Reference-Dependent Rational Choice Theory and the Attraction Effect*

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Abstract

The goal of this paper is to develop, axiomatically, a reference-dependent choice model that accounts fully for the famous *attraction effect*. Our model is formulated under the premises of revealed preference theory, and hence it does not take the "reference" for an agent as exogenously given in the description of a choice problem. Instead, by suitably relaxing the WARP, we *derive* the existence of reference alternatives, and the choice behavior conditioned on those alternatives. We consider choice under certainty and risk separately, and obtain fairly tractable models in each of these cases. As a genuine economic application, we reexamine the standard model of monopolistic (vertical) product differentiation where a fraction of the demand side of the market is subject to the attraction effect.

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1 Introduction

The canonical model of rational choice maintains that an individual has a well-defined manner of ranking alternatives according to their desirability (independently of any particular choice problem that she might face), and among *any* collection of feasible alternatives, chooses an item that she ranks highest. Despite its various advantages, such as its unifying structure, universal applicability, tractability, and predictive abilities, this model has recently been scrutinized on the basis of its descriptive strength. Indeed, various (experimental and market) evidence point persistently to certain types of choice behavior that are inconsistent with the premises of standard rational choice theory – it appears that human decision processes are often more intricate than this theory allows for. In particular, the consensus now seems to be that the presence of certain types of (observable) choice prospects – to wit, status quo choices, endowments, default options, etc. – may substantially affect individual choice behavior.[1]

Even more curious is the observation that the effects of *reference dependence* are evident in the behavior of individuals when no alternative is designated as a "natural" reference point. In certain contexts, it is observed that a seemingly ordinary feasible choice item may, for some reason or another, act as a reference for a decision-maker, thereby affecting her choice behavior. While it may seem to an outsider hard to "justify" the behavior of the agent, she may in fact be acting in an entirely predictable manner on the basis of her (subjectively determined) reference point.

One such phenomenon that is identified in the psychological literature is the famous *attraction effect* (also known as the *asymmetric dominance effect*). Discovered first by Huber, Payne and Puto (1982), and then corroborated in numerous studies, this effect may be described (in the language of random choice theory) as the actuality in which the probability of choice increases when an asymmetrically dominated alternative is introduced into a choice problem.[2] Unlike the status quo bias phenomenon and alike, it is not an easy matter to modify the standard choice model to incorporate this phenomenon, because here "reference dependence" of an individual arises from the structure of the choice problems she faces, and hence it cannot be appended to the description of the problem exogenously.

Put succinctly, our main objective in this paper is to reexamine the rational choice theory

[1] The status quo bias phenomenon has, in particular, received quite a bit of attention in the literature. The earlier experimental work on this topic is surveyed by Camerer (1995) and Rabin (1998). Among the individual choice models that are developed to represent this phenomenon are *loss aversion* models (Tversky and Kahneman (1991)) and *multi-utility* models (Masatlioglu and Ok (2005) and Sagi (2006)).

[2] *Wikipedia*, the free encyclopedia of the world wide web, describes this effect (in marketing terms) as "the phenomenon of greater consumer preference for an item in a two-item consideration set caused by the addition of a third item that is asymmetrically dominated. An asymmetrically dominated item is in some ways better than one of the items, but in no way better than the other item." We shall clarify the nature of this description shortly.

in a way to tackle such endogenously determined reference-dependent choice situations in general, and to develop a boundedly rational choice theory that incorporates the attraction effect in general. To clarify the nature of our contribution, however, we need to explain first what we exactly mean by the "attraction effect" in this paper.

1.1 The Attraction Effect

Broadly speaking, the *attraction effect* refers to the phenomenon in which, given a choice set of two feasible alternatives, the addition of a third alternative that is clearly inferior to *one* of the existing alternatives (but not to the other), may induce a shift of preference toward the item that dominates the new alternative. To illustrate, consider two alternatives, x and y, in a world in which each alternative is characterized by exactly two attributes (such as price and quality). Suppose, as shown in Figure 1, x is better (resp., worse) than y relative to the first (resp., second) attribute. Suppose also that the agent chooses x over y, which we denote as

$$\{x, y\} \longrightarrow x$$

Now suppose a third (decoy) alternative z becomes available to the agent; this alternative is inferior to y relative to *both* attributes, but it is still better than x with respect to the first attribute (Figure 1). It is thus natural that we have

$$\{y, z\} \longrightarrow y \quad \text{and} \quad \{x, z\} \longrightarrow x.$$

Thus, x is revealed to be the best alternative in $\{x, y, z\}$ in all pairwise comparisons. The *attraction effect* corresponds to the situation in which

$$\{x, y, z\} \longrightarrow y.$$

The idea is that, somehow, the asymmetrically dominated alternative z may act as a reference for the agent in the problem $\{x, y, z\}$, making the choice prospect that is unambiguously better than z (i.e., y) more attractive than it actually is in the absence of z.

Needless to say, this type of choice behavior conflicts with the weak axiom of revealed preference, and an economist may thus lean towards dismissing it simply as "irrational." Be that as it may, the attraction effect is documented in so many studies in the psychological literature, and in the context of a truly diverse set of choice situations, that taking this position seems unwarranted.[3]

[3] The attraction effect is demonstrated in the contexts of choice of political candidates (Pan, O'Curry and Pitts (1995)), risky alternatives (Wedell (1991) and Herne (1997)), medical decision-making (Schwartz and Chapman (1999), and Redelmeier and Shafir (1995)), investment decisions (Schwarzkopf (2003)), in job candidate evaluation (Highhouse (1996), Slaughter, Sinar and Highhouse (1999), and Slaughter (2007)), and

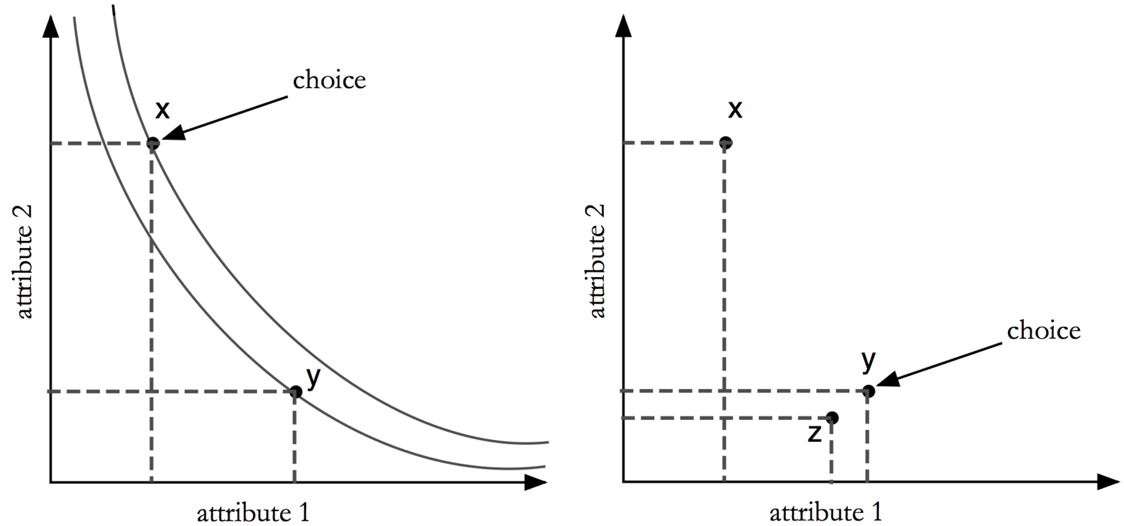

FIGURE 1

In addition, the findings of the psychological literature are thoroughly corroborated by several works in the marketing literature.[4] In passing, and to hint at the potential consequences of the attraction effect in the market place, let us briefly summarize one of the main findings of Doyle *et al.* (1999) whose field experiments have taken place in a real-world supermarket. First, the authors have recorded the sales of the Brands X and Y (of tins of baked beans) in the supermarket under study, and observed that Brand X has gotten 19% of the sales, and Y the rest, even though Brand X was cheaper. Doyle *et al.* have then introduced a third Brand Z to the supermarket, which is identical to Brand X in all attributes (including the price) except that the size of Brand Z was visibly smaller. Of course, the idea here is that Brand Z is asymmetrically dominated (i.e. it is dominated by X but not by Y). In accordance with the attraction effect, the sales for the following week showed that the sales of Brand X has

contingent evaluation of environmental goods (Bateman, Munro and Poe (2005)). While most of the experimental findings in this area are through questionnaire studies, some authors have confirmed the attraction effect also through experiments with incentives (Simonson and Tversky (1992), and Herne (1999)).

In the psychological literature, it is argued that the attraction effect may be due to simplifying decision heuristics (Wedell (1991)), or due to one's need to justify his/her decisions (Simonson (1999), and Simonson and Nowlis (2000)), or due to the ambiguity of the information about the attributes of products (Ratneshwar, Shocker and Stewart (1987) and Mishra, Umesh and Stem (1993)), or due to the comparative evaluation of goods (Simonson and Tversky (1992) and Bhargava, Kim and Srivastava (2000)), or dynamic formation of preferences in a dominance-seeking manner (Ariely and Wallsten (1995)), or evolutionary pressures (Shafir, Waite and Smith (2002)).

[4]See, *inter alia*, Burton and Zinkhan (1987), Lehmann and Pan (1994), Sivakumar and Cherian (1995), Sen (1998), Kivetz (1999), and Doyle, O'Connor, Reynolds and Bottomley (1999).

increased to 33% (while, of course, nobody has bought Brand Z).

1.2 The Revealed Preference Approach to the Attraction Effect

Given the amount of evidence on the presence of the attraction effect, and the obvious importance of this phenomenon for marketing, it is surprising that the literature does not provide a universally applicable model of individual decision-making that incorporates this phenomenon. Among other things, this makes it difficult to judge the implications (and hence importance) of this effect in terms of market predictions.[5] Moreover, it is not at all clear if, and to what extent, this effect can be reconciled with the canonical rational choice model of economic theory. The primary goal of this paper is, in fact, to investigate precisely this issue, thereby developing a reference-dependent choice model that accounts for the attraction effect.

As we wish to obtain a universally applicable choice model that parallels the standard decision making paradigm, we use the revealed preference method to construct this model. The primitives of our construction are identical to the classical choice theory, namely, a class of feasible sets of alternatives (choice problems), and a (set-valued) choice function defined on this class. This setup, by contrast to the existing experimental literature on the attraction effect, does not prescribe an alternative in terms of a given set of attributes.[6] The upside of this is that the resulting model is applicable also to situations in which the attributes of the choice prospects that are relevant to choice behavior are *unobservable*, for they are determined in the mind of the decision maker. The downside is that this approach makes it impossible to write down postulates about choice functions that correspond to the attraction effect directly. Consequently, we take an indirect route here, and concentrate rather on the choice functions that violate the weak axiom of revealed preference only due to the presence of a reference alternative that induces (potential) preference shifts in favor of certain alternatives. The nature of such reference points, their relation to the chosen alternatives and how they alter across choice problems are determined, implicitly, through our axiomatic framework.

1.3 An Outline of the Reference-Dependent Choice Model

It is perhaps a good idea to give an informal sketch of the choice model we derive in this paper. Roughly speaking, this model stipulates that the individual has a utility function

[5] For example, think of how one would model the optimal choices of a monopolist (or an oligopolist) who wishes to exploit this effect. Clearly, such a model would require an explicit model of the demand side of the market, and hence, without a suitable model of individual choice that incorporates the attraction effect, it is not possible to contemplate formally about the implications of this effect for marketing. (See Section 3 for more on this.)

[6] In fact, in the context of risky choice situations, we will be able to *derive* the standard "multi-attribute" choice model as a consequence of our behavioral postulates.

that represents her preferences (over pairwise comparisons of alternatives), and that, in every feasible set S of alternatives, she may (or may not) select an alternative $\mathbf{r}$ (in S) to serve as a reference point. If she does not single out a reference alternative, her behavior is entirely standard; she chooses the alternative in S that maximizes her utility function. If she does, however, rather than maximizing her utility function over the set S, the agent focuses on those members of S that are, in a sense, "unambiguously better than" $\mathbf{r}$ according to some criteria she deems relevant. Put differently, once $\mathbf{r}$ is determined as her reference point, the agent concentrates only on those outcomes in S that $\mathbf{r}$ "attracts" her to. In a formal sense, $\mathbf{r}$ induces an *attraction region* for the agent, say $Q(\mathbf{r})$, and she begins to view her choice problem not as S, but as $S \cap Q(\mathbf{r})$. Under this mental constraint, however, she acts fully rationally, that is, she finalizes her choice upon maximizing her utility function over $S \cap Q(\mathbf{r})$.

This is a (boundedly rational) reference-dependent choice model that allows for the presence of the attraction effect. The choice behavior exhibited in Figure 1 is, for instance, duly consistent with this model. Specifically, the model would "explain" this behavior by saying that it is "as if" the agent views z as a reference point in $\{x, y, z\}$, and the attraction region $Q(z)$ includes y but not z. So, while in the problem $\{x, y\}$, where there is no alternative to "use" as a reference for the agent to contrast x and y, the choice is x (as the (reference-free) utility of x exceeds that of y), the problem reduces in the mind of the agent to $\{y, z\} = \{x, y, z\} \cap Q(z)$, and hence the choice is y. (See Figure 2.)

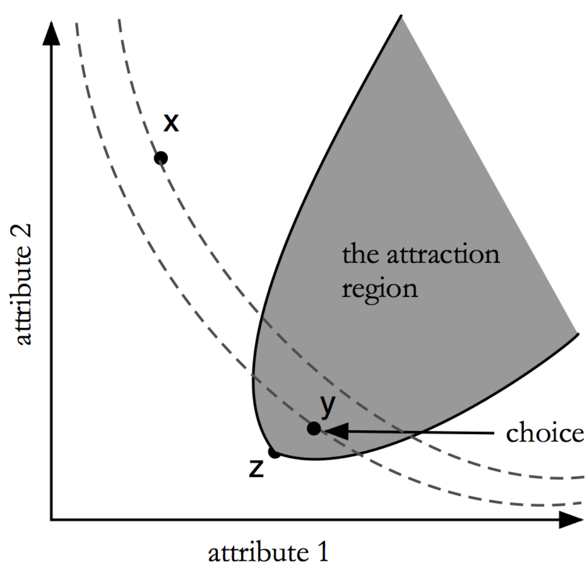

FIGURE 2

While it highlights the overall structure of the model, this discussion leaves a number of crucial issues open. Obviously, if the utility function, the reference function (the map that assigns a reference to any given feasible set) and the attraction regions can be chosen entirely freely, *any* choice behavior would be consistent with this model. This is, of course, not the case. The axiomatic setup we consider here connects these constructs in a number of ways, thereby yielding a choice model with considerable prediction power. This will be amply evident in the characterizations of our model (in Sections 2.3 and 4.3), and will be demonstrated by means of several examples (Section 2.2.5). Furthermore, in Section 3, we consider a genuine economic application that aptly shows that this model (when constricted to the context of the particular application under study) is more than capable of producing far reaching economic predictions.

1.4 The Structure of the Paper

In Section 2, we introduce our axiomatic setup and three relaxations of the weak axiom of revealed preference that are still consistent with the attraction effect. After looking at a few hypothetical examples that illustrate what is and is not allowed within this framework (Section 2.2), we introduce our reference-dependent choice model in Section 2.3 and 2.4. Our first main representation theorem is noted in Section 2.5, where we observe that our axiomatic setup and the reference-dependent choice model are, in a formal sense, one and the same.

In Section 3, we consider an application of our choice model to a genuine economic problem. In particular, we reexamine the classical (vertical) product differentiation problem of a monopoly, a lá Mussa and Rosen (1978), under the hypothesis that a given fraction of the demand side of the market is subject to the attraction effect. It appears that it is this sort of a scenario that the marketing scientists have in mind when talking about the importance of the attraction effect for designing marketing strategies, but, to our knowledge, no such formalization has appeared in the literature before. At any rate, and mainly for brevity, we only scratch the surface of economic insights that could be obtained by using our reference-dependent choice model (and its potential alternatives), as we only consider a very special scenario (where there is only one firm in the market, only low-type of consumers are subject to the attraction effect, and so on). But it will be clear from this application that the methods we use here can also be applied in more complicated settings and many other principal-agent problems.

Finally, in Section 4, we extend the coverage of our model to the case of risky choices. In that context, by using the suitable reformulations of the classical independence axiom, we refine our model further to make the structure of the utility functions, reference maps and attractions regions more tractable. This structure allows us to give a curious spin to our model. In particular, when the prize space is finite, we are able to show that an agent (who abides by our axioms) can be thought of as evaluating any given (riskless) alternative by aggregating a fixed number of criteria (known only to her). Moreover, the attraction sets of this agent for a given prospect q is obtained exactly as the set of all alternatives that dominate q relative to all criteria. This is a model of multi-attribute choice that the entire literature on the attraction effect seemingly have in mind. We are able to "derive" this model here from behavioral principles, and without assuming at the outset the observability of the attributes of the prospects that an agent deems relevant for finalizing her choices.

The proofs of our main representation theorems appear in the appendix.

2 Reference-Dependent Choice over Arbitrary Alternatives

2.1 Preliminaries

In this section X stands for an arbitrarily fixed separable metric space, and $\mathfrak{X}$ for the metric space of all nonempty compact subsets of X (under the Hausdorff metric). We think of X as the universal set of all distinct choice alternatives, and any $S \in \mathfrak{X}$ as a feasible set that an agent may need to choose an alternative from. (Some authors refer to any such S as a *choice problem* or a *choice situation.*)

The following concepts are basic.

Definition 1. A correspondence $\mathbf{c} : \mathfrak{X} \rightrightarrows X$ is said to be a **choice correspondence** on $\mathfrak{X}$ if $\emptyset \neq \mathbf{c}(S) \subseteq S$ for every $S \in \mathfrak{X}$. For any choice correspondence $\mathbf{c}$ on $\mathfrak{X}$, we define the binary relation $R_{\mathbf{c}}$ on X by

$$xR_{\mathbf{c}}y \quad \text{if and only if} \quad x \in \mathbf{c}\{x, y\}.$$

(We denote the asymmetric part of this relation by $P_{\mathbf{c}}$ and its symmetric part by $I_{\mathbf{c}}$.) This relation is called the **revealed preference relation** induced by $\mathbf{c}$.

As these notions are standard constructs of microeconomic theory, no further discussion about them is needed here. By way of preparation for the subsequent analysis, it remains only to introduce the following notational convention.

Notation. For any $S \in \mathfrak{X}$ and $x \in X$, we denote by $\mathcal{S}$ the class of all nonempty closed subsets of S, and by $\mathcal{S}_x$ the collection of all nonempty closed subsets of S that contain x, that is,

$$\mathcal{S} := \{T \in \mathfrak{X} : T \subseteq S\} \quad \text{and} \quad \mathcal{S}_x := \{T \in \mathfrak{X} : x \in T \subseteq S\}.$$

The notation $\mathcal{S}_{x,y}$ is understood similarly.

2.2 An Axiomatic Basis for Reference-Dependent Choice Theory

2.2.1 WARP and Revealed Preferences

The following is the classical *Weak Axiom of Revealed Preference* (WARP), as formulated by Arrow (1959).

WARP. For any $S \in \mathfrak{X}$ and $T \in \mathcal{S}$ with $\mathbf{c}(S) \cap T \neq \emptyset$, we have $\mathbf{c}(S) \cap T = \mathbf{c}(T)$.

The *fundamental theorem of revealed preference* says that any choice correspondence $\mathbf{c}$ on $\mathfrak{X}$ that satisfies WARP can be described as one that assigns to any feasible set $S \in \mathfrak{X}$ the elements of S that are maximal relative to a complete preorder, namely, the revealed preference relation $R_{\mathbf{c}}$. This fact is viewed as a foundation for rational choice theory, as it maintains that the choices of any individual can be modeled "as if" they stem from the maximization of a complete preference relation, provided that the choices of that individual do not conflict with the premise of WARP.

There is a substantial literature on weakening WARP. The major part of this literature concentrates on the extent of non-transitivity $R_{\mathbf{c}}$ would depict under various relaxations of this property.[7] While our main goal here too can be thought of as investigating the implications of relaxing WARP in a particular direction, we shall not focus on the potential non-transitivity of $R_{\mathbf{c}}$. As a matter of fact, in this paper we work exclusively with the following implication of WARP – dubbed the *weak WARP* (wWARP).

wWARP. $R_{\mathbf{c}}$ is transitive.

2.2.2 WARP and Awkward Choices

Strictly speaking, WARP is an independence-of-alternatives type condition that requires a given choice correspondence $\mathbf{c}$ render consistent choices for any given feasible set S and the subsets of S, whenever this is physically possible. We propose to refer to a feasible set S as $\mathbf{c}$-*awkward*, if $\mathbf{c}$ fails to satisfy this consistency property relative to S and at least one of the subsets of S.

Definition 2. Let $\mathbf{c}$ be a choice correspondence on $\mathfrak{X}$. For any given $S \in \mathfrak{X}$, we say that $x \in X$ is a **c-awkward choice** in S if $x \in \mathbf{c}(S)$ and there is a $T \in \mathcal{S}_x$ such that either $x \notin \mathbf{c}(T)$ or $\mathbf{c}(T)\backslash\mathbf{c}(S) \neq \emptyset$. We say that S is **c-awkward** if S contains at least one **c**-awkward choice.

[7] For collections of such results in this direction, see Schwartz (1976), Moulin (1985), Bandyopadhay and Sengupta (1991), among others. Suzumura (1983) and Aizerman and Aleskerov (1995) provide commendable surveys of the literature.

Note that the standard theory of revealed preference assumes away the possibility of $\mathbf{c}$-awkward choices. Indeed, the *absence* of such choices is just another formulation of WARP.

Observation 1. *Let* $\mathbf{c}$ *be a choice correspondence on* $\mathfrak{X}$ *and* $S \in \mathfrak{X}$. *Then,* S *is* $\mathbf{c}$*-awkward if, and only if, there exists a* $T \in \mathcal{S}$ *with* $\mathbf{c}(S) \cap T \neq \emptyset$ and $\mathbf{c}(S) \cap T \neq \mathbf{c}(T)$.

Proof. The "only if" part of the assertion is straightforward. To see the "if" part, suppose $\mathbf{c}(S) \cap T \neq \mathbf{c}(T)$ for some $T \subseteq S$ with $\mathbf{c}(S) \cap T \neq \emptyset$. If $y \in \mathbf{c}(S) \cap T$ for some $y \in X \backslash \mathbf{c}(T)$, then, obviously, y is a $\mathbf{c}$-awkward choice in S. If $y \in \mathbf{c}(T)$ for some $y \in X \backslash \mathbf{c}(S)$, then every x in $\mathbf{c}(S) \cap T$ must be a $\mathbf{c}$-awkward choice in S. □

It is also easily shown that being $\mathbf{c}$-awkward is hereditary under suitable *expansions* of feasible sets. That is, if a $\mathbf{c}$-awkward feasible set contains a choice from an expansion of it, then that expansion itself is bound to be $\mathbf{c}$-awkward.

Observation 2. *Let* $\mathbf{c}$ *be a choice correspondence on* $\mathfrak{X}$ *and* $R, S \in \mathfrak{X}$. *If* $S \subseteq R$, $\mathbf{c}(R) \cap S \neq \emptyset$, *and* S *is* $\mathbf{c}$*-awkward, then* R *is* $\mathbf{c}$*-awkward.*

Proof. Let S be $\mathbf{c}$-awkward, and suppose $S \subseteq R$ and $\mathbf{c}(R) \cap S \neq \emptyset$. If $\mathbf{c}(R) \cap S \neq \mathbf{c}(S)$, then the claim follows from Observation 1, so we assume $\mathbf{c}(R) \cap S = \mathbf{c}(S)$. Since S is $\mathbf{c}$-awkward, Observation 1 entails that there exists a $T \in \mathcal{S}$ such that $\mathbf{c}(T) \neq \mathbf{c}(S) \cap T \neq \emptyset$. But here we have $\mathbf{c}(S) \cap T = (\mathbf{c}(R) \cap S) \cap T = \mathbf{c}(R) \cap T$, which, by Observation 1, implies that R is also $\mathbf{c}$-awkward. □

2.2.3 The Potential Reference Relation

By far the most forthcoming arguments against WARP are based on the tendency of decision-makers to form their choices by using a certain alternative in a feasible set as a *reference.* Indeed, the violation of WARP in the case of the *attraction effect* phenomenon appears precisely due to such reference-dependent decision making. To study such violations systematically, therefore, we need a behavioral way of identifying which alternatives in a given feasible set is likely to play the role of a reference for an individual. The following definition introduces a proposal in this respect.

Definition 3. Let $\mathbf{c}$ be a choice correspondence on $\mathfrak{X}$ and $x, z \in X$. We say that z is a **potential reference for** x **relative to** $\mathbf{c}$ (or more simply, a **potential** $\mathbf{c}$**-reference for** x) if $xP_{\mathbf{c}}z$, and

$$\begin{cases} \{x\} = \mathbf{c}\{x, z, \omega\}, & \text{if } xP_{\mathbf{c}}\omega \\ x \in \mathbf{c}\{x, z, \omega\}, & \text{if } xI_{\mathbf{c}}\omega \end{cases} \quad \text{for any } \omega \in X \backslash \{x\}.$$

For any nonempty subset Y of X, we say that z is a **potential** $\mathbf{c}$**-reference for** Y if it is a potential $\mathbf{c}$-reference for each $x \in Y$.

Intuitively, by an alternative z being a potential $\mathbf{c}$-reference for x, we wish to capture the idea that the presence of z in a feasible set "enhances" in some way the appeal of x for the individual. We envision, as in the informal discussion of the *attraction effect*, this occurs because the superiority of x over z is transparent to the agent, and this somehow gives her a "reason" to choose x over the other feasible alternatives – something she would not do, had z been not available to her as an option. The definition above is a rather conservative formalization of this idea. It says that if z is a potential $\mathbf{c}$-reference for x, then, and only then, x is revealed to be strictly better than z for the agent in question, and if ω is another alternative that is worse than x (according to the revealed preference relation $R_{\mathbf{c}}$), then the agent does *not* abandon the choice x when a worse alternative ω becomes feasible, that is, she acts in a utility-maximizing manner to choose from the set $\{x, \omega, z\}$.[8]

This definition is "conservative" in the sense that it makes it rather easy for an alternative to be a potential $\mathbf{c}$-reference for another alternative. For instance, if $\mathbf{c}$ satisfies WARP, then z is a potential $\mathbf{c}$-reference for x iff $xP_{\mathbf{c}}z$. Be that as it may, it is only a definition – its usefulness will be realized when we invoke it to give content to $\mathbf{c}$ below. Before we proceed doing that, however, we make note of two fairly "natural" properties that can be imposed on the relation of being a potential $\mathbf{c}$-reference (and hence on $\mathbf{c}$). These are the usual properties of transitivity and continuity.

RT. For any $x, y, z \in X$, if z is a potential $\mathbf{c}$-reference for y, and y a potential $\mathbf{c}$-reference for x, then z is a potential $\mathbf{c}$-reference for x.

C. $R_{\mathbf{c}}$ is continuous. Moreover, for any convergent sequences (x^m) and (z^m) in X with distinct limits, if z^m is a potential $\mathbf{c}$-reference for x^m, $m = 1, 2, ...$, then $\lim z^m$ is a potential $\mathbf{c}$-reference for $\lim x^m$.

The interpretation of these properties is straightforward. We only wish to emphasize here that the property of RT (*reference transitivity*) can also be viewed as a relaxation of WARP. Indeed, if $\mathbf{c}$ satisfies WARP, then "being a potential $\mathbf{c}$-reference" relation on X coincides with $P_{\mathbf{c}}$, and hence, in this case, RT reduces to a mere implication of wWARP. Note also that C (*continuity*) property is satisfied automatically when X is a finite set (as in discrete choice theory).

[8] Implicit in this formulation is that "being a reference" is an all-or-nothing phenomenon. Put differently, we allow here for z to also be a $\mathbf{c}$-reference for ω, but "not more of a reference for ω than for x;" if z happens to be a $\mathbf{c}$-reference for both ω and x, the final arbiter of choice is the (reference-free) preference relation $R_{\mathbf{c}}$. This certainly simplifies the revealed preference theory that we are about to sketch. Moreover, we are not aware of any evidence that motivates the modeling of the notion of "being a reference" as a graded phenomenon.

2.2.4 WARP and Reference-Dependence

We now introduce our final weakening of WARP. There is a sense in which it is mainly this postulate that the present work is built on. We call it the *reference-dependent WARP* (rd-WARP).

rd-WARP. If $S \in \mathfrak{X}$ is $\mathbf{c}$-awkward, then there exists a potential $\mathbf{c}$-reference z for $\mathbf{c}(S)$ such that $z \in S \backslash \mathbf{c}(S)$ and

$$\mathbf{c}(T) = \mathbf{c}(S) \cap T \text{ for all } T \in \mathcal{S}_z \text{ with } \mathbf{c}(S) \cap T \neq \emptyset. \tag{1}$$

Suppose S is a feasible set such that a choice x in $\mathbf{c}(S)$ is not chosen from a set $T \subseteq S$ even though x was available in T. (Thus $\mathbf{c}$ violates WARP, and S is $\mathbf{c}$-awkward.) We wish to say that this happens because, in fact, x was chosen from S due to the presence of a reference point, say z, relative to which x appeared more desirable to the decision-maker. Now if this reference point is also available in T, then this interpretation runs into trouble, for then the elevated appeal of x is still present in the feasible set T, thereby making it difficult to understand why $x \notin \mathbf{c}(T)$. Similarly, if a choice x in $\mathbf{c}(S)$ is deemed "choosable" from a set $T \subseteq S$ along with an alternative $y \in T$, and yet $y \notin \mathbf{c}(S)$ – this means, again, that S is $\mathbf{c}$-awkward – we would like to interpret this by saying that there is a reference z in S that makes x more desirable than y in the choice situation S. Once again, this interpretation would not be tenable if the reference z was also available in T. In fact, rd-WARP eliminates precisely these difficulties. It says that if there is a $\mathbf{c}$-awkward choice x in S, then this is because there is a reference z in S which is itself unappealing (i.e. $z \notin \mathbf{c}(S)$) and which increases the appeal of x (over what is maintained by the revealed preference relation $R_{\mathbf{c}}$). In particular, in any subset of S that contains *both* x and z, the alternative x would remain to be a choice (as the desirability of x is also upgraded in T, thanks to the availability of z in T). More generally, that S being $\mathbf{c}$-awkward is justified by the presence of a reference point $z \in S$ (which is itself unchosen from S), and given this reference being available in any $T \subseteq S$, the agent's behavior abides precisely by the premises of WARP. This seems like quite a reasonable requirement for a reference-dependent *rational* choice theory.[9]

A number of remarks on the rd-WARP property are in order. First, note that this property is a weakening of WARP, as it is trivially satisfied by a choice correspondence $\mathbf{c}$ on $\mathfrak{X}$ such that no $S \in \mathfrak{X}$ is $\mathbf{c}$-awkward (Observation 1). Second, rd-WARP asks for only *some* potential

[9] If we have imposed the property of single-valuedness on the choice correspondence $\mathbf{c}$ on $\mathcal{X}$ – that is, consider $\mathbf{c}$ as a function, as it is done in some treatments of revealed preference theory – then the statement of rd-WARP would simplify to the following: If x is a $\mathbf{c}$-awkward choice in some $S \in \mathcal{X}$, then there exists a $z \in S \backslash \mathbf{c}(S)$ such that

(a) $x = \mathbf{c}(T)$ for all $T \in \mathcal{S}_{x,z}$; and (b) $x = \mathbf{c}\{x, \omega\}$ implies $x = \mathbf{c}\{x, z, \omega\}$ for any $\omega \in X \backslash \{x\}$.

c-reference to exist to be able to justify the **c**-awkward choices of the agent in a given feasible set S. Given that the relation of "being a potential **c**-reference" is fairly undemanding, then, rd-WARP is apparently not at all a stringent requirement. And yet, it borrows quite a bit from the classical WARP – it is really a reference-dependent version of WARP. Third, notice that rd-WARP demands that a potential **c**-reference that justifies the **c**-awkward choices of an agent not to be chosen from any set that includes at least one of those choices. Thus, it views the notion of a "reference" as something that *highlights* the appeal of other alternatives, but not itself. In this sense, it looks at a "reference" in a different light than, say, a status quo, endowment or aspiration. Its formulation is particularly suitable to model, and indeed inspired by, the *attraction effect* we discussed in Section 1.1.

In passing, we note that we know of no previous works that has examined this sort of a relaxation of WARP in particular, and a weakening of this property in a way to capture reference-dependent choice behavior, in general.

2.2.5 Examples

We have argued above that our axiomatic framework is weaker than that allowed in the standard theory (where WARP is assumed). In particular, this framework lets one model reference-dependent behavior that pertains to the attraction effect, as we illustrate next.

Example 1. Let $X := \{x, y, z\}$, and consider the choice function $\mathbf{c}$ on $\mathfrak{X}$ depicted below:

$$\begin{array}{rcl} \{x,y\} & \longrightarrow & x \\ \{x,z\} & \longrightarrow & x \\ \{y,z\} & \longrightarrow & y \\ \{x,y,z\} & \longrightarrow & y \end{array}$$

Obviously, this choice behavior is in violation of WARP – the alternative y is not revealed preferred to x even though it is deemed to be the choice from the feasible set $\{x, y, z\}$. (Thus $\{x, y, z\}$ is **c**-awkward.) As we discussed earlier, this is the prototypical behavior of the attraction effect phenomenon. The alternative x emerges as the best alternative with respect to pairwise comparisons, but presumably because z acts as a reference that "attracts" the attention of the individual to y when these two alternatives are both feasible, the agent chooses y from the set $\{x, y, z\}$. It is easily checked that this behavior is consistent with all four of the axioms we considered above; we find here that z is a potential **c**-reference for y. ‖

At the other end, one may be worried that the discipline that our axiomatic model instigates may be too lax, thereby leaving us with a theory with little predictive power. It is indeed not transparent what sort of behavior is outruled by the present axiomatic system.

To get an idea about this, we now consider a few instances of choice behavior that are *not* consistent with this system.

Example 2. Let $X := \{x, y, z\}$, and consider a choice function $\mathbf{c}$ on $\mathfrak{X}$ such that

$$\begin{aligned} \{x,y\} &\longrightarrow x \\ \{x,z\} &\longrightarrow z \\ \{y,z\} &\longrightarrow y \end{aligned}$$

Then, $\mathbf{c}$ is obviously outruled by the present axiomatic model, for it violates wWARP. ‖

Example 3. Let $X := \{x, y, z\}$, and consider the choice function $\mathbf{c}$ on $\mathfrak{X}$ depicted below:

$$\begin{aligned} \{x,y\} &\longrightarrow x \\ \{x,z\} &\longrightarrow x \\ \{y,z\} &\longrightarrow y \\ \{x,y,z\} &\longrightarrow z \end{aligned}$$

Then, $\mathbf{c}$ is outruled by the present axiomatic model, for it violates rd-WARP. Indeed, $\{x, y, z\}$ is $\mathbf{c}$-awkward, as z is chosen from $\{x, y, z\}$ but not from $\{x, z\}$. But since z is never chosen from a doubleton set, it is clear that there is no potential $\mathbf{c}$-reference for it, and hence $\mathbf{c}$ violates rd-WARP. ‖

Example 4. Let $X := \{v, w, x, y, z\}$, and consider a choice function $\mathbf{c}$ on $\mathfrak{X}$ such that $vP_{\mathbf{c}}wP_{\mathbf{c}}xP_{\mathbf{c}}yP_{\mathbf{c}}z$ and

$$\begin{aligned} \{v,w,y\} &\longrightarrow w \\ \{x,y,z\} &\longrightarrow y \\ \{w,y,z\} &\longrightarrow y \end{aligned}$$

Then, $\mathbf{c}$ is outruled by the present axiomatic model. To see this, suppose $\mathbf{c}$ satisfies wWARP, RT and rd-WARP. Then, by wWARP, we have $\mathbf{c}\{v, w\} = \{v\}$. Hence, $\{v, w, y\}$ is $\mathbf{c}$-awkward, and by rd-WARP, $\mathbf{c}\{v, w, y\} = \{w\}$ means that y is a potential $\mathbf{c}$-reference for w. A similar reasoning shows also that z is a potential $\mathbf{c}$-reference for y. So, by RT, z must be a potential $\mathbf{c}$-reference for w. Since $wP_{\mathbf{c}}y$ by wWARP, therefore, by definition of a potential $\mathbf{c}$-reference, we must have $\{w\} = \mathbf{c}\{w, y, z\}$ in contradiction to the data of the problem. ‖

Example 5. Let $X := \{w, x, y, z\}$, and consider a choice function $\mathbf{c}$ on $\mathfrak{X}$ such that

$$\begin{aligned} \{x,y\} &\longrightarrow x \\ \{x,y,z\} &\longrightarrow z \\ \{w,x,y\} &\longrightarrow y \\ \{w,x,y,z\} &\longrightarrow w \end{aligned}$$

Then, $\mathbf{c}$ is outruled by the present axiomatic model. To see this, suppose $\mathbf{c}$ satisfies wWARP, RT and rd-WARP. Notice that the first and third statements imply here that w is a potential $\mathbf{c}$-reference for y, and the third and fourth statements imply that z is a potential $\mathbf{c}$-reference for w. Then, by RT, z is a potential $\mathbf{c}$-reference for y so that $yP_{\mathbf{c}}z$. By the second statement above (and rd-WARP), therefore, x is a potential $\mathbf{c}$-reference for z, and hence $zP_{\mathbf{c}}x$. Since the first statement above means $xP_{\mathbf{c}}y$, this contradicts the transitivity of $P_{\mathbf{c}}$. $\|$

2.3 Mentally Constrained Utility Maximization

2.3.1 The Reference-Dependent Choice Model

We now depart from our axiomatic framework momentarily, and rather work towards introducing a reference-dependent choice model directly. The connection between this model and the indirect approach outlined in Section 2.2 will be clarified in Section 2.4.

We begin with a definition in which we reserve the symbol $\diamondsuit$ for an arbitrary object that does not belong to X.[10]

Definition 4. A function $\mathbf{r} : \mathfrak{X} \to X \cup \{\diamondsuit\}$ is said to be a **reference map** on $\mathfrak{X}$ if, for any $S \in \mathfrak{X}$, we have $\mathbf{r}(S) \in S$ whenever $\mathbf{r}(S) \neq \diamondsuit$, and $\mathbf{r}(S) = \diamondsuit$ whenever $|S| \leq 2$.

Given any choice problem S in $\mathfrak{X}$, a reference map r on $\mathfrak{X}$ either identifies an element $\mathbf{r}(S)$ of S to act as a reference point when solving this problem or it declares that no element in S qualifies to be a reference for the choice problem – we denote this situation by $\mathbf{r}(S) = \diamondsuit$. In particular, by definition, we posit that if S contains either a single element or only of two alternatives, then $\mathbf{r}$ declares that nothing serves as a reference point in S.[11]

The individual choice model that we wish to investigate in this paper is introduced next.

Definition 5. A **reference-dependent choice model** on $\mathfrak{X}$ is a triplet $\langle U, \mathbf{r}, Q \rangle$, where U is a continuous real function on X, $\mathbf{r}$ is a reference map on $\mathfrak{X}$, and $Q : X \cup \{\diamondsuit\} \rightrightarrows X$ is a correspondence with closed-graph such that, for any $S, T \in \mathfrak{X}$ with $T \subseteq S$,

1. $\mathbf{r}(S) = \diamondsuit$ and $T \cap \arg\max U(S) \neq \emptyset$ imply $\mathbf{r}(T) = \diamondsuit$,[12]

[10] In what follows, $X \cup \{\diamondsuit\}$ is viewed as a metric space upon adjoining $\diamondsuit$ to X as an isolated point.

[11] This highlights the fact that the notion of "reference alternative" that this definition is meant to capture is not related to, say, the status quo phenomenon. The latter notion would necessitate a default option to be thought of as a "reference" in dichotomous choice problems as well. To reiterate, our focus is on the notion of "reference" alternatives that are not desirable in themselves, but rather, affect the comparative desirability of other alternatives. Thus, the reference notion becomes meaningful in the present setup only when there are at least two alternatives in the choice situation at hand, in addition to the alternative designated as the reference point.

[12] *Notation.* For any set $A \subseteq X$, we write $\arg\max U(A)$ to denote the set of all $\omega \in A$ such that $U(\omega) = \max U(A)$.

2. $Q(\diamondsuit) = X$,

3. $Q \circ Q \subseteq Q$,

4. $S \cap Q(\mathbf{r}(S)) \neq \emptyset$, and

5. $Q(\omega) \subseteq \{x \in X : U(x) > U(\omega)\}$ for all $\omega \in X$.

For any given correspondence $\mathbf{c} : \mathfrak{X} \rightrightarrows X$, we say that this model **represents c** on $\mathfrak{X}$ if

$$\mathbf{c}(S) = \arg\max U(S \cap Q(\mathbf{r}(S))) \quad \text{for all } S \in \mathfrak{X}. \tag{2}$$

Let $\langle U, \mathbf{r}, Q \rangle$ be a reference-dependent choice model on $\mathfrak{X}$. Here U is interpreted as the utility function of the individual decision maker, *free of any referential considerations*. In particular, if the alternatives have various attributes that are relevant to the final choice – these attributes may be explicitly given, or may have a place in the mind of the agent – then U can be thought as aggregating the performance of all the attributes of any given alternative in a way that represents the preferences of the agent.

In turn, $\mathbf{r}$ serves as the reference map that tells us which alternative is viewed by the agent as the reference for a given choice situation. It seems fairly reasonable that if S and T are two choice problems with $T \subseteq S$, and the agent has a referential method of making a choice in the case of the smaller (hence easier) problem T, then she would also do so in the larger (hence harder) problem S. At first glance, this interpretation makes it appealing to assume that $\mathbf{r}(S) = \diamondsuit$ implies $\mathbf{r}(T) = \diamondsuit$. There is, however, a caveat. The reason why $\mathbf{r}(S) = \diamondsuit$ holds may be due to the presence of an alternative x in S that has an overwhelmingly high utility for the agent, leaving no room for comparisons relative to a reference point. Thus, if $x \notin T$, then it would be a mistake in this case to assume that referential considerations would not arise in T. By contrast, if we are given also that $T \cap \arg\max U(S) \neq \emptyset$, this objection disappears, and it becomes quite appealing to presume that $\mathbf{r}(S) = \diamondsuit$ implies $\mathbf{r}(T) = \diamondsuit$. This is exactly what condition (1) of the definition above does.

The interpretation of the correspondence Q is more subtle. For any $\omega \in X \cup \{\diamondsuit\}$, we interpret the set $Q(\omega)$ as telling us which alternatives in the universal set X look "better" to the agent *when compared to* ω – it may thus make sense to call $Q(\omega)$ the *attraction region* of ω. (For instance, if the agent deems a number of attributes of the alternatives as relevant for her choice, then $Q(\omega)$ may be thought of as the set of all alternatives that dominate ω with respect to all attributes.) This interpretation makes conditions (2) and (3) easy to understand. Condition (2) is actually trivial: "nothing" does not enhance the appeal of any alternative, or put differently, "nothing" does not attract the agent's attention to any particular set of alternatives. Condition (3), on the other hand, is a straightforward transitivity property: If y

makes alternative x look more attractive and z makes y more attractive, then z makes x look more attractive. (In the case of multi-attribute choice, this property becomes unexceptionable: if x dominates y with respect to all attributes, and y dominates z similarly, then x dominates z with respect to all attributes.)

Conditions (4) and (5), in turn, relate the structure of Q and those of $\mathbf{r}$ and U. In particular, condition (4) says that if, in a choice problem $S \in \mathfrak{X}$, the agent deems an alternative ω as a reference point (that is, $\mathbf{r}(S) = \omega$), then $S \cap Q(\omega)$ corresponds to the set of all alternatives in S that are (mentally) highlighted by the presence of ω in S. If $S \cap Q(\omega) = \emptyset$, then it would be odd that the agent has identified ω as a reference, for this alternative does not direct her attention to *any* feasible alternative in S. Since we would like to think that, once a reference alternative in S is determined, the agent would rather focus only on the feasible elements in the attraction region of ω, it seems unexceptionable that we have $S \cap Q(\omega) \neq \emptyset$ in the model. This is the gist of condition (4).

Finally, condition (5) says that an alternative accentuates the appeal of another alternative x only if the latter object is better than ω in an objective sense, that is, only if $U(x) > U(\omega)$. This is, again, in line with the attraction effect phenomenon. For the value of x to seem enhanced when ω is taken as a reference, x should seem clearly dominant relative to ω (in a sense that only the agent discerns). Hence, in particular, x should be ranked above ω according to the (aggregate) utility function U. But note that $U(x) > U(\omega)$ is not a sufficient condition for $x \in Q(\omega)$. Even though x may be a better alternative than ω (in terms of pairwise comparisons), its comparative appeal need not be upgraded in the presence of ω, that is, ω need not *attract* the attention of the decision maker to any x with $U(x) > U(\omega)$. (In the case of multi-attribute choice, for instance, if $x \in Q(\omega)$, that is, x is superior to ω with respect to all attributes relevant to choice, it is only natural to presume that $U(x) > U(\omega)$, but, of course, not conversely.)

Given these considerations, the content of the statement "$\langle U, \mathbf{r}, Q \rangle$ represents the choice correspondence $\mathbf{c}$ on $\mathfrak{X}$" becomes clear. Take any choice problem $S \in \mathfrak{X}$. The agent either evaluates this problem in a reference-independent manner or identifies a reference point in S and uses this point to finalize her choice. In the former case, $\mathbf{r}(S) = \diamondsuit$, so, by condition (2), $Q(\mathbf{r}(S)) = X$. Consequently, in this case (2) reads

$$\mathbf{c}(S) = \arg\max_{x \in S} U(x),$$

in concert with the standard theory of rational choice.[13] In the latter case, $\mathbf{r}(S)$ is an alternative in S, say z, and the agent is mentally "attracted" to the elements of S that belong to $Q(z)$. It is "as if" she faces the *mental constraint* that her choices from S must belong to $Q(z)$. As ilustrated in Figure 3, within this constraint, the agent acts fully rationally, and solves her

[13] The standard theory is thus captured by $\langle U, \mathbf{r}, Q \rangle$ upon setting $\mathbf{r}(S) = \diamondsuit$ for all $S \in \mathcal{X}$.

problem upon the maximization of U, that is,

$$\mathbf{c}(S) = \underset{x\in S\cap Q(z)}{\arg\max}\, U(x).$$

(Condition (4) guarantees that this equation is well-defined.) This is the basic content of the reference-dependent choice model $\langle U, \mathbf{r}, Q\rangle$.

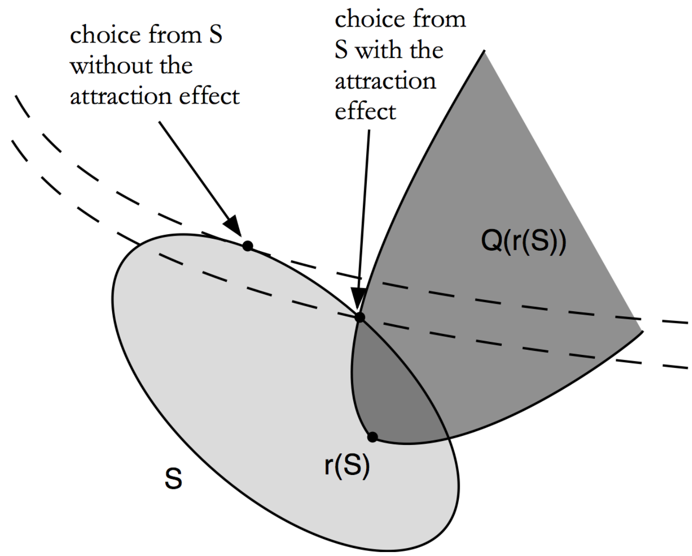

FIGURE 3

2.3.2 Properness

While its interpretation is somewhat promising, a reference-dependent choice model on X still lacks structure to be useful in applications. Indeed, in general, such a model does not at all restrict how the reference points of an agent relate to each other across varying choice situations. To illustrate, take a choice correspondence $\mathbf{c}$ on $\mathfrak{X}$ that is represented by such a model $\langle U, \mathbf{r}, Q\rangle$, and suppose that S is a choice problem in $\mathfrak{X}$ such that

$$\mathbf{r}(S) \neq \diamondsuit \quad \text{and} \quad x \in \mathbf{c}(S). \tag{3}$$

Then, x need not be a utility-maximizing alternative in S; it rather maximizes U on the subset of S that consists of all alternatives towards which the reference alternative $\mathbf{r}(S)$ "attracts" the agent (i.e. on $S \cap Q(\mathbf{r}(S))$). Now consider another choice problem $T \in \mathfrak{X}$ such that

$$\{x, \mathbf{r}(S)\} \subseteq T \subseteq S. \tag{4}$$

The model $\langle U, \mathbf{r}, Q\rangle$ does not put any restrictions on what would the reference point in T be, and hence, on how the choice from T would relate to that from the larger set S. It may be that a new alternative $\mathbf{r}(T)$ now acts as a reference in T and an alternative y – whose utility may be significantly below $U(x)$ – is thus chosen by the agent due to this (i.e., because $y \in Q(\mathbf{r}(T))$ and $x \notin Q(\mathbf{r}(T))$). Put differently, the arbitrariness of the $\mathbf{r}$ function allows for rather wild violations of WARP, thereby taking away significantly from the predictive strength of the model $\langle U, \mathbf{r}, Q \rangle$.

It thus makes good sense to impose some restraints on models of the form $\langle U, \mathbf{r}, Q \rangle$ that relate the references and choices across nested choice problems. In particular, a condition like "(4) implies $\mathbf{r}(T) = \mathbf{r}(S)$" would limit the violations of WARP significantly. Yet, while one may choose to impose such a restriction in the case of a specific application, this is really not a behavioral assumption – it cannot possibly be deduced from postulates on a given choice correspondence on $\mathfrak{X}$. After all, $\mathbf{c}$ cannot identify $\mathbf{r}$ uniquely. Consequently, so long as the choice behavior of the agent is consistent, say, in the sense of rd-WARP, varying reference points across S and T may be inescapable. What we need to do is, instead, to impose conditions on $\langle U, \mathbf{r}, Q \rangle$ that relate directly to the choice behavior that this model entails. In particular, we wish to impose that (3) and (4) imply that the model maintains x to be chosen from T as well. Conversely, if, in addition to (3) and (4), we know that y is deemed choosable from T (i.e. $y \in \mathbf{c}(T)$), then $\mathbf{r}(S)$ and $\mathbf{r}(T)$ should be related in such a way that y is also deemed choosable from S (i.e. $y \in \mathbf{c}(S)$). We propose to call those reference-dependent choice-models $\langle U, \mathbf{r}, Q \rangle$ that induce this kind of choice behavior *weakly proper*. The formal definition follows.

Definition 6. Take any $S \in \mathfrak{X}$ and let $\langle U, \mathbf{r}, Q \rangle$ be a reference-dependent choice model on $\mathfrak{X}$. We say that this model is **weakly proper** on S if, for any $x \in \arg\max U(S \cap Q(\mathbf{r}(S)))$ and $T \in \mathcal{S}_{x,\mathbf{r}(S)}$,[14] we have $x \in \arg\max U(T \cap Q(\mathbf{r}(T)))$ and

$$U(y) \geq U(x) \text{ and } y \in Q(\mathbf{r}(T)) \quad \text{imply} \quad y \in Q(\mathbf{r}(S)).$$

Finally, we say that $\langle U, \mathbf{r}, Q \rangle$ is **weakly proper** if it is weakly proper on any $S \in \mathfrak{X}$.

There is one more regularity condition we need to consider before stating our main characterization theorem. To motivate this condition, take again a reference-dependent choice model $\langle U, \mathbf{r}, Q \rangle$ and assume that $U(x) > U(z)$ and $x \notin Q(z)$. We wish to impose that there exists at least one choice problem S that contains x and z such that some alternative other than x is chosen from S due to the presence of z in S. If there was no such S, there is then no behavioral content of the statement $x \notin Q(z)$, as this fact is never used by $\mathbf{c}$ to justify x not being chosen in a problem where z acts as a reference. It thus seems quite reasonable to assume that there exists at least one alternative y in X with $U(y) \leq U(x)$ such that either

[14] *Notation.* For any $S \in \mathcal{X}$ and $y \in X$, by $\mathcal{S}_{y,\diamond}$ we simply mean $\mathcal{S}_y$.

$\mathbf{r}\{x,y,z\} = z$ and $y \in \mathbf{c}\{x,y,z\}$, or $\mathbf{r}\{x,y,z\} = y$ and $z \in \mathbf{c}\{x,y,z\}$. This assumption is formalized below.

Definition 7. Let $\langle U, \mathbf{r}, Q \rangle$ be a reference-dependent choice model on $\mathfrak{X}$. We say that this model is **proper** if it is weakly proper, and for any $x, z \in X$ with $U(x) > U(z)$ and $x \notin Q(z)$, there exists a $y \in X$, such that $U(x) \geq U(y)$ and $x \notin Q(\mathbf{r}\{x,y,z\})$.

One may wish to impose further conditions on a reference-dependent choice model $\langle U, \mathbf{r}, Q \rangle$ in order to increase the predictive power of this model. Indeed, especially in the case of applications, there may be natural restrictions that would refine this model considerably. At our level of abstraction, however, no further restriction is necessary, as a proper reference-dependent choice model corresponds precisely the choice behavior that abide by the behavior postulates we considered in Section 2.3.

2.4 A Representation Theorem

At present we have two different means of looking at reference-dependent choice behavior that is compatible with the attraction effect. The four axioms considered in Sections 2.2 correspond to the suitable violations of WARP that allows for such behavior. The idea that behavior may be reference-dependent is implicit in these axioms. The choice model considered in Section 2.3, on the other hand, takes reference-dependence explicitly, and specifies in precise terms how the decision-making is carried out once a reference point in a choice problem is identified by an individual. The following theorem shows that, at an abstract level, these two different ways of looking at reference-dependent choice behavior are one and the same.

Theorem 1. *A map $\mathbf{c} : \mathfrak{X} \to 2^X$ is a choice correspondence on $\mathfrak{X}$ that satisfies wWARP, C, RT and rd-WARP if, and only if, it is represented by a proper reference-dependent choice model.*

In particular, in a context where the alternative space is finite, the choice behavior of any individual that abides by the behavioral hypotheses of wWARP, RT and rd-WARP can be thought of "as if" this person solves her choice problems through a mentally constrained utility maximization (as specified by a reference-dependent choice model). Conversely, any sort of choice behavior that corresponds to some reference-dependent choice model is duly compatible with the behavioral postulates of wWARP, RT and rd-WARP. While such a model is only boundedly rational, its violations of rationality (i.e. WARP) is only due to the potential reference effects, and it is these effects that allow for the behavior of the agent being compatible with the attraction effect.

3 Application: Product Differentiation under the Attraction Effect

Our work so far provides a basic behavioral foundation for (proper) reference-dependent choice models, but does not make explicit how such models can be implemented in practice. In fact, especially in principal-agent type scenarios, such as contract design, nonlinear pricing and regulation, it is particularly easy to use such models to study if, and how, the principal would exploit the agents who may be subject to the attraction effect. In this section, we wish to outline a genuine economic application to illustrate this point.

The marketing literature on the attraction effect so far has been concentrated mainly on the importance of this phenomenon for market segmentation. Nonetheless, and perhaps due to the lack of a suitable model of consumer choice, the literature contains no formal discussion on how the supply side of the market may indeed exploit the demand side in the presence of this effect. In this section we shall use the reference-dependent choice model developed above to address this problem in the context of the classical product differentiation model of Mussa and Rosen (1978). On one hand, this application illustrates how the present choice theory (and its potential variants) can be used in applied economic analysis.[15] On the other, it provides a formal treatment of the marketing implications of the attraction effect, and yields a number of insights that would be difficult to obtain without such a treatment.

3.1 The Mussa-Rosen Model

Consider a monopolistic market for a single good, where there are two types of consumers with distinct taste parameters about the quality of the product. There are two types of consumers, H (for high) and L (for low), that are evenly distributed in the society. Types H and L evaluate the utility of one unit of the good of quality $q \geq 0$ at price $p \geq 0$, respectively, as

$$U_L(p,q) := \theta_L q - p \quad \text{and} \quad U_H(p,q) := \theta_H q - p,$$

where $\theta_H > \theta_L > 0$. For concreteness, we work here with a particular production technology by confining ourselves to the case of quadratic cost functions. That is, we assume that the cost of producing a unit good of quality $q \geq 0$ is q^2 (but, of course, the analysis below would extend to the case of any well-behaved cost function). The problem of the monopolist is, then, to choose (possibly differentiated) quality levels $q_H \geq 0$ and $q_L \geq 0$, the unit prices $p_H \geq 0$

[15] It will be obvious from the subsequent analysis that numerous other economic applications can be treated by similar methods. In particular, the subsequent analysis applies, with no essential modification, to the monopolistic nonlinear pricing model of Maskin and Riley (1984).

and $p_L \geq 0$ in order to

$$\begin{aligned} \text{Maximize} \quad & (p_L - q_L^2) + (p_H - q_H^2) \\ \text{such that} \quad & U_L(p_L, q_L) \geq 0, \\ & U_H(p_H, q_H) \geq U_H(p_L, q_L). \end{aligned}$$

Here, of course, the first constraint guarantees that (p_L, q_L) is indeed purchased by L types, and the second one is the usual incentive compatibility constraint that makes sure that H types do not wish to purchase the lower quality good. (These conditions are easily shown to imply the participation constraint for H types and the incentive compatibility constraint for L types.) In what follows, we work with the parametric restriction $\theta_L \geq \frac{\theta_H}{2}$. This condition guarantees the presence of a (unique) interior solution for the monopolist's problem. In the solution both constraints (that is, the participation constrint of the L type and the incentive compatibility constraint of the H type) are binding, and the optimal quality choices of the monopolist are found as:

$$\hat{q}_H := \frac{\theta_H}{2} \quad \text{and} \quad \hat{q}_L := \theta_L - \frac{\theta_H}{2}.$$

(The product produced by the monopolist for the H types is shown by $\hat{\text{H}}$ in Figure 4, and that for L types as $\hat{\text{L}}$.) Thus the monopolist finds using lower quality goods as a profitable method of market segmentation. As is apparent from Figure 4, and as is well-known, $\hat{q}_H$ is indeed the efficient (socially optimal) level of quality, while there is a downward distortion of the low valuation agent's quality with respect to the first-best outcome. In what follows, we shall refer to the menu $\{(\hat{p}_H, \hat{q}_H), (\hat{p}_L, \hat{q}_L)\}$ as the *Mussa-Rosen solution.*[16]

3.2 The Mussa-Rosen Model with the Attraction Effect

If the demand side of the market does not consist only of (standard) utility maximizing consumers, and includes, for instance, certain consumers that suffer from the attraction effect, then the constraints of the monopolist's problem considered above need to be modified. Consequently, incorporating the attraction effect into the model is likely to yield different market predictions in terms of market segmentation. In fact, evidently, the vast interest of the marketing literature in the attraction effect is based precisely on this sort of a reasoning.

We now use the reference-dependent choice model developed in Section 2 to address this problem within the confines of the Mussa-Rosen model. To simplify the exposition, and avoid

[16]It is easily computed from the constraints of the problem that

$$\hat{p}_H = \frac{\theta_H^2}{2} - (\theta_H - \theta_L)(\theta_L - \frac{\theta_H^2}{2}) \quad \text{and} \quad \hat{p}_L = \theta_L((\theta_L - \frac{\theta_H^2}{2}).$$

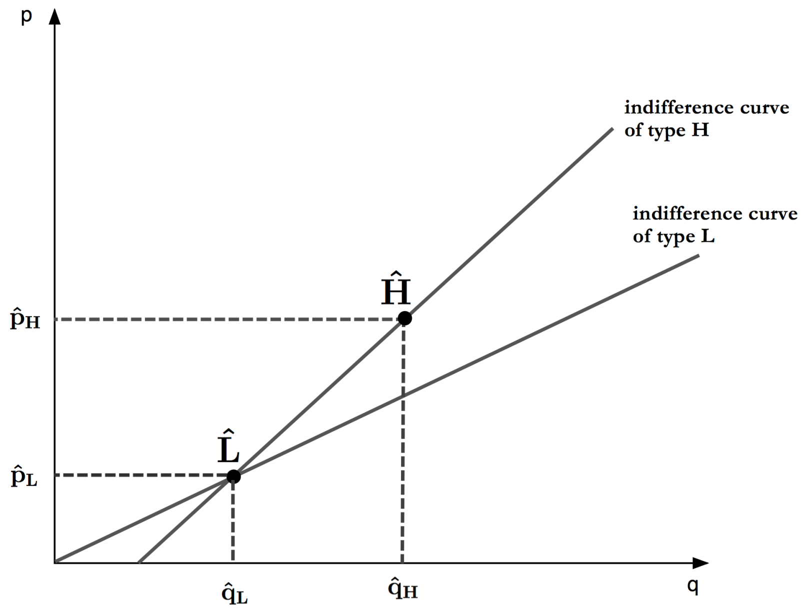

FIGURE 4

the consideration of certain trivial cases, we shall carry out our analysis under the assumption that types H and L are sufficiently different from each other. Specifically, we assume in what follows that

$$\frac{3\theta_H}{4} > \theta_L > \frac{\theta_H}{2}. \tag{5}$$

Suppose that a certain (known) proportion α of L types in the market are subject to the attraction effect – we refer to these consumers as type A – while, for simplicity, we model H types as standard utility-maximizers. Where we take X to be $\mathbb{R}^2_+$, the choice behavior of type A consumers is modeled by means of a reference dependent choice model $\langle U_A, \mathbf{r}, Q\rangle$ on the set $\mathfrak{X}$ of all compact subsets of $\mathbb{R}^2_+$ as follows. First, since types A and L are indistinguishable from each other absent potential reference effects, we take $U_A := U_L$. Second, we recall that a plausible interpretation of the attraction region $Q(\omega)$ of a choice alternative ω is as the set of all alternatives that dominate ω with respect to each attribute relevant for choice. This interpretation fits particularly well in the present setup where an alternative is identified on the basis of exactly two attributes, price and quality. It thus stands to good reason to define

$$Q(p,q) := \{(s,t) \in \mathbb{R}^2_+ \backslash \{(p,q)\} : s \leq p \text{ and } t \geq q\}$$

for any $(p,q) \in X$, while, of course, $Q(\diamondsuit) := \mathbb{R}^2_+$.[17]

[17] Other choices for the correspondence Q that yield narrower attraction regions (and hence weaker attraction effects) may also be considered here. While our definition of Q yields a particularly tractable model, the subsequent analysis would remain qualitatively unaltered with such alternative specifications.

The choice of $\mathbf{r}$ merits some discussion. First, observe that for $\langle U_A, \mathbf{r}, Q\rangle$ to be a reference-dependent choice model on $\mathfrak{X}$, we need to have $S \cap Q(\mathbf{r}(S)) \neq \emptyset$ for each $S \in \mathfrak{X}$ (Definition 5). It follows that $\mathbf{r}(S) = (p, q)$ only if there is a bundle in S that dominates (p, q). Put differently, we are forced to have $\mathbf{r}(S) = \diamondsuit$ for any $S \in \mathfrak{X}$ such that there is no dominated alternative in S (in terms of both price and quality). This suggests that we may define $\mathbf{r}(S)$ as an arbitrary member of the dominated alternatives in S, and set it to $\diamondsuit$ (and hence posit reference-free behavior) if there is no such alternative. For concreteness, for any such S, we shall take $\mathbf{r}(S)$ as a bundle with the smallest quality among all alternatives in S that are dominated by at least one bundle.

There is one conceptual difficulty with this modeling strategy, however. It allows, per force, the monopolist to attract any consumer of type A to buy a bundle at unrealistically high prices, and hence renders any such individual as unreasonably "irrational." It makes better sense to posit that reference effects become effective when this is not obviously "too costly" for the consumer, that is, when it does not cause too much of a utility loss for her. To keep our analysis in closer contact with the Mussa-Rosen model, therefore, we assume here that a type A agent is subject to the attraction effect only when this leads her to buy a bundle that yields a utility of $\beta := U_A(\hat{p}_H, \hat{q}_H)$, that is, a bundle that is at least as good as the bundle produced for type H in the standard model.[18] Thus, if no alternative in a feasible set S is dominated by a bundle (p', q') in S with

$$U_A(p', q') \geq \beta := U_A(\hat{p}_H, \hat{q}_H) = -(\theta_H - \theta_L)^2,$$

we set $\mathbf{r}(S) = \diamondsuit$.

These considerations lead us to the following formal prescription of the reference map $\mathbf{r}$. Define

$$\mathcal{D}_S := \{(p, q) \in S : p' \leq p \text{ and } q' \geq q \text{ for some } (p', q') \in S \text{ with } U_A(p', q') \geq \beta\}$$

for any $S \in \mathfrak{X}$, and let $\geq_{\text{lex}}$ denote the following lexicographic order on $\mathbb{R}^2_+$: $(p', q') \geq_{\text{lex}} (p, q)$ iff either $q < q'$, or $q = q'$ and $p \geq p'$. We define $\mathbf{r}(S) := \diamondsuit$ for any $S \in \mathfrak{X}$ with either $|S| \leq 2$ or $\mathcal{D}_S = \emptyset$, and let

$$\mathbf{r}(S) := \text{ the minimum element of } \mathcal{D}_S \text{ with respect to } \geq_{\text{lex}},$$

for any $S \in \mathfrak{X}$ with $|S| \geq 3$ and $\mathcal{D}_S \neq \emptyset$. (Thanks to the compactness of S, $\mathbf{r}(S)$ is well-defined.) This completes the specification of $\langle U_A, \mathbf{r}, Q\rangle$, that is, the choice behavior of a lower valuation consumer who suffers from the attraction effect. It is easy to check that $\langle U_A, \mathbf{r}, Q\rangle$

[18] While this assumption simplifies some of the computations, it is not essential to what follows. Suffices it to say that for larger values of $\beta < 0$ the influence of the attraction effect on the decisions of the monopolist diminishes (and vanishes at $\beta = 0$), while the converse holds for smaller values of β.

is indeed a proper reference-dependent model in the sense of Definitions 5-7.[19] We denote the choice correspondence that is represented by this model as $\mathbf{c}_A$.

Now that the demand side of the market is described, we turn to the problem of the monopolist. If we assume that the monopolist knows the fraction α of type L consumers that are in fact of type A, then it becomes duly plausible that it may consider exploiting this by producing a "decoy" bundle – denoted as (p_R, q_R) – and a "target" bundle – denoted as (p_A, q_A) – for such consumers, and attract them to buy the target (which they would not have bought in the absence of the decoy bundle). Of course, using decoys as such is costly for the monopolist; supplying (p_R, q_R) to the market costs q_R^2.[20]

Put precisely, then, the monopolist's current problem is to choose quality levels $q_A, q_H, q_L, q_R \geq 0$ and prices $p_A, p_H, p_L, p_R \geq 0$ in order to

$$\begin{array}{ll} \text{Maximize} & \alpha(p_A - q_A^2) + (p_H - q_H^2) + (1-\alpha)(p_L - q_L^2) - \alpha q_R^2 \\ \text{such that} & U_L(p_L, q_L) \geq 0, \\ & U_L(p_L, q_L) \geq U_L(p_A, q_A), \\ & U_H(p_H, q_H) \geq \max\{U_H(p_L, q_L), U_H(p_A, q_A)\}, \\ & (p_A, q_A) \in \mathbf{c}_A\{(p_A, q_A), (p_H, q_H), (p_L, q_L), (p_R, q_R), (0,0)\}. \end{array}$$

The first three constraints here guarantee that L types like to purchase (p_L, q_L) (and not (p_A, q_A)) and H types do not wish to purchase either (p_L, q_L) or (p_A, q_A). In turn, these imply that the participation constraint of H types and the incentive constraints of L types are also met. As it combines the participation and incentive constraints for type A, the interesting constraint here is obviously the fourth one.[21]

Due to the complex nature of the decision process of type A consumers, the problem above

[19] While this model is complete in that it specifies the choice behavior of type A agents across *all* possible feasible sets, only a small part of this description is relevant for the model at hand. After all, it is never beneficial for the monopolist to produce five or more types of bundles (just as it would never produce three or more bundles in the Mussa-Rosen model). So any choice problem S that consumers may face in the market contains at most four bundles, and again because it is never beneficial for the monopolist to offer more than two dominated items in a menu, we in fact have $|\mathcal{D}_S| = 1$. Thus, our choice of $\langle U_A, \mathbf{r}, Q \rangle$ is one of very many that would produce precisely the same results within this model.

[20] It is not unreasonable to assume that decoys are less costly to produce, especially because one needs them in the market only for show, and not for sale. (Put differently, there is no need to produce a different decoy bundle for every consumer of type A.) This would make the effects of the attraction effect even more pronounced, but as we shall see, even the high costs of decoy options will not prevent the use of them by the monopolist.

[21] Here we interpret the bundle $(0,0)$ as the option of "not buying." Consequently, choosing (p_A, q_A) in a menu S while $(0,0)$ is available means that A types prefer to buy (p_A, q_A) as opposed to buying nothing from S.

appears at first difficult to analyze. In fact, with a bit of work, we can write this problem in the format of a canonical optimization problem. Let us begin by observing that, under the restriction (5), solving this problem is bound to yield more than the level of profits $\hat{\Pi}$ that the monopolist would obtain in the Mussa-Rosen model.[22] Indeed, where $\{(\hat{p}_H, \hat{q}_H), (\hat{p}_L, \hat{q}_L)\}$ is the Mussa-Rosen solution, it is feasible for the monopolist to produce the same bundle $(\hat{p}_L, \hat{q}_L)$ for both types A and L, and offer $(\hat{p}_H, \hat{q}_H)$ to high types (and not produce a decoy option).[23] It follows that the optimal profits of the firm here, say Π^*, must satisfy $\Pi^* \geq \hat{\Pi}$. In fact, this inequality must hold strictly, for the monopolist has a strictly profitable deviation from offering the menu $\{(\hat{p}_H, \hat{q}_H), (\hat{p}_L, \hat{q}_L), (0, 0)\}$. To see this, consider the menu $S_\delta = \{(\hat{p}_H, \hat{q}_H), (\hat{p}_L, \hat{q}_L), (p_R, q_R), (0, 0)\}$, where $(p_R, q_R) = (\hat{p}_H, \hat{q}_L + \delta)$ with $\delta > 0$ being a sufficiently small number. It is easily checked that, this menu is feasible for the monopolist, and when offered this menu, L types purchase $(\hat{p}_L, \hat{q}_L)$, and *both* H *and* A types purchase $(\hat{p}_H, \hat{q}_H)$. It follows that, relative to the Mussa-Rosen solution, the monopolist gains $\alpha(\hat{p}_H - \hat{q}_H^2 - (\hat{p}_L - \hat{q}_L^2))$ while bearing the cost of producing the (decoy) option, namely, $\alpha\left(\hat{q}_L + \delta\right)^2$. But, by using the first inequality in (5), one can routinely verify that we have

$$(\hat{p}_H - \hat{q}_H^2 - (\hat{p}_L - \hat{q}_L^2)) > (\hat{q}_L + \delta)^2,$$

for small enough δ. It follows that offering S_δ (for small δ) is a profitable deviation from the Mussa-Rosen solution. Conclusion: $\Pi^* > \hat{\Pi}$.

An important consequence of this finding is that, in the solution to monopolist's problem, choosing (p_A, q_A) and (p_L, q_L) cannot both be optimal for a consumer of type A. Indeed, if this was the case, then we would have $(p_A - q_A^2) = (p_L - q_L^2)$, because otherwise it would not be optimal for the monopolist to bring to the market *both* (p_A, q_A) *and* (p_L, q_L). But this would imply that *an* optimum solution for the monopolist is to produce only (p_L, q_L) (for types A and L) and (p_H, q_H) (for H types). As (p_L, q_L) and (p_H, q_H) must satisfy the participation and incentive compatibility constraints of types L and H, it follows that doing this could not bring about higher profits than the Mussa-Rosen solution, that is, $\hat{\Pi} \geq \Pi^*$, a contradiction.

We now use this finding to understand the nature of the reference for a consumer of type A. Consider first the possibility that in the optimum menu S of the monopolist, we have $\mathbf{r}(S) = \diamondsuit$. In this case, the fourth constraint, and the fact that $U_A = U_L$, yield $U_A(p_A, q_A) \geq U_A(p_L, q_L) = U_L(p_L, q_L)$, whereas by the second constraint, we have $U_L(p_L, q_L) \geq U_L(p_A, q_A) = U_A(p_A, q_A)$.

[22] Formally speaking, there is no solution to the optimization problem above due to the fact that the constraint set of this problem is not closed. Yet, there is, of course, an ε-solution (i.e. a menu the value of which is at most ε away from the supremum value of the problem) for any $\varepsilon > 0$. By a "solution" in what follows, we mean an ε-solution for sufficiently small $\varepsilon > 0$.

[23] As there is no dominance among the bundles $(\hat{p}_L, \hat{q}_L)$, $(\hat{p}_H, \hat{q}_H)$ and $(0, 0)$, the type A agent acts in a reference free manner to choose from the menu $S := \{(\hat{p}_H, \hat{q}_H), (\hat{p}_L, \hat{q}_L), (0, 0)\}$. Since $U_L = U_A$, and $U_L(\hat{p}_L, \hat{q}_L) \geq \max\{U_L(\hat{p}_H, \hat{q}_H), 0\}$, therefore, we indeed have $(\hat{p}_L, \hat{q}_L) \in \mathbf{c}_A(S)$.

It follows that A types are indifferent between purchasing (p_A, q_A) and (p_L, q_L), a contradiction. Conclusion: We have $\mathbf{r}(S) \neq \diamondsuit$ in an optimal menu. As $(p_A, q_A) \in \mathbf{c}_A(S)$, this implies that (p_A, q_A) dominates at least one member of S and $U_A(p_A, q_A) \geq \beta$. It follows from the incentive compatibility constraints of types H and L that (p_A, q_A) dominates neither (p_H, q_H) nor (p_L, q_L). Conclusion: $\mathbf{r}(S) = (p_R, q_R)$.

Now suppose $(p_L, q_L) \in Q(p_R, q_R)$ in the optimum. Then since $(p_A, q_A) \in \mathbf{c}_A(S)$, we must have $U_A(p_A, q_A) \geq U_A(p_L, q_L)$. Using the incentive compatibility for L types and the fact that $U_A = U_L$, then, we have $U_A(p_A, q_A) = U_A(p_L, q_L)$. But this means that both (p_A, q_A) and (p_L, q_L) belong to $\mathbf{c}_A(S)$, which we have seen above to be impossible. It follows that $(p_L, q_L) \notin Q(p_R, q_R)$. Thus, we have either $\{(p_A, q_A)\} = Q(p_R, q_R)$ or $\{(p_A, q_A), (p_H, q_H)\} = Q(p_R, q_R)$. But, since $(p_A, q_A) \in \mathbf{c}_A(S)$, we must also have $U_A(p_A, q_A) \geq U_A(p_H, q_H)$ in the latter case. Consequently, we must have either

$$\{(p_A, q_A)\} = Q(p_R, q_R) \tag{6}$$

or

$$\{(p_A, q_A), (p_H, q_H)\} = Q(p_R, q_R) \quad \text{and} \quad U_A(p_A, q_A) \geq U_A(p_H, q_H). \tag{7}$$

This discussion shows that insofar as the solutions of the problems are concerned, the problem of the monopolist in the present framework may be viewed as choosing quality levels $q_A, q_H, q_L, q_R \geq 0$ and prices $p_A, p_H, p_L, p_R \geq 0$ in order to

$$\begin{aligned}
\text{Maximize} \quad & \alpha(p_A - q_A^2) + (p_H - q_H^2) + (1-\alpha)(p_L - q_L^2) - \alpha q_R^2 \\
\text{such that} \quad & U_L(p_L, q_L) \geq 0, \\
& U_L(p_L, q_L) \geq U_L(p_A, q_A), \\
& U_H(p_H, q_H) \geq \max\{U_H(p_L, q_L), U_H(p_A, q_A)\}, \\
& U_A(p_A, q_A) \geq \beta, \\
& \text{either (6) or (7) holds.}
\end{aligned}$$

Notice that the monopolist wishes to choose q_R as small as possible subject to the constraint $q_R > q_L$ here. Consequently, the supremum of the problem above is not attained. However, there is, of course, a sequence of menus the values of which converge to this supremum value (that is, there is an ε-solution for every $\varepsilon > 0$). Consequently, by a "solution" to the problem above, we shall understand the limit of such sequences of menus (that is, the limit of the ε-solutions as $\varepsilon \to 0$). In addition, we shall report the solution of this problem only in the case where α is not too large, as the cases for large α do not contain additional insights and have rather complicated closed-form solutions.[24]

[24] The complete solution to the monopolist's problem for any $0 \leq \alpha \leq 1$ is available from the authors upon request.

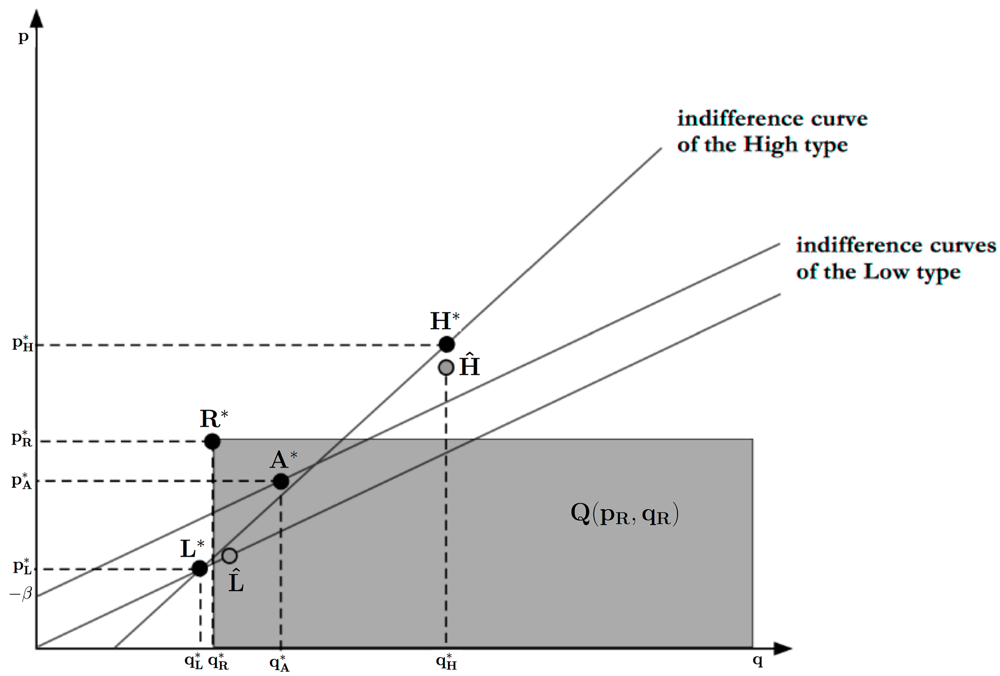

FIGURE 5

The most interesting case obtains when

$$0 \leq \alpha \leq \min\left\{\frac{\theta_H - \theta_L}{\theta_L}, \frac{2\theta_L - \theta_H}{\theta_L}\right\}. \tag{8}$$

(Depending on the values of θ_H and θ_L, this may leave room for quite a range for α. For instance, if we strengthen (5) to $\frac{2\theta_H}{3} < \theta_L < \frac{3\theta_H}{4}$, then (8) reduces to $0 \leq \alpha \leq \frac{\theta_H - \theta_L}{\theta_L}$, which is a range that contains the interval $[0, \frac{1}{3}]$.) Provided that (8) holds, we find that the optimal quality choices of the monopolist are as follows:

$$q_A^* := \frac{\theta_L}{2}, \quad q_H^* := \frac{\theta_H}{2} \quad \text{and} \quad q_L^* = q_R^* := (1-\alpha)\frac{\theta_L}{2} - \frac{\theta_H - \theta_L}{\theta_L}.$$

This solution is illustrated in Figure 5. As shown in that figure, there are sharp, and somewhat surprising, differences between the Mussa-Rosen solution and that of the monopolist in the present setup. An immediate observation is that, even though it is as costly to produce a decoy option (which will have to end up unsold) as the target brand in the present model, and the fraction of A types in the market is not overwhelming, the monopolist still finds it profitable to produce the decoy option (p_R^*, q_R^*). (This bundle is denoted as R* in Figure 5.) The reason is, of course, to shift the purchases of A types to a more profitable bundle, namely, (p_A^*, q_A^*). (This bundle is denoted as A* in Figure 5.)

Moreover, we see here that the presence of the A types affects the other types present in the market. While H types still purchase the efficient quality, they pay a higher price relative to the Mussa-Rosen solution (a further source of additional profits), and hence have lower welfare. By contrast, L types face a larger downward distortion in quality relative to the Mussa-Rosen solution, but their welfare is unaltered (as their participation constraint is binding).[25] Second, we find that A types in the market are also consume the efficient quality. This is a bit surprising, as in these types of principal-agent problems, it is often the case that only the highest type is offered the efficient bundle. The present situation is different, because A and H types differ from each other in ways that go beyond their taste parameters – a consumer of type A is unique in the way she solves her own choice problems. Indeed, what makes the lower types not receive the efficient amount in such problems, and certainly in the classical Mussa-Rosen model, is the incentive compatibility constraint of the higher valuation type. Here the monopolist is able to exploit the A type so significantly that this constraint does not bind relative to A types, that is, H types do not wish to consume what is offered to A types in the solution. (See Figure 5.) Consequently, there is no reason to offer an inefficient bundle to A types.

This is, however, not a general phenomenon. If α is sufficiently large, then it may well be the case that both of the incentive compatibility constraints of type H become binding, thereby forcing A types to consume a second-best quality. In particular, under the condition that

$$\frac{\theta_H - \theta_L}{\theta_L} \leq \alpha \leq \frac{2\theta_L - \theta_H}{\theta_L},$$

(which is possible, for instance, when $\frac{2\theta_H}{3} < \theta_L < \frac{3\theta_H}{4}$), we find that the optimal quality choices of the monopolist are as follows:

$$q_A^{**} := q_L^{**} + \theta_H - \theta_L, \quad q_H^{**} := \frac{\theta_H}{2} \quad \text{and} \quad q_L^{**} = q_R^{**} := \frac{1}{1+\alpha}\left(\theta_L - \frac{\theta_H}{2} - \alpha(\theta_H - \theta_L)\right).$$

In this case, then, both L and A types purchase goods of suboptimal levels of quality.

4 Choice over Lotteries

In this section we go back to the theoretical development sketched in Section 2, and explore how one may be able to extend our reference-dependent choice model, and its axiomatic foundations thereof, to the context of choice under risk.

[25] As there are fewer L types in the present model, this is quite intuitive. The monopolist loses money on L types here by providing them with the bundle L^* (see Figure 5), but it is more than compensated for this loss, as L^* relaxes a bit the incentive compatibility constraint of the high valuation types (and hence allowing the monopolist charge a higher price to them) and, in addition, makes the production of the decoy option less costly.

4.1 Preliminaries

In what follows Z stands for an arbitrarily fixed compact metric space, and $\mathbb{P}(Z)$ the set of all Borel probability measures on Z. As usual, we metrize $\mathbb{P}(Z)$ by the Prokhorov metric to make it a compact metric subspace of the space $ca(Z)$ of countably additive signed Borel measures on Z. (*Note.* The topology of $\mathbb{P}(Z)$ is that of weak convergence.) Finally, we denote the set of all closed subsets of $\mathbb{P}(Z)$ as $\mathfrak{X}_Z$. As is standard, we metrize $\mathfrak{X}_Z$ by means of the Hausdorff metric in order to make it a compact metric space.

The following development can be viewed as a special case of the theory we developed in Section 2 upon setting X as $\mathbb{P}(Z)$ and $\mathfrak{X}$ as $\mathfrak{X}_Z$. However, as there is substantially more structure in the present setting, we are able to impose further rationality properties here on the choice correspondences under consideration, and hence obtain more powerful representation theorems by way of using the expected utility theory.

4.2 Independence Axioms in Reference-Dependent Choice Theory

A natural way of making use of the additional structure that the framework of lotteries provides is to bring to the fore the independence axiom. By contrast to the classical expected utility theory, however, in the present setting there are two dimensions in which such a linearity postulate can be imposed. First, we may ask for the invariance of the potential references of an individual (Definition 3) with respect to mixing with a fixed lottery. This idea is formalized in the next axiom (which is imposed on a choice correspondence $\mathbf{c}$ on $\mathfrak{X}_Z$).

rIND. Let $p, q, r \in \mathbb{P}(Z)$ and $0 < a \leq 1$. If

$$q \text{ is a potential } \mathbf{c}\text{-reference for } p,$$

then

$$aq + (1-a)r \text{ is a potential } \mathbf{c}\text{-reference for } ap + (1-a)r.$$

The interpretation of this property, which may be called *referential independence*, is straightforward. If the lottery q is a potential $\mathbf{c}$-reference for the lottery p, and thus "enhances" the appeal of p for the individual, it seems quite reasonable that any mixing of q with another lottery r should do the same for that mixing of p with r. For instance, if q is a potential $\mathbf{c}$-reference for the lottery p because q is unambiguously worse than p in the sense of stochastic dominance, then, for any $0 < a \leq 1$, the lottery $aq + (1-a)r$ would indeed be a potential $\mathbf{c}$-reference for $ap + (1-a)r$. For a linear formulation of the choice theory sketched in Section 2, rIND thus appears to be a suitable property.

The second dimension in which the independence axiom can be posited here concerns directly the choice behavior of the individual – this is identical to how this axiom is used in the classical revealed preference theory (under risk). Suppose the individual has chosen q from a given feasible set S in $\mathfrak{X}_Z$, that is, $q \in \mathbf{c}(S)$. What would she choose from the set

$$aS + (1-a)p := \{as + (1-a)p : s \in S\},$$

where $0 < a \leq 1$? Whether or not the agent makes her decisions by using a reference point, the usual (normative) justification of the independence axiom suggests that she would choose $aq+(1-a)p$ from this set. Conversely, if $aq+(1-a)p$ is chosen from $aS+(1-a)p$, then, again, the expected utility theory would maintain that this must be because q is deemed choosable over anything else in S (with or without reference effects). As in standard revealed preference theory (under risk), we are thus led to impose the property of *choice independence.*

cIND. Let $S \in \mathfrak{X}_Z$ and $p \in \mathbb{P}(Z)$. Then

$$\mathbf{c}\left(aS + (1-a)p\right) = a\mathbf{c}(S) + (1-a)p$$

for any $0 < a \leq 1$.

Like rIND, this postulate too seems quite natural for a "linearization" of the choice theory we sketched so far. Nevertheless, it should be noted at the outset that the appeal of cIND stems mainly from a normative viewpoint. It is well-known that the independence axiom of expected utility theory has serious descriptive difficulties, and there is no reason to expect that these difficulties would not be pressing in the present framework. Be that as it may, our main concern here is on bounded rationality of an individual as caused by reference-dependence, not by the violation of the independence axiom. From this standpoint, it seems only reasonable that we explore the joint implications of rIND and cIND in our framework. Combining the theories of reference dependent choice and non-expected utility is an interesting matter which is at this stage best relegated to future research.

4.3 The Reference-Dependent Multi-Utility Maximization Model

In this section we introduce the "linear" version of the reference-dependent choice model. This version will be seen to not only generalize the standard expected utility maximization model, but also to provide a precise manner in which one can think of the attraction effect in general.

Our discussion is facilitated by the following notational conventions.

Notation. Let p and q be any two Borel probability measures on Z. In what follows, we denote the expected value of any given continuous real map u on Z with respect to p by $\mathbf{E}(u,p)$, that is,

$$\mathbf{E}(u,p) := \int_Z u dp.$$

If $\mathbf{u} := (u_1, u_2, ...)$ is a sequence in $\mathbf{C}(Z)$, then we define

$$\mathsf{E}(\mathbf{u}, p) := (\mathsf{E}(u_1, p), \mathsf{E}(u_2, p), ...),$$

which is a member of $\mathbb{R}^\infty$, and write $\mathsf{E}(\mathbf{u}, p) > \mathsf{E}(\mathbf{u}, q)$ to mean that

$$\mathsf{E}(u_i, p) \geq \mathsf{E}(u_i, q) \text{ for each } i \in \mathbb{N} \text{ and } \mathsf{E}(u_i, p) > \mathsf{E}(u_i, q) \text{ for some } i \in \mathbb{N}.$$

We are now prepared to introduce our second reference-dependent choice model.

Definition 8. We say that the list $[U, \mathbf{r}, \mathbf{u}]$ is a **reference-dependent multi-utility choice model** on $\mathfrak{X}_Z$ if U is a continuous real function on $\mathbb{P}(Z)$, $\mathbf{r}$ is a reference map on $\mathfrak{X}_Z$, and $\mathbf{u}$ is a bounded sequence in $\mathbf{C}(Z)$. This model is said to **represent** the correspondence $\mathbf{c} : \mathfrak{X}_Z \rightrightarrows \mathbb{P}(Z)$ if $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$ is a reference-dependent choice model on $\mathfrak{X}_Z$ that represents $\mathbf{c}$, where

$$Q_{\mathbf{u}}(q) := \begin{cases} \{p \in \mathbb{P}(Z) : \mathsf{E}(\mathbf{u}, p) > \mathsf{E}(\mathbf{u}, q)\}, & \text{if } q \neq \diamondsuit \\ \mathbb{P}(Z), & \text{otherwise.} \end{cases}$$

We say that this model is **(weakly) proper** if $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$ is a (weakly) proper reference-dependent choice model on $\mathfrak{X}_Z$ (Definitions 6 and 7).

Let $[U, \mathbf{r}, \mathbf{u}]$ be a reference-dependent multi-utility model on $\mathfrak{X}_Z$. The interpretation of U and $\mathbf{r}$ are exactly as in the case of the reference-dependent choice model (Definition 5). That is, U is the utility function of the agent that dictates her pairwise comparisons of the choice prospects, and $\mathbf{r}$ is the reference map that assigns to any given feasible set S of alternatives either $\diamondsuit$ (the case in which the agent does not conceive a reference point in S) or an alternative in S (the case in which the agent determines $\mathbf{r}(S)$ as a reference point in S).

The novelty of $[U, \mathbf{r}, \mathbf{u}]$ arises from its third ingredient, namely, $\mathbf{u}$. We would like to interpret each term u_i of the sequence $\mathbf{u}$ as measuring the performance of (riskless) alternatives in Z with respect to a certain criterion. For instance, we may think of the agent as deeming relevant a number of attributes (physical or otherwise) for her decision-making, and measuring the peformance of $z \in Z$ with respect to the ith attribute by u_i.[26] Perhaps a more compelling interpretation is that the agent is unsure of her own tastes at the time of choice. (This is particularly reasonable if there is a gap between the time of choice and time of consumption.) In particular, she may be sure that her tastes (at the time of consumption) would be represented by one of the von Neumann-Morgenstern utility functions in $\{u_1, u_2, ...\}$, but may feel uncertain about exactly which u_i she should base her decisions on. Adopting the commonly used terminology of behavioral economics, we may say that, in this interpretation, each u_i

[26] Of course, this interpretation becomes more appealing in the special case where $\mathbf{u}$ is eventually constant, that is, when there are only finitely many distinct u_is.

would correspond to the von Neumann-Morgenstern utility function of a "potential self" of the individual.

The sequence $\mathbf{u}$ of von Neumann-Morgenstern utility functions induces a dominance notion in the obvious way: For any $z, z' \in Z$, the alternative z $\mathbf{u}$-*dominates* z' iff $u_i(z) \geq u_i(z')$ for each i, with strict inequality for at least one i. More generally, for any $p, q \in \mathbb{P}(Z)$, the lottery p $\mathbf{u}$-*dominates* q iff $\mathsf{E}(\mathbf{u}, p) > \mathsf{E}(\mathbf{u}, q)$. The model $[U, \mathbf{r}, \mathbf{u}]$ uses precisely this dominance notion to define an attraction region for any choice prospect q:

$$Q_{\mathbf{u}}(q) = \{p \in \mathbb{P}(Z) : p \ \mathbf{u}\text{-}dominates \ q\}.$$

In our interpretation of $\mathbf{u}$, therefore, $p \in Q_{\mathbf{u}}(q)$ means that lottery p is weakly preferred to q by all "potential selves" of the agent, and strictly so for at least one "potential self" of her.

No relation between U and $\mathbf{u}$ is stipulated in the definition of $[U, \mathbf{r}, \mathbf{u}]$. Obviously, among other things, this calls to question our interpretation of $\mathbf{u}$. After all, there is no reason to presume that the agent would be uncertain about her potential tastes only when comparing more than two alternatives. Yet, the model says that $\mathbf{u}$ does not play a role in the settlement of pairwise comparisons, the agent rather uses U for that purpose. Fortunately, this difficulty disappears when $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$ is actually a reference-dependent choice model. Indeed, in that case, condition 4 of Definition 5 (applied to $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$) maintains that U and $\mathbf{u}$ are consistent in the following sense:

$$\mathsf{E}(\mathbf{u}, p) > \mathsf{E}(\mathbf{u}, q) \quad \text{implies} \quad U(p) > U(q)$$

for any $p, q \in \mathbb{P}(Z)$. Also worth noting is that, when $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$ is a reference-dependent choice model, we are ensured that $\mathbf{r}(S)$ is an alternative in S that is necessarily dominated by another feasible prospect with respect to all "potential selves" of the agent:

$$\mathbf{r}(S) \neq \diamondsuit \quad \text{implies} \quad \mathsf{E}(\mathbf{u}, p) > \mathsf{E}(\mathbf{u}, \mathbf{r}(S)) \text{ for some } p \in S$$

for any $S \in \mathfrak{X}_Z$.

In general, however, there is little discipline on the structure of the reference map $\mathbf{r}$, even when $\langle U, \mathbf{r}, Q_{\mathbf{u}} \rangle$ is a proper reference-dependent choice model. In the present framework we can in fact demand a bit more in this respect.

Definition 9. Let $\mathbf{r}$ be a reference map on $\mathfrak{X}_Z$. For any $S \in \mathfrak{X}_Z$, we say that $\mathbf{r}$ is **quasi-affine at** S if, for every $p \in \mathbb{P}(Z)$ and $0 < a \leq 1$,

$$\mathbf{r}\left(aS + (1-a)p\right) = a\mathbf{r}(S) + (1-a)p$$

when $\mathbf{r}(S) \neq \diamondsuit$ and

$$\mathbf{r}\left(aS + (1-a)p\right) = \diamondsuit$$

when $\mathbf{r}(S) = \diamondsuit$. We say that $\mathbf{r}$ is **quasi-affine** if it is quasi-affine at every $S \in \mathfrak{X}_Z$.

Let $\langle U, \mathbf{r}, Q \rangle$ be a reference-dependent choice model on $\mathfrak{X}_Z$ such that $\mathbf{r}$ is quasi-affine. Take any $S \in \mathfrak{X}_Z$, $0 < a \leq 1$ and $p \in \mathbb{P}(Z)$. Obviously, $aS + (1-a)p$ is none other than a feasible set in which all members of S are affinely transformed in exactly the same way (that is, by the same type of mixing with p). Quasi-affinity of $\mathbf{r}$ means that the reference assignments of the agent whose choice behavior is represented by $\langle U, \mathbf{r}, Q \rangle$ are consistent across such transformations. That is, if the agent does not perceive an alternative in S as a reference, neither does she in $aS + (1-a)p$, and if she views $q \in S$ as a reference point in S, she designates $aq + (1-a)p$ as a reference point in $aS + (1-a)p$. Among other things, this brings some discipline to how reference points are related across choice problems, thereby refining the predictive power of the choice model under consideration.

4.4 Two Representation Theorems

Our objective now is to refine the class of choice correspondences characterized by Theorem 1 by adding the axioms rIND and cIND into the axiomatic makeup of that result. It turns out that any choice correspondence on $\mathfrak{X}_Z$ that satisfy these six axioms can be represented by means of a reference-dependent multi-utility model $[U, \mathbf{r}, \mathbf{u}]$ in which $U(p)$ is a strictly increasing affine function of $\mathbf{E}(u_1, p)$, $\mathbf{E}(u_2, p)$,.... More precisely, we have the following:

Theorem 2A. *Let Z be a compact metric space. A correspondence $\mathbf{c} : \mathfrak{X}_Z \rightrightarrows \mathbb{P}(Z)$ is a choice correspondence on $\mathfrak{X}_Z$ that satisfies wWARP, RT, C, rd-WARP, rIND and cIND if, and only if, there exist a bounded sequence $\mathbf{u}$ in $\mathbf{C}(Z)$, an affine, strictly increasing and continuous real map Ψ on the convex set $\mathbf{E}(\mathbf{u}, \mathbb{P}(Z)) \subseteq \mathbb{R}^\infty$,[27] and a quasi-affine reference map $\mathbf{r}$ on $\mathfrak{X}_Z$ such that $[\Psi(\mathbf{E}(\mathbf{u}, \cdot)), \mathbf{r}, \mathbf{u}]$ is a proper reference-dependent multi-utility model on $\mathfrak{X}_Z$ that represents $\mathbf{c}$.*

The structure of the representation obtained in this theorem becomes more transparent in the special case where the set Z of (riskless) prizes is finite. In that case Theorem 2A may be restated as follows:

Theorem 2B. *Let Z be a nonempty finite set. Then, the correspondence $\mathbf{c} : \mathfrak{X}_Z \rightrightarrows \mathbb{P}(Z)$ is a choice correspondence on $\mathfrak{X}_Z$ that satisfies wWARP, RT, C, rd-WARP, rIND and cIND if, and only if, there exist a bounded sequence $\mathbf{u}$ in $\mathbf{C}(Z)$, a real sequence $(\lambda_m) \in \ell^1_{++}$, and a quasi-affine reference map $\mathbf{r}$ on $\mathfrak{X}_Z$ such that $[\mathbf{E}(\sum^\infty \lambda_i u_i, \cdot), \mathbf{r}, \mathbf{u}]$ is a proper reference-dependent multi-utility model on $\mathfrak{X}_Z$ that represents $\mathbf{c}$.*

[27] Continuity of Ψ is understood here relative to the product topology.

Take any nonempty finite set Z and suppose $\mathbf{c} : \mathfrak{X}_Z \rightrightarrows \mathbb{P}(Z)$ is a choice correspondence on $\mathfrak{X}_Z$ that satisfies the six axioms stated in Theorem 2B. This theorem then says that the choice behavior of an individual (as captured by $\mathbf{c}$) can be thought of as follows. First, the agent discerns (subjectively) countably many ways to rank the lotteries in any given feasible set S. Each of these rankings i have a representation in terms of a von Neumann-Morgenstern utility function u_i. (Again, we interpret each u_i as representing the preferences of a "potential self" of the agent in question.) In a pairwise comparison, the agent ranks lottery p over q by using a von Neumann-Morgenstern utility function that is a weighted average of all the u_is. That is, there is a summable sequence $(\lambda_1, \lambda_2, ...)$ of strictly positive weights such that, for any $p, q \in \mathbb{P}(Z)$,

$$p \in \mathbf{c}\{p, q\} \quad \text{iff} \quad \mathbf{E}(u, p) \geq \mathbf{E}(u, q)$$

where $u := \lambda_1 u_1 + \lambda_2 u_2 + \cdots$. (Clearly, u may be thought of as arising from the individual's (linear) aggregation of the utility functions $u_1, u_2, ...$ of her "potential selves.") Furthermore, given any choice situation $S \in \mathfrak{X}_Z$, the agent either evaluates the problem in a reference-free manner ($\mathbf{r}(S) = \diamondsuit$), or designates an alternative $\mathbf{r}(S) \in S$ to guide her as a reference in her decision-making process. In the former case, we simply have

$$\mathbf{c}(S) = \arg\max \{\mathbf{E}(u, p) : p \in S\}, \qquad S \in \mathfrak{X}_Z,$$

that is, the agent chooses from S those alternatives that maximize the expectation of her (aggregated) utility function $u := \sum^\infty \lambda_i u_i$. In the latter case, she feels mentally "attracted" to the prospects in S that dominate $\mathbf{r}(S)$ with respect to each individual criterion u_i. (In our interpretation, it is these prospects that every "potential type" of the agent would prefer to $\mathbf{r}(S)$, or put differently, it is clear to the agent that any such prospect will be better than $\mathbf{r}(S)$ however she may feel at the time of consumption.) The model thus maintains that the individual conditions herself to choose an alternative p from S *only if* $\mathbf{E}(\mathbf{u}, p) > \mathbf{E}(\mathbf{u}, \mathbf{r}(S))$. Given this mental constraint, she acts as an expected utility maximizer, that is, among those alternatives p in S that satisfy this condition – the choice of $\mathbf{r}$ must have been such that there is at least one such alternative in S – she picks a lottery that yields her the highest expected (aggregate) utility. Thus, in this case,

$$\mathbf{c}(S) = \arg\max \{\mathbf{E}(u, p) : p \in S \text{ and } \mathbf{E}(\mathbf{u}, p) > \mathbf{E}(\mathbf{u}, \mathbf{r}(S))\}$$

for any $S \in \mathfrak{X}_Z$.

To see how this choice model refines the one we derived in Theorem 1, consider a set $S \in \mathfrak{X}_Z$ that contains only degenerate lotteries. Then, indentifying a degenerate lottery with the only element of its support, we find that $\mathbf{c}(S)$ obtains upon solving the following optimization problem:

$$\text{Maximize} \sum_{i=1}^{\infty} \lambda_i u_i(x) \quad \text{such that} \qquad x \in S,$$

if $\mathbf{r}(S) = \diamondsuit$, and

$$\begin{aligned} &\text{Maximize } \sum_{i=1}^{\infty} \lambda_i u_i(x) \\ &\text{such that } u_i(x) \geq u_i(\mathbf{r}(S)) \text{ for all } i \in \mathbb{N} \\ &\qquad\qquad u_i(x) > u_i(\mathbf{r}(S)) \text{ for some } i \in \mathbb{N} \\ &\qquad\qquad x \in S \end{aligned}$$

if $\mathbf{r}(S) \neq \diamondsuit$. If the sequence $(u_1, u_2, ...)$ is eventually constant here, and each u_i corresponds to a measure of an attribute, then this model corresponds exactly to the conceptualization of the attraction effect found in the literature on behavioral psychology and marketing, albeit with endogenously determined reference points.

Appendix: Proofs

Proof of Theorem 1.

[*Necessity of the Axioms*] Let $\langle U, \mathbf{r}, Q\rangle$ be a proper reference-dependent choice model on $\mathfrak{X}$ that represents $\mathbf{c}$. It is obvious that $\mathbf{c}$ is a choice correspondence on $\mathfrak{X}$ that satisfies wWARP. The rest of the proof is based on the following observation.

Claim 1.1. For any $z \in X$,

$$Q(z) = \{x \in X : z \text{ is a potential } \mathbf{c}\text{-reference for } x\}.$$

Moreover, if $S \in \mathfrak{X}$ is $\mathbf{c}$-awkward, then $\mathbf{r}(S) \neq \diamondsuit$.

Proof of Claim 1.1. Fix an arbitrary $z \in X$, and suppose $x \in Q(z)$. Since $\langle U, \mathbf{r}, Q\rangle$ is a reference-dependent choice model on $\mathfrak{X}$, we have $U(x) > U(z)$. So, if z was not a potential $\mathbf{c}$-reference for x, we could find an $\omega \in X\backslash\{x\}$ such that either (i) $U(x) > U(\omega)$ and $\{x\} \neq \mathbf{c}\{x, z, \omega\}$, or (ii) $U(x) = U(\omega)$ and $x \notin \mathbf{c}\{x, z, \omega\}$. Therefore, by the representation of $\mathbf{c}$, $x \notin Q(\mathbf{r}(x, z, w\})$, which is possible only if $\mathbf{r}\{x, z, \omega\} \in \{x, \omega\}$. But, since $U(x) \geq \max\{U(z), U(\omega)\}$, if we were to have $\mathbf{r}\{x, z, \omega\} = x$, then

$$Q(\mathbf{r}(x, z, \omega\}) \cap \{x, z, \omega\} = Q(x) \cap \{x, z, \omega\} = \emptyset,$$

a contradiction to $\langle U, \mathbf{r}, Q\rangle$ being a reference-dependent choice model on $\mathfrak{X}$. Thus, we have $\mathbf{r}\{x, z, \omega\} = \omega$. Then, (i) and (ii) entail that $x \notin Q(\omega)$. Since $\omega \notin Q(\omega)$ trivially, and $Q(\mathbf{r}(x, z, \omega\}) \cap \{x, z, \omega\} \neq \emptyset$, therefore, we find $z \in Q(\omega)$. But then $Q \circ Q \subseteq Q$ entails $x \in Q(\omega)$, so we again arrive at a contradiction. Conclusion: z is a potential $\mathbf{c}$-reference for x.

Conversely, suppose z is a potential $\mathbf{c}$-reference for some $x \in X\backslash Q(z)$. Then $U(x) > U(z)$, so, by properness of $\langle U, \mathbf{r}, Q\rangle$, there exists a $y \in X$ such that $U(x) \geq U(y)$ and $x \notin Q(\mathbf{r}\{x, y, z\})$. But by the representation of $\mathbf{c}$ this implies that $x \notin \mathbf{c}\{x, y, z\}$, contradicting z being a potential $\mathbf{c}$-reference for x.

To prove the second assertion, take any $S \in \mathfrak{X}$ with $\mathbf{r}(S) = \diamondsuit$. Then, since $\langle U, \mathbf{r}, Q\rangle$ is a reference-dependent choice model, we have $\mathbf{r}(T) = \diamondsuit$ for any $T \in \mathcal{S}$ with $T \cap \arg\max U(S) \neq \emptyset$. Since this model represents $\mathbf{c}$, then,

$$\mathbf{c}(T) = \arg\max U(T) = T \cap \arg\max U(S) = \mathbf{c}(S) \cap T$$

for every $T \in \mathcal{S}$ with $\mathbf{c}(S) \cap T \neq \emptyset$. So, by Observation 1, S is not $\mathbf{c}$-awkward. $\|$

Given Claim 1.1, that $\mathbf{c}$ satisfies C is immediate from the closedness of the graph of Q. Similarly, $\mathbf{c}$ satisfies RT because $Q \circ Q \subseteq Q$. To show that $\mathbf{c}$ satisfies rd-WARP, assume that S is a $\mathbf{c}$-awkward set in $\mathfrak{X}$. Then, it follows from the second part of Claim 1.1 that $\mathbf{r}(S) \in S$. Thus, since $\mathbf{r}(S) \notin Q(\mathbf{r}(S))$, we have $\mathbf{r}(S) \in S \backslash \mathbf{c}(S)$.

Now take any $T \in \mathcal{S}_{\mathbf{r}(S)}$ with $\mathbf{c}(S) \cap T \neq \emptyset$. The weak properness of $\langle U, \mathbf{r}, Q \rangle$ implies readily that $\mathbf{c}(S) \cap T \subseteq \mathbf{c}(T)$. To establish the converse containment, take any $y \in \mathbf{c}(T)$. We then pick an arbitrary $x \in \mathbf{c}(S) \cap T$, and observe that, by what is just noted, $x \in \mathbf{c}(T)$. Thus, by definition of $\mathbf{c}$, we have $x \in Q(\mathbf{r}(T))$ while $y \in \arg\max U(T \cap Q(\mathbf{r}(T)))$. Thus $U(y) \geq U(x)$, and it follows that $y \in Q(\mathbf{r}(S))$ by weak properness of $\langle U, \mathbf{r}, Q \rangle$. Thus $y \in \arg\max U(S \cap Q(\mathbf{r}(S))) = \mathbf{c}(S)$, that is, $\mathbf{c}(T) \subseteq \mathbf{c}(S) \cap T$. Conclusion:

$$\mathbf{c}(T) = \mathbf{c}(S) \cap T \text{ for all } T \in \mathcal{S}_{\mathbf{r}(S)} \text{ with } \mathbf{c}(S) \cap T \neq \emptyset.$$

Moreover, it follows from the representation of $\mathbf{c}$ that $\mathbf{c}(S) \subseteq Q(\mathbf{r}(S))$. Therefore, by Claim 1.1, $\mathbf{r}(S)$ is a potential $\mathbf{c}$-reference for $\mathbf{c}(S)$. Conclusion: $\mathbf{c}$ satisfies rd-WARP.

[*Sufficiency of the Axioms*] Let $\mathbf{c} : \mathfrak{X} \rightrightarrows X$ be a choice correspondence on $\mathfrak{X}$ that satisfies wWARP, C, RT and rd-WARP. Since X is a separable metric space and $R_{\mathbf{c}}$ is a continuous and complete preorder by wWARP and C, we can use Debreu's Utility Representation Theorem to find a map $U \in \mathbf{C}(X)$ such that $xR_{\mathbf{c}}y$ iff $U(x) \geq U(y)$, for every $x, y \in X$. Next, we define the correspondence $\mathfrak{R} : \mathfrak{X} \rightrightarrows X$ by

$$\mathfrak{R}(S) := \{z \in S \backslash \mathbf{c}(S) : z \text{ is a potential } \mathbf{c}\text{-reference for } \mathbf{c}(S) \text{ that satisfies (1)}\},$$

and note that rd-WARP is equivalent to say that $\mathfrak{R}(S) \neq \emptyset$ for each $\mathbf{c}$-awkward $S \in \mathfrak{X}$. We may then use the Axiom of Choice to define the reference map $\mathbf{r} : \mathfrak{X} \to X \cup \{\diamondsuit\}$ by

$$\mathbf{r}(S) := \begin{cases} z, & \text{if } S \text{ is } \mathbf{c}\text{-awkward} \\ \diamondsuit, & \text{otherwise,} \end{cases}$$

where z is chosen arbitrarily from $\mathfrak{R}(S)$.

Claim 1.2. $\mathbf{r}$ is a reference map on $\mathfrak{X}$, and $\mathbf{r}(S) \in \mathfrak{R}(S)$ for every $\mathbf{c}$-awkward $S \in \mathfrak{X}$.

Proof of Claim 1.2. This follows readily from the definition of $\mathbf{r}$. $\|$

Claim 1.3. If $S \in \mathfrak{X}$ is $\mathbf{c}$-awkward, then

$$\mathbf{r}(S) \in S \backslash \mathbf{c}(S) \quad \text{ and } \quad U(x) > U(\mathbf{r}(S)) \text{ for any } x \in \mathbf{c}(S).$$

Proof of Claim 1.3. Let S be a $\mathbf{c}$-awkward set in $\mathfrak{X}$. Set $z := \mathbf{r}(S)$, and notice that $z \in \mathfrak{R}(S) \subseteq S \backslash \mathbf{c}(S)$, so the first assertion here is immediate. Moreover, if $x \in \mathbf{c}(S)$ but $z \in \mathbf{c}\{x, z\}$, then the fact that z satisfies (1) implies $z \in \mathbf{c}\{x, z\} = \mathbf{c}(S) \cap \{x, z\}$, which contradicts $z \in S \backslash \mathbf{c}(S)$. Thus $xP_{\mathbf{c}}z$, and hence $U(x) > U(z)$, for each $x \in \mathbf{c}(S)$. $\|$

Claim 1.4. If $S \in \mathfrak{X}$ is not $\mathbf{c}$-awkward, then $\mathbf{c}(S) = \arg\max U(S)$.

Proof of Claim 1.4. Take any $S \in \mathfrak{X}$ that is not $\mathbf{c}$-awkward set in $\mathfrak{X}$.Then, by Observation 1, we have $x \in \mathbf{c}(S) \cap \{x, y\} = \mathbf{c}\{x, y\}$ for any $(x, y) \in \mathbf{c}(S) \times S$. This implies that $\mathbf{c}(S) \subseteq \arg\max U(S)$. Conversely, let $x \in \arg\max U(S)$ and pick any $y \in \mathbf{c}(S)$. Then $U(x) \geq U(y)$, that is, $xR_{\mathbf{c}}y$. It follows that $x \in \mathbf{c}\{x, y\} = \mathbf{c}(S) \cap \{x, y\}$, that is, $x \in \mathbf{c}(S)$. Thus: $\arg\max U(S) \subseteq \mathbf{c}(S)$. $\|$

Let $Q(\diamondsuit) := X$, and for any $z \in X$, define $Q(z)$ as the collection of all $x \in X$ such that z is a potential $\mathbf{c}$-reference for x. That is, for any $z \in X$, we have $x \in Q(z)$ iff $U(x) > U(z)$ and

$$\begin{cases} \{x\} = \mathbf{c}\{x, z, \omega\}, & \text{if } U(x) > U(\omega) \\ x \in \mathbf{c}\{x, z, \omega\}, & \text{if } U(x) = U(\omega) \end{cases} \quad \text{for any } \omega \in X \backslash \{x\}. \tag{9}$$

It is plain that the property C ensures that the graph of $Q|_X$ is a closed subset of $X \times X$. Since $\diamondsuit$ is an isolated point of the domain of Q, then, Q satisfies the closed-graph property.

Claim 1.5. $\langle U, \mathbf{r}, Q\rangle$ is a reference-dependent choice model on $\mathfrak{X}$.

Proof of Claim 1.5. Given Observation 2 and Claim 1.2, it is enough to show that $Q \circ Q \subseteq Q$ and $S \cap Q(\mathbf{r}(S)) \neq \emptyset$ for all $S \in \mathfrak{X}$. The former property of Q is an immediate consequence of RT. To establish the latter, take an arbitrary $S \in \mathfrak{X}$. If S is not $\mathbf{c}$-awkward, then $\mathbf{r}(S) = \diamondsuit$, so $S \cap Q(\mathbf{r}(S)) = S$, that is, there is nothing to prove. Suppose S is $\mathbf{c}$-awkward, and set $z := \mathbf{r}(S)$. We pick any $x \in \mathbf{c}(S)$ and note that $U(x) > U(z)$ by Claim 1.3. Moreover, $z \in \mathfrak{R}(S)$, so z is a potential $\mathbf{c}$-reference for $\mathbf{c}(S)$, and hence (9). Thus $x \in Q(z)$, that is, $x \in S \cap Q(\mathbf{r}(S))$. $\|$

Claim 1.6. $\mathbf{c}(S) = \arg\max U(S \cap Q(\mathbf{r}(S)))$ for any $S \in \mathfrak{X}$.

Proof of Claim 1.6. Fix any $S \in \mathfrak{X}$. If S is not $\mathbf{c}$-awkward, then $Q(\mathbf{r}(S)) = X$, so the assertion follows from Claim 1.4. Suppose, then, S is $\mathbf{c}$-awkward. Take any $x \in \mathbf{c}(S)$, and set $z := \mathbf{r}(S)$. By Claim 1.2, z is a potential $\mathbf{c}$-reference for $\mathbf{c}(S)$, it belongs to $S \backslash \mathbf{c}(S)$, and it satisfies (1). Moreover. by the argument sketched while proving Claim 1.5, we have $x \in Q(z)$. So, if $x \notin \arg\max U(S \cap Q(z))$, then $U(w) > U(x)$ for some $w \in S \cap Q(z)$. But, by definition of Q, $w \in Q(z)$ implies $\{w\} = \mathbf{c}\{x, w, z\}$, while we have $\{x, w, z\} \in \mathcal{S}_{x,z}$, in contradiction to (1). Thus $x \in \arg\max U(S \cap Q(z))$, and we conclude that $\mathbf{c}(S) \subseteq \arg\max U(S \cap Q(\mathbf{r}(S)))$.

Conversely, take any $x \in \arg\max U(S \cap Q(z))$. Pick any $y \in \mathbf{c}(S)$, and note that, by what we have just shown, $y \in \arg\max U(S \cap Q(z))$, and hence, $U(x) = U(y)$. So, since $x \in Q(z)$, we have $x \in \mathbf{c}\{x, y, z\} = \{x, y\}$ by definition of Q. Then, by (1), $x \in \mathbf{c}\{x, y, z\} = \mathbf{c}(S) \cap \{x, y, z\}$, that is, $x \in \mathbf{c}(S)$, as we sought. $\|$

Claim 1.7. $\langle U, \mathbf{r}, Q\rangle$ is a proper reference-dependent choice model that represents $\mathbf{c}$.

Proof of Claim 1.7. In view of the previous two claims, we only need to verify that $\langle U, \mathbf{r}, Q\rangle$ is proper. We first verify that this model is weakly proper. Fix an arbitrary $S \in \mathfrak{X}$, $x \in \arg\max U(S \cap Q(\mathbf{r}(S)))$ and $T \in \mathcal{S}_{x,\mathbf{r}(S)}$. If $\mathbf{r}(S) = \diamondsuit$, the assertion follows from the fact that $\mathbf{r}(T) = \diamondsuit$ for any $T \in \mathcal{S}_x$. Suppose $\mathbf{r}(S) \in S$, so there is a $\mathbf{c}$-awkward choice in S. Notice that, by Claim 1.6, we have $x \in \mathbf{c}(S)$, so, by using (1) and Claim 1.6, we obtain $x \in \mathbf{c}(T) = \arg\max U(T \cap Q(\mathbf{r}(T)))$. Moreover, if $y \in Q(\mathbf{r}(T))$ and $U(y) \geq U(x)$, then, using Claim 1.6, (1) and Claim 1.6 again, we get

$$y \in \mathbf{c}(T) = \mathbf{c}(S) \cap T = \arg\max U(S \cap Q(\mathbf{r}(S))) \cap T,$$

which implies $y \in Q(\mathbf{r}(S))$. Thus $\langle U, \mathbf{r}, Q\rangle$ is weakly proper.

It remains to show that $\langle U, \mathbf{r}, Q\rangle$ is proper. Take any $x, z \in X$ with $U(x) > U(z)$ and $x \notin Q(z)$. We need to find a $y \in X$ with $U(x) \geq U(y)$ and $x \notin Q(\mathbf{r}\{x, y, z\})$. But, since $x \notin Q(z)$, the very definition of Q maintains that there is some $y \in X \backslash \{x\}$ such that either (i) $U(x) > U(y)$ and $\{x\} \neq \mathbf{c}\{x, y, z\}$, or (ii) $U(x) = U(y)$ and $x \notin \mathbf{c}\{x, y, z\}$. Moreover, by Claim 1.6, this is possible only if $x \notin Q(\mathbf{r}\{x, y, z\})$, as we sought. Conclusion: $\langle U, \mathbf{r}, Q\rangle$ is proper. $\|$

Proof of Theorem 2A.

We begin with a few preliminary definitions and an auxiliary lemma. First, define

$$ca_o(Z) := \{\mu \in ca(Z) : \mu(Z) = 0\}$$

and

$$\Gamma_Z := \mathbb{R}_{++} \times ca_o(Z).$$

Let us also agree to denote the affine hull of $\mathbb{P}(Z)$ in $ca(Z)$ as $\mathbb{M}(Z)$. For any $a \in \mathbb{R}$, the *mixing* operation $\oplus_a$ on $\mathbb{M}(Z)$ is defined in the usual way:

$$\nu \underset{a}{\oplus} \upsilon := a\nu + (1-a)\upsilon.$$

In turn, for any $(a, \mu) \in \Gamma_Z$, the *mixing-translation* operation $\oplus_{a,\mu}$ is defined on $\mathbb{M}(Z)$ as

$$\nu \underset{a,\mu}{\oplus} \upsilon := \left(\nu \underset{a}{\oplus} \upsilon \right) + \mu.$$

For any such (a, μ), we also define the binary operation $\ominus_{a,\mu} := \oplus_{\frac{1}{a}, -\frac{1}{a}\mu}$, that is,

$$\nu \underset{a,\mu}{\ominus} \upsilon := \frac{1}{a}\nu + \left(1 - \frac{1}{a}\right)\upsilon - \frac{1}{a}\mu \quad \text{ for any } \nu, \upsilon \in \mathbb{M}(Z).$$

Note that the binary relations $\ominus_{a,\mu}$ and $\oplus_{a,\mu}$ are inverses of each other in the following sense: For any $\nu, \upsilon, \eta \in \mathbb{M}(Z)$,

$$\nu = \upsilon \underset{a,\mu}{\oplus} \eta \quad \text{ iff } \quad \nu \underset{a,\mu}{\ominus} \eta = \upsilon. \tag{10}$$

Consequently,

$$\left(\nu \underset{a,\mu}{\oplus} \upsilon \right) \underset{a,\mu}{\ominus} \upsilon = \nu = \left(\nu \underset{a,\mu}{\ominus} \upsilon \right) \underset{a,\mu}{\oplus} \upsilon \quad \text{ for any } \nu, \upsilon \in \mathbb{M}(Z),$$

and

$$T = S \underset{a,\mu}{\oplus} \eta \quad \text{ iff } \quad T \underset{a,\mu}{\ominus} \eta = S \quad \text{ for any } \eta \in \mathbb{M}(Z) \text{ and } T, S \subseteq \mathbb{M}(Z).$$

Lemma 1. Take any (a, μ) and (b, σ) in Γ_Z. For any $\nu, \upsilon \in \mathbb{M}(Z)$ and $S \subseteq \mathbb{P}(Z)$ if

$$S \underset{a,\mu}{\oplus} \nu = S \underset{b,\sigma}{\oplus} \upsilon, \tag{11}$$

then

$$s \underset{a,\mu}{\oplus} \nu = s \underset{b,\sigma}{\oplus} \upsilon \quad \text{ for every } s \in S. \tag{12}$$

Proof of Lemma 1. Fix any $\nu, \upsilon \in \mathbb{M}(Z)$ and $S \subseteq \mathbb{P}(Z)$ such that (11) holds, and without loss of generality, assume $a \geq b$. Evidently, (11) entails $S = S \oplus_{b,\sigma} \upsilon \ominus_{a,\mu} \nu$, or more precisely,

$$S = \frac{b}{a} S + \kappa \tag{13}$$

where

$$\kappa := \frac{1}{a}\left((1-b)\nu - (1-a)\upsilon + \sigma - \mu\right) \in ca(Z).$$

Now take any $s \in S$ and use (13) recursively to find a sequence $(s^1, s^2, ...) \in S^\infty$ such that

$$s = \frac{b}{a} s^1 + \kappa \quad \text{ and } \quad s^k = \frac{b}{a} s^{k+1} + \kappa, \; k = 1, 2, ...$$

If $a = b$, this implies that $s = s^k + k\kappa$ for each positive integer k, which is possible only if $\kappa = 0$, as s and each s^k are probability measures. But $\kappa = 0$ means $(1-a)\nu + \mu = (1-b)\upsilon + \sigma$, so (12) is trivially true in that case. If, on the other hand, $a > b$, then

$$s = \left(\frac{b}{a}\right)^k s^k + \left(\left(\frac{b}{a}\right)^{k-1} + \cdots + \frac{b}{a} + 1 \right) \kappa = \left(\frac{b}{a}\right)^k s^k + \left(\frac{1 - \left(\frac{b}{a}\right)^k}{1 - \frac{b}{a}} \right) \kappa.$$

Letting $k \to \infty$, then, we find

$$s = \frac{a}{a-b}\kappa = \frac{1}{a-b}\left((1-b)\nu - (1-a)\upsilon + \sigma - \mu\right),$$

which is equivalent to say that $s \oplus_{a,\mu} \nu = s \oplus_{b,\sigma} \upsilon$. ■

We are now ready to prove Theorem 2A.

[*Necessity of the Axioms*] Take any $\mathbf{u} := (u_1, u_2, ...) \in \mathbf{C}(Z)^\infty$ with $\sup\{\|u_i\|_\infty : i \in \mathbb{N}\} < \infty$, and let $\Psi : \mathbf{E}(\mathbf{u}, \mathbb{P}(Z)) \to \mathbb{R}$ be a map that is strictly increasing, continuous and affine. Define $U := \Psi(\mathbf{E}(\mathbf{u}, \cdot))$, and note that U is a continuous and affine real map on $\mathbb{P}(Z)$. Given that Z is compact, therefore, by a well-known duality theorem of convex analysis, there exists a $u \in \mathbf{C}(Z)$ such that

$$U(p) = \mathbf{E}(u,p) \quad \text{for all } p \in \mathbb{P}(Z). \tag{14}$$

Let $\mathbf{r}$ be a quasi-affine reference map on $\mathfrak{X}_Z$. Finally, suppose that $[U, \mathbf{r}, \mathbf{u}]$ is a proper reference-dependent multi-utility model on $\mathfrak{X}_Z$ that represents the map $\mathbf{c} : \mathfrak{X}_Z \to 2^{\mathbb{P}(Z)}$. By definition, then, $\langle U, \mathbf{r}, Q_\mathbf{u}\rangle$ is a proper reference-dependent choice model that represents $\mathbf{c}$. Therefore, by Theorem 1, $\mathbf{c}$ is a choice correspondence on $\mathfrak{X}_Z$ that satisfies wWARP, C, RT and rd-WARP. Moreover, by Claim 1.1, we have, for any $q \in \mathbb{P}(Z)$,

$$Q_\mathbf{u}(q) = \{p \in \mathbb{P}(Z) : q \text{ is a potential } \mathbf{c}\text{-reference for } p\}. \tag{15}$$

Now fix $0 < a \leq 1$. Since each of the maps $\sigma \mapsto \int_Z u_i d\sigma$ is linear on $\mathbb{P}(Z)$, we obviously have $\mathbf{E}(\mathbf{u}, p) > \mathbf{E}(\mathbf{u}, q)$ iff $\mathbf{E}(\mathbf{u}, p \oplus_a r) > \mathbf{E}(\mathbf{u}, q \oplus_a r)$, that is,

$$p \in Q_\mathbf{u}(q) \quad \text{if and only if} \quad p \underset{a}{\oplus} r \in Q_\mathbf{u}\left(q \underset{a}{\oplus} r\right),$$

for any $0 < a \leq 1$ and $p, q, r \in \mathbb{P}(Z)$. Applying (15), then, establishes that $\mathbf{c}$ satisfies rIND.

Finally, take any $(S, p) \in \mathfrak{X}_Z \times \mathbb{P}(Z)$ and $0 < a \leq 1$. Note first that,

$$s \in Q_\mathbf{u}(\mathbf{r}(S)) \quad \text{if and only if} \quad s \underset{a}{\oplus} p \in Q_\mathbf{u}\left(\mathbf{r}\left(S \underset{a}{\oplus} p\right)\right). \tag{16}$$

(Indeed, when $\mathbf{r}(S) \neq \diamondsuit$, the definition of $Q_\mathbf{u}$ warrants that $s \in Q_\mathbf{u}(\mathbf{r}(S))$ iff $s \oplus_a p \in Q_\mathbf{u}(\mathbf{r}(S) \oplus_a p)$, and hence (16) follows from the quasi-affinity of $\mathbf{r}$. On the other hand, if $\mathbf{r}(S) = \diamondsuit$, then quasi-affinity of $\mathbf{r}$ implies $\mathbf{r}(S \oplus_a p) = \diamondsuit$ and thus (16) is trivially true.) Now take any $r \in \mathbf{c}(S \oplus_a p)$. Then, $r = s^* \oplus p$ for some $s^* \in S$, and we have

$$U\left(s^* \underset{a}{\oplus} p\right) \geq U\left(s \underset{a}{\oplus} p\right) \quad \text{for all } s \in S \text{ with } s \underset{a}{\oplus} p \in Q_\mathbf{u}\left(\mathbf{r}\left(S \underset{a}{\oplus} p\right)\right).$$

In view of (14) and (16), this implies

$$\mathbf{E}(u, s^*) \geq \mathbf{E}(u, s) \quad \text{for all } s \in S \cap Q_\mathbf{u}(\mathbf{r}(S)).$$

Thus: $s^* \in \mathbf{c}(S)$. But then,

$$r = s^* \underset{a}{\oplus} p \in \mathbf{c}(S) \underset{a}{\oplus} p,$$

establishing that $\mathbf{c}(S \oplus_a p) \subseteq \mathbf{c}(S) \oplus_a p$. As the converse containment is proved analogously, we have that $\mathbf{c}(S \oplus_a p) = \mathbf{c}(S) \oplus_a p$. Conclusion: $\mathbf{c}$ satisfies cIND.

[*Sufficiency of the Axioms*] Let $\mathbf{c} : \mathfrak{X}_Z \rightrightarrows \mathbb{P}(Z)$ be a choice correspondence on $\mathfrak{X}_Z$ that satisfies the six axioms stated in Theorem 2A. We begin with applying the classical von Neumann-Morgenstern Expected Utility Theorem to find a map $u \in \mathbf{C}(Z)$ such that

$$pR_{\mathbf{c}}q \quad \text{iff} \quad \mathbf{E}(u,p) \geq \mathbf{E}(u,q) \quad \text{for any } p,q \subseteq \mathbb{P}(Z). \tag{17}$$

We need the following fact.

Claim 2.1. For any $p,q,r \in \mathbb{P}(Z)$, and $a > 0$ such that both $p \oplus_a r$ and $q \oplus_a r$ belong to $\mathbb{P}(Z)$,

$$q \text{ is a potencial } \mathbf{c}\text{-reference for } p \tag{18}$$

if, and only if,

$$q \underset{a}{\oplus} r \text{ is a potencial } \mathbf{c}\text{-reference for } p \underset{a}{\oplus} r \tag{19}$$

Also, for any $S \in \mathfrak{X}_Z$, $a > 0$ and $r \in \mathbb{P}(Z)$ such that $S \oplus_a r \in \mathfrak{X}_Z$,

$$\mathbf{c}\left(S \underset{a}{\oplus} r\right) = \mathbf{c}(S) \underset{a}{\oplus} r$$

Proof of Claim 2.1. Take any $p,q,r \in \mathbb{P}(Z)$, and $a > 0$ with $\{p \oplus_a r,\ q \oplus_a r\} \subseteq \mathbb{P}(Z)$. Assume first that $a \in (0,1]$. Then, the "only if" part of the claim coincides with rIND. To see the "if" part, assume that $q \oplus_a r$ is a potencial $\mathbf{c}$-reference for $p \oplus_a r$. We wish to show that q is a potencial $\mathbf{c}$-reference for p. To this end, define

$$a^* := \sup\left\{b \in [0,1] : q \underset{b}{\oplus} r \text{ is a potencial } \mathbf{c}\text{-reference for } p \underset{b}{\oplus} r\right\}$$

Clearly $a^* \geq a > 0$. Moreover, using C, it is easily verified that $q \oplus_{a^*} r$ is a potencial $\mathbf{c}$-reference for $p \oplus_{a^*} r$. Thus, if we can show that $a^* = 1$, it will follow that q is a potencial $\mathbf{c}$-reference for p. Define $b^* := \frac{1}{1+a^*}$ and observe that, by rIND, $(q \oplus_{a^*} r) \oplus_{b^*} p$ is a potencial $\mathbf{c}$-reference for $(p \oplus_{a^*} r) \oplus_{b^*} p$. Similarly, $(q \oplus_{a^*} r) \oplus_{b^*} q$ is a potencial $\mathbf{c}$-reference for $(p \oplus_{a^*} r) \oplus_{b^*} q$. Since

$$\left(q \underset{a^*}{\oplus} r\right) \underset{b^*}{\oplus} p = \left(p \underset{a^*}{\oplus} r\right) \underset{b^*}{\oplus} q$$

and, therefore, $(q \oplus_{a^*} r) \oplus_{b^*} q$ is a potential $\mathbf{c}$-reference for $(q \oplus_{a^*} r) \oplus_{b^*} p$. By RT, then, $(q \oplus_{a^*} r) \oplus_{b^*} q$ is a potencial $\mathbf{c}$-reference for $(p \oplus_{a^*} r) \oplus_{b^*} p$. This is equivalent to say that

$$q \underset{\frac{2a^*}{1+a^*}}{\oplus} r \text{ is a potencial } \mathbf{c}\text{-reference for } p \underset{\frac{2a^*}{1+a^*}}{\oplus} r$$

By construction we must have $\frac{2a^*}{1+a^*} \leq a^*$, which implies that $a^* = 1$, as we sought. Conclusion: (18) and (19) are equivalent for any $0 < a \leq 1$.

Now assume $a > 1$. From rIND and what we have just proved we know that for any $p,q,r \in \mathbb{P}(Z)$ such that $p \oplus_a r$ and $q \oplus_a r$ belong to $\mathbb{P}(Z)$, $q \oplus_a r$ is a potencial $\mathbf{c}$-reference for $p \oplus_a r$ iff $(q \oplus_a r) \oplus_{1/a} r$ is a potential $\mathbf{c}$-reference for $(p \oplus_a r) \oplus_{1/a} r$. Since $(q \oplus_a r) \oplus_{1/a} r = q$ and $(p \oplus_a r) \oplus_{1/a} r = p$, we obtain the first part of the claim.

Finally, let $a > 1$, $r \in \mathbb{P}(Z)$ and $S \in \mathfrak{X}_Z$ be such that $S \oplus_a r \in \mathfrak{X}_Z$. By cIND we know that

$$\mathbf{c}(S) = \mathbf{c}\left(\left(S \underset{a}{\oplus} r\right) \underset{\frac{1}{a}}{\oplus} r\right) = \mathbf{c}\left(S \underset{a}{\oplus} r\right) \underset{\frac{1}{a}}{\oplus} r$$

which implies that $\mathbf{c}(S) \oplus_a r = \mathbf{c}(S \oplus_a r)$. This completes the proof of the claim. $\|$

Our next objective is to construct a quasi-affine reference map on $\mathfrak{X}_Z$. In doing this we shall use the correspondence $\mathfrak{R}$ defined in the previous subsection (with $X := \mathbb{P}(Z)$ and $\mathfrak{X} := \mathfrak{X}_Z$).[28] The following property of $\mathfrak{R}$ is essential.

Claim 2.2. Take any $(S, q) \in \mathfrak{X}_Z \times \mathbb{P}(Z)$ and $a > 0$ such that $S \oplus_a q \subseteq \mathbb{P}(Z)$. Then,

$$\mathfrak{R}\left(S \underset{a}{\oplus} q\right) = \mathfrak{R}(S) \underset{a}{\oplus} q.$$

Proof of Claim 2.2. (To simplify the notation, we write $\oplus$ for $\oplus_a$, and $\ominus$ for $\ominus_a$ throughout the argument.) Pick any $p \in \mathfrak{R}(S)$, and let $r := p \oplus q$. (By hypothesis, $r \in \mathbb{P}(Z)$.) Since p is in $\mathfrak{R}(S)$, it is a potential $\mathbf{c}$-reference for each $s \in \mathbf{c}(S)$ and we have $p \notin \mathbf{c}(S)$. So, by Claim 2.1, r is a potential $\mathbf{c}$-reference for $\mathbf{c}(S \oplus q)$ and $r \notin \mathbf{c}(S \oplus q)$. Thus, to verify that $r \in \mathfrak{R}(S \oplus q)$, it is enough to show that, for an arbitrarily fixed $T \subseteq S \oplus q$ with $r \in T$ and $\mathbf{c}(S \oplus q) \cap T \neq \emptyset$, we have $\mathbf{c}(T) = \mathbf{c}(S \oplus q) \cap T$. Note first that $r \in T$ implies $p \in T \ominus q$. Moreover, if $t \in \mathbf{c}(S \oplus q) \cap T$, then, again by Claim 2.1, $t \in (\mathbf{c}(S) \oplus q) \cap T$, so there exists an $s \in \mathbf{c}(S)$ such that $t = s \oplus q$, while, of course, $t \in T$. But then $t \ominus q \in \mathbf{c}(S)$ and $t \ominus q \in T \ominus q$, and it follows that $\mathbf{c}(S) \cap (T \ominus q) \neq \emptyset$. Since p belongs to $\mathfrak{R}(S)$, therefore, we have

$$\mathbf{c}(S) \cap (T \ominus q) = \mathbf{c}(T \ominus q).$$

Using this fact and Claim 2.1, then, we find

$$\begin{aligned}
\mathbf{c}(S \oplus q) \cap T &= (\mathbf{c}(S) \oplus q) \cap T \\
&= (\mathbf{c}(S) \oplus q) \cap ((T \ominus q) \oplus q) \\
&= (\mathbf{c}(S) \cap (T \ominus q)) \oplus q \\
&= \mathbf{c}(T \ominus q) \oplus q \\
&= \mathbf{c}(T),
\end{aligned}$$

which proves that $r \in \mathfrak{R}(S \oplus q)$. Conclusion: $\mathfrak{R}(S) \oplus q \subseteq \mathfrak{R}(S \oplus q)$.

The converse containment is obtained by applying this conclusion with $S \oplus q$ playing the role of S and $\ominus$ that of $\oplus$:

$$\begin{aligned}
\mathfrak{R}(S \oplus q) &= (\mathfrak{R}(S \oplus q) \ominus q) \oplus q \\
&\subseteq \mathfrak{R}((S \oplus q) \ominus q) \oplus q \\
&= \mathfrak{R}(S) \oplus q.
\end{aligned}$$

The proof of the claim is now complete. $\|$

We also need the following claim:

Claim 2.3. Take any $(S, q) \in \mathfrak{X}_Z \times \mathbb{P}(Z)$ and $a > 0$ such that $S \oplus_a q \subseteq \mathbb{P}(Z)$. Then, S is $\mathbf{c}$-awkward if, and only if, $S \oplus_a q$ is $\mathbf{c}$-awkward.

[28] *Reminder.* For any $\mathbf{c}$-awkward $S \in \mathfrak{C}$, the set $\mathfrak{R}(S)$ consists of all elements z of $S \backslash \mathbf{c}(S)$ that satisfies (1) and that is a potential $\mathbf{c}$-reference for each $p \in \mathbf{c}(S)$. (This set is nonempty, thanks to rd-WARP.)

Proof of Claim 2.3. (Again, we write $\oplus$ for $\oplus_a$, and $\ominus$ for $\ominus_a$ throughout the argument.) Suppose that S is $\mathbf{c}$-awkward. By Observation 1, there exists a subset T of S such that $\mathbf{c}(T) \neq \mathbf{c}(S) \cap T \neq \emptyset$. By Claim 2.1, then,

$$\mathbf{c}(T \oplus q) = \mathbf{c}(T) \oplus q \neq (\mathbf{c}(S) \cap T) \oplus q \neq \emptyset$$

It follows that

$$\mathbf{c}(T \oplus q) \neq \mathbf{c}(S \oplus q) \cap (T \oplus q) \neq \emptyset$$

Conclusion: If S is $\mathbf{c}$-awkward, then so is $S \oplus_a q$. The converse claim is obtained, again, by applying this conclusion with $S \oplus_a q$ playing the role of S and $\ominus_a$ that of $\oplus_a$. $\|$

Let

$$\mathfrak{X}_Z^{\text{awk}} := \{S \in \mathfrak{X}_Z : S \text{ is } \mathbf{c}\text{-awkward}\}$$

and define the binary relation $\hat{\approx}$ on $\mathfrak{X}_Z^{\text{awk}}$ by

$$S \mathrel{\hat{\approx}} T \quad \text{ iff } \quad T = S \underset{a}{\oplus} \nu \text{ for some } a > 0 \text{ and } \nu \in \mathbb{P}(Z).$$

Obviously $S \mathrel{\hat{\approx}} S$ and it is easy to see that if $S \mathrel{\hat{\approx}} T$, then $T \mathrel{\hat{\approx}} S$. Define $\approx$ to be the transitive closure of $\hat{\approx}$. Then, $\approx$ is an equivalence relation on $\mathfrak{X}_Z^{\text{awk}}$.

For each $S \in \mathfrak{X}_Z^{\text{awk}}$, let

$$[S] := \{T \in \mathfrak{X}_Z^{\text{awk}} : S \approx T\}$$

and define

$$\mathfrak{Y} := \{[S] : S \in \mathfrak{X}_Z^{\text{awk}}\},$$

which is a partition of $\mathfrak{X}_Z^{\text{awk}}$. By the Axiom of Choice, there exists a map $\psi : \mathfrak{Y} \to \mathfrak{X}_Z^{\text{awk}}$ such that $\psi([S]) \in [S]$ for each $S \in \mathfrak{X}_Z^{\text{awk}}$. Since $\mathfrak{R}(\psi([S])) \neq \emptyset$ for each $S \in \mathfrak{X}_Z^{\text{awk}}$ by rd-WARP, we may apply the Axiom of Choice again to find a map $\mathbf{r} : \{\psi([S]) : S \in \mathfrak{X}_Z^{\text{awk}}\} \to \mathbb{P}(Z)$ such that

$$\mathbf{r}(\psi([S])) \in \mathfrak{R}(\psi([S])) \qquad \text{for all } S \in \mathfrak{X}_Z^{\text{awk}}.$$

We extend this map to $\mathfrak{X}_Z^{\text{awk}}$ by defining

$$\mathbf{r}(S) := \mathbf{r}(\psi([S])) \underset{a_1}{\oplus} \nu_1 \underset{a_2}{\oplus} \nu_2 ... \underset{a_n}{\oplus} \nu_n, \tag{20}$$

for any positive integer n, $a_1, a_2, ..., a_n > 0$ and $\nu_1, \nu_2, ..., \nu_n \in \mathbb{P}(Z)$ such that $S = \psi([S]) \oplus_{a_1} \nu_1 \oplus_{a_2} \nu_2 ... \oplus_{a_n} \nu_n$. Let us check that this extension is well-defined. Note that, for any $a_1, a_2, ..., a_n > 0$ and $\nu_1, \nu_2, ..., \nu_n \in \mathbb{P}(Z)$, we have

$$S = \psi([S]) \underset{a_1}{\oplus} \nu_1 \underset{a_2}{\oplus} \nu_2 ... \underset{a_n}{\oplus} \nu_n = \psi([S]) \underset{a,\mu}{\oplus} \nu_1$$

where

$$\Gamma_Z \ni (a, \mu) = \left(\left(\prod_{i=1}^{n} a_i \right), \sum_{i=2}^{n} (1 - a_i) \left(\prod_{j=i+1}^{n} a_j \right) (\nu_i - \nu_1) \right)$$

and, by convention,

$$\prod_{j=n+1}^{n} a_j := 1.$$

Clearly, it is also the case that for any $s \in S$ and $t \in \psi([S])$ such that $s = t \oplus_{a_1} \nu_1 \oplus_{a_2} \nu_2 ... \oplus_{a_n} \nu_n$ we must have $s = t \oplus_{a,\mu} \nu_1$. Given these observations, an application of Lemma 1 ensures that, for any positive integers m and n, and any vectors $(a_1, \nu_1), (a_2, \nu_2), ..., (a_n, \nu_n)$ and $(b_1, \upsilon_1), (b_2, \upsilon_2), ..., (b_m, \upsilon_m)$ in $\mathbb{R}_{++} \times \mathbb{P}(Z)$, if

$$S = \psi([S]) \underset{a_1}{\oplus} \nu_1 \underset{a_2}{\oplus} \nu_2 ... \underset{a_n}{\oplus} \nu_n$$

and

$$S = \psi([S]) \underset{b_1}{\oplus} \upsilon_1 \underset{b_2}{\oplus} \upsilon_2 ... \underset{b_m}{\oplus} \upsilon_m,$$

then we must necessarily have

$$\mathbf{r}(\psi([S])) \underset{a_1}{\oplus} \nu_1 \underset{a_2}{\oplus} \nu_2 ... \underset{a_n}{\oplus} \nu_n = \mathbf{r}(\psi([S])) \underset{b_1}{\oplus} \upsilon_1 \underset{b_2}{\oplus} \upsilon_2 ... \underset{b_m}{\oplus} \upsilon_m.$$

Conclusion: $\mathbf{r} : \mathfrak{X}_Z^{\mathrm{awk}} \to \mathbb{P}(Z)$ is well-defined.

We next verify that $\mathbf{r}(S) \in \mathfrak{R}(S)$ for every $S \in \mathfrak{X}_Z^{\mathrm{awk}}$. Fix any $S \in \mathfrak{X}_Z^{\mathrm{awk}}$, and pick any positive integer n and $(a_1, \upsilon_1), ... (a_n, \upsilon_n) \in \mathbb{R}_{++} \times \mathbb{P}(Z)$ such that, for any $k \in \{1, ..., n\}$, $\psi([S]) \oplus_{a_1} \nu_1 \oplus_{a_2} \nu_2 ... \oplus_{a_k} \nu_k \in \mathfrak{X}_Z^{\mathrm{awk}}$ and $S = \psi([S]) \oplus_{a_1} \nu_1 \oplus_{a_2} \nu_2 ... \oplus_{a_n} \nu_n$. A simple induction argument shows that Claim 2.2 implies

$$\mathfrak{R}(S) = \mathfrak{R}(\psi([S])) \underset{a_1}{\oplus} \nu_1 \underset{a_2}{\oplus} \nu_2 ... \underset{a_n}{\oplus} \nu_n$$

and, therefore, we have $\mathbf{r}(S) \in \mathfrak{R}(S)$ by definition of $\mathbf{r}$.

Finally, we extend $\mathbf{r}$ to $\mathfrak{X}_Z$ by setting $\mathbf{r}(S) := \diamondsuit$ for any $S \in \mathfrak{X}_Z \backslash \mathfrak{X}_Z^{\mathrm{awk}}$. By construction, $\mathbf{r}$ is a reference map on $\mathfrak{X}_Z$ and $\mathbf{r}(S) \in \mathfrak{R}(S)$ for any $S \in \mathfrak{X}_Z^{\mathrm{awk}}$. We next show that $\mathbf{r}$ is quasi-affine. To this end, take any $S \in \mathfrak{X}_Z$, $0 < a \leq 1$ and $p \in \mathbb{P}(Z)$. If $\mathbf{r}(S) = \diamondsuit$ (so that $S \in \mathfrak{X}_Z \backslash \mathfrak{X}_Z^{\mathrm{awk}}$), Claim 2.3 implies that $\mathbf{r}(S \oplus_a p) = \diamondsuit$ as we sought. Suppose that $\mathbf{r}(S) \neq \diamondsuit$ (so that $S \in \mathfrak{X}_Z^{\mathrm{awk}}$). By the argument given above, we know that there exist $(b, \mu) \in \Gamma_Z$ and $\nu \in \mathbb{P}(Z)$ such that $S = \psi([S]) \oplus_{b,\mu} \nu$. Moreover, for any such (b, μ) and ν, it must be the case that

$$\mathbf{r}(S) = \mathbf{r}\left(\psi([S])\right) \underset{b,\mu}{\oplus} \nu$$

Now notice that we can write $S \oplus_a p$ as

$$S \underset{a}{\oplus} p = \psi([S]) \underset{ab, a\mu + (1-a)(p-\nu)}{\oplus} \nu$$

which, by definition of $\mathbf{r}$, implies that

$$\mathbf{r}\left(S \underset{a}{\oplus} p\right) = \mathbf{r}\left(\psi([S])\right) \underset{ab, a\mu + (1-a)(p-\nu)}{\oplus} \nu.$$

Consequently,

$$\begin{aligned} \mathbf{r}\left(S \underset{a}{\oplus} p\right) &= \mathbf{r}\left(\psi([S])\right) \underset{ab, a\mu + (1-a)(p-\nu)}{\oplus} \nu \\ &= \left(\mathbf{r}\left(\psi([S])\right) \underset{b,\mu}{\oplus} \nu\right) \underset{a}{\oplus} p \\ &= \mathbf{r}(S) \underset{a}{\oplus} p \end{aligned}$$

Conclusion: $\mathbf{r}$ is quasi-affine.

This discussion may be summarized as follows:

Claim 2.4. $\mathbf{r}$ is a quasi-affine reference map on $\mathfrak{X}_Z$. Moreover, for any $\mathbf{c}$-awkward $S \in \mathfrak{X}_Z$, we have $\mathbf{r}(S) \in \mathfrak{R}(S)$ and

$$\mathbf{E}(u, p) > \mathbf{E}(u, \mathbf{r}(S)) \quad \text{for all } p \in \mathbf{c}(S).$$

Proof of Claim 2.4. All assertions but the last one were proved above. In view of (17), the argument for the last assertion is, on the other hand, identical to the one given for Claim 1.3. ‖

We now define the correspondence $Q : \mathbb{P}(Z) \cup \{\diamondsuit\} \rightrightarrows \mathbb{P}(Z)$ exactly as in the proof of Theorem 1, that is, let $Q(\diamondsuit) := X$, and for any $q \in \mathbb{P}(Z)$, define $Q(q)$ as the collection of all $p \in \mathbb{P}(Z)$ such that q is a potential $\mathbf{c}$-reference for p. Then, the arguments given for Claims 1.5, 1.6 and 1.7 remain valid in the present setting, and culminate in the following observation.

Claim 2.5. $\langle \mathsf{E}(u,\cdot), \mathbf{r}, Q\rangle$ is a proper reference-dependent choice model on $\mathfrak{X}_Z$ that represents $\mathbf{c}$.

Now define the reflexive binary relation $\trianglerighteq_Q$ on $\mathbb{P}(Z)$ by

$$p \trianglerighteq_Q q \quad \text{if and only if} \quad p \in Q(q) \text{ or } p = q.$$

Since $p \in Q(q)$ implies $\mathsf{E}(u,p) > \mathsf{E}(u,q)$ for any $p, q \in \mathbb{P}(Z)$, it is clear that $\trianglerighteq_Q$ is antisymmetric. Since the graph of Q is closed, and $Q \circ Q \subseteq Q$, this relation is thus a continuous partial order on $\trianglerighteq_Q$. Finally, an immediate application of rIND shows that $\trianglerighteq_Q$ satisfies the classical von Neumann-Morgenstern independence axiom. Therefore, by the Expected Multi-Utility Theorem (cf. Dubra, Maccheroni and Ok, 2004), there exists a nonempty closed and convex subset $\mathcal{V}$ of $\mathbf{C}(Z)$ such that, for any $p, q \in \mathbb{P}(Z)$,

$$p \trianglerighteq_Q q \quad \text{if and only if} \quad \mathsf{E}(v,p) \geq \mathsf{E}(v,q) \text{ for all } v \in \mathcal{V}.$$

Since Z is compact, $\mathbf{C}(Z)$ is separable, so $\mathcal{V}$ is a separable metric subspace of $\mathbf{C}(Z)$. We pick any countable dense subset $\{w_1, w_2, ...\}$ of $\mathcal{V}$, and let

$$u_i := \frac{w_i}{\|w_i\|_\infty}, \qquad i = 1, 2, ...$$

(*Note.* $u_i(Z) \subseteq [-1, 1]$ for each i.) Then, letting $\mathbf{u} := (u_1, u_2, ...)$, it is readily verified that, for any $p, q \in \mathbb{P}(Z)$,

$$p \trianglerighteq_Q q \quad \text{if and only if} \quad \mathsf{E}(\mathbf{u},p) \geq \mathsf{E}(\mathbf{u},q).$$

In particular, we have $Q = Q_\mathbf{u}$.

To complete the proof, note that $\mathsf{E}(\mathbf{u}, \mathbb{P}(Z))$ is a convex subset of $\mathbb{R}^\infty$, and define the real map Ψ on this set by the equations

$$\mathsf{E}(u,p) = \Psi(\mathsf{E}(\mathbf{u},p)), \qquad p \in \mathbb{P}(Z).$$

Ψ is well-defined, because $\mathsf{E}(\mathbf{u},\cdot)$ is an injection from $\mathbb{P}(Z)$ into $\mathsf{E}(\mathbf{u},\mathbb{P}(Z))$. (Indeed, if $\mathsf{E}(\mathbf{u},p) = \mathsf{E}(\mathbf{u},q)$ for some $p, q \in \mathbb{P}(Z)$, then $p \trianglerighteq_Q q \trianglerighteq_Q p$, which is possible only if $p = q$ by the antisymmetry of $\trianglerighteq_Q$.) Moreover, by Claim 2.5, $\langle \Psi(\mathsf{E}(\mathbf{u},\cdot)), \mathbf{r}, Q_\mathbf{u}\rangle$ is a proper reference-dependent choice model on $\mathfrak{X}_Z$ that represents $\mathbf{c}$. The following claim, then, completes the proof of Theorem 2A.

Claim 2.6. Ψ is an affine, strictly increasing and continuous real map on $\mathsf{E}(\mathbf{u}, \mathbb{P}(Z))$.

Proof of Claim 2.6. The affineness of Ψ is a fairly straightforward consequence of that of $\mathsf{E}(u,\cdot)$. Moreover, for any $p, q \in \mathbb{P}(Z)$, $\mathsf{E}(\mathbf{u},p) > \mathsf{E}(\mathbf{u},q)$ implies $p \in Q(q)$, and hence $\Psi(\mathsf{E}(\mathbf{u},p)) = \mathsf{E}(u,p) > \mathsf{E}(u,q) = \Psi(\mathsf{E}(\mathbf{u},q))$, which shows that Ψ is strictly increasing. Finally, we show that Ψ is continuous (relative to the product topology). To this end, take any $p, p^m \in \mathbb{P}(Z)$, $m = 1, 2, ...$, such that

$$\mathsf{E}(u_i, p^m) \to \mathsf{E}(u,p), \qquad i = 1, 2, ...$$

Pick any subsequence of (p^m), and denote this subsequence again by (p^m) for convenience. Since Z is compact, so is $\mathbb{P}(Z)$, and hence there exist positive integers $m_1 < m_2 < \cdots$ such that (p^{m_k}) converges weakly to some $q \in \mathbb{P}(Z)$. Then, since each u_i is continuous on Z, we have

$$\mathsf{E}(u_i, p^{m_k}) \to \mathsf{E}(u,q), \qquad i = 1, 2, ...,$$

that is, $\mathbf{E}(\mathbf{u},p)=\mathbf{E}(\mathbf{u},q)$. Since $\mathbf{E}(\mathbf{u},\cdot)$ is injective, therefore, $p=q$. Conclusion: Every subsequence of (p^m) has a subsequence that converges weakly to p. This means that (p^m) converges weakly to p, and hence

$$\Psi(\mathbf{E}(\mathbf{u},p^m))=\mathbf{E}(u,p^m)\to\mathbf{E}(u,p)=\Psi(\mathbf{E}(\mathbf{u},p)),$$

as we sought. ‖

Proof of Theorem 2B.

[*Necessity of the Axioms*] Apply Theorem 2A.

[*Sufficiency of the Axioms*] Let $\mathbf{c}:\mathfrak{X}_Z\to 2^{\mathbb{P}(Z)}$ be a choice correspondence on $\mathfrak{X}_Z$ that satisfies the six axioms stated in Theorem 2A. Note first that $\mathbf{C}(Z)=\mathbb{R}^Z$ as $|Z|<\infty$, and then define u, $\mathbf{r}$, Q and $\trianglerighteq_Q$ exactly as in the proof of the sufficiency part of Theorem 2A. As we have shown above, there exists a sequence $\mathbf{w}:=(w_m)$ in $\mathbb{R}^Z$ such that $p\trianglerighteq_Q q$ iff $\mathbf{E}(\mathbf{w},p)\geq\mathbf{E}(\mathbf{w},q)$ for any $p,q\in\mathbb{P}(Z)$. Let

$$v_1:=-\mathbf{1}_Z,\quad v_2:=\mathbf{1}_Z,\quad v_{i+2}:=\frac{w_i+\|w_i\|_\infty}{\|w_i+\|w_i\|_\infty\|_\infty},\ i=1,2,...,$$

and define $\mathbf{v}:=(v_1,v_2,...)$. Evidently, $v_i(Z)\subseteq[0,1]$ for each $i\geq 2$, and $p\trianglerighteq_Q q$ iff $\mathbf{E}(\mathbf{v},p)\geq\mathbf{E}(\mathbf{v},q)$ for any $p,q\in\mathbb{P}(Z)$. Finally, let $\mathcal{F}$ be the closed convex cone in $\mathbb{R}^Z$ generated by $\{v_1,v_2,...\}$. It follows from the uniqueness part of the Expected Multi-Utility Theorem (cf. Dubra, Maccheroni and Ok, 2004) that $\mathcal{F}$ is the largest subset of $\mathbb{R}^Z$ such that, for any $p,q\in\mathbb{P}(Z)$,

$$p\trianglerighteq_Q q\quad\text{if and only if}\quad\mathbf{E}(f,p)\geq\mathbf{E}(f,q)\text{ for all }f\in\mathcal{F}.$$

Since, for any $p,q\in\mathbb{P}(Z)$,

$$p\trianglerighteq_Q q\quad\text{implies}\quad\mathbf{E}(u,p)>\mathbf{E}(u,q)\text{ or }p=q,\tag{21}$$

therefore, we have $u\in\mathcal{F}$.

We next show that u lies in the relative interior of $\mathcal{F}$. Suppose this is false. Then, by the Minkowski Supporting Hyperplane Theorem, there exists a nonzero map $\alpha\in\mathbb{R}^Z$ such that

$$\sum_{z\in Z}\alpha(z)u(z)=0\geq\sum_{z\in Z}\alpha(z)f(z)\quad\text{for all }f\in\mathcal{F}.\tag{22}$$

The latter inequalities imply that $0\geq\theta\sum_{z\in Z}\alpha(z)$ for every real number θ, so we must have $\sum_{z\in Z}\alpha(z)=0$. Now let α^+ and α^- stand for the positive and negative parts of α, respectively. Since $\alpha=\alpha^+-\alpha^-$, the previous observation yields

$$a:=\sum_{z\in Z}\alpha^+(z)=\sum_{z\in Z}\alpha^-(z)\geq 0.$$

As $\alpha\neq 0$, we have $a>0$. It follows that both $p:=\frac{1}{a}\alpha^+$ and $q:=\frac{1}{a}\alpha^-$ belong to $\mathbb{P}(Z)$. So, by (22), we have

$$\sum_{z\in Z}u(z)p(z)=\sum_{z\in Z}u(z)q(z)\quad\text{and}\quad\sum_{z\in Z}p(z)f(z)\leq\sum_{z\in Z}q(z)f(z)\text{ for all }f\in\mathcal{F},$$

that is, $\mathbf{E}(u,p)=\mathbf{E}(u,q)$ and $q\trianglerighteq_Q p$. By (21), therefore, we must have $p=q$. But this means that $\alpha=\alpha^+-\alpha^-=0$, a contradiction. Conclusion: u belongs to the relative interior of $\mathcal{F}$.

Now define

$$v:=\sum_{i=1}^{\infty}\frac{1}{2^i}v_i,$$

and note that $v \in \mathcal{F}$. Since u is in the relative interior of $\mathcal{F}$, therefore, there exists a real number $b > 0$ small enough to guarantee that $v_0 := u + b(u - v) \in \mathcal{F}$. Then

$$u = \frac{1}{1+b} v_0 + \frac{b}{1+b} \sum_{i=1}^{\infty} \frac{1}{2^i} v_i,$$

so, letting $\lambda_1 := \frac{1}{1+b}$, and $\lambda_{i+1} := \frac{b}{2^i(1+b)}$ and $u_i := v_{i-1}$, $i = 1, 2, \ldots$, we find $u = \sum^{\infty} \lambda_i u_i$ where (λ_m) is a strictly positive sequence with $\sum^{\infty} \lambda_i = 1$. But it is obvious that $p \trianglerighteq_Q q$ iff $\mathsf{E}(\mathbf{u}, p) \geq \mathsf{E}(\mathbf{u}, q)$ for any $p, q \in \mathbb{P}(Z)$, where $\mathbf{u} := (u_1, u_2, \ldots)$. So $Q = Q_{\mathbf{u}}$, and invoking Claim 2.5 completes the proof.[29]

[29] It may be worth noting here that the finiteness of Z is needed for the argument above only to be able to use the Minkowski Supporting Hyperplane Theorem. When $|Z| = \infty$, that part of the argument would require the relative interior of F be nonempty, and it is not at all clear why this should be the case.

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