

# REPUTATION WITH LONG RUN PLAYERS

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## ABSTRACT.

Previous work shows that reputation results may fail in repeated games with long-run players with equal discount factors. We restrict attention to stage games where two players, with equal discount factors, move sequentially. A pure Stackelberg action is assumed to exist. One and two sided reputation results are provided. If one of the players is a Stackelberg type with positive probability, then that player receives the highest individually rational payoff in all perfect equilibria, as agents become patient. If both players are Stackelberg types with positive probability, then equilibrium payoffs converge to a unique payoff vector; and the equilibrium play converges to the unique equilibrium of a continuous time war of attrition. All results generalize to simultaneous move stage games, if the stage game is a game of strictly conflicting interest.

Keywords: Repeated Games, Reputation, Equal Discount Factor, Long-run Players, War of Attrition.

JEL Classification Numbers: C73, D83.

## 1. INTRODUCTION AND RELATED LITERATURE

This paper proves one and two sided reputation results for a class of two-player repeated games. The players have equal discount factors. Our first main result shows that if there is incomplete information about the type of only one player, then that player receives the highest possible payoff, in any equilibrium, as the discount factors converge to one. Our second main result establishes that if there is incomplete information about both players'

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types, then the equilibrium path of play, in any equilibrium, resembles a war of attrition, as the time between repetitions of the stage game shrinks to zero. Also, all equilibrium payoffs converge to the unique equilibrium payoff of an appropriately defined continuous time war of attrition.

Reputation effects were first established for finitely repeated games by Kreps and Wilson (1982) and Milgrom and Roberts (1982); and extended to infinitely repeated games by Fudenberg and Levine (1989). Fudenberg and Levine (1989, 1992) focused on repeated games where one long-run player faces a sequence of short-run players. The authors considered an incomplete information version of the repeated game where the long-run player is a “Stackelberg” (or commitment) type who always plays the “Stackelberg” (or commitment) action. They proved a *one-sided reputation result* that shows if the long-run player is sufficiently patient, then the long-run player’s equilibrium payoff is at least as large as what the player could get by publicly committing to the Stackelberg action. However, most reputation results in the literature are for repeated games where one of the players is a short-run player (as in Fudenberg and Levine (1989, 1992)); or for repeated games where the player building the reputation is infinitely more patient than his rival and so the rival is essentially a short-run player, at the limit (for example, see Schmidt (1993b) or Celantani, Fudenberg, Levine, and Pesendorfer (1996)).

Previous research has also shown that reputation results may fail when long-run players with equal discount factors play an infinitely repeated game in which the stage game is a simultaneous-move game in normal form. In particular, one-sided reputation results obtain only if the stage game is a game of *strictly conflicting interest* (Cripps, Dekel, and Pesendorfer (2005)), or the commitment action is a dominant action in the stage (Chan (2000)).<sup>1</sup> For all other simultaneous move games, folk theorems by Chan (2000) and Cripps and Thomas (1997) have shown that any individually rational and feasible payoff can be

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<sup>1</sup>A game has strictly conflicting interests (Chan (2000)) if a best reply to the commitment action of player 1 yields the best feasible and individually rational payoff for player 1 and the minimax for player 2.

sustained in perfect equilibria of the infinitely repeated game, if the players are sufficiently patient.

In what follows, a particular strategy for the extensive form stage game is termed a *Stackelberg action* for player 1 (him) if, whenever player 2 (her) best replies to this stage game strategy, player 1 receives his highest individually rational payoff.<sup>2</sup> A *Stackelberg type* is a player who always plays the Stackelberg action in every repetition of the stage game. A game is termed a *generic Stackelberg game* if there exists a pure Stackelberg action and the Stackelberg payoff vector is unique. *One-sided reputation* refers to a repeated game with one-sided incomplete information, where player 1 is a Stackelberg type with positive probability; and *two-sided reputation* refers to a repeated game with two-sided incomplete information, where each player can be Stackelberg type with positive probability.

Almost all the recent work on reputation has focused on simultaneous move stage games. In sharp contrast, we restrict attention to stage games which are extensive form games where the two agents move sequentially. Also, we assume that the normal form representation of the extensive form stage game is a generic Stackelberg game. These games include common interest games, the battle of the sexes, the chain store game as well as all strictly conflicting interest games (see section 2.1). For the class of games we consider, without incomplete information, the folk theorem of Fudenberg and Maskin (1986) applies, under a full dimensionality condition (see Sorin (1995) and Rubinstein and Wolinsky (1995)). Also, if the normal form representation of the extensive form game we consider is played simultaneously in each period, then under one-sided incomplete information, a folk theorem applies for a subset of the class of games we consider (see Chan (2000)).

Our *one sided reputation result* shows that any perfect equilibrium payoff for player 1 converges to his highest individually rational payoff, as the discount factor converges to one. The result does not depend on the order of moves within the stage game but is sensitive to the assumption that the players do not move simultaneously. Sequential moves by the players enables perfection to preclude folk theorems. The one-sided reputation

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<sup>2</sup>In our set-up a pure stage game action is a pure strategy profile for the extensive form stage game.

result remains valid without alteration, if the stage is a simultaneous move game of strictly conflicting interest. Also, our one-sided reputation result is robust to a small positive probability that player 1 is any other arbitrary behavioral type.

Our *two sided reputation result* establishes that equilibrium play in the repeated game resembles a war of attrition and all equilibrium payoffs converge to the unique equilibrium payoff vector of a continuous time war of attrition, as the time between the repetitions of the stage game shrinks to zero. In the war of attrition, the two players insist on their Stackelberg action, revealing rationality slowly over time, according to a mixed strategy. Once one of the players reveals rationality by playing an action that differs in outcome from the Stackelberg action, then the one-side reputation result applies. The player who is not known to be rational with certainty receives the highest individually rational payoff for that agent, in the continuation. The continuation payoff of the agent that has revealed rationality is equal to the payoff the player gets from best responding to the Stackelberg action. The unique limit payoff entails the “weak” player receiving a payoff equal to the payoff from best responding to the Stackelberg action. If the game is not a common interest game, then the “strong” player receives a payoff that is less than the highest individually rational payoff for this agent.<sup>3</sup> This is because it takes time for the “weak” player to reveal rationality. Consequently, the limit equilibrium is inefficient. Strength in the repeated game is determined by the relative costs of not best responding to the Stackelberg actions, and by the relative initial probabilities of being a Stackelberg type. Also, our two sided reputation results remain valid, without alteration, if the stage game is a simultaneous move strictly conflicting interest game.

The two sided reputation result we obtain is closely related to previous work by Kreps and Wilson (1982), Abreu and Gul (2000) and Abreu and Pearce (2007). In their seminal paper, Kreps and Wilson (1982) partially showed that the equilibrium set of a finitely repeated game, with two-sided incomplete information, resembles equilibrium play in a continuous

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<sup>3</sup>If the game is a common interest game, then both players are “strong” and they receive their highest individually rational payoffs.

time war of attrition. Our approach is closest to Abreu and Gul (2000). In this paper the authors show that in a two-sided alternating offer bargaining game, as the frequency of offers increases, the equilibria of the (two-sided) incomplete information game converges to the unique equilibrium of a continuous time war of attrition. They use a one-sided reputation result for bargaining games with frequent offers due to Myerson (1991).

As stated earlier, in a repeated game with one-sided incomplete information, with a simultaneous move stage game, many payoffs can be supported in perfect equilibria. Therefore there are also many perfect equilibria of the two-sided incomplete information repeated game. Abreu and Pearce (2007) deal with this multiplicity by allowing players to write enforceable contracts that determine uniquely continuation payoffs. They consider a repeated interaction where every period a player chooses a stage game action, and is allowed to offer a contract that determines the continuation equilibrium payoff vector. They show that the equilibrium strategies converge to the unique equilibrium of a continuous time war of attrition as the period length shrinks to zero. If the support of the commitment types is rich enough, then as the commitment type probabilities approach zero, the limit of the equilibrium payoff vectors is the unique solution of the “Nash demand game.” Their approach differs significantly from ours. We do not assume enforceable contracts. In our set-up, once one of the players is known not to be a commitment type, then the continuation payoff is uniquely determined as a direct consequence of our one-sided reputation result.

The paper proceeds as follows: section 2 describes the model and discusses some examples that satisfy our assumptions; section 3 presents the main one-sided reputation result; section 4 outlines the continuous time war of attrition and presents the two-sided reputation result as well as some comparative statics; and section 5 concludes. All proofs that are not in the main text are in the appendix.

## 2. THE MODEL

The stage game  $\Gamma$  is an extensive form game of perfect recall. The set of players in the game is  $I = \{1, 2\}$ .

**Assumption 1.** *In the stage game  $\Gamma$  player 1 and player 2 never move simultaneously.*

$\Gamma_N$  is the normal form of the extensive form game  $\Gamma$ . The set of pure stage game actions,  $a_i$ , for player  $i$  in the game  $\Gamma_N$  is denoted  $A_i$  and the set of mixed stage game strategies  $\alpha_i$  is denoted  $\mathcal{A}_i$ . The payoff function of player  $i$  for the game  $\Gamma_N$  is  $g_i : A_1 \times A_2 \rightarrow \mathbb{R}$ .

The minimax for player  $i$ ,  $\hat{g}_i = \min_{\alpha_j} \max_{\alpha_i} g_i(\alpha_i, \alpha_j)$ . The set of feasible payoffs  $F = \text{co}\{g_1(a_1, a_2), g_2(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2\}$ ; and the set of feasible and individually rational payoffs  $G = F \cap \{(g_1, g_2) : g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$ . Largest feasible and individually rational payoff for player  $i$ ,  $\bar{g}_i = \max\{g_i : (g_i, g_j) \in G\} = 1$ . Also, let  $M > \max_{g \in F} |g_i|$ .

**Assumption 2** (Generic Stackelberg Game for Player  $i$ ). *There exists  $a_i^s \in A_i$  such that if  $a_j$  is a best response for player  $j$  to action  $a_i^s$ , then  $g_i(a_i^s, a_j) = \bar{g}_i = 1$ . Also, for any two payoff vectors  $g \in G$  and  $g' \in G$ , if  $g_i = g'_i = 1$ , then  $g_j = g'_j$ .*

Assumption 2 requires that there exists a pure Stackelberg action for player 1, denoted  $a_1^s$ , which gives player 1 his highest individually rational payoff if player 2 best replies to  $a_1^s$ . The assumption also requires that the Stackelberg payoff profile is unique, more precisely, whenever player 1 receives his largest individually rational payoff  $\bar{g}_1$ , player 2 receives the same payoff. Put another way, there can be more than one Stackelberg action for player 1, however, player 1 cannot guarantee  $\bar{g}_1$  unless player 2 best replies. Moreover, player 2's best response to player 1's Stackelberg actions need not be unique, however, player 2's payoff from best responding to player 1's Stackelberg actions is assumed to be unique. If Assumption 2 holds for player 1, then there is a finite constant  $\rho \geq 0$  such that  $(1 - g_1)\rho + g_2(a_1^s, a_2^b) \geq g_2 \geq g_2(a_1^s, a_2^b) - (1 - g_1)\rho$  for any  $(g_1, g_2) \in G$ , where  $a_2^b$  denotes a best reply to  $a_1^s$ .

**Assumption 3** (Strictly Conflicting Interest for Player  $i$ ). *There exists  $a_i^s \in A_i$  such that if  $a_j$  is a best response for player  $j$  to action  $a_i^s$ , then  $g_i(a_i^s, a_j) = \bar{g}_i = 1$ . Also, for all  $(\bar{g}_i, g_j) \in G$ ,  $g_j = \hat{g}_j$ .*

Assumption 3 implies Assumption 2 and so it is a strict strengthening of Assumption 2. If Assumption 3 holds for player 1, then there exists a finite constant  $\rho \geq 0$  such that  $\hat{g}_2 \leq g_2 \leq \hat{g}_2 + \rho(\bar{g}_1 - g_1)$  for all  $(g_1, g_2) \in G$ .

In the repeated game  $\Gamma^\infty$ , the stage game  $\Gamma$  is played in each of periods  $t = 0, 1, 2, \dots$ . Players have perfect recall and can observe past outcomes.  $H$  is the set of all possible histories for the stage game  $\Gamma$  and  $Z \subset H$  the set of all terminal histories of the stage game.  $H_t \equiv Z^t$  denotes the set of partial histories at time  $t$ . A behavior strategy is a function  $\sigma_i : \bigcup_{t=0}^\infty H_t \rightarrow \mathcal{A}_i$ . A behavior strategy chooses a mixed stage game strategy given the partial history  $h_t$ . Players discount payoffs using their discount factor  $\delta = e^{-r\Delta}$ . The players' continuation payoffs in the repeated game given the partial history  $h_t$  are given by the normalized discounted sum of the continuation stage-game payoffs

$$u_i(h_\infty, h_t) = (1 - \delta_i) \sum_{s=t}^{\infty} \delta^{s-t} g(a_i, a_j)$$

where  $h_\infty = (h_t, h_{-t})$  is any infinite history where the play in the first  $t-1$  periods conforms to  $h_t$ .

Before time 0 nature selects each player  $i$ , with probability  $z_i$ , to be a “commitment” type that always plays action  $a_i^s$  or to be a normal type. A belief for player  $i$  is a function  $z_j : \bigcup_{t=0}^\infty H_t \rightarrow [0, 1]$ , that is, the belief,  $z_j(h_t)$ , is the probability that player  $i$  places on player  $j$  being a commitment type given any history  $h_t$ . Beliefs are obtained using Bayes' rule given the initial probabilities  $z_i$  and players' strategies. A player's expected continuation utility, following a partial history  $h_t$ , when the normal players use the strategy profile  $\sigma$ , is given by

$$U_i(\sigma|h_t) = (1 - z_j(h_t))\mathbb{E}[u_i(h_\infty, h_t)|\sigma] + z_j(h_t)\mathbb{E}[u_i(h_\infty, h_t)|\sigma_i, a_j^s]$$

where the expectation is taken over all histories  $h_\infty = (h_t, h_{-t})$ , generated when normal players use strategy profile  $\sigma$  and the commitment type always plays  $a_j^s$ .

The repeated game where the initial probability that the players are commitment types  $z = (z_1, z_2)$  and the discount factor is  $\delta$  is denoted  $\Gamma^\infty(z, \delta)$ . The analysis in the paper focuses on perfect Bayesian equilibria of the game of incomplete information  $\Gamma^\infty(z, \delta)$ . In particular, a strategy profile  $\sigma$  comprises a perfect Bayesian equilibrium for the repeated game of incomplete information if the strategy profile is a Bayesian Nash equilibrium for any subgame of the repeated game given beliefs  $z_i$ .

**2.1. Examples.** The following are examples of stage games that satisfy Assumption 2. These examples are not games of strictly conflicting interest, and also, the players do not have strictly dominant actions in these games. Consequently, if in any of these stage games the two players move simultaneously (i.e, subgame perfection is not imposed within the stage game), then Folk theorems apply implying that there is no scope for reputation effects. (See Cripps and Thomas (1997) for Example 1, and Chan (2000) for Examples 1 through 4). That is, in any of these games, any individually rational payoff profile can be sustained as a perfect equilibrium of the repeated game, for sufficiently high discount factor and sufficiently small probability of facing a commitment type. In marked contrast, if the players move sequentially in these examples, (i.e., if the games satisfy Assumption 1), then Theorem 1 and Theorem 2 provide one and two sided reputation results. It should be noted that in these examples if the repeated game is played under complete information and if the players move sequentially, that is, the stage game is viewed as an extensive form game, then the usual folk theorems continue to apply and all (strictly) individually rational payoffs can be sustained in perfect equilibria for sufficiently high discount factors. (see for example, Rubinstein and Wolinsky (1995) or Sorin (1995) for more on repeated extensive-form games.)

**2.1.1. Common Interest Games.** Consider the following simplified common interest game as depicted in Figure 1 and suppose that player 2 chooses between  $L$  and  $R$  first and player 1 chooses between  $U$  and  $D$  after observing player 2's chosen action.



		Player 2	
		<i>L</i>	<i>R</i>
P1	<i>U</i>	1, 1	0, 0
	<i>D</i>	$\epsilon, 0$	0, 0

 FIGURE 1. Common Interest Game ( $\epsilon < 1$ ).

Assume that there is a (possibly small) probability that one of the two players (but not both) is a commitment type that always plays the Stackelberg action (action  $U$  for 1 and  $L$  for 2). In this case Theorem 1 implies that the player who is potentially a commitment type can guarantee a payoff arbitrarily close to 1 in any perfect equilibrium of the repeated game, for sufficiently high discount factors. Instead assume that there is a (possibly small) probability that both players are commitment types that always play the Stackelberg action. In this case, Theorem 2 implies that the equilibrium path of play involves both players always choosing the Stackelberg action in every period and equilibrium payoffs equal to (1,1), for discount factors close to 1.

2.1.2. *Battle of the Sexes*. In the version of the battle of the sexes depicted in Figure 2 suppose that player 1 chooses his action first and player 2 follows. In this game both players prefer action profiles on the diagonal of the payoff matrix but Player 1 prefers ( $U, L$ ) while player 2 prefers ( $D, R$ ). The minimax payoff for each player is  $1/2$  and is attained by the action profile ( $M, M$ ).

		Player 2		
		<i>L</i>	<i>R</i>	<i>M</i>
P1	<i>U</i>	2, 1	0, 0	0, 0
	<i>D</i>	0, 0	1, 2	0, 0
	<i>M</i>	0, 0	0, 0	1/2, 1/2

FIGURE 2. Battle of the Sexes.

Theorem 1 and 2 provide one and two sided reputation results for this example. In particular, if each of the two players is a commitment type with probability  $z_i > 0$ , then Theorem 2 implies that the equilibrium path of play for this game resembles a war of attrition. During the “war” player 1 insists on playing  $U$  while player 2 insists on playing

$R$  and both receive per period payoff equal to 0. The “war” ends when one of the players reveals rationality by playing a best reply and accepting a continuation payoff equal to 1, while the opponent, who wins the war of attrition, receives continuation payoff equal to 2.

2.1.3. *Hawk-Dove*. In the hawk-dove game outlined in Figure 3, action  $T$ (ough) is preferable only if the opponent plays  $W$ (eak) while  $D$ (estroy) is a dominated action and results in hurting both players.

		Player 2		
		$T$	$W$	$D$
P1	$T$	$-1, -1$	$2, 0$	$-1/2, -1/2$
	$W$	$0, 2$	$0, 0$	$-1/2, -1/2$
	$D$	$-1/2, -1/2$	$-1/2, -1/2$	$-1, -1$

FIGURE 3. Hawk - Dove.

In the unique subgame perfect equilibrium of the static stage game, player 1 moves first and plays  $T$  and player 2 observes  $T$  and best responds by playing  $W$ . In contrast, in the perturbed repeated game where there is a chance that player 2 is a commitment type, player 2 can guarantee a payoff of 2 in every equilibrium by feigning irrationality and playing  $T$ . With two sided incomplete information, each players insist on play  $T$ , in a war of attrition like behavior, until one of them reveals rationality by switching to play  $W$ .

2.1.4. *A Game in Extensive Form*. In the static version of the extensive form game depicted in Figure 4, the subgame perfect equilibrium involves player 1 choosing to declare  $W$ (ar) and player 2 best responding by choosing to  $S$ (urrender). Suppose that player 2 is a Stackelberg type who always plays  $F$  if player 1 chooses  $W$  and plays  $S$  if player 2 plays  $P$ , with probability  $z > 0$ . (The Stackelberg type in this example is a “tit-for-tat” type.) In this case, Theorem 1 implies that player 2 can guarantee an outcome where player 1 chooses to play  $P$  and player 2 best replies by choosing  $S$ . In the two-sided reputation version the two players will be locked into playing  $(W, F), (W, F)$  until one of them gives in. The player that gives in will reveal rationality by choosing to play  $P$ , in the case of player 1, and  $S$ , in the case of player 2.

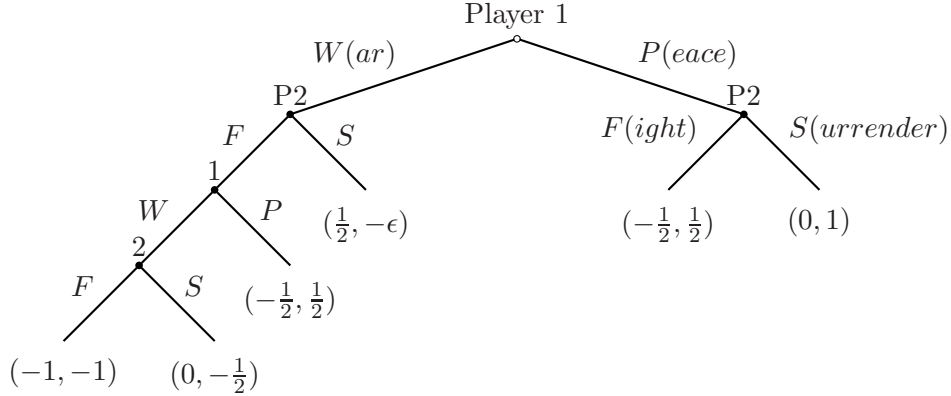


FIGURE 4. A Game in Extensive Form

### 3. ONE SIDED REPUTATION

The central finding of this section, Theorem 1, establishes a one sided reputation result. The theorem maintains Assumption 1 and Assumption 2, and shows that if the probability that player 1 is a commitment type is positive while the probability that player 2 is a commitment type is zero, then player 1 can guarantee a payoff close to his highest individually rational payoff for sufficiently high discount factors. The result in Theorem 1 does not depend on the order of moves in the extensive form game but crucially depends on the assumption that the two players do not move simultaneously.

This section also presents two corollaries to Theorem 1. Corollary 1 shows that the one sided reputation result can also be established without Assumption 1, if Assumption 2 is strengthened by maintaining Assumption 3, that is, under the stronger strictly conflicting interest stage game assumption. Corollary 2 shows that the one-sided reputation result continues to hold in the case where player 1, who builds reputation, is any other arbitrary type, different from the Stackelberg type, with sufficiently small probability.

In order to provide some intuition for Theorem 1 and to establish the contrast with previous literature suppose that the stage game is given by the common interest game as outlined in Figure 1 and that the players discount the future at a common rate  $\delta < 1$ . Note that  $(D, R)$  is a Nash equilibrium of the stage game and the minimax value for both players is zero. In the complete information repeated game, for sufficiently high  $\delta < 1$ , any

(strictly) individually rational payoff can be sustained in a perfect equilibrium. This is true when the stage game is played simultaneously (Fudenberg and Maskin (1986)) as well as when it is played sequentially (Sorin (1995)).

Suppose that there is a small chance  $z$  that player 1 is a commitment type that plays  $U$  in every period of the repeated game and suppose, in contrast to Assumption 1, that the players move simultaneously in the stage game. Cripps and Thomas (1997) have shown that there are many perfect equilibrium payoffs in this repeated game. In particular, they construct equilibria where players' payoffs are close to 0 when  $z$  is close to 0 and  $\delta$  is close to 1. In the equilibrium that they construct, in the first  $K$  periods player 2 plays  $R$ . As  $\delta$  converges to 1,  $K$  increases to ensure that the discounted payoffs converge to 0. To make  $K$  large, Player 1's equilibrium strategy is chosen to be similar to the commitment type strategy; this ensures that player 1 builds a reputation very slowly. If this strategy exactly coincided with the commitment strategy, player 2 would not have the incentives to play  $R$ . Therefore this strategy is a mixed strategy that plays  $D$  with small probability. To ensure that player 2 has an incentive to play  $R$ , she is punished when she plays  $L$ . Punishment entails a continuation payoff for player 2 that is close to 0, if player 2 plays  $L$  and player 1 plays  $D$  (and hence reveals rationality). Observe player 1 is willing to mix between  $U$  and  $D$  in the first  $K$  periods since player 2 only plays  $R$  on the equilibrium path. Also, the punishment that follows  $(D, L)$  is subgame perfect since, after such a history, the players are in a repeated game of complete information and any continuation payoff between 0 and 1 can be sustained in equilibrium, by a standard folk theorem.

Instead suppose that Assumption 1 is satisfied and player 1 moves after player 2. When players move sequentially, the "follower" (player 1) observes the outcome of the behavior strategy used by his opponent. For the payoff of player 1 to be low, there should be many periods in which player 2 plays  $R$ . To give her an incentive to play  $R$ , player 1 must punish player 2 if she plays  $L$ . After any history where player 1 has not revealed rationality yet, punishing player 2 is also costly for player 1. Following a play of  $L$  by player 2, in order for player 1 to punish player 2, he must be indifferent between  $U$  and  $D$ . However, this is

not possible since playing  $U$  gives player 1 a payoff of 1 for the period and improves his reputation. On the other hand, a play of  $D$  gives a payoff of zero for the period and moves the game into a punishment phase. Consequently, subgame perfection rules out player 1 punishing player 2 for playing  $L$ .

The reason Cripps and Thomas (1997)'s construction works and avoids the previous logic is as follows: If the stage game is a simultaneous move game, then subgame perfection has no "bite" within the stage game. Consequently, because player 1 does not observe that player 2 has deviated and played  $L$ , and this never actually happens on the equilibrium path of play in the first  $K$  periods, player 1 is willing to randomize between  $U$  and  $D$ , and so, the required punishments can be sustained.

**3.1. The Main One Sided Reputation Result.** Theorem 1 considers a repeated game  $\Gamma^\infty(z, \delta)$  where  $z_1 > 0$  and  $z_2 = 0$ , that is player 2 is known to be the normal type and player 1 is potentially a commitment type. Attention is restricted to sequential move stage games (Assumption 1) where player 1 has a pure Stackelberg action and the Stackelberg payoff profile is unique (Assumption 2). Within this class of games, the theorem demonstrates that a normal type for player 1 can secure a payoff arbitrarily close to the Stackelberg payoff  $\bar{g}$  by mimicking the commitment type, in any equilibrium of the repeated game, for a sufficiently large discount factor.

**Theorem 1.** *Suppose that the extensive form stage game  $\Gamma$  satisfies Assumption 1 and the normal form representation,  $\Gamma_N$ , of the extensive form game satisfies Assumption 2 for player 1. For any  $z = (z_1 > 0, z_2 = 0)$  and any  $\gamma > 0$ , there exists a  $\bar{\delta}$  such that, for any  $\delta > \bar{\delta}$  and any equilibrium strategy profile  $\sigma$  for the repeated game  $\Gamma^\infty(z, \delta)$ ,  $U_1(\sigma) > \bar{g}_1 - \gamma = 1 - \gamma$ .*

*Proof.*

*Step 1.* For this proof let  $\Gamma^\infty(z, \delta)$  denote a repeated game where  $z_1 = \underline{z}$  and  $z_2 = 0$ ; and let  $z(h)$  denote the probability that player 1 is a commitment type given history  $h$ . Observe

that following any history in which player 1 has only played  $a_1^s$ ,  $z(h) \geq \underline{z}$ , that is  $z(h)$  must be at least as large as the initial probability,  $\underline{z}$ , that player 1 is the commitment type. Let  $a_2^b$  denote a best reply to  $a_1^s$ . By Assumption 2,  $g_1(a_1^s, a_2^b) = \bar{g}$ . Normalize payoffs such that  $\bar{g} = 1$ ,  $g_2(a_1^s, a_2^b) = 0$  and  $g_2(a_1^s, a_2) \leq -l$  for any  $a_2$  that is not a best reply to  $a_1^s$ , for some  $l > 0$ . Also, let  $\min_{a_2 \in A_2} g_1(a_1^s, a_2) = 0$ .

*Step 2.* For an action profile  $a = (a_1, a_2)$ , let  $i(a) = 1$  if  $a_2$  is not a best reply to  $a_1^s$  and  $i(a) = 0$ , otherwise. Fix a history  $h_\infty$  and let  $i(\delta, h_\infty) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t i(a^t)$ . Let  $\sigma^*$  denote a strategy where player 1 always plays the Stackelberg action  $a_1^s$  and player 2 follows strategy  $\sigma_2$ . Let  $w(\delta, \sigma_2) = \mathbb{E}_{\sigma^*}[i(\delta, h_\infty)]$  where the expectation is taken over all histories generated when the players use strategy profile  $\sigma^*$ . In words,  $w(\delta, \sigma_2)$  is the expectation of the normalized discounted sum of the number of non-best replies player 2 will play against a commitment type who always plays the Stackelberg action  $a_1^s$ .

*Step 3.* Note that  $U_1(\sigma) \geq 1 - w(\delta, \sigma_2)$ . This is because the payoff to player 1 of always taking the Stackelberg action is at least  $1 - w(\delta, \sigma_2)$  by definition.

*Step 4.* Fix  $\delta < 1$ ,  $K > 0$  and let for each  $n \geq 0$

$$z_n = \sup\{z : \exists \text{ equilibrium } \sigma \text{ of } \Gamma^\infty(z, \delta) \text{ such that } w(\delta, \sigma_2) \geq K^n \epsilon\}$$

For any  $\xi > 0$ , let  $z_\xi = z_1 - \xi$  and

$$K_\xi = \sup\{k : \exists \text{ equilibrium } \sigma \text{ of } \Gamma^\infty(z, \delta) \text{ such that } w(\delta, \sigma_2) \geq k\epsilon \text{ and } z_1 \geq z \geq z_\xi\}.$$

Observe that by the definition of  $K_\xi$ , there exists  $z_1 \geq z \geq z_\xi$  and an equilibrium strategy profile  $\sigma$  such that  $w(\delta, \sigma_2) \geq (K_\xi - \xi)\epsilon$ , in the game  $\Gamma^\infty(z, \delta)$ . Also, by the definition of the supremum  $K_\xi \geq K$ .

Define  $q_1$  such that  $\frac{z_1}{1-q_1} = z_0$  and similarly define  $q_\xi$  such that  $\frac{z_\xi}{1-q_\xi} = z_1$ . To interpret  $q_1$ , suppose that player 2 believes player 1 to be the commitment type with probability  $z$ . Also, suppose that the probability that player 1 plays an action different than  $a_1^s$  at least

once over the next  $M$  periods is  $q$ . Now note that if player 1 plays the Stackelberg action  $a_1^s$  in each of the  $M$  periods, then the posterior probability that player 2 places on player 1 being the commitment type is  $\frac{z}{1-q}$ .

*Step 5.* For any reputation level  $z_1 \geq z \geq z_\xi$  and for any equilibrium  $\sigma$ ,

$$(LB) \quad U_2(\sigma) \geq -q_\xi K_\xi \epsilon \rho - q_1 K \epsilon \rho - (1 - z - q_1 - q_\xi) \epsilon \rho - 2(1 - \delta)(1 + M)\rho.$$

*Proof of Step 5.* Suppose that player 2 always plays  $a_2^b$ . Using this strategy player 2 will receive payoff equal to zero in any period where player 1 plays  $a_1^s$ . Let

$$T_\xi = \min\{T : \mathbb{P}\{\exists n \leq T : (a_1)_n \neq a_1^s\} > q_\xi\}.$$

That is,  $T_\xi$  is the smallest time  $T$  such that, the probability with which player 1 is expected to take an action different than the Stackelberg action  $a_1^s$  at least once, in any period  $n \leq T$ , exceeds  $q_\xi$ . Note that by definition for any  $T < T_\xi$ ,  $\mathbb{P}\{\exists n \leq T : (a_1)_n \neq a_1^s\} \leq q_\xi$ . Also, let

$$T_1 = \min\{T : \mathbb{P}\{\exists n \leq T : (a_1)_n \neq a_1^s\} > q_1 + q_\xi\}.$$

The definition implies that  $T_1 \geq T_\xi$  and also that

$$\frac{z_\xi}{1 - q_1 - q_\xi} > \frac{z_\xi}{(1 - q_1)(1 - q_\xi)}.$$

In any period  $n < T_\xi$ , where player 1 has always played as an action compatible with  $a_s^1$  in history  $h_n$ ,

$$U_1(\sigma|h_n) \geq 1 - w(\sigma, \delta) \geq 1 - K_\xi \epsilon$$

where the first inequality follows from step 2 and the second from the definition of  $z_\xi$ . Consequently, in any such period  $n < T_\xi$ , if player 1 plays an action different from  $a_1^s$ , then player 2 will receive continuation payoff

$$U_2(\sigma|h_{n < T_\xi}, a_1) \geq -\rho K_\xi \epsilon \triangleq \underline{U}_2(\sigma|h_{n < T_\xi}, a_1)$$

by Assumption 2, and otherwise if player 1 plays the Stackelberg action  $a_1^s$ , then player 2 will receive zero for the period. In period  $T_\xi$ , player 1's equilibrium payoff  $U_1(\sigma)$  must be at least as large as  $\delta(1 - K\epsilon) - (1 - \delta)M$ , if he has not revealed rationality. This is because player 1 can play  $a_1^s$  in this period, lose at most  $M$  for the period, increase his reputation to at least  $z_1$  and thereby guarantee a continuation payoff of at least  $1 - K\epsilon$ . This implies that if player 1 plays an action different from  $a_1^s$ , then player 2 will receive continuation payoff

$$U_2(\sigma|h_{T_\xi}, a_1) \geq -\rho(1 - \delta(1 - K\epsilon) + (1 - \delta)M) \geq -\rho K\epsilon - (1 - \delta)(1 + M)\rho \triangleq \underline{U}_2(\sigma|h_{T_\xi}, a_1)$$

and otherwise if player 1 plays the Stackelberg action  $a_1^s$ , then player 2 will receive zero for the period. In any period,  $T_\xi < n < T_1$  player 1's continuation payoff is at least  $1 - K\epsilon$  and so player 2's continuation payoff

$$U_2(\sigma|h_{T_\xi < n < T_1}, a_1) \geq -\rho K\epsilon \triangleq \underline{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1),$$

if player 1's action for the period  $a_1 \neq a_1^s$ . In period  $T_1$ , by the same logic as above, player 1's payoff is at least  $\delta(1 - \epsilon) - (1 - \delta)M$  and so

$$U_2(\sigma|h_{T_1}, a_1) \geq -\rho(1 - \delta(1 - \epsilon) + (1 - \delta)M) \geq -\rho\epsilon - (1 - \delta)(1 + M)\rho \triangleq \underline{U}_2(\sigma|h_{T_1}, a_1),$$

if player 1's action for the period  $a_1 \neq a_1^s$ . Also, for any period,  $n > T_1$ ,

$$U_2(\sigma|h_{n > T_1}, a_1) \geq -\rho\epsilon \triangleq \underline{U}_2(\sigma|h_{n > T_1}, a_1).$$

Observe that the probability that player 1 takes action  $a_1 \neq a_1^s$  for the first time in any period  $n < T_\xi$  is less than  $q_\xi$  and in any period  $n < T_1$  is less than  $q_\xi + q_1$ . Also,  $\underline{U}_2(\sigma|h_{n > T_1}, a_1) \geq \underline{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1) \geq \underline{U}_2(\sigma|h_{n < T_\xi}, a_1)$  and so

$$U_2(\sigma) \geq q_\xi \underline{U}_2(\sigma|h_{n < T_\xi}, a_1) + q_1 \underline{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1) + (1 - z - q_\xi - q_1) \underline{U}_2(\sigma|h_{n > T_1}, a_1) - 2(1 - \delta)(1 + M)\rho$$



delivering the required inequality. Observe that if  $T_\xi = \infty$  and  $T_1 = \infty$  the bound is still valid.  $\square$

*Step 6.* Choose  $z_1 \geq z \geq z_\xi$  and an equilibrium strategy profile  $\sigma$  of the game  $\Gamma^\infty(z, \delta)$  such that  $w(\delta, \sigma) \geq (K_\xi - \xi)\epsilon$ . For the chosen  $z$  and  $\sigma$ ,

$$(UB) \quad U_2(\sigma) \leq q_\xi K_\xi \epsilon \rho + q_1 K \epsilon \rho + (1 - z - q_1 - q_\xi) \epsilon \rho + 2(1 - \delta)(1 + M)\rho - z(K_\xi - \xi)\epsilon l.$$

*Step 7.* Let  $K$  be such that  $\frac{z_1 l}{2\rho} - \frac{2}{K} > 0$  and let  $\bar{\delta} = 1 - \frac{\epsilon}{2(1+M)}$ . For all  $\delta > \bar{\delta}$ , if  $z_1 \geq \underline{z}$ , then there exists an  $r > 0$  such that  $z_1 \leq z_0 \frac{1}{1+2r}$ .

*Proof of Step 7.* Combining the lower bound for  $U_2(\sigma)$ , given Equation (LB) established in Step 5, and the upper bound for  $U_2(\sigma)$ , given by Equation (UB) established in Step 6, and simplifying by canceling  $\epsilon$  delivers

$$z(K_\xi - \xi)l \leq 2\rho(q_\xi K_\xi + q_1 K + (1 - z - q_1 - q_\xi) + \frac{2(1 - \delta)(1 + M)}{\epsilon}).$$

and so,

$$z \leq \frac{2\rho(q_\xi K_\xi + q_1 K + 2 - z - q_1 - q_\xi)}{(K_\xi - \xi)l}.$$

However  $\xi$  is arbitrary, thus  $q_\xi \rightarrow 0$  and  $z \rightarrow z_1$ , as  $\xi \rightarrow 0$ . Also,  $K_\xi \geq K$  and so  $\lim_{\xi \rightarrow 0} (K_\xi - \xi)l \geq Kl$ . So

$$z_1 \leq \frac{2\rho(q_1 K + 2 - z_1 - q_1)}{Kl} \leq \frac{2\rho(q_1 K + 2)}{Kl}$$

Rearranging,

$$q_1 \geq \frac{z_1 l}{2\rho} - \frac{2}{K}.$$

Remember  $K$  is such that  $\frac{z_1 l}{2\rho} - \frac{2}{K} > 0$ . Note that  $K$  is independent of  $\delta$ . If  $z_1 \geq \underline{z}$ , then pick  $r > 0$  such that

$$\frac{z_1 l}{2\rho} - \frac{2}{K} \geq \frac{\underline{z} l}{2\rho} - \frac{2}{K} > r \geq r z_1 \geq r \underline{z}.$$

By definition  $\frac{z_1}{1-q_1} = z_0$  or  $q_1 = 1 - \frac{z_1}{z_0}$ , and so  $1 - \frac{z_1}{z_0} \geq z_1 r$  and consequently,

$$z_1 \leq \frac{z_0}{1 + rz_0} \leq \frac{z_0}{1 + r\underline{z}},$$

delivering the required inequality.  $\square$

*Step 8.* The argument in Step 5 through 7 implies that if  $z_n \geq \underline{z}$ , then

$$z_n \leq z_{n-1} \frac{1}{1 + r\underline{z}}.$$

The constants  $K$  and  $r > 0$  are independent of  $\delta$  and depend only on  $\underline{z}$ . So for some finite  $n^*$ , also independent of  $\delta$ ,  $z_{n^*} < \underline{z}$ . However, in any subgame where player 1 is yet to reveal his rationality, i.e., is yet to play an action different than  $a_1^s$ , the reputation level is at least  $\underline{z}$ . Consequently,  $w(\delta, \sigma_2) \leq K^{n^*} \epsilon$  for all  $\delta > \bar{\delta}$  and all equilibria  $\sigma$  of the repeated game  $\Gamma^\infty(\underline{z}, \delta)$ . Pick  $\epsilon = \frac{\gamma}{K^{n^*}}$ . Consequently, for all  $\delta > \bar{\delta} = 1 - \frac{\gamma}{2K^{n^*}(1+M)}$ , player 1's equilibrium payoff, in any equilibrium, will be at least  $1 - \gamma$  completing the proof.  $\square$

As first demonstrated by Cripps, Dekel, and Pesendorfer (2005), it is possible to obtain reputation results for simultaneous move stage games by strengthening Assumption 2. In particular, if Assumption 2 is strengthened to assume that the stage game is a game of strictly conflicting interest (Assumption 3), then Assumption 1 can be omitted. The following corollary to Theorem 1 maintains Assumption 3 and shows that Player 1 can guarantee a payoff arbitrarily close to  $\bar{g}_1$ .<sup>4</sup>

**Corollary 1.** *Assume that a normal form stage game  $\Gamma_N$  satisfies Assumption 3 for player 1. For any  $z = (z_1 > 0, z_2 = 0)$  and any  $\gamma > 0$ , there exists a  $\bar{\delta}$  such that, for any  $\delta > \bar{\delta}$  and any equilibrium strategy profile  $\sigma$  for the repeated game  $\Gamma^\infty(z, \delta)$ ,  $U_1(\sigma) > \bar{g}_1 - \gamma = 1 - \gamma$ .*

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<sup>4</sup>This provides an alternative argument, albeit by imposing subgame perfection, for the result in Cripps, Dekel, and Pesendorfer (2005). Cripps, Dekel, and Pesendorfer (2005) proved a reputation result in Nash equilibria for repeated games where the stage game is a game of strictly conflicting interest.

*Proof.* Normalize the minimax for player 2,  $\hat{g}_2 = 0$ . The argument is identical to Theorem 1. However, the lower bound for Player 2's payoff, as in step 6, is  $U_2(\sigma) \geq 0$ , since player 2 can guarantee her minimax value. The upper bound for player 2's payoff, given by Equation (UB), is again

$$U_2(\sigma) \leq q_\xi K_\xi \epsilon \rho + q_1 K \epsilon \rho + (1 - z - q_1 - q_\xi) \epsilon \rho + 2(1 - \delta)(1 + M)\rho - z(K_\xi - \xi)\epsilon l.$$

Observe this is a game of conflicting interest and so  $g_2 \leq \rho(\bar{g}_1 - g_1)$  for all  $(g_1, g_2) \in G$  which enables us to impose the upper bound. Combining the lower and upper bound and working through step 8 gives

$$q_1 \geq \frac{z_1 K l - 2}{\rho K}$$

and the rest proceeds again according to the theorem above.  $\square$

The following corollary shows that the one sided reputation result extends to a situation where player 1 is another arbitrary type with probability  $\phi$ . In particular, the corollary demonstrates that a normal type for player 1 can secure a payoff arbitrarily close to the Stackelberg payoff by mimicking the commitment type, in any equilibrium of the repeated game, for a sufficiently large discount factor, and for sufficiently low  $\phi$ . Let  $\Gamma^\infty(z, \phi, \delta)$  denote a repeated game where the probability that player 1 is an arbitrary type is equal to  $\phi$  and consequently the probability that player 1 is the normal type is  $1 - z - \phi$ .

**Corollary 2.** *Suppose that the extensive form stage game  $\Gamma$  satisfies Assumption 1 and the normal form representation,  $\Gamma_N$ , of the extensive form game satisfies Assumption 2 for player 1. For any  $z = (z_1 > 0, z_2 = 0)$  and any  $\gamma > 0$ , there exists a  $\bar{\delta}$  and  $\phi^*$  such that, for any  $\delta > \bar{\delta}$ , any  $\phi < \phi^*$  and any equilibrium strategy profile  $\sigma$  for the repeated game  $\Gamma^\infty(z, \phi, \delta)$ ,  $U_1(\sigma) > \bar{g}_1 - \gamma = 1 - \gamma$ .*

*Proof.* The lower bound, Equation (LB), can be altered as follows

$$U_2(\sigma) \geq -q_\xi K_\xi \epsilon \rho - q_1 K \epsilon \rho - (1 - z - \phi - q_1 - q_\xi) \epsilon \rho - 2(1 - \delta)(1 + M)\rho - \phi M.$$

since player 2 can loose at most  $M$  against another type and this happens with probability  $\phi$ . The upper bound, Equation (UB) can be altered as follows:

$$U_2(\sigma) \leq q_\xi K_\xi \epsilon \rho + q_1 K \epsilon \rho + (1 - z - \phi - q_1 - q_\xi) \epsilon \rho + 2(1 - \delta)(1 + M)\rho + M\phi - z(K_\xi - \xi)\epsilon l.$$

since player 2 can make at most  $M$  against another type and this happens with probability  $\phi$ . Given these bounds, choosing  $\phi^*$  and  $\bar{\delta}$  such that  $\epsilon = \phi^* M + (1 - \bar{\delta})(1 + M)$  and following the steps in Theorem 1 gives the result.  $\square$

**3.2. Examples with no Reputation Effects.** As mentioned earlier previous researchers have shown that reputation results may fail without Assumption 1, for the class of games that we consider. (See for example, Chan (2000)). In this subsection we show that Assumption 2 is essential for reputation even when Assumption 1 is maintained. In particular, we provide two games that do not satisfy Assumption 2, where reputation results can not be obtained, and a wide range of payoffs can be sustained in perfect equilibria of the repeated game.

**3.2.1. Another Common Interest Game.** In the following altered version of the common interest game outlined in Figure 5 suppose that player 2 moves first and player 1 moves second.

		Player 2		
		$L$	$M$	$R$
P1	$U$	1, 1	1, $\epsilon$	0, 0
	$D$	$\epsilon$ , 0	0, 0	0, 0

FIGURE 5. Another Common Interest Game.

This example does not satisfy Assumption 2 since the Stackelberg payoff profile is not unique; when Player 1 receives 1, player 2 may receive 1 or  $\epsilon$ . In this game many payoff vectors can be sustained in equilibria. This game differs significantly from the Common Interest Game as outlined in Figure 1 in that player 1 can actually punish player 2 without

hurting himself by playing the action profile  $(U, M)$ . In this game the Cripps and Thomas (1997) construction will work since if player 2 deviates from the equilibrium strategy and plays  $L$  or  $M$ , then player 1 can be made indifferent between playing  $U$  and  $D$ . This is accomplished by designing the punishment phase so that the players receive payoff close to  $(1, \epsilon)$ . This is made possible by the introduction of the action profile  $(U, M)$  which punishes player 2 but is not costly for player 1.

**3.2.2. Moral Hazard Game.** In the game demonstrated in Figure 6 suppose that player 1 moves first and player 2 moves second. In this game, player 1's highest individually rational payoff is 1.5. However, player 1 does not have a pure Stackelberg action that would give his highest individually rational payoff. In order to receive 1.5 player 1 would have to use a mixed strategy that plays  $U$  and  $D$  with equal probability and have player 2 best respond by playing  $B$ . Let  $\alpha_1^s$  denote the mixed action where player 1 plays  $U$  and  $D$  with equal probability. Even if a pure action that replicated  $\alpha_1^s$  was available to player 1, the game still would not satisfy Assumption 2. This is because player 2's best reply to  $\alpha_1^s$  is not unique and if player 2 chooses to play  $N$ , which is a best reply to  $\alpha_1^s$ , player 1 receives payoff 0, which is not his highest individually rational payoff. Consequently, even if player 1 convinces player 2 that he is the Stackelberg type, many equilibrium payoffs can be sustained.

		Player 2	
		$B$	$N$
P1	$U$	1, 1	0, 0
	$D$	2, -1	0, 0

FIGURE 6. Moral Hazard Game.

#### 4. TWO SIDED REPUTATION AND A WAR OF ATTRITION

The main finding presented in this section, Theorem 2, establishes a two sided reputation result. Throughout the discussion we assume that  $z_1 > 0$  and  $z_2 > 0$ , that is the initial probability of being the Stackelberg type is strictly positive for both of the players. Recall

that  $\delta = e^{-r\Delta}$ . The focus of analysis is on sequences of repeated games  $\Gamma^\infty(z, \Delta_n)$  parameterized by the time  $\Delta_n$  between repetitions of the stage game. Theorem 2 maintains Assumption 1 and Assumption 2 for both players and demonstrated that as  $\Delta_n \rightarrow 0$ , for any sequence of perfect equilibrium strategy profiles  $\sigma_n$  for the repeated game  $\Gamma^\infty(z, \Delta_n)$ , any equilibrium payoff vector  $(U_1(\sigma_n), U_2(\sigma_n))$  converges to a unique limit. This unique limit is the unique equilibrium payoff vector of a continuous time war of attrition which is outlined below. Corollary 3 shows that Assumption 1 can be discarded and the players can be allowed to move simultaneously in the stage game if the stage game is a game of conflicting interest (Assumption 3).

In what follows we *assume that  $a_1^s$  is not a best response for  $a_2^s$*  (and consequently  $a_2^s$  is not a best response for  $a_1^s$ ). This assumption is maintained out of convenience in order to focus attention on the interesting cases. If  $a_1^s$  is a best response for  $a_2^s$  and consequently,  $a_2^s$  is a best response for  $a_1^s$ , then Theorem 1 immediately implies that  $(U_1(\sigma_n), U_2(\sigma_n)) \rightarrow (\bar{g}, \bar{g})$  for any sequence of perfect equilibria  $\sigma_n$  for the sequence of repeated games  $\Gamma^\infty(z, \Delta_n)$  where  $z_1 > 0$  and  $z_2 > 0$ .<sup>5</sup> For the remainder of the discussion normalize  $g_1(a_1^b, a_2^s) = g_2(a_1^s, a_2^b) = 0$  and define  $l_i \triangleq -g_i(a_1^s, a_2^s) > 0$ , i.e.,  $l_i$  is the loss incurred from playing the Stackelberg action against the Stackelberg action.

**4.1. The War of Attrition.** The War of Attrition is played over continuous time by the two players. At time zero, both players simultaneously choose either to concede or insist. If both players choose to insist, then the continuous time war ensues. The game continues until one of the two players concedes. Each player can concede at any time  $t \in [0, \infty]$ . If player  $i$  concedes at time  $t \in [0, \infty]$  and player  $j$  continues to play insist through time  $t$ , then player  $i$ 's payoff is  $-l_i (1 - e^{-rt})$  and player  $j$ 's payoff is  $e^{-rt}\bar{g} - l_j (1 - e^{-rt})$ . If both players concede concurrently at time  $t$ , then they receive payoff  $e^{-rt}g_i(t) - l_i (1 - e^{-rt})$  and  $e^{-rt}g_j(t)$

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<sup>5</sup>Observe that given the definition of  $w(\delta, \sigma_2)$  in Theorem 1, Step 2, the lower bound for player 1's payoff in Step 3 is unchanged. This is because the commitment type of Player 2 always best responds to the Stackelberg action of Player 1. Consequently, the Equations (LB) and (UB) are also unchanged. So, the theorem implies that  $w(\delta, \sigma_2)$  converges to zero. Going through the symmetric argument for Player 2 would show  $w(\delta, \sigma_1)$  converges to zero. So,  $(U_1(\sigma_n), U_2(\sigma_n)) \rightarrow (\bar{g}, \bar{g})$ .

$-l_i(1 - e^{-rt})$  where  $(g_i(t), g_j(t)) \in G$ , and consequently,  $-\rho(\bar{g} - g_j(t)) \leq g_i(t) \leq \rho(\bar{g} - g_j(t))$ . Before the game begins at time 0, nature chooses a type for each player independently. A player is chosen as either a Stackelberg type that never concedes, with probability  $z_i > 0$ , or a normal type, with probability  $1 - z_i$ .

This War of Attrition is closely related to the repeated game  $\Gamma^\infty(z, \Delta)$  for  $\Delta \approx 0$ : Insisting corresponds to playing the Stackelberg action in each period, and conceding corresponds to switching to any other action that reveals that the player is not the Stackelberg type. Players incur cost  $l_i$  from insisting on the Stackelberg action against the Stackelberg action.<sup>6</sup> They insist on the Stackelberg action in the hope that their rival will give in and play another action. If one of the players deviates from the Stackelberg action and the other does not, then the player that deviated from the Stackelberg action is known to be the rational type with certainty. After such a history, Theorem 1 implies that the player known as normal receives payoff zero and the rival receives payoff  $\bar{g}$ . This corresponds exactly to the payoffs when one of the players conceding at time  $t$  in the War of Attrition. Both players incur the cost  $l_i$  for  $t$  units of time, i.e.,  $l_i(1 - e^{-rt})$ ; the conceding player receives continuation payoff of zero; and the player that wins receives continuation payoff of  $\bar{g}$ , i.e.,  $e^{-rt}\bar{g}$ . If both players reveal rationality concurrently in period  $t$ , that is, if both players play concede in the War of Attrition in period  $t$ , then Theorem 1 puts no restrictions on continuation payoffs. So, agents receive an arbitrary payoff from the set of individually rational and feasible repeated game payoffs.

The War of Attrition outlined above differs from the game analyzed in Abreu and Gul (2000). In the War of Attrition presented here, the payoffs that the players receive, if they concede concurrently, depend on  $t$  and are potentially non-stationary. In contrast, in Abreu and Gul (2000) concurrent concessions involve stationary payoffs. Nevertheless, the argument below (for condition (i)) shows that the non-stationarity of payoffs does not introduce any new complications and Abreu and Gul (2000)'s analysis applies without alteration. In particular, the unique equilibrium of the War of Attrition satisfies three

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<sup>6</sup>Best replying gives a per-period payoff of zero.

conditions: (i) at most one agent concedes with positive probability at time zero, (ii) after time zero each player concedes with constant hazard rate  $\lambda_i$ , (iii) the normal types finish conceding at some finite time  $T$ . Consequently, at time  $T$  the posterior probability that an agent faces a Stackleberg type equals one.

In order to provide a rationale for condition (i) suppose that both players were to concede with positive probability at time  $t$ . If they concede concurrently, then player 1's payoff is  $g_1(t)$  and player 2's payoff is  $g_2(t)$ . By Assumption 2,  $g_i(t) \leq \rho(\bar{g} - g_j(t))$ . Consequently, for one the of the two players  $g_i(t) < \bar{g}$ . But for this player  $i$  waiting to see whether player  $j$  quits at time  $t$  and then quitting immediately afterwards does strictly better than quitting at time  $t$ . Consequently, both players cannot concede with positive probability at any time  $t$ ; and in particular, at most one of the players can concede with positive probability at time zero.

For some insight into condition (ii) note that player  $i$  cannot concede with certainty at any time. If player  $i$  were to concede with certainty at time  $t$ , then by not conceding player  $i$  would ensure that player  $j$  believes that player  $i$  is the Stackelberg type with probability one. But this would induce player  $j$  to concede immediately improving  $i$ 's payoff.

Condition (ii) implies that the hazard rate  $\lambda_i$  must leave  $j$  indifferent between conceding immediately and waiting for an additional  $\Delta$  units of time and then conceding. Conceding immediately guarantees player  $j$  zero. By waiting for  $\Delta$  units of time player  $j$  incurs cost  $l_j(1 - e^{-r\Delta})$ , but receives  $\bar{g}$  if  $i$  quits which happens with probability  $\Delta \lambda_i$ . Consequently,

$$0 = \lim_{\Delta \rightarrow 0} \Delta \lambda_i \bar{g} - l_j(1 - e^{-r\Delta})$$

$$\lambda_i = \lim_{\Delta \rightarrow 0} \frac{(1 - e^{-r\Delta})l_j}{\bar{g} \Delta} = \frac{rl_j}{\bar{g}}$$

Once one of the players' normal type has finished conceding and so the player is known as the Stackelberg type with certainty, then the normal type of the other player should also concede immediately. Because a Stackelberg type never concedes, the normal player has no



incentive to insist. Consequently, condition (iii) must hold and both players must complete their concession by the same finite time  $T$ . Given the concession rate  $\lambda_i$ , let  $T_i = \frac{-\ln z_i}{\lambda_i}$ . If player  $i$  does not concede with positive probability at time zero, then by time  $T_i$  the normal type of player  $i$  concedes with probability one, i.e., if  $F_i(0) = 0$ , then  $F_i(T_i) = 1 - z_i$ . The common concession time  $T$  must equal  $\min\{T_1, T_2\}$ . And so, if  $T_i > T_j$ , then  $F_i(0) > 0$ . In words, the player with the larger  $T_i$ , the player that concedes more slowly or who is more likely to be the normal type, is the weaker player that concedes with positive probability at time zero.

Let  $F_i(t)$  denote the cumulative probability that player 1 concedes by time  $t$ . That is,  $F_i(t)$  is  $1 - z_i$  multiplied by the probability that the normal type of player  $i$  quits by time  $t$ . Conditions (i) through (iii) imply that  $(F_1, F_2)$  is an equilibrium of the War of Attrition if and only if

$$F_i(t) = 1 - c_i e^{-\lambda_i t} \text{ for all } t \leq T < \infty$$

$$\lambda_i = \frac{r l_j}{\bar{g}}$$

$$(1 - c_1)(1 - c_2) = 0 \text{ for } c_i \in [0, 1],$$

$$F_i(T) = 1 - z_i.$$

**Proposition 1.** *The unique sequential equilibrium of the War of Attrition is  $(F_1, F_2)$ . Also, the unique equilibrium payoff vector for the War of Attrition is  $((1 - c_2)\bar{g}, (1 - c_1)\bar{g})$ .*

*Proof.* Observe if  $F_1$  jumps at time  $t$ , then  $F_2$  does not jump at time  $t$ . This follows from the argument provided for condition (i). The rest of the argument in Abreu and Gul (2000) applies verbatim. Thus  $F_1$  and  $F_2$  comprise the unique equilibrium for the War of Attrition.

Suppose that player 1 is the player that concedes with positive probability at time zero. Since player 1 concedes with positive probability at time zero, he is indifferent between conceding immediately and receiving a payoff equal to zero and continuing. Consequently, player 1's equilibrium payoff must equal zero. Player 2 is also indifferent between quitting

and conceding at any time after time zero. This implies that player 2's payoff is equal to zero at any time  $t > 0$ . Consequently, player 2's equilibrium payoff at the start of the game must equal  $(1 - c_1)\bar{g}$ .  $\square$

**4.2. The Main Two Sided Reputation Result.** Let the function  $G_1^n(t)$  denote the cumulative probability that player 1 reveals that he is rational by time  $t$ , if he is playing against the Stackelberg type of player 2, in an equilibrium  $\sigma$  of the repeated game  $\Gamma^\infty(z, \Delta^n)$ . Theorem 2 demonstrates that the distributions  $G_i^n$  have a limit and proves that this limiting distribution solves the War of Attrition and is thus equal to  $F_i$ . The Theorem then proceeds to show that convergence of  $G_i^n$  to  $F_i$  implies that the equilibrium payoffs in the repeated game also converge to the unique equilibrium payoff for the War of Attrition.

**Theorem 2.** *Suppose that  $a_1^s$  is not a best response to  $a_2^s$ . Also,  $\Gamma$  satisfies Assumption 1 and  $\Gamma_N$  satisfies Assumption 2 for both players. For any  $z = (z_1 > 0, z_2 > 0)$  and any  $\epsilon > 0$ , there exists a  $\Delta^*$  such that, for any  $\Delta < \Delta^*$ , any equilibrium strategy profile  $\sigma$  for the repeated game  $\Gamma^\infty(z, \Delta)$ ,  $|U_1(\sigma) - (1 - c_2)\bar{g}| < \epsilon$  and  $|U_2(\sigma) - (1 - c_1)\bar{g}| < \epsilon$ .*

**4.2.1. Comparative Statics.** Before proving the theorem we discuss some comparative statics. Theorem 2 establishes that the the unique limit equilibrium payoff for the repeated game is  $(1 - c_2)\bar{g}$  and  $(1 - c_1)\bar{g}$ . Suppose that  $1 - c_2 = 0$ , and make the generic assumption,  $\frac{\ln z_1}{\lambda_1} \neq \frac{\ln z_2}{\lambda_2}$ . Since player 2 never concedes at time zero, player 1's payoff is equal to zero, and player 2's payoff is equal to  $(1 - c_1)\bar{g}$ . Hence, player 1 is the weaker player. The identity of the agent who quits with positive probability at time zero, player 1 in this case, is determined by the quitting time if the agent was not expected to quit at time zero, that is,  $T_i = \frac{-\ln z_i}{\lambda_i}$  which solves the equation  $F_i(T_i) = 1 - z_i$  under the assumption  $c_i = 1$ . Since  $c_2 = 1$ ,  $T_1 > T_2$ . The time zero quitting probability  $c_1$  is calculated to ensure that  $F_1(T_2) = 1 - z_1$ . Consequently, player 2's payoff is decreasing in  $\lambda_1$  and  $z_1$ ; and increasing in  $\lambda_2$  and  $z_2$ . The concession rate of player 2, in turn, is increasing in player 1's cost of resisting the Stackelberg action,  $l_1$ . Thus, player 2's payoff is increasing in  $l_1$  and  $z_2$ ; and

decreasing in  $l_2$  and  $z_1$ . Also note that because  $c_1 > 0$ , player 2's payoff is strictly less than  $\bar{g}$  and so the limit equilibrium payoff is inefficient.

Another comparative statics exercise of possible interest is the limit of the equilibrium payoffs as the probabilities of irrationality  $z_1$  and  $z_2$  converge to zero at the same rate. Suppose that  $\lambda_1 < \lambda_2$ . Observe that as  $z_1$  and  $z_2$  converge to zero at the same rate,  $T_1/T_2$  converges to  $\lambda_2/\lambda_1 > 1$ . So, player 1 is the weak player close to the limit. Note that  $c_1 = z_1 z_2^{-\frac{\lambda_1}{\lambda_2}}$ , and so  $c_1$  converges to zero as the  $z$ 's converge to zero. This shows that the equilibrium payoff vector converges to zero for player 1 and  $\bar{g}$  for player 2 as the model converges to full rationality. Hence the limiting payoff at full rationality is  $(0, \bar{g})$  if  $\lambda_1 < \lambda_2$  and  $(\bar{g}, 0)$  if  $\lambda_1 > \lambda_2$ . This argument suggests that the equilibrium payoffs become efficient as the incomplete information in the model disappears. Also, as incomplete information becomes negligible, the prediction of the model is similar to the one-sided incomplete information case; the stronger player attains his/her highest individually rational payoff.

**4.2.2. Proof of Theorem 2.** After any history  $h$  where player  $i$  has always played only  $a_i^s$  and player  $j$  has played an action different than  $a_j^s$  at least once  $z_i(h) > \underline{z}_i$  and  $z_j(h) = 0$ . Consequently, after such a history Theorem 1 implies that  $U_i(\sigma|h) \geq \bar{g} - 2(1+M)(1 - e^{-r_i\Delta})K^{n^*}$  where both  $K$  and  $n^*$  are independent of  $\Delta$ . Let  $K(\Delta) = K^{n^*} 2(1+M)(1 - e^{-r\Delta})$  and so  $U_i(\sigma|h) \geq \bar{g} - K(\Delta)$  after any history where  $i$  has only played  $a_i^s$ . Also, let  $M(\Delta) = (\bar{g} + M)(1 - e^{-r\Delta})$  which converges to zero when  $\Delta$  goes to zero. Take  $\Delta$  sufficiently small so that  $\rho K(\Delta) + M(\Delta) < l_i$  and  $\bar{g} - K(\Delta) > 0$ .

*Step 1.* Let  $\sigma$  denote a mixed (not behavioral) repeated game strategy profile. Consequently,  $\sigma_i$  is a probability distribution over all pure repeated game strategies for player  $i$ . Let  $R_i(n)$  denote the set of pure repeated game strategies that have the following properties:

- (1) In all periods  $l < n$ , at each decision node in the extensive form stage game where player  $i$  moves, the strategy picks an action compatible with the Stackelberg type,

if only actions compatible with the Stackelberg type have been played up to that point in the game.

- (2) In period  $n$ , the strategy picks an action incompatible with the Stackelberg type, at some decision node in the extensive form stage game where player  $i$  moves, if only actions compatible with the Stackelberg type have been played up to that point in the game.

Given this definition,  $\sigma_i(R_i(n))$  is the probability that player  $i$  chooses a pure repeated game strategy profile in the set  $R_i(n)$ . Observe that the sets  $R_i(n)$  are disjoint and their union  $\bigcup_n R_i(n)$  gives all pure repeated game strategies excluding the set of strategies  $N_i$  that never play an action incompatible with the Stackelberg type, if only actions compatible with the Stackelberg type have been played up to that point in the game.

Let

$$G_i(t) = (1 - z_i) \sum_{\Delta n \leq t} \sigma_i(R_i(n)).$$

This is just the probability that player  $i$  will reveal rationality by playing an action incompatible with the Stackelberg type by time  $t$ . Step 4 given below shows that for all equilibria  $\sigma$  of the repeated game  $\Gamma^\infty(z, \Delta)$ , there exists a time  $T$  such that  $G_i(T) = 1 - z_i$ , that is, every normal player eventually reveals rationality, if faced with a sufficiently long history of play compatible with the Stackelberg type. Consequently, for any equilibrium  $\sigma$ ,  $\sigma_i(N_i) = 0$ .

If  $\sigma_i(R_i(n)) > 0$ , then let  $U_i(\sigma|R_i(n))$  denote the expected repeated game payoff for player  $i$ , under mixed strategy profile  $\sigma$ , conditional on player  $i$  having picked a strategy in the set  $R_i(n)$ . If  $\sigma_i(R_i(n)) = 0$ , then let  $U_i(\sigma|R_i(n)) = \sup_{a_i^\infty \in R_i(n)} U_i(a_i^\infty, \sigma_j)$ . Also, for any real number  $t$ , let  $U_i(\sigma|t) = U_i(\sigma|R_i(\bar{n}))$  where  $\bar{n} = \max_n \{\Delta n \leq t\}$ . Observe that for

any equilibrium mixed strategy profile  $\sigma$ ,

$$\begin{aligned} U_i(\sigma) &= \sum_{n=0}^{\infty} \sigma_i(R_i(n)) U_i(\sigma | R_i(n)) + \sigma_i(N_i) U_i(\sigma | N_i) \\ &= \sum_{n=0}^{\infty} \sigma_i(R_i(n)) U_i(\sigma | R_i(n)) = \frac{1}{1 - z_i} \int_{t=0}^{\infty} U_i(\sigma | t) dG_i(t). \end{aligned}$$

*Step 2.* Define

$$\begin{aligned} \bar{U}_i(t, k) &= e^{-r \min\{t, k\}} \bar{g} - l_i(1 - e^{-r \min\{t, k\}}) & \text{if } t \geq k \\ &= -l_i(1 - e^{-r \min\{t, k\}}) & \text{if } t < k. \end{aligned}$$

In this definition  $t, k \in \mathbb{R}_+$ . For any equilibrium profile  $\sigma^n$  for the repeated game  $\Gamma^\infty(z, \Delta^n)$

$$U_i(\sigma^n) \leq \frac{1}{1 - z_i} \int_t \int_k \bar{U}_i(t, k) dG_j^n(k) dG_i^n(t) + \rho K(\Delta).$$

*Proof of Step 2.* Fix an equilibrium strategy  $\sigma$  and suppose that  $j$  behaves according to  $\sigma_j$ . Consider a strategy for player  $i$  that chooses to play the Stackelberg action up to period  $t = \Delta n$ , if no player has played an action that is incompatible with the Stackelberg type. Then, in period  $t$  chooses any stage game action  $a_i$  that reveals rationality with certainty in that period, if player  $j$  does not reveal rationality before  $i$  in that period. Also, let this strategy behave according to the equilibrium profile after one of the players reveals rationality. Note that player  $i$  is choosing a strategy in  $R_i(n)$ . Let  $U_i(\sigma | t, a_i)$  denote the utility to  $i$  from using any such strategy. If  $j$  does not reveal rationality in any period  $k \leq t$ , then player  $i$  will reveal rationality. Consequently, the continuation utility, by Theorem 1, for player  $j$  will be at least  $\bar{g} - K(\Delta)$ . This implies that player  $i$ 's continuation utility after period  $t$  is at most  $\rho K(\Delta)$ . In period  $t$  player  $i$  will get at most zero since  $j$  plays  $a_j^s$  and player  $i$  will incur  $-(1 - e^{-rt})l_i$  since both players will play the Stackelberg action up to period  $t$ . This event occurs with probability  $1 - G_2(t)$ . If player  $j$  reveals rationality at any time  $\Delta m = k \leq t$ , then player 1 will receive payoff at most  $\bar{g}$  from that period onwards and

will incur  $-(1 - e^{-rk})l_i$  up to time  $k$ . Consequently,

$$U_i(\sigma|t, a_i) \leq \int_{\{t \geq k\}} (e^{-rk}\bar{g} - l_i(1 - e^{-rk}))dG_j(k) + (1 - G_j(t))(\rho K(\Delta) - l_i(1 - e^{-rt}))$$

$$U_i(\sigma|t, a_i) \leq \int \bar{U}_i(t, k)dG_j(k) + \rho K(\Delta)$$

However,  $U_i(\sigma|t, a_i) \leq \int \bar{U}_i(t, k)dG_j(k) + \rho K(\Delta)$  implies that  $U_i(\sigma|t) \leq \int \bar{U}_i(t, k)dG_j(k) + \rho K(\Delta)$ . Hence,

$$U_i(\sigma) \leq \frac{1}{1 - \underline{z}_i} \int \int \bar{U}_i(t, k)dG_j(k)dG_i(t) + \rho K(\Delta)$$

□

*Step 3.* Define

$$\begin{aligned} \underline{U}_i(t, k) &= e^{-r \min\{t, k\}}\bar{g} - l_i(1 - e^{-r \min\{t, k\}}) && \text{if } t > k \\ &= -l_i(1 - e^{-r \min\{t, k\}}) && \text{if } t \leq k \end{aligned}$$

In this definition  $t, k \in \mathbb{R}_+$ . For any equilibrium profile  $\sigma^n$  for the repeated game  $\Gamma^\infty(z, \Delta^n)$

$$U_i(\sigma^n) \geq \frac{1}{1 - \underline{z}_i} \int \int \underline{U}_i(t, k)dG_j^n(k)dG_i^n(t) - K(\Delta)(1 + \rho) - 2M(\Delta)$$

*Step 4.* There exists a  $T$  such that  $G_i^n(T) = 1 - \underline{z}_i$ .

*Step 5.* There exists a subsequence  $\{n_k\} \subset \{n\}$  such that  $(G_1^{n_k}(t), G_2^{n_k}(t)) \rightarrow (\hat{G}_1(t), \hat{G}_2(t))$ .

*Proof of Step 5.* Since  $G_1^n$  and  $G_2^n$  are distribution functions, by Helly's theorem they have a (possibly) subsequential limit  $\hat{G}_1(t), \hat{G}_2(t)$ . Also, since the support of the  $G_1^n$  and  $G_2^n$ 's is uniformly bounded by the previous lemma, the limiting functions  $\hat{G}_1(t), \hat{G}_2(t)$  are also distribution functions. □

*Step 6.* The distribution functions  $\hat{G}_1(t)$  and  $\hat{G}_2(t)$  do not have any common points of discontinuity.

*Step 7.* If  $(G_1^n(t), G_2^n(t)) \rightarrow (\hat{G}_1(t), \hat{G}_2(t))$ , then

$$\lim U_i(\sigma^n) = \int \int \bar{U}_i(t, k) d\hat{G}_j(k) d\hat{G}_i(t) = \int \int \underline{U}_i(t, k) d\hat{G}_j(k) d\hat{G}_i(t).$$

*Step 8.* The distribution functions  $(\hat{G}_1(t), \hat{G}_2(t))$  solve the War of Attrition and consequently  $(\hat{G}_1(t), \hat{G}_2(t)) = (F_1(t), F_2(t))$

*Step 9.* Observe that  $\lim U_i(\sigma^n) = \int \int \bar{U}_i(t, k) dF_j(k) dF_i(t)$ . However,  $\int \int \bar{U}_i(t, k) dF_j(k) dF_i(t)$  is just the expected utility of player  $i$  from playing the War of Attrition. Consequently,  $\int \int \bar{U}_i(t, k) dF_j(k) dF_i(t) = (1 - c_j)\bar{g}$  thus completing the argument.

Assumption 1 and Assumption 2 were used for two purposes in the previous Theorem. First, to bound the continuation payoff of the two agents. Second, to show that the two agents never concede concurrently, at the limit. Both of these can be achieved if we drop Assumption 1, but strengthen Assumption 2 by assuming Assumption 3 instead. The bound on continuation payoffs continue to hold since Theorem 1 is valid under Assumption 3. Also, to prove that the two players never concede concurrently, only the bound of continuation payoffs and Assumption 2 is required. Since, Assumption 3 is a strict strengthening of Assumption 2, this result also holds.

**Corollary 3.** *Suppose that  $\Gamma_N$  satisfies Assumption 3 for both players. For any  $z = (z_1 > 0, z_2 > 0)$  and any  $\epsilon > 0$ , there exists a  $\Delta^*$  such that, for any  $\Delta < \Delta^*$ , any equilibrium strategy profile  $\sigma$  for the repeated game  $\Gamma^\infty(z, \Delta)$ ,  $|U_1(\sigma) - (1 - c_2)\bar{g}| < \epsilon$  and  $|U_2(\sigma) - (1 - c_1)\bar{g}| < \epsilon$ .*

## 5. CONCLUSION AND DISCUSSION

In this paper we proved one and two sided reputation results for extensive form stage games that satisfy assumptions 1 and 2. Also, we showed that our results continue to hold without alteration if assumption 1 is dropped, but assumption 2 is strengthened by assuming that the stage is a game of strictly conflicting interest.

The assumption that the agents do not move simultaneously is an essential ingredient for our findings. This restriction gives perfection arguments considerable selection power. It should be noted, however, that none of our results depend on the order of moves, they just depend on non-simultaneity. For example, in a two-stage extensive form stage game if the identity of the player that moves first was selected at random, or the two players alternated in moving first, our results would not be affected. More generally, the one and two sided reputation results would continue to hold in a stochastic game where the game played in each period is selected at random, according to a stationary probability, from a finite subset of games that satisfy assumptions 1 and 2.<sup>7</sup>

## APPENDIX A. OMITTED PROOFS

### A.1. Omitted Steps in the Proof of Theorem 1.

*Proof of Step 6. Statement.* Choose  $z_1 \geq z \geq z_\xi$  and equilibrium  $\sigma$  such that  $w(\delta, \sigma) \geq (K_\xi - \xi)\epsilon$ . For the chosen  $z$  and equilibrium strategy profile  $\sigma$ ,

$$U_2(\sigma) \leq q_\xi K_\xi \epsilon \rho + q_1 K \epsilon \rho + (1 - z - q_1 - q_\xi) \epsilon \rho + 2(1 - \delta)(1 + M) \rho - z(K_\xi - \xi) \epsilon l.$$

*proof.* In any period, player 2 will receive payoff less than or equal to zero if player 1 takes action  $a_1^s$ . By the same reasoning as in Step 6, in any period  $n < T_\xi$ , with player 1 only playing action  $a_1^s$  in history  $h_n$ ,

$$U_1(\sigma|h_n) \geq 1 - w(\delta, \sigma_2) \geq 1 - K_\xi \epsilon.$$

Consequently, in any such period  $n < T_\xi$ , player 2 will receive payoff equal to at most zero for the period if player 1 takes action  $a_1^s$ , and otherwise will receive continuation payoff

$$U_2(\sigma|h_{n < T_\xi}, a_1) \leq \rho K_\xi \epsilon \triangleq \bar{U}_2(\sigma|h_{n < T_\xi}, a_1),$$

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<sup>7</sup>In this case  $\hat{g}$  and  $l_i$  would have to be appropriately defined as the expectation over the set of finite stage games.



by Assumption 2. Likewise, for periods  $T_\xi$ ,  $T_\xi < n < T_1$ ,  $T_1$  and  $n > T_1$ ,

$$\begin{aligned}
U_2(\sigma|h_{T_\xi}, a_1) &\leq \rho(1 - \delta(1 - K\epsilon) + (1 - \delta)M) \leq \rho K\epsilon + \rho(1 - \delta)(1 + M) \triangleq \bar{U}_2(\sigma|h_{T_\xi}, a_1) \\
U_2(\sigma|h_{T_\xi < n < T_1}, a_1) &\leq \rho K\epsilon \triangleq \bar{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1) \\
U_2(\sigma|h_{T_1}, a_1) &\leq \rho\epsilon + \rho(1 - \delta)(1 + M) \triangleq \bar{U}_2(\sigma|h_{T_1}, a_1) \\
U_2(\sigma|h_{n > T_1}, a_1) &\leq \rho\epsilon \triangleq \bar{U}_2(\sigma|h_{n > T_1}, a_1)
\end{aligned}$$

Player 1 is a commitment type with probability  $z$  and in this case player 2 will lose at least  $(K_\xi - \xi)\epsilon l$  by not best replying to  $a_1^s$  since  $w(\delta, \sigma_2) \geq (K_\xi - \xi)\epsilon$  in the equilibrium  $\sigma$  under consideration. Also, as before the probability that player 1 takes action  $a_1 \neq a_1^s$  for the first time in any period  $n < T_\xi$  is less than  $q_\xi$  and in any period  $T_\xi < n < T_1$  is less than  $q_\xi + q_1$ . Also,  $\bar{U}_2(\sigma|h_{n > T_1}, a_1) \leq \bar{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1) \leq \bar{U}_2(\sigma|h_{n < T_\xi}, a_1)$  and so

$$\begin{aligned}
U_2(\sigma) &\leq q_\xi \bar{U}_2(\sigma|h_{n < T_\xi}, a_1) + q_1 \bar{U}_2(\sigma|h_{T_\xi < n < T_1}, a_1) \\
&\quad + (1 - z - q_\xi - q_1) \bar{U}_2(\sigma|h_{n > T_1}, a_1) + 2\rho(1 - \delta)(1 + M) - z(K_\xi - \xi)\epsilon l
\end{aligned}$$

delivering the required inequality. Observe that if  $T_\xi = \infty$  and  $T_1 = \infty$  the bound is still valid.  $\square$

## A.2. Omitted Steps in the Proof of Theorem 2.

*Proof of Step 3. Statement.* Let

$$\begin{aligned}
\underline{U}_i(t, k) &= e^{-r \min\{t, k\}} \bar{g} - l_i(1 - e^{-r \min\{t, k\}}) && \text{if } t > k \\
&= -l_i(1 - e^{-r \min\{t, k\}}) && \text{if } t \leq k
\end{aligned}$$

For any equilibrium profile  $\sigma^n$  for the repeated game  $\Gamma^\infty(z, \Delta^n)$

$$U_i^n(\sigma^n) \geq \frac{1}{1 - z_i} \int \int \underline{U}_i(t, k) dG_j^n(k) dG_i^n(t) - K(\Delta)(1 + \rho) - 2M(\Delta)$$

*proof.* Fix an equilibrium strategy  $\sigma$  and suppose that  $j$  behaves according to  $\sigma_j$ . As in step 2, consider a strategy for player  $i$  that chooses to play the Stackelberg action up to time  $t = \Delta n$ , if both players have played actions compatible with the Stackelberg type. Then, at time  $t$  chooses any stage game action  $a_i$  that reveals rationality with certainty in that period, if player  $j$  does not reveal rationality before  $i$  in that period. Also, once rationality is revealed, then let the strategy behave according to the equilibrium strategy. Note that player  $i$  is choosing a strategy in  $R_i(n)$ . Let  $U_i(\sigma|t, a_i)$  denote the utility to  $i$  from using any such strategy. If  $j$  reveals rationality in any period  $\Delta m = k < t$ , then player  $i$  incurs  $-l_i$  up to that time, loses at most  $-M$  in period  $k$  and receives continuation payoff  $\bar{g} - K(\Delta)$ . This exceeds  $-(1 - e^{-rk})l_i + e^{-rk}(\bar{g} - K(\Delta)) - (\bar{g} + M)(1 - e^{-r\Delta})$ . If player  $i$  reveals rationality first in period  $t$ , then she loses at most  $M$  for the period and receives as a continuation  $-\rho K(\Delta) \leq 0$ . If player  $j$  reveals first in period  $t$ , then player  $i$  loses again at most  $M$  for the period and receives in continuation  $\bar{g} - K(\Delta) > 0$ . Consequently,

$$U_i(\sigma|t, a_i) \geq \int_{k < t} (e^{-rk}\bar{g} - (1 - e^{-rk})l_i) dG_j(k) - (1 - G_j(t^-))(1 - e^{-rt})l_i - (1 + \rho)K(\Delta) - 2M(\Delta)$$

where  $1 - G_j(t^-)$  denotes the probability that player  $j$  reveals at a time  $k \geq t$ . This implies that

$$U_i(\sigma|t, a_i) \geq \int \underline{U}_i(t, k) dG_j(k) - (1 + \rho)K(\Delta) - 2M(\Delta)$$

However,  $U_i(\sigma|t, a_i) \geq \int \underline{U}_i(t, k) dG_j(k) - (1 + \rho)K(\Delta) - 2M(\Delta)$  for all  $a_i$  implies that  $U_i(\sigma|t) \geq \int \underline{U}_i(t, k) dG_j(k) - (1 + \rho)K(\Delta) - 2M(\Delta)$ . Hence,

$$U_i(\sigma) \geq \frac{1}{1 - z_i} \int \int \underline{U}_i(t, k) dG_j(k) dG_i(t) - (1 + \rho)K(\Delta) - 2M(\Delta)$$

proving the result.  $\square$

*Proof of Step 4. Statement.* There exists a  $T$  such that  $G_i^n(T) = 1 - z_i$ .

*proof.* For some time  $s$  suppose that  $(a_i^s, a_j^s)$  has been played in all periods  $\Delta k < s$  and consider the strategy of  $i$  that continues to play  $a_i^s$  for all periods  $n$  such that  $\Delta n \in [s, s + t]$ , given that  $(a_i^s, a_j^s)$  has been played in all prior periods. For this strategy to be considered,

it must do better for player  $i$  than revealing rationality which guarantees her  $-\rho K(\Delta)$  by loosing at most  $M(\Delta)$  for the period. For this strategy to do better than revealing rationality for  $i$ , the probability with which player  $j$  plays  $a_j^s$  in all periods  $\{n : \Delta n \in [s, s+t]\}$ , given that  $(a_i^s, a_j^s)$  has been played in all prior periods,  $\mathbb{P}\{a_{n,j} = a_j^s \forall \Delta n \in [s, s+t] | a_{k < n} = (a_i^s, a_j^s)\}$ , must satisfy the following:

$$-\rho K(\Delta) - M(\Delta) \leq \mathbb{P}\{a_{n,j} = a_j^s \forall \Delta n \in [s, s+t] | a_{k < n} = (a_i^s, a_j^s)\}(\bar{g}e^{-rt} - l_i(1 - e^{-rt})) + (1 - \mathbb{P}\{a_{n,j} = a_j^s \forall \Delta n \in [s, s+t] | a_{k < n} = (a_i^s, a_j^s)\})\bar{g}$$

and consequently:

$$\mathbb{P}\{a_{n,j} = a_j^s \forall \Delta n \in [s, s+t] | a_{k < n} = (a_i^s, a_j^s)\} \leq \frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})}$$

Observe that for  $t$  large,  $\frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})} < 1$ . This implies that for  $i$  to be willing to play  $a_i^s$  for all  $\Delta n \in [0, tk]$

$$z_j \leq \mathbb{P}\{a_j^s \forall tk\} = \prod_{s=0}^k \mathbb{P}\{a_{n,j} = a_j^s \forall \Delta n \in [s, s+t] | a_{k < n} = (a_i^s, a_j^s)\} \leq \left( \frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})} \right)^k$$

However for  $t$  and  $k$  sufficiently large this is not possible.  $\square$

*Proof of Step 6. Statement.* The distribution functions  $\hat{G}_1(t)$  and  $\hat{G}_2(t)$  do not have any common points of discontinuity.

*proof.* Assume that  $\hat{G}_1$  and  $\hat{G}_2$  have a common point  $t$  where they are both discontinuous. Let  $J_1 = \hat{G}_1(t) - \lim_{s \nearrow t} \hat{G}_1(s)$  and let  $J_2 = \hat{G}_2(t) - \lim_{s \nearrow t} \hat{G}_2(s)$ . We can pick  $\zeta$ , arbitrarily close to  $t$ , such that both  $\hat{G}_1$  and  $\hat{G}_2$  are continuous at  $t + \zeta$  and  $t - \zeta$ . This implies that for each  $\epsilon > 0$ , there is a  $N$  such that the game is played at least once in each interval of length  $2\zeta$  and  $G_i^n[t - \zeta, t + \zeta] = G_i^n(t + \zeta) - G_i^n(t - \zeta) \geq J_i - \epsilon > 0$ , for all  $n \geq N$ . In words, the probability that 1 plays an action different than  $a_i^s$  in the interval  $[t - \zeta, t + \zeta]$  is greater than  $J_1 - \epsilon$ . Also, pick  $N$  such that the value to any player after she has played an

action other than the commitment action is less than  $\epsilon$  and the payoff to any player who has not played an action different than the commitment action against an opponent known to be rational is greater than  $\bar{g} - \epsilon$ . Pick the first period  $k$  such that  $\Delta^n k \in [t - \zeta, t + \zeta]$  and  $\mathbb{P}\{a_{k1} \neq a_1^s\} > 0$  or  $\mathbb{P}\{a_{k2} \neq a_2^s\} > 0$ . Since  $G_i^n[t - \zeta, t + \zeta] > 0$  such a period exists for all  $n > N$ .

Without loss of generality assume that  $\mathbb{P}\{a_{k1} \neq a_1^s\} > 0$ . Observe that this implies that player 1 is willing to play something other than  $a_1^s$  in period  $k$ . Consequently, the payoff that player 1 receives from this strategy must do at least as well as playing  $a_1^s$  throughout the interval  $[t - \zeta, t + \zeta]$ . Let  $U_i$  denote the payoff that player  $i$  receives in equilibrium conditional on both players not playing  $a_i^s$  in period  $k$ . Consequently,

$$\mathbb{P}\{a_{k2} \neq a_2^s\}U_1 + (1 - \mathbb{P}\{a_{k2} \neq a_2^s\})\epsilon \geq e^{-r4\zeta}(\bar{g} - \epsilon)G_2^n[t - \zeta, t + \zeta] - l(1 - e^{-r4\zeta})$$

Redefine  $\epsilon' = \epsilon + \frac{\epsilon}{e^{-r4\zeta}(J_i - \epsilon)} + \frac{l(1 - e^{-r4\zeta})}{e^{-r4\zeta}(J_i - \epsilon)}$  and rewrite the above equation as follows:

$$\mathbb{P}\{a_{k2} \neq a_2^s\}U_1 \geq e^{-r4\zeta}(\bar{g} - \epsilon')G_2^n[t - \zeta, t + \zeta].$$

For  $\epsilon$  and  $\zeta$  sufficiently small, the right hand side of the equation is approximately  $J_2\bar{g}$  and the left hand side is close to  $\mathbb{P}\{a_{k2} \neq a_2^s\}U_1$ . Consequently, for this inequality to hold,  $\mathbb{P}\{a_{k2} \neq a_2^s\} > 0$ . Also, by definition,  $\mathbb{P}\{a_{k2} \neq a_2^s\} \leq G_2^n[t - \zeta, t + \zeta]$ , and consequently,  $U_1 \geq e^{-r(4\zeta)}(\bar{g} - \epsilon')$ .

$\mathbb{P}\{a_{k2} \neq a_2^s\} > 0$  implies, by a symmetric argument as in the case of player 1, that

$$\mathbb{P}\{a_{k1} \neq a_1^s\}U_2 \geq e^{-r4\zeta}(\bar{g} - \epsilon')G_1^n[t - \zeta, t + \zeta]$$

Consequently,  $U_2 \geq e^{-r4\zeta}(\bar{g} - \epsilon')$ . However,  $U_1 \geq e^{-r4\zeta}(\bar{g} - \epsilon')$ , implies that  $U_2 \leq \rho(\bar{g} - U_1) = \rho(\bar{g}(1 - e^{-r4\zeta}) + \epsilon'e^{-r4\zeta})$ . So,

$$\rho(\bar{g}(1 - e^{-r4\zeta}) + \epsilon'e^{-r4\zeta}) \geq e^{-r4\zeta}(\bar{g} - \epsilon')$$

Taking the limit first with respect to  $\epsilon$  and then with respect to  $\zeta$  gives  $0 \geq \bar{g}$ , which is a contradiction.  $\square$

*Proof of Step 7. Statement.* If  $(G_1^n(t), G_2^n(t)) \rightarrow (\hat{G}_1(t), \hat{G}_2(t))$ , then

$$\lim U_i(\sigma^n) = \int \int \bar{U}_i(t, k) d\hat{G}_j(k) d\hat{G}_i(t) = \int \int \underline{U}_i(t, k) d\hat{G}_j(k) d\hat{G}_i(t).$$

*proof.* If  $G_1^n$  converges to  $\hat{G}_1$  and  $G_2^n$  converges to  $\hat{G}_2$ , then the product measure  $G_1^n \times G_2^n$  converges to  $\hat{G}_1 \times \hat{G}_2$ , see Billingsley (1995), Page 386, Exercise 29.2. Observe that the functions  $\bar{U}_1(t, k)$  and  $\underline{U}_1(t, k)$  are continuous at all points except on the set  $\{t = k\}$ . By the previous lemma,  $\int_{\mathbb{R}^2} 1_{\{t=k\}} d(\hat{G}_1 \times \hat{G}_2) = 0$ . Consequently, the  $\hat{G}_1 \times \hat{G}_2$  measure of the points of discontinuity of  $\bar{U}_1(t, k)$  and  $\underline{U}_1(t, k)$  is zero. Billingsley (1995), Theorem 29.2, shows that if the set of discontinuities of a measurable function  $h$ ,  $D_h$ , has  $\mu$  measure zero, i.e.,  $\mu(D_h) = 0$  and  $\mu_n \rightarrow \mu$ , then  $\int h d\mu_n \rightarrow \int h d\mu$ . So,

$$\lim_n \int_{t_1} \left( \int_{t_2} \bar{U}_1(t_1, t_2) dG_2^n(t_2) \right) dG_1^n(t_1) = \lim_n \int_{\mathbb{R}^2} \bar{U}_1 d(G_1^n \times G_2^n) = \int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2)$$

and similarly for  $\underline{U}_1$ . Also, since  $\bar{U}_1$  and  $\underline{U}_1$  differ only on a set of zero measure,  $\int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2) = \int_{\mathbb{R}^2} \underline{U}_1 d(\hat{G}_1 \times \hat{G}_2)$ .  $\square$

*Proof of Step 8. Statement.* The distribution functions  $(\hat{G}_1(t), \hat{G}_2(t))$  solve the War of Attrition and consequently  $(\hat{G}_1(t), \hat{G}_2(t)) = (F_1(t), F_2(t))$

*proof.* In the continuous time war of attrition, if player 1 is behaving according to  $\hat{G}_1$ , then for each  $\epsilon$ , there is a  $N$  such that for all  $n > N$ ,  $G_2^n$  is an  $\epsilon$  best response to  $\hat{G}_1$  and consequently, since  $\epsilon$  is arbitrary  $\hat{G}_2$  is a best response to  $\hat{G}_1$ . Also, the symmetric argument

is true for player 2 showing that  $\hat{G}_1$  is a best response to  $\hat{G}_2$ . Proving that  $\hat{G}_1$  and  $\hat{G}_2$  form an equilibrium for the continuous time war of attrition. Since the war of attrition has a unique equilibrium  $\hat{G}_1 = F_1$  and  $\hat{G}_2 = F_2$ . This argument is identical to Abreu and Gul (2000), proof of Proposition 4, on page 114 where a more detailed proof may be found.  $\square$

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