cover of S_j with a finite family of sets of the form $\{int(cone(\{s_{1,i}, ..., s_{N,i}\})) \cap S_j\}_{i \in I_j}$ such that for any $i \in I_j$, $\{s_{1,i}, ..., s_{N,i}\}$ are linearly independent and $-s_* \notin int(cone(\{s_{1,i}, ..., s_{N,i}\}))$. Let $j(i)$ be the index j such that $i \in I_j$ and let $I \equiv \cup_{j \in J} I_j$. Note that by construction, I is a finite set. Take some arbitrary lottery $y \in int(\Delta(Z))$.

For each $i \in I$, since $\{s_{1,i}, ..., s_{N,i}\}$ are linearly independent we can employ Lemma 43 as in the proof of Lemma 44 from Proposition 35 to conclude that $\cap_{k=1}^N L_{s_{k,i}}(y) \neq \varnothing$ and then immediately that $\cap_{j=1}^N L_{-s_{k,i}}(y) \neq \varnothing$. Therefore, using again Lemma 43 for the set $\{-s_{1,i}, ..., -s_{N,i}\}$ and $-s_*$ we will have that $\left(\cap_{k=1}^N L_{-s_{k,i}}(y)\right) \cap L_{-s_*}(y) \neq \varnothing$ so $\left(\cap_{k=1}^N U_{s_{k,i}}(y)\right) \cap L_{-s_*}(y) \neq \varnothing$ because $L_{-s_{k,i}}(y) = U_{s_{k,i}}(y)$. Now, for each $i \in I$, take $x_i' \in \left(\cap_{k=1}^N U_{s_{k,i}}(y)\right) \cap L_{-s_*}(y)$ and note that by an argument similar to the one from Lemma 45 we will have $x_i' \in \left(\cap_{s \in int(cone(\{s_{1,i}, ..., s_{N,i}\})) \cap S_{j(i)} \cap (S^N \backslash N_\varepsilon(-s_*))} U_s(y)\right) \cap L_{-s_*}(y)$. Since x_i' as chosen above is not necessarily in $\Delta(Z)$ consider the set $\{\lambda x_i' + (1-\lambda)y : \lambda \in (0,1)\}$ and note that since $y \in int(\Delta(Z))$ which is algebraically open, for a small enough λ we will have $x_i \equiv \lambda x_i' + (1-\lambda)y \in int(\Delta(Z))$. In addition, since $(\lambda x_i' + (1-\lambda)y) \cdot s \geq y \cdot s$ when $x_i' \cdot s \geq y \cdot s$ and $(\lambda x_i' + (1-\lambda)y) \cdot (-s_*) < y \cdot (-s_*)$ when $x_i' \cdot (-s_*) < y \cdot (-s_*)$ it follows that

$$x_i \in \left(\cap_{s \in int(cone(\{s_{1,i}, ..., s_{N,i}\})) \cap S_{j(i)} \cap (S^N \backslash N_\varepsilon(-s_*))} U_s(y)\right) \cap L_{-s_*}(y) \cap int(\Delta(Z)). \tag{33}$$

Let $A' \equiv \cup_{i \in I} x_i$. Firstly, note that since $x_i \in L_{-s_*}(y)$ for each i we have $x_i \cdot (-s_*) < y \cdot (-s_*) \Rightarrow \sup_{i \in I}(x_i \cdot (-s_*)) = \max_{i \in I}(x_i \cdot (-s_*)) < y \cdot (-s_*)$ so $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$. On the other hand, the family $\{int(\{cone(\{s_{1,i}, ..., s_{N,i}\})) \cap S_{j(i)} \cap \left(S^N \backslash N_\varepsilon(-s_*)\right)\}_{i \in I}$ being a cover of $S^N \backslash N_\varepsilon(-s_*)$, for any $s \in S^N \backslash N_\varepsilon(-s_*)$ we will have $s \in int(cone(\{s_{1,i}, ..., s_{N,i}\})) \cap S_{j(i)}$ for some $i \in I$. Therefore, $\sigma_{A'}(s) \geq \sigma_{\{x_i\}}(s) > \sigma_{\{y\}}(s)$ which completes the proof of the Lemma 48. ■

Let $A \equiv hull(A')$ and $B \equiv hull(A \cup y)$, where A' and y are given by the Lemma 48 and we will show that A and B thus defined will satisfy the conditions of the Axiom CEB-2 from Appendix B which we already proved that is equivalent to Axiom CEB-2. Firstly, since $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$ it follows that $\sigma_B(-s_*) = \sigma_{A \cup \{y\}}(-s_*) = \sigma_A(-s_*) \vee \sigma_{\{y\}}(-s_*) = \sigma_{A'}(-s_*) \vee \sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*) = \sigma_A(-s_*)$ where we employed repeatedly the fact that $\sigma_C(\cdot) = \sigma_{hull(C)}(\cdot)$. Using similar steps and the monotonicity of $\sigma_{(\cdot)}(s)$, it can be shown that $\sigma_{A'}(s) > \sigma_{\{y\}}(s)$ implies $\sigma_A(s) = \sigma_B(s)$ for $s \in S^N \backslash N_\varepsilon(-s_*)$. Secondly, we want to show that for any lottery $z \in \Delta(Z)$ with $\lambda \sigma_{\{z\}}(-s_*) \geq$

$\lambda\sigma_B(-s_*)$, we have $W(\sigma_{hull(B\cup\{z\})}) = W(\sigma_{hull(A\cup\{z\})})$.

For $C \in \{A, B\}$ we have

$$W(\sigma_{hull(C\cup\{z\})}) = \int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(C\cup\{z\})}(s)\mu(ds) + \int_{N_\varepsilon(-s_*)} \sigma_{hull(C\cup\{z\})}(s)\mu(ds).$$

Since $\mu(N_\varepsilon(-s_*)) = 0$ and $|\sigma_{hull(C\cup\{z\})}(s)| \leq |\sigma_{\Delta(Z)}(s)| < \infty$ because $\Delta(Z)$ is compact, we have $W(\sigma_{hull(C\cup\{z\})}) = \int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(C\cup\{z\})}(s)\mu(ds)$. On the other hand,

$$\int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(A\cup\{z\})}(s)\mu(ds) = \int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(B\cup\{z\})}(s)\mu(ds)$$

because $\sigma_A(s) = \sigma_B(s)$ for $s \in S^N\setminus N_\varepsilon(-s_*)$. Thus, we have $W(\sigma_{hull(B\cup\{z\})}) = W(\sigma_{hull(A\cup\{z\})})$ as desired. Finally, note that standard results guarantee that A and B are compact sets since A' is finite and $A\cup\, y$ is compact.

This completes the proof of the necessity of the axioms for the representation. We mention here that this slightly elaborate construction of the set A is necessary. Thus, it would have not been enough to select a lottery $x_s \in U_s(y) \cap L_{-s_*}(y)$ for each $s \in S^N\setminus N_\varepsilon(-s_*)$ appealing to the Axiom of Choice and then define $A \equiv cl\left(\cup_{s\in S^N\setminus N_\varepsilon(-s_*)} x_s\right)$, because $x_s \in L_{-s_*}(y)$ for all $s \in S^N\setminus N_\varepsilon(-s_*)$ would not necessarily imply $\sup_{x\in A}(x\cdot(-s_*)) < y\cdot(-s_*)$ as needed in order to show the required condition that $\sigma_{\{y\}}(-s_*) > \sigma_A(-s_*)$. On the other hand, $\cap_{s\in S^N\setminus N_\varepsilon(-s_*)} U_s(y)$ is in general not necessarily nonempty so we cannot just take an element in the intersection of this set with $L_{-s_*}(y)$ and let A be that element. An alternative approach would be to take some element $y' \in L_{-s_*}(y) \cap int(\Delta(Z))$ and then to try take elements $x_s \in U_s(y) \cap L_{-s_*}(y')$ with the aim of obtaining the strict condition $\sigma_{\{y\}}(-s_*) > \sigma_{\{y'\}}(-s_*) \geq \sigma_A(-s_*)$. However, this approach also runs into problems because even though $U_s(y) \cap L_{-s_*}(y') \neq \varnothing$ we cannot insure in general that $U_s(y) \cap L_{-s_*}(y') \cap \Delta(Z) \neq \varnothing$ as necessary to obtain $A \subset \Delta(Z)$. ■

B3. Proof of Theorem 29

Since the preferences $\succeq$ satisfy Weak Order, Independence and Continuity, Proposition 2 in DLR(2001) shows that the function that represents these preferences must be unique up to an affine transformation. Thus, if $V_i(A) = \int_S \left[\max_{z\in A}(z\cdot s)\right]\mu_i(ds) - \theta_i \min_{x\in A}\left[\int_S (x\cdot s)\mu_i(ds)\right]$ are two normalized reference-dependent representations of $\succeq$, then $V_1 = \alpha V_2 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. If $v_i(z) \equiv V_i(\{z\})$ are the corresponding restrictions to the singletons, we must have $v_1 = \alpha v_2 + \beta$. As argued in Appendix A, for each $i \in \{1, 2\}$ there exists $s^i_* \in S^N$ and $\lambda_i \geq 0$ such that $v_i(z) = \lambda_i(z \cdot s^i_*)$ for all $z \in \Delta(Z)$. Moreover, as argued in Appendix B, we must actually have $\lambda_i > 0$. Therefore, for any $z \in \Delta(Z)$ we have $\lambda_1(z \cdot s_*^1) = \alpha\lambda_2(z \cdot s_*^2) + \beta$. Because of the normalization $\sum_{k=1}^N s^k = 0$ for all elements in S^N, if we take $z^* = (\frac{1}{N}, ..., \frac{1}{N}) \in \Delta(Z)$ we will have $z^* \cdot s_*^i = 0$. Thus, $\lambda_1(z^* \cdot s_*^1) = \alpha\lambda_2(z^* \cdot s_*^2) + \beta$ implies $\beta = 0$.

Therefore, $z \cdot (\lambda_1 s_*^1) = z \cdot (\alpha\lambda_2 s_*^2)$ for any $z \in \Delta(Z)$ which in turn implies that $\lambda_1 s_*^1 = \alpha\lambda_2 s_*^2$. To see this, for each $k \in \{1, ..., N\}$ take $z_k = (0, ..., 0, 1, 0, ..., 0) \in \Delta(Z)$ with the 1 on k^{th} position and note that $z_k \cdot (\lambda_1 s_*^1) = z_k \cdot (\alpha\lambda_2 s_*^2)$ implies $(\lambda_1 s_*^1)_k = (\alpha\lambda_2 s_*^2)_k$ where by $(w)_k$ we denote the k^{th} coordinate of a finite dimensional vector w. Thus, s_*^1 is an affine transformation of s_*^2 which immediately implies that $s_*^1 = s_*^2$, because $s_*^i \in S^N$ for $i \in \{1, 2\}$ and we know that S^N contains the *unique* normalization of any affine function. On the other hand, as shown in Appendix A for any $A \in \widetilde{K}(\Delta(Z))$ we have $V_i(A) = W_i(\sigma_A) = \int_{S^N} \sigma_A(s)\mu_i(ds)$ where μ_i is the measure from the DLR representation. Since $V_1(A) = \alpha V_2(A)$ we have $\int_{S^N} \sigma_A(s)\mu_1(ds) = \int_{S^N} \sigma_A(s)(\alpha\mu_2)(ds)$ and then Lemma 14 in Sarver(2005) shows that this implies that $\mu_1 = \alpha\mu_2$. But μ_1 and μ_2 are both normalized to be probability measures so it must be that $\alpha = 1$ and then $\mu_1 = \mu_2$. Finally, $\alpha = 1$ together with $\lambda_1 s_*^1 = \alpha\lambda_2 s_*^2$ and $s_*^1 = s_*^2$ imply $\lambda_1 = \lambda_2$.

Now, recall that at the end of the sufficiency part of the proof of Theorem 27 we used the elements of the DLR representation to define the elements of our normalized reference-dependent representation. More specifically, with a slight abuse of notation we have $\theta_i \equiv \frac{\mu_i(\{-s_*^i\})}{\lambda_i + \mu_i(\{-s_*^i\})}$ and $\widetilde{\mu}_i(ds) \equiv \frac{\mu_i(ds)}{\mu_i(S^N) - \mu_i(\{-s_*^i\})}$ for $i \in \{1, 2\}$. Since $s_*^1 = s_*^2$, $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$ it follows that $\theta_1 = \theta_2$ and $\widetilde{\mu}_1 = \widetilde{\mu}_2$ which completes the proof of Theorem 29. ■

C. Proof of Proposition 30

Throughout the proof of this Proposition, we will denote by RHS and LHS the right hand side and left hand side of the inequation that we will be dealing with at that moment. Also, we will use RHS^a and LHS^a to denote the corresponding expressions with the *adjustments* that we will consider at that moment. Finally, we will use the same upperscript to refer to the adjusted incentive constraints of the problem in (17).

Lemma 49 *The optimal solution of the seller's problem in (17) requires* $u_l(q_l', p_l') = 0$.

Proof. Assume by contradiction that $u_l(q_l', p_l') > 0$ and note that by (ICP_l) it follows $u_l(q_l, p_l) > 0$. Take now ε satisfying the following conditions: *(i)* $0 < \varepsilon < u_l(q_l', p_l')$, *(ii)* $\varepsilon < \frac{1}{1-\theta}\left[u_h(q_l', p_l') - u_h(q_l, p_l)\right]$, whenever $u_h(q_l', p_l') > u_h(q_l, p_l)$. Consider the following adjustment: $p_l^a = p_l + \theta\varepsilon$ and $p_l'^a = p_l' + \varepsilon$ so that we will also have $u_l(q_l, p_l^a) > 0$ and $u_l(q_l', p_l'^a) > 0$. We want to show that all incentive restrictions of problem (17) will still be satisfied. Since the value of the objective function will be strictly higher this is enough to prove the lemma. Note firstly that the adjustment does not affect in any way equations (IRR_h^a) and (ICP_h^a) while (IRR_l^a) would continue to hold by the choice of ε in *(i)*. For (ICM_l^a) we have $LHS^a = LHS$, while the RHS is not affected. For (ICP_l^a), note that $LHS^a = LHS - \theta\varepsilon > LHS - \varepsilon > RHS - \varepsilon = RHS^a$, where we used the fact that $\theta \in (0,1)$ and (ICP_l). We will show now that (ICM_h^a) also holds. Note that $u_h(q_l, p_l^a) > u_l(q_l, p_l^a) > 0$ and $u_h(q_l', p_l'^a) > u_l(q_l', p_l'^a) > 0$ so the products (q_l, p_l^a) and $(q_l', p_l'^a)$ satisfy the necessary incentive rationality conditions to act as selected options or as reference points. We have two cases to consider. First, if $u_h(q_l, p_l) \geq u_h(q_l', p_l')$, then we will also have $u_h(q_l, p_l^a) > u_h(q_l', p_l'^a)$ so $RHS^a = RHS$ and then immediately $LHS^a > RHS^a$ because LHS is not affected by the adjustment. For the case when $u_h(q_l', p_l') > u_h(q_l, p_l)$, note that the condition *(ii)* implies that $u_h(q_l', p_l'^a) > u_h(q_l, p_l^a)$. Therefore, $RHS^a = RHS + \theta^2\varepsilon - \varepsilon < RHS \leq LHS^a$. ■

Lemma 50 *The optimal solution of problem (17) requires* $u_h(q_h', p_h') = 0$.

Proof. Assume $u_h(q_h', p_h') > 0$, which by (ICP_h) also implies $u_h(q_h, p_h) > 0$ and take ε such that: *(i)* $0 < \varepsilon < u_h(q_h', p_h')$, *(ii)* $\varepsilon < \frac{1}{\theta} u_l(q_h, p_h)$, when $u_l(q_h, p_h) > 0$, *(iii)* $\varepsilon < u_l(q_h', p_h')$, when $u_l(q_h', p_h') > 0$ and *(iv)* $\varepsilon < \frac{1}{1-\theta}\left[u_l(q_h', p_h') - u_l(q_h, p_h)\right]$, when $u_l(q_h', p_h') > u_l(q_h, p_h)$. Consider the following adjustment: $p_h^a = p_h + \theta\varepsilon$ and $p_h'^a = p_h' + \varepsilon$ so that $u_h(q_h, p_h^a) > u_h(q_h', p_h'^a) > 0$. Using arguments similar to ones from the proof of the previous lemma, it follows that the only condition that we need to check is (ICM_l^a). Note that since here the signs of $u_l(q_h, p_h)$ and $u_l(q_h', p_h')$ are not determined, we need to consider more cases. Firstly, if $u_l(q_h, p_h) > u_l(q_h', p_h') > 0$, then $RHS^a = RHS$ and (ICM_l^a) will be satisfied. Also, if $\max(u_l(q_h, p_h), u_l(q_h', p_h')) \leq 0$ then $RHS^a = RHS = 0$ so we are done. If $u_l(q_h, p_h) > 0 \geq u_l(q_h', p_h')$, then $RHS^a = RHS - \theta\varepsilon + \theta^2\varepsilon < RHS \leq LHS^a$, where we used the fact that the condition *(ii)* implies $u_l(q_h, p_h^a) > 0$. If $u_l(q_h', p_h') > 0 \geq u_l(q_h, p_h)$, then using the condition *(iii)*, we have $RHS^a = RHS - \varepsilon + \theta\varepsilon < RHS \leq LHS^a$. Finally, if $u_l(q_h', p_h') > u_l(q_h, p_h) > 0$, then *(iv)* ensures $u_l(q_h', p_h'^a) > u_l(q_h, p_h^a)$ while *(ii)* ensures $u_l(q_h, p_h^a) > 0$. Therefore, $RHS^a = RHS - \varepsilon + \theta^2\varepsilon < RHS \leq LHS^a$ so the proof is complete. ■

Note that $\max(u_i(q_{-i}, p_{-i}), u_i(q_{-i}', p_{-i}')) - \theta \min(u_i(q_{-i}, p_{-i}), u_i(q_{-i}', p_{-i}')) \geq 0$ implies that the RHS of (ICM_i) is at least as high as the RHS of (IRP_i) which is zero by the results of Lemma 49 and Lemma 50. Since the LHS of both (ICM_i) and (IRP_i) are equal to $u_i(q_i, p_i)$, (ICM_i) implies (IRP_i) for $i \in \{h, l\}$ so we may also disregard both (IRP)s.

Lemma 51 *The solution of problem (17) requires that* (ICM_h) *binds and* $u_h(q_l, p_l) = u_h(q_l', p_l')$.

Proof. Consider (ICM_h) and note that $u_l(q_l', p_l') = 0$ implies $u_h(q_l', p_l') > 0$, while (IRT_l) implies $u_h(q_l, p_l) > 0$. Assume now by contradiction that (ICM_h) does not bind. But then note that the principal could increase p_h slightly and thus increase its profits without violating (ICM_h). p_h appears also in (ICM_l), but an increase in p_h would only relax that constraint. Assume that $u_h(q_l, p_l) > u_h(q_l', p_l')$ so that $RHS = u_h(q_l, p_l) - \theta u_h(q_l', p_l')$. Using the result of Lemma 49, we have $u_h(q_l', p_l') = v_h(q_l') - v_l(q_l')$. Consider now a slight increase in q_l' coupled with a corresponding increase in p_l' so that $u_l(q_l'^a, p_l'^a) = 0$ and a corresponding increase in p_h so that (ICM_h^a) would continue to bind. As argued above, this would increase the value of the objective function without violating any of the restrictions so we must have $u_h(q_l, p_l) \leq u_h(q_l', p_l')$. Assume now that $u_h(q_l, p_l) < u_h(q_l', p_l')$ so that the RHS is $v_h(q_l') - v_l(q_l') - \theta u_h(q_l, p_l)$. Note that the single crossing property implies that the RHS is increasing in q_l'. Therefore, by decreasing q_l' and p_l' so that $u_l(q_l'^a, p_l'^a) = 0$ and increasing p_h so that (ICM_h^a) continues to bind, the objective function is strictly higher while all constraints would continue to hold. This completes the proof of the Lemma. ■

Note now that Lemma 50 implies that $u_l(v_h', p_h') < 0$ so the RHS of (ICM_l) is $(1-\theta)\max(0, u_l(v_h, p_h))$. Also note that (ICM_l) must bind because otherwise the principal could slightly increase p_l and thus increase the value of the objective function without violating any of the constraints. Therefore, the restrictions in problem in (17) basically reduce to the following: *(i)* $(ICM_h) : u_h(q_h, p_h) = (1-\theta)u_h(q_l, p_l)$ and *(ii)* $(ICM_l) : u_l(q_l, p_l) = (1-\theta)\max(0, u_l(q_h, p_h))$.

Lemma 52 *The optimal solution of problem (17) requires* $u_l(q_l, p_l) = 0$.

Proof. Assume by contradiction that the claim is not true so that $u_l(q_l, p_l) = (1-\theta)\, u_l(q_h, p_h) > 0$. But then, solving for p_l and p_h from the system given by (ICM_h) and (ICM_l), plugging the values into the objective function of problem (17) and taking the first order conditions with respect to q_h and q_l we obtain: $c'(q_h) = [v_h'(q_h) - (1-\theta)^2\, v_l'(q_h)] + \frac{\lambda}{1-\lambda}(1-\theta)\,[v_h'(q_h) - v_l'(q_h)]$ and $c'(q_l) = [v_l'(q_l) - (1-\theta)^2\, v_h'(q_l)] - \frac{1-\lambda}{\lambda}(1-\theta)\,[v_h'(q_l) - v_l'(q_l)]$. Clearly, $c'(q_h) > c'(q_l)$ so since $c(\cdot)$ is convex, it follows that $q_h > q_l$. Now consider the following adjustment: p_l^a is chosen such that $u_l(q_l, p_l^a) = 0$ and p_h^a such that $u_h(q_h, p_h^a) = (1-\theta)u_h(q_l, p_l^a)$. Note that $u_l(q_h, p_h^a) < 0$ because $q_h > q_l$ so for (ICM_l^a) we have $LHS^a = RHS^a = 0$. Therefore under the above adjustment, the value of the objective function is strictly higher because $p_l^a > p_l$ and $p_h^a > p_h$, while both restrictions are still satisfied. ■

Using the above results to solve for the optimal solution of problem (17) we obtain: $c'(q_h) = v_h'(q_h)$ and $c'(q_l) = v_l'(q_l) - \frac{1-\lambda}{\lambda}(1-\theta)\,[v_h'(q_l) - v_l'(q_l)]$. Since in the benchmark model we have $c'(q_h^b) = v_h'(q_h^b)$ and $c'(q_l^b) = v_l'(q_l^b) - \frac{1-\lambda}{\lambda}\left[v_h'(q_l^b) - v_l'(q_l^b)\right]$, it follows that q_l is closer to the value under perfect information given by $c'(q_l^b) = v_l'(q_l^b)$. Therefore, since the objective function is strictly concave, its value is strictly higher for problem in (17) than in the benchmark model. Also, since $c'(q_l) > c'(q_l^b)$ and $c(\cdot)$ is convex we have $q_l > q_l^b$. Finally, it is clear that the higher is θ, the smaller is q_l so the higher are the profits of the principal. This completes the proof of Proposition 30. ■

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