Computation of Moral-Hazard Problems

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Abstract

We study computational aspects of moral-hazard problems. In particular, we consider deterministic contracts as well as contracts with action and/or compensation lotteries, and formulate each case as a mathematical program with equilibrium constraints. We investigate and compare solution properties of the MPEC approach to that of the linear programming (LP) approach with lotteries. We propose a hybrid procedure that combines the best features of both. The hybrid procedure obtains a solution that is, if not global, at least as good as an LP solution. It also preserves the fast local convergence property by applying an SQP algorithm to MPECs. The numerical results on an example show that the hybrid procedure outperforms the LP approach in both computational time and solution quality in terms of the optimal objective value.

1 Introduction

We study mathematical programming approaches to solve moral-hazard problems. More specifically, we formulate moral-hazard problems with finitely many action choices, including the basic deterministic models and models with lotteries, as mathematical programs with equilibrium constraints. One advantage of using an MPEC formulation is that the size of resulting program is often orders of magnitude smaller than the linear programs derived from the LP lotteries approach [17, 18]. This feature makes the MPEC approach an appealing alternative when solving a large-scale linear program is computationally infeasible because of limitations on computer memory or computing time.

The moral-hazard model studies the relationship between a principal (leader) and an agent (follower) in situations in which the principal can neither observe nor verify an agent's action. The model is formulated as a bilevel program, in which the principal's upper-level

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decision takes the agent's best response to the principal's decision into account. Bilevel programs are generally difficult mathematical problems, and much research in the economics literature has been devoted to analyzing and characterizing solutions of the moral-hazard model (see Grossman and Hart [6] and the references therein). When the agent's set of actions is a continuum, an intuitive approach to simplifying the model is to assume the agent's optimal action lies in the interior of the action set. One then can treat the agent's problem as an unconstrained maximization problem and replace it by the first-order optimality conditions. This is called the *first-order approach* in the economics literature. However, Mirrlees [11, 12] showed that the first-order approach may be invalid because the lower-level agent's problem is not necessarily a concave maximization program, and that the optimal solution may fail to be unique and interior. Consequently, a sequence of papers [19, 7, 8] has developed conditions under which the first-order approach is valid. Unfortunately, these conditions are often more restrictive than is desirable.

In general, if the lower-level problem in a bilevel program is a convex minimization (or concave maximization) problem, one can replace the lower-level problem by the first-order optimality conditions, which are both necessary and sufficient, and reformulate the original bilevel problem as an MPEC. This idea is similar to the first-order approach to the moral-hazard problem with one notable difference: MPEC formulations include complementarity constraints. The first-order approach assumes that the solution to the agent's problem lies in the interior of the action set, and hence, one can treat the agent's problem as an unconstrained maximization problem. This assumption may also avoid issues associated with the failure of the constraint qualification at a solution. General bilevel programs do not make an interior solution assumption. As a result, the complementarity conditions associated with the Karush-Kuhn-Tucker multipliers for inequality constraints would appear in the first-order optimality conditions for the lower-level program. MPECs also arise in many applications in engineering (e.g., transportation, contact problems, mechanical structure design) and economics (Stackelberg games, optimal taxation problems). One well known theoretical difficulty with MPECs is that the standard constraint qualifications, such as the linear independence constraint qualification and the Mangasarian-Fromovitz constraint qualification, fail at every feasible point. A considerable amount of literature is devoted to refining constraint qualifications and stationarity conditions for MPECs; see Scheel and Scholtes [21] and the references therein. We also refer to the two-volume monograph by Facchinei and Pang [2] for theory and applications of equilibrium problems and to the monographs by Luo et al. [10] and Outrata et al. [15] for more details on MPEC theory and applications.

The failure of the constraint qualification conditions means that the set of Lagrange multipliers is unbounded and that conventional numerical optimization software may fail to converge to a solution. Economists have avoided these numerical problems by reformulating the moral-hazard problem as a linear program involving lotteries over a finite set of outcomes. See Townsend [23, 24] and Prescott [17, 18]. While this approach avoids the constraint qualification problems, it does so by restricting aspects of the contract, such as consumption, to a finite set of possible choices even though a continuous choice formulation would be economically more natural.

The purpose of this chapter is twofold: (1) to introduce to the economics community the MPEC approach, or more generally, advanced equilibrium programming approaches, to the moral-hazard problem; (2) to present an interesting and important class of incentive problems in economics to the mathematical programming community. Many incentive problems, such as contract design, optimal taxation and regulation, and multiproduct pricing, can be naturally formulated as an MPEC or an *equilibrium problem with equilibrium constraints* (EPEC). This greatly extends the applications of equilibrium programming to one of the most active research topics in economics in past three decades. The need for a global solution for these economical problems provides a motivation for the optimization community to develop efficient global optimization algorithms for MPECs and EPECs.

The remainder of this chapter is organized as follows. In the next section, we describe the basic moral-hazard model and formulate it as a mixed-integer nonlinear program and as an MPEC. In Section 3, we consider moral-hazard problems with action lotteries, with compensation lotteries, and with a combination of both. We derive MPEC formulations for each of these cases. We also compare the properties of the MPEC approach and the LP lottery approach. In Section 4, we develop a hybrid approach that preserves the desired global solution property from the LP lottery approach and the fast local convergence of the MPEC approach. The numerical efficiency of the hybrid approach in both computational speed and robustness of the solution is illustrated in an example in Section 5.

2 The Basic Moral-Hazard Model

2.1 The deterministic contract

We consider a moral-hazard model in which the agent chooses an action from a finite set $\mathcal{A} = \{a_1, \ldots, a_M\}$. The outcome can be one of N alternatives. Let $\mathcal{Q} = \{q_1, \ldots, q_N\}$ denote the outcome space, where the outcomes are dollar returns to the principal ordered from smallest to largest.

The principal can only observe the outcome, not the agent's action. However, the stochastic relationship between actions and outcomes, which is often called a *production technology*, is common knowledge to both the principal and the agent. Usually, the production technology is exogenously described by the probability distribution function, $p(q \,|\, a)$, which presents the probability of outcome $q \in \mathcal{Q}$ occurring given that action a is taken. We assume $p(q \,|\, a) > 0$ for all $q \in \mathcal{Q}$ and $a \in \mathcal{A}$; this is called the *full-support assumption.*

Since the agent's action is not observable to the principal, the payment to the agent is only based on the outcome observed by the principal. Let $\mathcal{C} \subset R$ be the set of all possible compensations.

Definition 1. A *compensation schedule* $c = (c(q_1), \ldots, c(q_N)) \in R^N$ is an agreement between the principal and the agent such that $c(q) \in \mathcal{C}$ is the payoff to the agent from the principal if outcome $q \in \mathcal{Q}$ is observed.

The agent's utility $u(x, a)$ is a function of the payment $x \in R$ received from the principal and of his action $a \in \mathcal{A}$. The principal's utility $w(q - x)$ is a function over net income $q - x$ for $q \in \mathcal{Q}$. We let $W(c, a)$ and $U(c, a)$ denote the *expected utility* to the principal and agent, respectively, of a compensation schedule $c \in R^N$ when the agent chooses action $a \in \mathcal{A}$, i.e.,

$$\begin{aligned} W(c, a) &= \sum_{q \in \mathcal{Q}} p(q \,|\, a)\, w\left(q - c(q)\right), \\ U(c, a) &= \sum_{q \in \mathcal{Q}} p(q \,|\, a)\, u\left(c(q), a\right). \end{aligned} \tag{1}$$

Definition 2. A *deterministic contract* (proposed by the principal) consists of a recommended action $a \in \mathcal{A}$ to the agent and a compensation schedule $c \in R^N$.

The contract has to satisfy two conditions to be accepted by the agent. The first condition is the *participation constraint*. It states that the contract must give the agent an expected utility no less than a required utility level U^*:

$$U(c, a) \geq U^*. \tag{2}$$

The value U^* represents the highest utility the agent can receive from other activities if he does not sign the contract.

Second, the contract must be *incentive compatible* to the agent; it has to provide incentives for the agent not to deviate from the recommended action. In particular, given the compensation schedule c, the recommended action a must be optimal from the agent's perspective by maximizing the agent's expected utility function. The *incentive compatibility constraint* is given as follows:

$$a \in \operatorname{argmax}\{U(c, a) : \ a \in \mathcal{A}\}. \tag{3}$$

For a given U^*, a *feasible* contract satisfies the participation constraint (2) and the incentive compatibility constraint (3). The objective of the principal is to find an *optimal deterministic contract*, a feasible contract that maximizes his expected utility. A mathematical program for finding an optimal deterministic contract (c^*, a^*) is

$$\begin{aligned} \underset{(c,a)}{\text{maximize}} \quad & W(c, a) \\ \text{subject to} \quad & U(c, a) \geq U^*, \\ & a \in \operatorname{argmax}\{U(c, a) : \ a \in \mathcal{A}\}. \end{aligned} \tag{4}$$

Since there are only finitely many actions in $\mathcal{A}$, the incentive compatibility constraint (3) can be presented as the following set of inequalities:

$$U(c, a) \geq U(c, a_i), \quad \text{for } \ i = 1, \ldots, M. \tag{5}$$

These constraints ensure that the agent's expected utility obtained from choosing the recommendation action is no worse than that from choosing other actions. Replacing (3) by the set of inequalities (5), we have an equivalent formulation of the optimal contract problem:

$$
\begin{array}{ll}
\underset{(c,a)}{\text{maximize}} & W(c,a) \\
\text{subject to} & U(c,a) \geq U^*, \\
& U(c,a) \geq U(c,a_i), \quad \text{for } i = 1, \ldots, M, \\
& a \in \mathcal{A} = \{a_1, \ldots, a_M\}.
\end{array} \tag{6}
$$

2.2 A mixed-integer NLP formulation

The optimal contract problem (6) can be formulated as a mixed-integer nonlinear program. Associated with each action $a_i \in \mathcal{A}$, we introduce a binary variable $y_i \in \{0,1\}$. Let $y = (y_1, \ldots, y_M) \in R^M$ and let e_M denote the vector of all ones in R^M. The mixed-integer nonlinear programming formulation for the optimal contract problem (6) is

$$
\begin{array}{ll}
\underset{(c,y)}{\text{maximize}} & W(c, \sum_{i=1}^{M} a_i y_i) \\
\text{subject to} & U(c, \sum_{i=1}^{M} a_i y_i) \geq U^*, \\
& U(c, \sum_{i=1}^{M} a_i y_i) \geq U(c, a_j), \quad \forall\, j = 1, \ldots, M, \\
& e_M^{\mathrm{T}} y = 1, \\
& y_i \in \{0,1\} \quad \forall\, i = 1, \ldots, M.
\end{array} \tag{7}
$$

The above problem has N nonlinear variables, M binary variables, one linear constraint and $(M+1)$ nonlinear constraints. To solve a mixed-integer nonlinear program, one can use MINLP [3], BARON [20] or other solvers developed for this class of programs. For (7), since the agent will choose one and only one action, the number of possible combinations on the binary vector y is only M. One then can solve (7) by solving M nonlinear programs with $y_i = 1$ and the other $y_j = 0$ in the i-th nonlinear program, as Grossman and Hart suggested in [6] for the case where the principal is risk averse. They further point out that each nonlinear program can be transformed into an equivalent convex program if the agent's utility function $u(x,a)$ can be written as $G(a) + K(a)V(x)$, where (1) V is a real-valued, strictly increasing, concave function defined on some open interval $\mathcal{I} = (\underline{I}, \bar{I}) \subset R$; (2) $\lim_{x \to \underline{I}} V(x) = -\infty$; (3) G, K are real-valued functions defined on $\mathcal{A}$ and K is strictly positive; (4) $u(x,a) \geq u(x,\hat{a}) \Rightarrow u(\hat{x},a) \geq u(\hat{x},\hat{a})$, for all a, $\hat{a} \in \mathcal{A}$, and x, $\hat{x} \in \mathcal{I}$. The above assumption implies that the agent's preferences over income lotteries are independent of his action.

2.3 An MPEC formulation

In general, a mixed-integer nonlinear program is a difficult optimization problem. Below, by considering a mixed-strategy reformulation of the incentive compatibility constraints for the agent, we can reformulate the optimal contract problem (6) as a mathematical program with equilibrium constraints (MPEC); see [10].

For $i = 1, \ldots, M$, let δ_i denote the probability that the agent will choose action a_i. Then, given the compensation schedule c, the agent chooses a mixed strategy profile $\delta^* = (\delta_1^*, \ldots, \delta_M^*) \in R^M$ such that

$$\delta^* \in \operatorname{argmax} \left\{ \sum_{k=1}^{M} \delta_k U(c, a_k) \; : \; e_M^{\mathrm{T}} \delta = 1, \; \delta \geq 0 \right\}. \tag{8}$$

Observe that the agent's mixed-strategy problem (8) is a linear program, and hence, its optimality conditions are necessary and sufficient.

The following lemma states the relationship between the optimal pure strategy a_i and the optimal mixed strategy δ^*. To ease the notation, we define

$$\begin{aligned} U(c) &= (U(c, a_1), \ldots, U(c, a_M)) \in R^M, \\ W(c) &= (W(c, a_1), \ldots, W(c, a_M)) \in R^M. \end{aligned} \tag{9}$$

Lemma 3. *Given a compensation schedule $\bar{c} \in R^N$, the agent's action $a_i \in \mathcal{A}$ is optimal for problem (3) iff there exists an optimal mixed strategy profile δ^* for problem (8) such that*

$$\begin{gathered} \delta_i^* > 0, \\ \sum_{k=1}^{M} \delta_k^* U(\bar{c}, a_k) = U(\bar{c}, a_i), \\ e_M^{\mathrm{T}} \delta^* = 1, \quad \delta^* \geq 0. \end{gathered}$$

Proof. If a_i is an optimal action of (3), then let $\delta^* = e_i$, the i-th column of the identity matrix of order M. It is easy to verify that all the conditions for δ^* are satisfied. Conversely, if a_i is not an optimal solution of (3), then there exists an action a_j such that $U(\bar{c}, a_j) > U(\bar{c}, a_i)$. Let $\tilde{\delta} = e_j$. Then $\tilde{\delta}^{\mathrm{T}} U(c) = U(\bar{c}, a_j) > U(\bar{c}, a_i) = \delta^{*\mathrm{T}} U(c)$. We have a contradiction. ■

An observation following from Lemma 3 is stated below.

Lemma 4. *Given a compensation schedule $c \in R^N$, a mixed strategy profile δ is optimal for the linear program (8) iff*

$$0 \leq \delta \perp \left(\delta^{\mathrm{T}} U(c)\right) e_M - U(c) \geq 0, \qquad e_M^{\mathrm{T}} \delta = 1. \tag{10}$$

Proof. This follows from the optimality conditions and the strong duality theorem for the LP (8). ■

Substituting the incentive compatibility constraint (5) by the system (10) and replacing $W(c,a)$ and $U(c,a)$ by $\delta^{\mathrm{T}} W(c)$ and $\delta^{\mathrm{T}} U(c)$, respectively, we derive an MPEC formulation of the principal's problem (6):

$$\begin{array}{ll} \underset{(c,\delta)}{\text{maximize}} & \delta^{\mathrm{T}} W(c) \\ \text{subject to} & \delta^{\mathrm{T}} U(c) \geq U^*, \\ & e_M^{\mathrm{T}} \delta = 1, \\ & 0 \leq \delta \perp \left(\delta^{\mathrm{T}} U(c)\right) e_M - U(c) \geq 0. \end{array} \tag{11}$$

To illustrate the failure of constraint qualification at any feasible point of an MPEC, we consider the feasible region $\mathcal{F}_1 = \{(x,y) \in R^2 \,|\, x \geq 0,\, y \geq 0,\, xy = 0\}$. At the point $(\bar{x}, \bar{y}) = (0,2)$, the first constraint $x \geq 0$ and the third constraint $xy = 0$ are binding. The gradients of the binding constraints at $(\bar{x}, \bar{y})$ are $(1,0)$ and $(2,0)$, which are dependent. It is easy to verify that the gradient vectors of the binding constraints are indeed dependent at other feasible points.

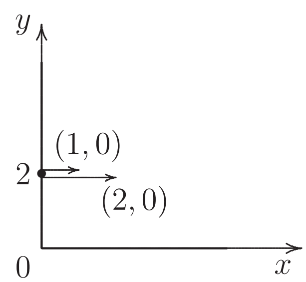

Figure 1: The feasible region $\mathcal{F}_1 = \{(x,y) \,|\, x \geq 0,\, y \geq 0,\, xy = 0\}$.

The following theorem states the relationship between the optimal solutions for the principal-agent problems (6) and the corresponding MPEC formulation (11).

Theorem 5. *If* (c^*, δ^*) *is an optimal solution for the MPEC (11), then* (c^*, a_i^*)*, where* $i \in \{j : \delta_j^* > 0\}$*, is an optimal solution for the problem (6). Conversely, if* (c^*, a_i^*) *is an optimal solution for the problem (6), then* (c^*, e_i) *is an optimal solution for the MPEC (11).*

Proof. The statement follows directly from Lemma 4. ■

The MPEC (11) has $(N+M)$ variables, 1 linear constraint, 1 nonlinear constraint, and M complementarity constraints. Hence, the size of the problem grows linearly in the number of the outcomes and actions. As we will see in Section 4, this feature is the main advantage of using the MPEC approach rather than the LP lotteries approach.

3 Moral-Hazard Problems with Lotteries

In this section, we study moral-hazard problems with lotteries. In particular, we consider action lotteries, compensation lotteries, and a combination of both. For each case, we first give definitions for the associated lotteries and then derive the nonlinear programming or MPEC formulation.

3.1 The contract with action lotteries

Definition 6. A *contract with action lotteries* is a probability distribution over actions, $\pi(a)$, and a compensation schedule $c(a) = (c(q_1, a), \ldots, c(q_N, a)) \in R^N$ for all $a \in \mathcal{A}$. The compensation schedule $c(a)$ is an agreement between the principal and the agent such that $c(q, a) \in \mathcal{C}$ is the payoff to the agent from the principal if outcome $q \in \mathcal{Q}$ is observed and the action $a \in \mathcal{A}$ is recommended by the principal.

In the definition of a contract with action lotteries, the compensation schedule $c(a)$ is contingent on both the outcome and the agent's action. Given this definition, one might raise the following question: if the principal can only observe the outcome, not the agent's action, is it reasonable to have the compensation schedule $c(a)$ contingent on the action chosen by the agent? After all, the principal does not know which action is implemented by the agent. One economic justification is as follows. Suppose that the principal and the agent sign a total of M contracts, each with different recommended action $a \in \mathcal{A}$ and compensation schedule $c(a)$ as a function of the recommended action, a. Then, the principal and the agent would go to an authority or a third party to conduct a lottery with probability distribution function $\pi(a)$ on which contract would be implemented on that day. If the i-th contract is drawn from the lottery, then the third party would inform both the principal and the agent that the recommended action for that day is a_i with the compensation schedule $c(a_i)$.

Arnott and Stiglitz [1] use *ex ante randomization* for action lotteries. This terminology refers to the situation that a random contract occurs before the recommended action is chosen. They demonstrate that the action lotteries will result in a welfare improvement if the principal's expected utility is nonconcave in the agent's expected utility. However, it is not clear what sufficient conditions would be needed for the statement in the assumption to be true.

3.2 An NLP formulation with star-shaped feasible region

When the principal proposes a contract with action lotteries, the contract has to satisfy the participation constraint and the incentive compatibility constraints. In particular, for a given contract $(\pi(a), c(a))_{a\in\mathcal{A}}$, the participation constraint requires the agent's expected utility to be at least U^*:

$$\sum_{a\in\mathcal{A}} \pi(a)U(c(a),a) \geq U^*. \tag{12}$$

For any recommended action a with $\pi(a) > 0$, it has to be incentive compatible with respect to the corresponding compensation schedule $c(a) \in R^N$. Hence, the incentive compatibility constraints are

$$\forall\, a \in \{\hat{a} : \pi(\hat{a}) > 0\} : \quad a = \operatorname{argmax}\{U(c(a),\tilde{a}) : \tilde{a} \in \mathcal{A}\}, \tag{13}$$

or equivalently,

$$\text{if } \pi(a) > 0, \text{ then } U(c(a),a) \geq U(c(a),a_i), \quad \text{for } i = 1,\ldots,M. \tag{14}$$

However, we do not know in advance whether $\pi(a)$ will be strictly positive at an optimal solution. One way to overcome this difficulty is to reformulate the solution-dependent constraints (14) as:

$$\forall\, a \in \mathcal{A} : \quad \pi(a)U(c(a),a) \geq \pi(a)U(c(a),a_i), \quad \text{for } i = 1,\ldots,M, \tag{15}$$

or in a compact presentation,

$$\pi(a)\left(U(c(a),a) - U(c(a),\tilde{a})\right) \geq 0, \quad \forall\, (a,\tilde{a}(\neq a)) \in \mathcal{A}\times\mathcal{A}. \tag{16}$$

Finally, since $\pi(\cdot)$ is a probability distribution function, we need

$$\begin{aligned} &\sum_{a\in\mathcal{A}} \pi(a) = 1, \\ &\pi(a) \geq 0, \quad \forall\, a \in \mathcal{A}. \end{aligned} \tag{17}$$

The principal chooses a contract with action lotteries that satisfies participation constraint (12), incentive compatibility constraints (16), and the probability measure constraint (17) to maximize his expected utility. An optimal contract with action lotteries $(\pi^*(a), c^*(a))_{a\in\mathcal{A}}$ is then a solution to the following nonlinear program:

$$\begin{aligned} \text{maximize} \quad & \sum_{a\in\mathcal{A}} \pi(a)W(c(a),a) \\ \text{subject to} \quad & \sum_{a\in\mathcal{A}} \pi(a)U(c(a),a) \geq U^*, \\ & \sum_{a\in\mathcal{A}} \pi(a) = 1, \\ \forall\, (a,\tilde{a}(\neq a)) \in \mathcal{A}\times\mathcal{A} : \quad & \pi(a)\left(U(c(a),a) - U(c(a),\tilde{a})\right) \geq 0, \\ & \pi(a) \geq 0, \quad \forall\, a \in \mathcal{A}. \end{aligned} \tag{18}$$

The nonlinear program (18) has $(N * M + M)$ variables and $(M * (M-1) + 2)$ constraints. In addition, its feasible region is highly nonconvex because of the last two sets of constraints in (18). As shown in Figure 2, the feasible region $\mathcal{F}_2 = \{(x,y) \,|\, xy \geq 0,\ x \geq 0\}$ is the union of the first quadrant and the y-axis. Furthermore, the standard nonlinear programming constraint qualification fails to hold at every point on the y-axis.

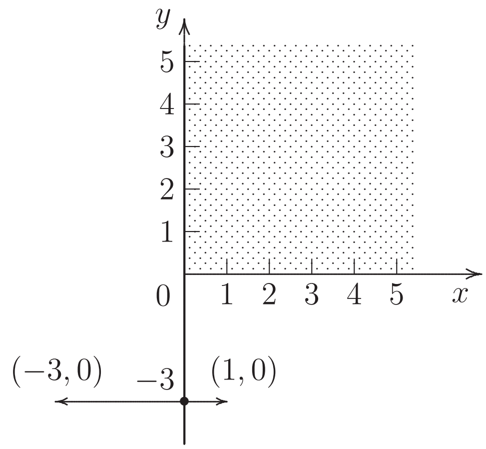

Figure 2: The feasible region $\mathcal{F}_2 = \{(x, y) \,|\, xy \geq 0,\, x \geq 0\}$.

3.3 MPEC formulations

Below, we introduce an MPEC formulation for the star-shaped problem (18). We first show that constraints of a star-shaped set $\mathcal{Z}_1 = \{z \in R^n \,|\, g_i(z) \geq 0,\, g_i(z)\, h_i(z) \geq 0,\, i = 1, \dots, m\}$ can be rewritten as complementarity constraints if we introduce additional variables.

Proposition 7. *A point z is in $\mathcal{Z}_1 = \{z \in R^n \,|\, g_i(z) \geq 0,\, g_i(z)\, h_i(z) \geq 0,\, i = 1, \dots, m\}$ iff there exists an s such that (z, s) is in $\mathcal{Z}_2 = \{(z, s) \in R^{n+m} \,|\, 0 \leq g(z) \perp s \geq 0,\, h(z) \geq -s\}$.*

Proof. Suppose that z is in $\mathcal{Z}_1$. If $g_i(z) > 0$, choose $s_i = 0$; if $g_i(z) = 0$, choose $s_i = -h_i(z)$. Then (z, s) is in $\mathcal{Z}_2$. Conversely, if (z, s) is in $\mathcal{Z}_2$, then $g_i(z)h_i(z) \geq g_i(z)(-s_i) = 0$ for all $i = 1, \dots m$. Hence, the point z is in $\mathcal{Z}_1$. ■

Following Proposition 7, we introduce a variable $s(a, \tilde{a})$ for each pair $(a, \tilde{a}) \in \mathcal{A} \times \mathcal{A}$ for the incentive compatibility constraints in (18). We then obtain the following MPEC formulation with variables $(\pi(a), c(a), s(a, \tilde{a}))_{(a,\tilde{a}) \in \mathcal{A} \times \mathcal{A}}$ for the optimal contract with action lottery problem:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{a \in \mathcal{A}} \pi(a) W(c(a), a) \\
\text{subject to} \quad & \sum_{a \in \mathcal{A}} \pi(a) U(c(a), a) \geq U^*, \\
& \sum_{a \in \mathcal{A}} \pi(a) = 1, \\
\forall\, (a, \tilde{a}(\neq a)) \in \mathcal{A} \times \mathcal{A}: \quad & U(c(a), a) - U(c(a), \tilde{a}) + s(a, \tilde{a}) \geq 0, \\
\forall\, (a, \tilde{a}(\neq a)) \in \mathcal{A} \times \mathcal{A}: \quad & 0 \leq \pi(a) \perp s(a, \tilde{a}) \geq 0.
\end{aligned}
\tag{19}
$$

Allowing the compensation schedules to be dependent on the agent's action will increase the principal's expected utility; see Theorem 8 below. The difference between the optimal objective value of the NLP (18) (or the MPEC(19)) and that of the MPEC (11) characterizes the principal's improved welfare from using an optimal contract with action lotteries.

Theorem 8. *The principal prefers an optimal contract with action lotteries to an optimal deterministic contract. His expected utility from choosing an optimal contract with action lotteries will be at least as good as that from choosing an optimal deterministic contract.*

Proof. This is clear. ∎

3.4 The contract with compensation lotteries

Definition 9. For any outcome $q \in \mathcal{Q}$, a *randomized compensation* $\tilde{c}(q)$ is a random variable on the set of compensations $\mathcal{C}$ with a probability measure $F(\cdot)$.

Remark If the set of compensations $\mathcal{C}$ is a closed interval $[\underline{c}, \bar{c}] \in R$, then the measure of $\tilde{c}(q)$ is a cumulative density function (cdf) $F : [\underline{c}, \bar{c}] \rightarrow [0, 1]$ with $F(\underline{c}) = 0$ and $F(\bar{c}) = 1$. In addition, $F(\cdot)$ is nondecreasing and right-continuous.

To simplify the analysis, we assume that every randomized compensation $\tilde{c}(q)$ has finite support.

Assumption 10 (Finite support for randomized compensation.)**.** *For all $q \in \mathcal{Q}$, the randomized compensation $\tilde{c}(q)$ has finite support over an unknown set $\{c_1(q), c_2(q), \ldots, c_L(q)\}$ with a known L.*

An immediate consequence of Assumption 10 is that we can write $\tilde{c}(q) = c_i(q)$ with probability $p_i(q) > 0$ for all $i = 1, \ldots, L$ and $q \in \mathcal{Q}$. In addition, we have $\sum_{i=1}^{L} p_i(q) = 1$ for all $q \in \mathcal{Q}$. Notice that both $(c_i(q))_{i=1}^{L} \in R^L$ and $(p_i(q))_{i=1}^{L} \in R^L$ are endogenous variables and will be chosen by the principal.

Definition 11. A *compensation lottery* is a randomized compensation schedule $\tilde{c} = (\tilde{c}(q_1), \ldots, \tilde{c}(q_N)) \in R^N$, in which $\tilde{c}(q)$ is a randomized compensation satisfying Assumption 10 for all $q \in \mathcal{Q}$.

Definition 12. A *contract with compensation lotteries* consists of a recommended action a to the agent and a randomized compensation schedule $\tilde{c} = (\tilde{c}(q_1), \ldots, \tilde{c}(q_N)) \in R^N$.

Let $c^q = (c_i(q))_{i=1}^{L} \in R^L$ and $p^q = (p_i(q))_{i=1}^{L} \in R^L$. Given that the outcome q is observed by the principal, we let $\mathbf{w}(c^q, p^q)$ denote the principal's expected utility with respect to a randomized compensation $\tilde{c}(q)$, i.e.,

$$\mathbf{w}(c^q, p^q) = \mathbb{E}\, w(q - \tilde{c}(q)) = \sum_{i=1}^{L} p_i(q) w(q - c_i(q)).$$

With a randomized compensation schedule $\tilde{c}$ and a recommended action a, the principal's expected utility then becomes

$$\mathbb{E}\,W(\tilde{c}, a) = \sum_{q \in \mathcal{Q}} p(q \mid a) \left(\sum_{i=1}^{L} p_i(q) w(q - c_i(q)) \right) = \sum_{q \in \mathcal{Q}} p(q \mid a) \mathbf{w}(c^q, p^q). \tag{20}$$

Similarly, given a recommended action a, we let $\mathbf{u}(c^q, p^q, a)$ denote the agent's expected utility with respect to $\tilde{c}(q)$ for the observed outcome q:

$$\mathbf{u}(c^q, p^q, a) = \mathbb{E}\,u(\tilde{c}(q), a) = \sum_{i=1}^{L} p_i(q) u(c_i(q), a).$$

The agent's expected utility with a randomized compensation schedule $\tilde{c}$ and a recommended action a is

$$\mathbb{E}\,U(\tilde{c}, a) = \sum_{q \in \mathcal{Q}} p(q \mid a) \left(\sum_{i=1}^{L} p_i(q) u(c_i(q), a) \right) = \sum_{q \in \mathcal{Q}} p(q \mid a) \mathbf{u}(c^q, p^q, a). \tag{21}$$

To further simply to notation, we use $c_{\mathcal{Q}} = (c^q)_{q \in \mathcal{Q}}$ and $p_{\mathcal{Q}} = (p^q)_{q \in \mathcal{Q}}$ to denote the collection of variables c^q and p^q, respectively. We also let $\mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a)$ denote the principal's expected utility $\mathbb{E}\,W(\tilde{c}, a)$ as defined in (20), and similarly, $\mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a)$ for $\mathbb{E}\,U(\tilde{c}, a)$ as in (21).

An optimal contract with compensation lotteries $(c^*_{\mathcal{Q}}, p^*_{\mathcal{Q}}, a^*)$ is a solution to the following problem:

$$\begin{aligned} \text{maximize} \quad & \mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a) \\ \text{subject to} \quad & \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a) \geq U^*, \\ & \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a) \geq \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_i), \quad \forall\, i = 1, \ldots, M, \\ & a \in \mathcal{A} = \{a_1, \ldots, a_M\}. \end{aligned} \tag{22}$$

Define

$$\begin{aligned} \mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) &= (\mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_1), \ldots, \mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_M)) \in R^M, \\ \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) &= (\mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_1), \ldots, \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_M)) \in R^M. \end{aligned}$$

Following the derivation as in Section 2, we can reformulate the program for an optimal contract with compensation lotteries (22) as a mixed-integer nonlinear program with decision variables $(c_{\mathcal{Q}}, p_{\mathcal{Q}})$ and $y = (y_i)_{i=1}^M$:

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, \sum_{i=1}^{M} a_i y_i) \\
\text{subject to} \quad & \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, \sum_{i=1}^{M} a_i y_i) \geq U^*, \\
& \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, \sum_{i=1}^{M} a_i y_i) \geq \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}, a_j), \quad \forall\, j = 1, \ldots, M, \\
& e_M^{\mathrm{T}} y = 1, \\
& y_i \in \{0, 1\} \quad \forall\, i = 1, \ldots, M,
\end{aligned}
\tag{23}
$$

Similarly, the MPEC formulation with decision variables $(c_{\mathcal{Q}}, p_{\mathcal{Q}})$ and $\delta \in R^M$ is

$$
\begin{aligned}
\text{maximize} \quad & \delta^{\mathrm{T}} \mathbf{W}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) \\
\text{subject to} \quad & \delta^{\mathrm{T}} \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) \geq U^*, \\
& e_M^{\mathrm{T}} \delta = 1, \\
& 0 \leq \delta \perp \left(\delta^{\mathrm{T}} \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) \right) e_M - \mathbf{U}(c_{\mathcal{Q}}, p_{\mathcal{Q}}) \geq 0.
\end{aligned}
\tag{24}
$$

Arnott and Stiglitz [1] call the compensation lotteries *ex post randomization*; this refers to the situation where the random compensation occurs after the recommended action is chosen or implemented. They show that if the agent is risk averse and his utility function is separable, and if the principal is risk neutral, then the compensation lotteries are not desirable.

3.5 The contract with action and compensation lotteries

Definition 13. A *contract with action and compensation lotteries* is a probability distribution over actions, $\pi(a)$, and a randomized compensation schedule $\tilde{c}(a) = (\tilde{c}(q_1, a), \ldots, \tilde{c}(q_N, a)) \in R^N$ for every $a \in \mathcal{A}$. The randomized compensation schedule $c(a)$ is an agreement between the principal and the agent such that $\tilde{c}(q, a) \in \mathcal{C}$ is a randomized compensation to the agent from the principal if outcome $q \in \mathcal{Q}$ is observed and the action $a \in \mathcal{A}$ is recommended by the principal.

Assumption 14. *For every action $a \in \mathcal{A}$, the randomized compensation schedule $\tilde{c}(q, a)$ satisfies the finite support assumption (Assumption 10) for all $q \in \mathcal{Q}$.*

With Assumption 14, the notation $c^q(a), p^q(a), c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a)$ is analogous to what we have defined in Section 3.1 and 3.2. Without repeating the same derivation process described earlier, we give the star-shaped formulation with variables $(\pi(a), c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a))_{a\in\mathcal{A}}$ for the optimal contract with action and compensation lotteries problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{a\in\mathcal{A}} \pi(a)\mathbf{W}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) & \\ \text{subject to} & \displaystyle\sum_{a\in\mathcal{A}} \pi(a)\mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) \geq U^*, & \\ & \displaystyle\sum_{a\in\mathcal{A}} \pi(a) = 1, & \\ \forall\,(a,\tilde{a}) \in \mathcal{A}\times\mathcal{A}: & \pi(a)\left(\mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) - \mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), \tilde{a})\right) \geq 0, & \\ & \pi(a) \geq 0. & \end{array} \tag{25}$$

Following the derivation in Section 3.3, an equivalent MPEC formulation is with variables $(\pi(a), c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), s(a,\tilde{a}))_{(a,\tilde{a})\in\mathcal{A}\times\mathcal{A}}$:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{a\in\mathcal{A}} \pi(a)\mathbf{W}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) & \\ \text{subject to} & \displaystyle\sum_{a\in\mathcal{A}} \pi(a)\mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) \geq U^*, & \\ & \displaystyle\sum_{a\in\mathcal{A}} \pi(a) = 1, & \\ \forall\,(a,\tilde{a}) \in \mathcal{A}\times\mathcal{A}: & \mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), a) - \mathbf{U}(c_{\mathcal{Q}}(a), p_{\mathcal{Q}}(a), \tilde{a}) \geq -s(a,\tilde{a}), & \\ \forall\,(a,\tilde{a}) \in \mathcal{A}\times\mathcal{A}: & 0 \leq \pi(a) \perp s(a,\tilde{a}) \geq 0. & \end{array} \tag{26}$$

3.6 Linear programming approximation

Townsend [23, 24] was among the first to use linear programming techniques to solve static incentive constrained problems. Prescott [17, 18] further apply linear programming specifically to solve moral-hazard problems. A solution obtained by the linear programming approach is an approximation to a solution to the MPEC (26). Instead of treating $c_{\mathcal{Q}}(a)$ as unknown variables, one can construct a grid Ξ with elements ξ to approximate the set $\mathcal{C}$ of compensations. By introducing probability measures associated with the action lotteries on $\mathcal{A}$ and compensation lotteries on Ξ, one can then approximate a solution to the moral-hazard problem with lotteries (26) by solving a linear program. More specifically, the principal chooses probability distributions $\pi(a)$, and $\pi(\xi|\, q, a)$ over the set of actions $\mathcal{A}$, the set of outcomes $\mathcal{Q}$, and the compensation grid Ξ. One then can reformulate the resulting nonlinear program as a linear program with decision variables $\pi = (\pi(\xi, q, a))_{\xi \in \Xi, q \in \mathcal{Q}, a \in \mathcal{A}}$:

$$
\begin{aligned}
\text{maximize}_{(\pi)} \quad & \sum_{\xi, q, a} w(q - \xi)\pi(\xi, q, a) \\
\text{subject to} \quad & \sum_{\xi, q, a} u(\xi, a)\pi(\xi, q, a) \geq U^*, \\
\forall\, (a, \tilde{a}) \in \mathcal{A} \times \mathcal{A} : \quad & \sum_{\xi, q} u(\xi, a)\pi(\xi, q, a) \geq \sum_{\xi, q} u(\xi, \tilde{a}) \frac{p(q|\tilde{a})}{p(q|a)} \pi(\xi, q, a) \\
\forall\, (\tilde{q}, \tilde{a}) \in \mathcal{Q} \times \mathcal{A} : \quad & \sum_{\xi} \pi(\xi, \tilde{q}, \tilde{a}) = p(\tilde{q}|\tilde{a}) \sum_{\xi, q} \pi(\xi, q, \tilde{a}), \\
& \sum_{\xi, q, a} \pi(\xi, q, a) = 1, \\
& \pi(\xi, q, a) \geq 0 \quad \forall\, (\xi, q, a) \in \Xi \times \mathcal{Q} \times \mathcal{A}.
\end{aligned}
\tag{27}
$$

Note that the above linear program has $(|\Xi| * N * M)$ variables and $(M * (N + M - 1) + 2)$ constraints. The size of the linear program will grow enormously when one chooses a fine grid. For example, if there are 50 actions, 40 outputs, and 500 compensations, then the linear program has one million variables and 4452 constraints. It will become computationally intractable because of the limitation on computer memory, if not the time required. On the other hand, a solution of the LP obtained from a coarse grid will not be satisfactory if an accurate solution is needed. Prescott [18] points out that the constraint matrix of the linear program (27) has block angular structure. As a consequence, one can apply Dantzig-Wolfe decomposition to the linear program (27) to reduce the computer memory and computational time. Recall that the MPEC (11) for the optimal contract problem has only $(N + M)$ variables and M complementarity constraints with one linear constraint and one nonlinear constraint. Even with the use of the Dantzig-Wolfe decomposition algorithm to solve LP (27), choosing the "right" grid is still an issue. With the advances in both theory and numerical methods for solving MPECs in the last decade, we believe that the MPEC approach has greater advantages in solving a much smaller problem and in obtaining a more accurate solution.

The error from discretizing set of compensations $\mathcal{C}$ is characterized by the difference between the optimal objective value of LP (27) and that of MPEC (26).

Theorem 15. *The optimal objective value of MPEC (26) is at least as good as that of LP (27).*

Proof. It is sufficient to show that given a feasible point of LP (27), one can construct a feasible point for MPEC (26) with objective value equal to that of the LP (27).

Let $\pi = (\pi(\xi, q, a))_{\xi \in \Xi, q \in \mathcal{Q}, a \in \mathcal{A}}$ be a given feasible point of LP (27). Let

$$\pi(a) = \sum_{\xi \in \Xi, q \in \mathcal{Q}} \pi(\xi, q, a).$$

For every $q \in \mathcal{Q}$ and $a \in \mathcal{A}$, we define

$$\begin{aligned}
\mathcal{S}(q,a) &:= \{\xi \in \Xi \,|\, \pi(\xi,q,a) > 0\}, \\
L^q(a) &:= |\mathcal{S}(q,a)|, \\
c^q(a) &:= (\xi)_{\xi \in \mathcal{S}(q,a)} \\
p^q(a) &:= (\pi(\xi,q,a))_{\xi \in \mathcal{S}(q,a)}
\end{aligned}$$

It is easy to check that $\pi(a)$, $c^q(a)$ and $p^q(a)$ is a feasible for MPEC (26). Furthermore, its objective value is the same as that of π for the LP (27). ■

4 A Hybrid Approach toward Global Solution

One reason that nonconvex programs are not popular among economists is the issue of the need for global solutions. While local search algorithms for solving nonconvex programs have fast convergence properties near a solution, they are designed to find a local solution. Algorithms for solving MPECs are no exception. One heuristic in practice is to solve the same problem with several different starting points. It then becomes a trade-off between the computation time and the quality of the "best" solution found.

Linear programming does not suffer from the global solution issue. However, to obtain an accurate solution to a moral-hazard problem via the linear programming approach, one needs to use a very fine compensation grid. This often leads to large-scale linear programs with millions of variables and tens or hundreds of thousands of constraints, which might require excessive computer memory or time.

Certainly, there is a need to develop a global optimization method with fast local convergence for MPECs. Below, we propose a hybrid approach combining both MPECs and linear programming approaches to find a global solution (or at least better than the LP solution) of an optimal contract problem. The motivation for this hybrid method comes from the observation that the optimal objective value of the LP approach from a coarse grid could provide a lower bound on the optimal objective value of the MPEC as well as a good guess on the final recommended action a^*. We can then use this information to exclude some undesired local minimizers and to provide a good starting point when we solve the MPEC (11). This heuristic procedure toward a global solution of the MPEC (11) leads to the following algorithm.

A hybrid method for the optimal contract problem as MPEC (11)

Step 0: Construct a coarse grid Ξ over the compensation interval.
Step 1: Solve the LP (27) for the given grid Ξ.

$$\textbf{Step 2:} \quad \begin{cases} (2.1): & \text{Compute } p(a) = \sum_{\xi \in \Xi} \sum_{q \in \mathcal{Q}} \pi(\xi, q, a), \quad \forall\, a \in \mathcal{A}; \\ (2.2): & \text{Compute } \mathbb{E}[\xi(q)] = \sum_{\xi \in \Xi} \xi \pi(\xi, q, a), \quad \forall\, q \in \mathcal{Q}; \\ (2.3): & \text{Set initial point } c^0 = (\mathbb{E}[\xi(q)])_{q \in \mathcal{Q}} \text{ and } \delta^0 = (p(a))_{a \in \mathcal{A}}; \\ (2.4): & \text{Solve the MPEC (11) with starting point } (c^0, \delta^0). \end{cases}$$

Step 3: Refine the grid and repeat Step 1 and Step 2.

Remark If the starting point from an LP solution is close to the optimal solution of the MPEC (11), then the sequence of iterates generated by an SQP algorithm converges Q-quadratically to the optimal solution. See Proposition 2 in Fletcher et al. [4].

One can also develop similar procedures to find global solutions for optimal contract problems with action and/or compensation lotteries. However, the MPECs for contracts with lotteries are much more numerically challenging problems than the MPEC (11) for deterministic contracts.

5 An Example and Numerical Results

To illustrate the use of the mixed-integer nonlinear program (7), the MPEC (11) and the hybrid approaches, and to understand the effect of discretizing the set of compensations $\mathcal{C}$, we only consider problems of deterministic contracts without lotteries. We consider a two-outcome example in Karaivanov [9]. Before starting the computational work, we summarize in Table 1 the problem characteristics of various approaches to computing the optimal deterministic contracts.

Table 1: Problem characteristics of various approaches.

	MINLP (7)	MPEC (11)	LP (27)
Regular Variables	N	$N+M$	$\lvert\Xi\rvert * N * M$
Binary Variables	M	–	–
Constraints	$M+2$	2	$M*(N+M-1)+2$
Complementarity Const.	–	M	–

Example 1: No Action and Compensation Lotteries

Assume the principal is risk neutral with utility $w(q-c(q)) = q-c(q)$, and the agent is risk averse with utility

$$u(c(q),a) = \frac{c^{1-\gamma}}{1-\gamma} + \kappa \frac{(1-a)^{1-\delta}}{1-\delta}.$$

Suppose there are only two possible outcomes, e.g., a coin-flip. If the desirable outcome (high sale quantities or high production quantities) happens, then the principal receives $q_H = \$3$; otherwise, he receives $q_L = \$1$. For simplicity, we assume that the set of actions $\mathcal{A}$ consists of M equally-spaced effort levels within the closed interval $[0.01, 0.99]$. The production technology for the high outcome is described by $p(q = q_H|a) = a^{\alpha}$ with $0 < \alpha < 1$. Note that since 0 and 1 are excluded from the action set $\mathcal{A}$, the full-support assumption on production technology is satisfied.

The parameter values for the particular instance we solve are given in Table 2.

Table 2: The value of parameters used in Example 1.

γ	κ	δ	α	U^*	M
0.5	1	0.5	0.7	1	10

We solve this problem first as a mixed-integer nonlinear program (7) and then as an MPEC (11). For the LP lotteries approach, we start with 20 grid points in the compensation grid (we evenly discretize the compensation set $\mathcal{C}$ into 19 segments) and then increase the size of the compensation grid to $50, 100, 200, \ldots, 5000$.

We submitted the corresponding AMPL programs to the NEOS server [14]. The mixed-integer nonlinear programs were solved using the MINLP solver [3] on the computer host `newton.mcs.anl.gov`. To obtain fair comparisons between the LP, MPEC, and hybrid approaches, we chose SNOPT [5] to solve the associated mathematical programs. The AMPL programs were solved on the computer host `tate.iems.northwestern.edu`.

Table 3 gives the solutions returned by the MINLP solver to the mixed-integer nonlinear program (7). We use $y = 0$ and $y = e_M$ as starting points. In both cases, the MINLP solver returns a solution very quickly. However, it is not guaranteed to find a global solution.

Table 3: Solutions of the MINLP approach.

Starting Point	Regular Variables	Binary Variables	Constraints	Solve Time (in sec.)	Objective Value
$y = 0$	2	10	12	0.01	1.864854251
$y = e_M$	2	10	12	0.00	1.877265189

For solving the MPEC (11), we try two different starting points to illustrate the possibility of finding only a local solution. The MPEC solutions are given in Table 4 below.

Table 4: Solutions of the MPEC approach with two different starting points.
(22 variables and 10 complementarity constraints)

Starting Point	Read Time (in sec.)	Solve Time (in sec.)	# of Major Iterations	Objective Value
$\delta = 0$	0	0.07	45	1.079621424
$\delta = e_M$	0	0.18	126	1.421561553

The solutions for the LP lottery approach with different compensation grids are given in Table 5. Notice that the solve time increases faster than the size of the grid when $|\Xi|$ is of order 10^5 and higher, while the number of major iterations only increases about 3 times when we increase the grid size 250 times (from $|\Xi| = 20$ to $|\Xi| = 5000$).

Table 5: Solutions of the LP approach with 8 different compensation grids.

$\lvert\Xi\rvert$	# of Variables	Read Time (in sec.)	Solve Time (in sec.)	# of Iterations	Objective Value
20	400	0.01	0.03	31	1.876085819
50	1000	0.02	0.06	46	1.877252488
100	2000	0.04	0.15	53	1.877252488
200	4000	0.08	0.31	62	1.877254211
500	10000	0.21	0.73	68	1.877263962
1000	20000	0.40	2.14	81	1.877262184
2000	40000	0.83	3.53	71	1.877260460
5000	100000	2.19	11.87	101	1.877262793

Finally, for the hybrid approach, we first use the LP solution from a compensation grid with $|\Xi| = 20$ to construct a starting point for the MPEC (11). As one can see in Table 6, with a good starting point, it takes SNOPT only 0.01 seconds to find a solution to the example formulated as the MPEC (11). Furthermore, the optimal objective value is higher than that of the LP solution from a fine compensation grid with $|\Xi| = 5000$.

Table 6: Solutions of the hybrid approach for Example 1.

LP $\lvert\Xi\rvert$	Read Time (in sec.)	Solve Time (in sec.)	# of Iterations	Objective Value
20	0.01	0.03	31	1.876085819
MPEC Starting Point	Read Time (in sec.)	Solve Time (in sec.)	# of Major Iterations	Objective Value
$\delta_6 = 1, \delta_{i(\neq 6)} = 0$	0.02	0.01	13	1.877265298

6 Conclusions and Future Work

The purpose of this chapter is to introduce the MPEC approach and apply it to moral-hazard problems. We have presented MPEC formulations for optimal deterministic contract problems and optimal contract problems with action and/or compensation lotteries. We also formulated the former problem as a mixed-integer nonlinear program. To obtain a global solution, we have proposed a hybrid procedure that combines the LP lottery and the MPEC approaches. In this procedure, the LP solution from a coarse compensation grid provides a good starting point for the MPEC. We can then apply specialized MPEC algorithms with fast local convergence rate to obtain a solution. In a numerical example, we have demonstrated that the hybrid method is more efficient than using only the LP lottery approach, which requires the solution of a sequence of large-scale linear programs. Although we cannot prove that the hybrid approach will guarantee to find a global solution, it always finds one better than the solution from the LP lottery approach. We plan to test the numerical performance of the hybrid procedure on other examples such as the bank regulation example in [17] and the two-dimensional action choice example in [18].

One can extend the MPEC approach to single-principal multiple-agent problems without any difficulty. For multiple-principal multiple-agent models [13], it can be formulated as an equilibrium problem with equilibrium constraint. We will investigate these two topics in our future research.

Another important topic we plan to explore is the dynamic moral-hazard problem; see Phelan and Townsend [16]. In the literature, dynamic programming is applied to solve this model. Analogous to the hybrid procedure proposed in Section 4, we believe an efficient method to solve this dynamic model is to combine dynamic programming and nonlinear programming.

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