Can Voting Solve the Information Problem in Large Elections?

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Abstract

Starting with Feddersen-Pesendorfer (1997), most of the work examining the question of information aggregation under strategic voting has assumed that as the state changes, everyone becomes more (or less) inclined towards one alternative ("common values"). We show that this assumption is necessary and sufficient in delivering the result that voting always aggregates information for any rule. We examine the "non-common values" setting where such a correlation among voter preferences and the state does not hold, and show that for all economically important voting rules, there must be multiple equilibria where, in at least one equilibrium, we get an outcome different from the full information outcome with a probability arbitrarily close to one. And, for certain voting rules, there is no equilibrium where the full-information outcome is achieved. This result does not depend on the accuracy of the signal, which means that even when there is a small uncertainty about the state, voting can deliver a sure outcome that is different from what would happen, had the state been known for certain.

[0] While writing this paper, I have benefited immensely from discussions with David Austen-Smith, Tim Feddersen, Steve Callander, Marciano Siniscalchi, Sean Gailmard, Jaehoon Kim, Alexandre Debs and all participants in the Voting and Information panel at the Econometric Society Summer Conference at Minnesota, June 2006. Siddarth Madhav has helped a lot with the figures. This is a preliminary draft of this paper and it is possible that there are errors. All responsibility for such errors is mine.

1 Introduction

Groups often choose by voting. Choosing the vote share required by the winning alternative has two deep problems. The first concerns how best to aggregate preferences when voters have diverse preferences over alternatives. The second concerns how to aggregate information when voters have different information about some decision relevant aspect of the choice problem. This paper deals with *informational efficiency* of voting as a mechanism in general and of plurality rules[1] in particular.

If an individual is uncertain about some decision relevant feature of the environment - the *state* - then his preference over alternatives is in principle sensitive to change in his information. Consider the canonical example used in the literature to illustrate this. Suppose that a jury has to vote on whether to acquit or convict a defendant. If the evidence of guilt is overwhelmingly large, then the jury members would want to convict him, otherwise they prefer to acquit him. In many cases like this, we do not know the state for sure and only have noisy evidence about it. If the state were known for sure, given a voting rule, we would get a certain voting outcome for each state. Would the voting mechanism deliver the same outcome when the state is known to every individual only with a certain probability? If it does, then individual uncertainty is irrelevant in the aggregate. The question addressed in this paper concerns identifying the rules that aggregate information.

The Condorcet Jury Theorem claims that when the electorate is large and everyone gets an independent private signal about the state that is correct with a probability greater than half, the majority rule always aggregates information. If everyone votes according to his signal, then the correct alternative almost always receives more than half the votes. Over the last decade or so, the theorem has been subjected to renewed scrutiny. Earlier proofs of the theorem relied on the "sincere voting" assumption, but Austen-Smith and Banks (1996) pointed out that one's vote matters only when he is "pivotal", i.e. when the others are tied or almost tied. Conditioning on the event of being pivotal, one may find more information about the state, and may as a result vote against his signal in equilibrium. Feddersen and Pesendorfer (1997), henceforth F-P, showed that with such strategic voting, under conditions of reasonable generality, we get full information aggregation for any voting rule. In their model, the state represents the "commonality" of preferences in the sense that for a given change in the state variable, everyone becomes more prone to voting for exactly one of the two alternatives. Subsequently, Myerson (1998, 2000), Wit(1998), Meirowitz (2002) and others examined the issue of information aggregation in very similar settings (maintaining the "common values" assumption) and found positive results for most plurality rules with respect to information efficiency. Notable exceptions are Razin (2003) which demonstrates that aggregation can break down if the voters use their vote as a signal of their preference to the can-

[1] In this paper we focus only on two-candidate elections with plurality rules or q-rules, according to which the candidate getting more than q share of the votes wins the elections, where $q \in (0, 1)$. We, however, denote a voting rule in this paper by θ.

didates, and Martinelli (2006) which shows that aggregation may not happen if information is not cheap enough.

Kim (2005) looks at a non-common values setting with two groups of voters that have opposed ranking over the alternatives in each state. In this paper, the preferences in each group are identical, and he too finds that information is fully aggregated for most voting rules as long as the.voter cares enough about mistakes in each state[2]. Kim and Fey (2006) works with a similar setting, and find that aggregation may break down when abstention is allowed. Meirowitz (2005) examines the issue of aggregation in the same setting with the addition of a communication stage. Oliveros (2005) examines the same issue in a similar setting, allowing for both abstention and costly information.

In order to demonstrate that the source of informational inefficiency in the democratic institution of voting lies in the nature of relationship between the uncertainty and voter preferences, we do not allow for abstention, communication or signaling motivation, and consider an environment with a more general correlation between the state variable and preferences. In our setting, contrary to F-P, commonly perceived information about the true state need not lead to a common shift in induced preferences, and contrary to Kim, types with the same *ranking* over the alternatives under full information need not have the same *intensity* of preference for them, leading to different behaviour under uncertainty. Our claim is that elections may involve voter preferences which look neither like adversarial committees nor like jury boards. Our model is similar to Kim (2005) and Meirowitz (2005), in that we allow two competing groups of voters.with opposed preferences in either state, but since we use a spatial model of voting, we can allow for differing preferences within each group. Our main result is that common values is necessary and sufficient for information to be fully aggregated.

To understand the "common values" assumption, consider the canonical jury example where voters vary over what counts as "reasonable doubt". All the individual rankings are similar in the sense that everyone wants to acquit for low levels of guilt and convict for high levels, but they vary with the precise level at which they switch from acquittal to conviction. So, as the level of guilt increases, more and more members favour the guilty verdict. However, the common values assumption may be violated in other situations. Consider the following example.

Suppose a country has so far been isolated and now is voting on whether to allow free trade by joining the WTO. Because of their isolation, they have developed both an industrial sector and an agricultural sector that suits their own consumption needs. If they allow free trade, the sector in which they have comparative advantage will grow and the other will die. Assume that they do not know where their comparative advantage lies. If their advantage lay in industry and this was commonly known, those in the industrial sector would

[2] What is erroneously claimed as a failure of aggregation in Proposition 3 in Kim (2005) is really a full information equivalent outcome under what we call a $\mathcal{Q}$-trivial rule in this paper. We can indeed have a failure of aggregation in Kim's set up if the *exogenous* utility loss from a mistake in one state is low enough for the majority group (see his Proposition 4).

vote in favour of joining WTO while those engaged in the agricultural sector would vote against, and conversely if their advantage lay in agriculture, people engaged in agriculture. Note here that, it is as if there are two opposing interest groups. Their rankings over alternatives change with the state, but given a state the ranking of each is opposed to that of the other. In other words, as the state changes, one set of types switch from status quo to the alternative, while another set switches the other way round. We call this type of situation one of non-common values. In contrast, in a common value situation, as the state changes, there is switching in only one direction.

For a second example consider an election with an incumbent candidate and a challenger. Assume for now that a candidate cannot commit to any location other than his own most preferred point on the policy space. The incumbent's best point is known to be $\mathcal{Q}$, but there is some uncertainty about that of the challenger, it can be one of two locations: L or R. If L is to the left of $\mathcal{Q}$ and R to the right, then we are in a non-common values situation similar to the ones described above. However, if L is to the left of R, but both locations are to the left of $\mathcal{Q}$, then, for all practical purposes we have a leftist challenger and a rightist incumbent. As the challenger becomes more extremely leftist (state changes from R to L), he loses support of some of the moderates but does not have anyone new switching to him. Thus we are back in the common values situation.

The rest of the paper is organized as follows. In Section 2, we provide an intuitive discussion of our model, show how it works in an illustrative example, and discuss the main results. In Section 3, we set up the formal model to be used throughout the rest of the paper. Section 4 discusses the benchmark common value case and Section 5 analyses the non-common values situation. Section 6 compares and contrasts the two settings and discusses the implications and an extension. Most of the proofs are relegated to the appendix.

2 Discussion of the Model and Results

In this paper we develop a model that allows for both the common values and the non-common values situation depending on parameters and compare equilibria across the two cases. Voters have quadratic preference over a policy space which is a compact subset of the real line. An extension of the model considering a policy space with a higher dimensions is discussed later. A voter's type is identified by his bliss point on this space. There is a status quo $\mathcal{Q}$ whose location on the space is known. The state variable is the location of the alternative policy $\mathcal{P}$. The state space is binary – $\mathcal{P}$ can be located at one of two given points on the policy space. Based on where these two points are located, we may have a non-common value or a common value set-up, as illustrated in the candidate competition example above. The voting rule θ is the share of votes requited for $\mathcal{P}$ to win. We study the limiting outcomes as the number of voters becomes large.

Our main result is that while in the common values case, information is aggregated with a very high probability in the unique limiting equilibrium for any voting rule, the property breaks down in the non-common values case. In the latter case, we have multiple limiting equilibria depending on beliefs, and for all economically important voting rules, one or more of the equilibria reach a "wrong" outcome with a very high probability. To understand why information aggregation can break down, note that, in equilibrium, there is always a set of types voting informatively while others do not pay heed to the signal. In most equilibria of the one-dimensional model, the set of responsive voters is an interval of types, with types on either side of the interval voting in opposite directions uninformatively. Based on the full information outcome, there are two kinds of voting rules - the ones which, under full information, lead to different outcomes in the two states (the *consequential rules*), and those which lead to the same outcome in both states (the *trivial rules*).

For voting with consequential rules, we need the following to happen in equilibrium for information to aggregated:

1. The responsive types should be *influential*, i.e. the overall voting outcome should change as the responsive types vote differently in the different states. For a voting rule θ, this condition is satisfied if the θ-quantile type lies in the responsive set.[3]

2. The responsive types are *aligned* with the society, i.e. they vote the same way as the full information mapping from the states to the outcomes demands. This always happens under common values, but under non-common values it happens only if the responsive types belong to the larger interest group.

On the other hand, for voting with trivial rules, we need the responsive types *not* to be influential for information to be aggregated.

All these conditions are satisfied under equilibrium in the common values situation, but in the non-common values situation, each of these conditions can individually fail in the limiting equilibrium.

To see how the model works, let us first consider the common values case. Assume that the policy space is $[-1, 1]$, $\mathcal{Q}$ is located at 0, and the policy $\mathcal{P}$ is located under states L and R at -0.8 and -0.2 respectively. A voter votes for whichever alternative is located closer to his bliss point. If the state is known to be L, everyone left of -0.4 votes for $\mathcal{P}$ and if it is known to be R, and everyone left of -0.1 votes for $\mathcal{P}$. Everyone else votes for $\mathcal{Q}$. Therefore, if the state is known, the types between -0.4 and -0.1 vote for $\mathcal{P}$ when it the state is moderate (R) and switch to $\mathcal{Q}$ when it is extreme (L). These are the types that have an incentive to change their vote based on information about the state. The responsive set is thus a subset of $[-0.4, -0.1]$ for any voting rule. Assume that 35% of the voters have bliss points left of -0.4, and 55% have bliss points to the left of -0.1. Under full information, for $\theta < 35\%$, $\mathcal{P}$ obtains as the

[3] We define the θ-quantile type as the type which has exactly θ proportion of types below it, when the types are ranked in order of their bliss points.

winner under both states, for $35\% < \theta < 55\%$ we get $\mathcal{Q}$ under state L and $\mathcal{P}$ under state R, and for $\theta > 55\%$ we always get $\mathcal{Q}$. To follow what happens in equilibrium under incomplete information for different values of θ, we need to track the responsive set in equilibrium and whether the θ-quantile type belongs to the set.

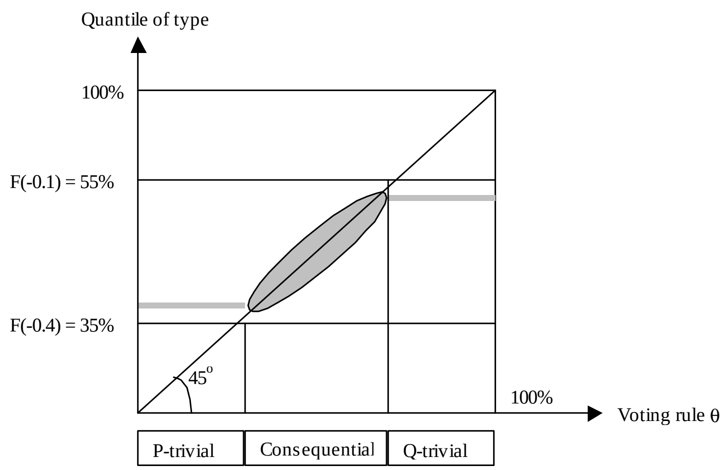

Figure 1a: Responsive set of types in limiting equilibrium under common values

We show the responsive set of types in the limiting equilibrium for each voting rule in Figure 1a. The voting rule θ is plotted on the horizontal axis and the quantile of types (considered from the left) on the vertical axis. For all $\theta < 35\%$ ($\mathcal{P}$-trivial rules), the θ-quantile type is to the left of -0.4, and the responsive set is stuck at a small interval of types just right of -0.4, so the responsive types are never influential. $\mathcal{P}$ gets slightly more than 35% votes in both states and wins. Similarly, for all $\theta > 55\%$ ($\mathcal{Q}$-trivial rules), the responsive set is a small set of types just left of -0.1, and $\mathcal{P}$ receives less than 55% share of votes in both states and loses. .For rules between 35% and 55% (consequential rules), the responsive set includes the θ-quantile type. Hence the responsive set is influential for these rules. Also, the responsive types are aligned with the society – they vote for $\mathcal{Q}$ when L and $\mathcal{P}$ when R. Therefore, information is aggregated in the common values set up under any voting rule θ .

Next, consider the case where the locations of the alternative at L and R are on two sides of $\mathcal{Q}$ - say at -0.8 and at 0.8 respectively. If the state is known, under L, only the types to the left of -0.4 (say 35% of the voters) vote for $\mathcal{P}$, while under R, only those to the right of 0.4 vote for $\mathcal{P}$ (say 55% of the voters). We shall call these two groups of types as the L-group and R-group respectively. Note that both these groups are sensitive to information about the state, while the others, i.e. the types in $(-0.4, 0.4)$ always vote for $\mathcal{Q}$. Under full information, here too, $\theta < 35\%$ is $\mathcal{P}$-trivial, $\theta > 55\%$ is $\mathcal{Q}$-trivial, while

$\theta \in (35\%, 55\%)$ is consequential with $\mathcal{P}$ winning in state R and losing in state L. Note that R-group, the majority interest group, is aligned with the society while the L-group is not.

In this setting, for any distribution of bliss points, we can identify voting equilibria only for rules greater than some minimum threshold $\underline{\theta}< 35\%$. For each $\theta >\underline{\theta}$, there are are at least two generic[4] equilibria, one with the responsive set in the L-group and one with the responsive set in the R-group. If everyone believes that the responsive set will be in the L-group, then conditional on being pivotal, there is a larger probability that the state is L, i.e. the policy is at -0.8. Under state L, although the utility difference between $\mathcal{Q}$ and $\mathcal{P}$ is positive in the R- group and negative in the L-group, due to quadratic preferences, the absolute value of the difference is larger for the R-group since it is farther away from the location of the policy. As a result, under uncertainty, all the members of the R-group vote for $\mathcal{Q}$ without paying attention to the signal, while some in group L vote informatively – thus confirming the belief. Similarly, if the responsive set is believed to lie in the R-group, members of the L-group vote uninformatively for $\mathcal{Q}$ while some in R-group have incentive to vote responsively. It is this phenomenon that gives us multiple equilibria based on the different beliefs.

Given a θ, to examine the equilibrium where the responsive set is in the L-group, consider the θ-quantile when types are considered from the left. We illustrate this equilibrium in figure 1b.

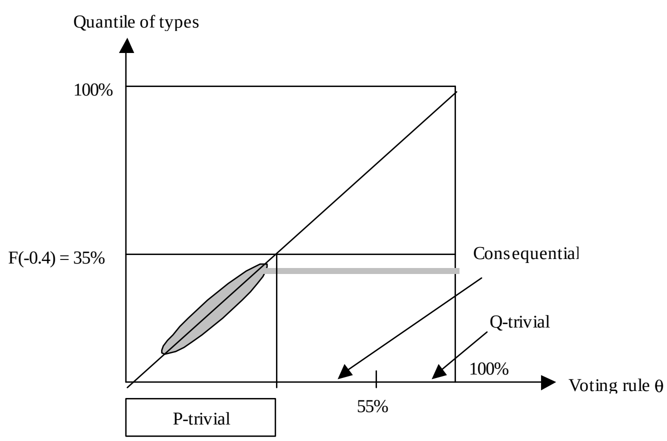

Figure 1b: Limiting equilibrium under non-common values: responsive set in L-group

For $\theta > 35\%$, the respective θ-quantile is greater than -0.4 and hence is outside the L-group. The responsive set for all these rules is a small set just left of -0.4, and hence $\mathcal{Q}$ always wins. For $\theta < 35\%$, the θ-quantile is in the

[4] These equilibria are generic in the sense that they hold true for any distribution of bliss points. There can be other equilibria for specific distributions.

L-group. The responsive set for these rules (above $\underline{\theta}$) always contain the θ-quantile, and hence is influential. As a result, in this equilibrium, we have various ways in which information aggregation fails – for the consequential rules due to the responsive set not being influential when it should be, and for the $\mathcal{P}$-trivial rules due to the responsive set being influential when it should not be. This happens because the L-group is the smaller interest group.

Given a voting rule, to examine the other equilibrium where the responsive set is in the R-group, we need to consider the θ-quantile when types are considered from the right. We illustrate this equilibrium in figure 1c.

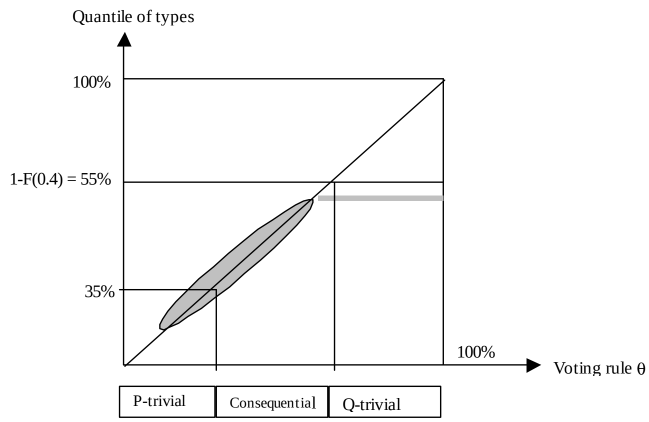

Figure 1c: Limiting equilibrium under non-common values: responsive set in R-group

As illustrated by the figure, $\mathcal{Q}$ is the outcome under both states for $\theta >$ 55%. For $\theta <$ 55%, the θ-quantile is inside the group and is contained by the responsive set. The responsive set is thus influential, and because it belongs to the larger interest group, is aligned with the social objective too. So, the outcome for these rules (above $\underline{\theta}$) is $\mathcal{P}$ in state R and $\mathcal{Q}$ in state L. In these equilibria, although information is aggregated for consequential rules and $\mathcal{Q}$-trivial rules, we have a failure for $\mathcal{P}$-trivial rules because the responsive set is influential when it should not be.

Two things are to be noted here. In all equilibria in both settings, information is fully aggregated for all $\mathcal{Q}$-trivial rules, including the unanimity rule in particular, about which there is some debate in the literature. This can be read as a *status quo bias* in this setting. Also, we have a breakdown of information aggregation even for the majority rule – if we have a non-common values setting and the rule is not trivial.

One important thing to note here that is although the failure of aggregation happens only when there is noise in the signals, it does not depend on the level of noise. Thus, for a large set of rules in the non-common values situation, with a slight noise in the signals, we can get outcomes different from those under full information with a very high probability. This is explained by the fact that the two different equilibria are a result of a peculiar kind of voting strategies based on learning from the event of being pivotal that does not happen under full information. When states are known, no one is pivotal, so we do not have the pivotal voting outcomes under full information.

The above, somewhat technical issue apart, our major inference from the above analysis is the difference between information aggregation properties between the common values and the non-common values situation. The common values situation is essentially a discrete version of F-P and hence it is no wonder that we find full information aggregation in the unique equilibrium. But more interestingly, in the non-common values situation, while for the consequential rules there may be one equilibrium that has the aggregation property, there is always one where we get the status quo in both states with a very high probability. And for all $\mathcal{P}$-trivial rules above some minimum, neither equilibrium aggregates information. This singles out the common values condition as the necessary and sufficient condition for information aggregation for any voting equilibrium with any voting rule and any distribution of single peaked preferences.

Lastly, how important is this non-common values condition empirically? Is it really common enough for us to bother about the informational efficiency of voting as a mechanism? At least in the incumbent-challenger example, it appears a little far-fetched. Although we may not know the exact policy preferences of a challenger, we at least know whether he is to the right or to the left of the incumbent. Thus, we seem to be getting back to the common values world in the model with one-dimensional policy space. But, elections are most often fought over many issues. Real policy spaces are often multidimensional. We show in an extension that the above analysis holds true even in the case of a policy space with finitely many dimensions. Moreover, the common values condition is more difficult to obtain in a multidimensional context. Hence our claim is that there is a very real problem with voting as a mechanism of information aggregation.

3 The Set-up

Suppose there is an electorate composed of a finite number $(n+1)$ of people who are voting for or against a policy $\mathcal{P}$. If the policy gets more than θ proportion of the votes[5], then $\mathcal{P}$ wins; otherwise the status quo $\mathcal{Q}$ wins. Assume that the policy space is $[-1, 1]$. While $\mathcal{Q}$ is known known to be located at 0, there is uncertainty about the location of the alternative $\mathcal{P}$ on the policy space. $\mathcal{P}$ is located at $L \in [-1, 1]$ or $R \in [-1, 1]$ with equal probability. This event that $\mathcal{P}$ is located at S, where $S \in \{L, R\}$ is referred to as state S. To give a natural meaning to the names of the state, we assume that $L < R$, which is without loss of generality[6]. We also assume that the policy never coincides with the status

[5] To simplify the analysis, we assume the tie breaking rule that if the policy receives exactly θ proportion of votes, the status quo wins.

[6] There is some loss of generality - by this assumption, we exclude the case that in the policy location is state invariant, i.e. $L = R$. Thus assuming $L < R$ is tantamount to assuming that

quo, i.e both L and R are non-zero numbers[7]. Everyone receives a private signal $\sigma \in \{l, r\}$ about the state. Signals are independent and identically distributed conditional on the state, with the distribution being:

$$\Pr(l|L) = \Pr(r|R) = q \in (\frac{1}{2}, 1)$$

Voters have single peaked preference defined on the policy space. Every individual has a privately known bliss point x that is drawn independently from a commonly known distribution $F(\cdot)$ that has the entire policy space $[-1, 1]$ as its support and admits a density $f(\cdot)$. The utility function from the alternative A, when the location of the alternative is at a, is given by:

$$U(x, A) = -(x-a)^2, \; A \in \{\mathcal{Q}, \mathcal{P}\}$$

Given a draw of x and S, we define $v(x, S)$ as the difference in utility between the policy alternative and the status quo:

$$v\,(x, S) = U(x, \mathcal{P}) - U(x, \mathcal{Q}) = x^2 - (x-S)^2, \; S \in \{L, R\} \tag{1}$$

From here onwards, we shall use $v(x, S)$, the utilitity difference between the two alternatives as given by (1) for all further analysis. If the state S is known, a voter votes for $\mathcal{P}$ if and only if $v(x, S)$ is non-negative. If S is not known, a voter calculates the expected value of this function using the relevant probability distribution over the states and votes $\mathcal{P}$ if the expectation is non-negative. The equilibrium concept we employ is symmetric Bayesian Nash equilibrium in undominated strategies.

Given an individual's private information (bliss point x and signal σ), the strategy specifies a probability of voting for $\mathcal{P}$:

$$\pi(x, \sigma) : [-1, 1] \times \{r, l\} \to [0, 1]$$

Thus, under state S,the expected share of votes is:

$$t(S, \pi) = \int_{-1}^{1} \Pr(l|S)\pi(x, l)dF(x) + \int_{-1}^{1} \Pr(r|S)\pi(x, r)dF(x), \; S = L, R \tag{2}$$

Exapanding (2) we can write

$$\begin{aligned} t(L, \pi) &= q\int_{-1}^{1} \pi(x, l)dF(x) + (1-q)\int_{-1}^{1} \pi(x, r)dF(x) \\ t(R, \pi) &= (1-q)\int_{-1}^{1} \pi(x, l)dF(x) + q\int_{-1}^{1} \pi(x, r)dF(x) \end{aligned}$$

Under a rule θ a voter is pivotal if $n\theta$ votes are cast for the policy $\mathcal{P}$ from the remaining n voters. So, the probability of being pivotal under state S is given by[8]:

$$\Pr(piv|\pi, S) = \binom{n}{n\theta} \left(t(S, \pi)\right)^{n\theta} \left(1 - t(S, \pi)\right)^{n-n\theta}, \; S = L, R \tag{3}$$

there is always some uncertainty about the policy location.

[7] In other words, we assume that if the state were known, then there will always be a positive interval of types that would strictly prefer to vote for the policy in either state.

[8] For technical convenience, we assume that $n\theta$ is an integer.

Note that (3) actually denotes a pair of equations, one for each state. Call these the *pivot equations*. Note that if $t(S,\pi) \in (0,1)$ then $\Pr(piv|\pi, S) > 0$. We later show that in any equilibrium of our model, we must have $t(S,\pi) \in (0,1)$. Had the probabilistic belief on the state conditional on being pivotal been well defined, it would be given by:

$$\beta\left(S|piv,\pi\right) = \frac{\Pr(piv|\pi, S)}{\Pr(piv|\pi, L) + \Pr(piv|\pi, R)}, \;\; S = L, R \tag{4}$$

Since $\Pr(piv|\pi, S) > 0$ for both states, we have $\beta\left(S|piv,\pi\right) \in (0,1)$. The strategies played in equilibrium determine the pivot probabilities in each state through (2) and (3). In return,the probability of state L conditional on being pivotal is determined by Bayes rule by (4). We call $\beta(L|piv,\pi)$ the *induced prior* and denote it as β_L. The posterior beliefs given a signal are:

$$\left.\begin{array}{l} \beta(L|piv,\pi,l) = \frac{q\beta_L}{q\beta_L+(1-q)(1-\beta_L)} \\ \beta(L|piv,\pi,r) = \frac{(1-q)\beta_L}{(1-q)\beta_L+q(1-\beta_L)} \\ \beta(R|piv,\pi,l) = \frac{q(1-\beta_L)}{q\beta_L+(1-q)(1-\beta_L)} \\ \beta(R|piv,\pi,r) = \frac{q(1-\beta_L)}{(1-q)\beta_L+q(1-\beta_L)} \end{array}\right\} \tag{5}$$

We refer to $\beta(L|piv,\pi,l)$ as p_l and to $\beta(L|piv,\pi,r)$ as p_r. Note that while both p_l and p_r are increasing functions of the induced prior, p_l is concave and p_r is convex throughout. This, coupled with their equality at the extreme values of β_L, i.e. $p_l = p_r = \beta_L$ at $\beta_L = 0$ and $\beta_L = 1$, implies that $p_l > p_r$ for all other values of β_L. Figure 2 graphs the posteriors as functions of the induced prior β_L.

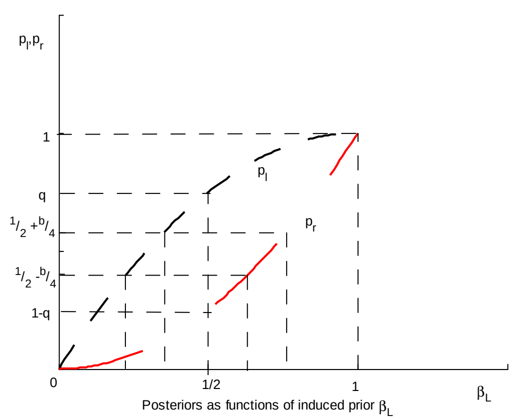

Figure 2

Before identifying equilibrium strategies, we provide some important definitions.

Definition 1 *A voting strategy is a* cut-off *strategy if given a signal and an induced belief, the type space* $[-1, 1]$ *can be partitioned into exactly two intervals (one possibly empty) such that every type votes for* $\mathcal{Q}$ *in one interval and for* $\mathcal{P}$ *in the other. The cut-offs are said to be* ordered[9] *if, as the location of the cutpoint changes due to changes in the induced belief, only the types to the left (or right) of the cutpoint vote for* $\mathcal{P}$.

In other words, a voting strategy is a cut-off strategy if given $\sigma \in \{l, r\}$ and $\beta_L \in (0, 1)$, there is some $x_\sigma(\beta_L)$ such that for any $x_1 < x_\sigma(\beta_L)$ and $x_2 > x_\sigma(\beta_L)$, the absolute value of $\pi(x_2, \sigma) - \pi(x_1, \sigma)$ is 1. If $x_\sigma(\beta_L) \in \{0, 1\}$, a cut off strategy requires that $\pi(x, \sigma)$ be 0 or 1 for all x. A cut-off strategy is said to be ordered if, given any two types x and x' with $x \neq x'$, the sign of $\pi(x, \sigma) - \pi(x', \sigma)$ is either always nonnegative or always nonpositive for any value of β_L.

The nature of the cut-off strategies vary on the basis of the possible locations of the uncertain alternative. Based on the location of the policy $\mathcal{P}$, we distinguish between two situations with a condition that is very important for this paper.

Definition 2 *Define* $\mathbb{P}(S)$ *to be the set of types that (weakly) prefer the alternative policy to the status quo if they know that the state is* S*:*

$$\mathbb{P}(S) = \{x : v(x, S) \geq 0\}$$

$\mathbb{P}(S)$ *exhibits* common values *if* $\mathbb{P}(L) \subset \mathbb{P}(R)$ *or* $\mathbb{P}(L) \supset \mathbb{P}(R)$, *and* non-common values *otherwise.*

Denote $\mathbb{P}(L) \cap \mathbb{P}(R)$ as $\mathbb{P}_{LR}$, and note that it can be empty. This set of types always votes for the policy irrespective of the state. They are *committed types,* or *type-*$\mathcal{P}$ *partisans* according to the nomenclature in Feddersen-Pesendorfer (1996)[10]. Now consider the sets $\mathbb{P}(L) \backslash \mathbb{P}_{LR}$ and $\mathbb{P}(R) \backslash \mathbb{P}_{LR}$. These are the *independent types*, as they change their vote based on the state. The above definition says that when S changes, if the independent types switch their votes in only one direction ($\mathcal{P}$ to $\mathcal{Q}$ or $\mathcal{Q}$ to $\mathcal{P}$), then it is a common values situation. If some (a positive mesaure of) independents switch from $\mathcal{P}$ to $\mathcal{Q}$ and some from $\mathcal{Q}$ to $\mathcal{P}$ for a change in S then it is a non-common values situation.

The intuition behind Definition (2) can be clarified by looking at the common values condition in F-P. Their assumption is that $v(x, S)$ is strictly increasing in S for every value of x. If $x \in \mathbb{P}(L) \backslash \mathbb{P}_{LR}$, then $v(x, L) > 0$, but $v(x, R) < 0$. And if $x \in \mathbb{P}(R) \backslash \mathbb{P}_{LR}$, the $v(x, L) < 0$, but $v(x, R) > 0$. If $\mathbb{P}(L) \subset \mathbb{P}(R)$ or

[9] Note that the definition of ordering of cut-offs is different here from the one in F-P (page 1035) where ordering is defined based on whether cut-offs are monotonic in signals. Here, for any location of $\mathcal{P}$, we always have cut-offs monotonic in the signals. However, it is possible that for some values of the cutpoint, those to the left of the cut-off vote for $\mathcal{P}$ while for other values of the cut-point, those to the right of the cut-off vote for $\mathcal{P}$. We distinguish those situations as unordered.

[10] Given that the location of Q is known and $b_i \neq 0$, there is always an interval of types around 0 that are *Q-partisans*.

$\mathbb{P}(L) \supset \mathbb{P}(R)$, exactly one of $\mathbb{P}(L)\backslash\mathbb{P}_{LR}$ or $\mathbb{P}(R)\backslash\mathbb{P}_{LR}$ is empty. So, for all independent types, $v(x, L) - v(x, R)$ takes the same sign. In other words, for all independent types, the F-P condition holds, at least in the weak sense. That justifies the name common value. On the other hand, if neither of $\mathbb{P}(L)\backslash\mathbb{P}_{LR}$ and $\mathbb{P}(R)\backslash\mathbb{P}_{LR}$ are empty, then we are considering independent types that do not satisfy the F-P condition. Although the ranking for each of the two sets of types changes with the state, their preferences are opposed to each other in both states.

Remark 1 *If L and R have the same sign, we have a common values situation, and if they have different signs, we have a non-common values situation.*

Proof. In Appendix. ■

The intuition behind this remark is illustrated by the example in Section 1.

Definition 3 *If the vote of an individual with type x changes with the signal, i.e. if $\pi(x, l) \neq \pi(x, r)$, then type x is said to be* responsive. *Suppose that given a consequential rule θ, under full information, $\mathcal{P}$ wins under state L. Then the responsive type x is said to be* aligned *with the society if $\pi(x, l) = 1$ and $\pi(x, r) = 0$. Similarly if, under a consequential rule θ, the full information outcome is $\mathcal{P}$ under state R, the responsive type x is said to be* aligned *with the society if $\pi(x, l) = 0$ and $\pi(x, r) = 1$*

The responsiveness and alignment conditions have been discussed in the introduction in detail.

4 Common values

We start by looking at the benchmark case with common values. As we shall see, this turns out to be the discrete version of the F-P model. A common value game is defined by its parameters $(F(\cdot), q, L, R, n, \theta)$. Since we shall look for the sequence of limiting equilibria of this game for all values of θ as $n \to \infty$, we denote a common value setting as a collection $(F(\cdot), q, L, R)$.

4.1 Strategies and equilibria

Lemma 1 *In the common values case, all equilibrium strategies are ordered cut-off strategies.*

Proof. A voter with signal $\sigma, (\sigma \in \{l, r\})$ evaluates the state using the distribution $\beta(S|piv, \pi, \sigma)$ and votes for the policy if and only if the expected value of the function $v(.,.)$ is non-negative. Assume for now that $\beta(S|piv, \pi, \sigma)$ is well-defined. Define $x(p_\sigma)$ as the solution of the equation $E(v(x, S)|piv, \pi, \sigma) = 0$. Solving,

$$x(p_\sigma) = \frac{1}{2}\left(\frac{(L)^2 p_\sigma + (R)^2 (1 - p_\sigma)}{Lp_\sigma + R(1 - p_\sigma)}\right) \in \left[\frac{L}{2}, \frac{R}{2}\right]$$

Thus, $x(p_\sigma)$ always exists uniquely. Also, since $\frac{\partial Ev(x,S)}{\partial x} = 2\left(Lp_\sigma + R(1-p_\sigma)\right)$, $R > L > 0 \Rightarrow \frac{\partial Ev(x,S)}{\partial x} > 0 \Rightarrow Ev(x,S) > 0$ iff $x > x(p_\sigma)$.Similarly, if $L < R < 0$, $Ev(x,S) > 0$ iff $x < x(p_\sigma)$. This establishes the cut off nature of strategies. Given L and R, the strategies do not depend on the precise location of $x(p_\sigma)$.If $L < R < 0$, types to the left of the cut-off $x(p_\sigma)$ vote for $\mathcal{P}$, while if $0 < L < R$, types to the right of the cut-off $x(p_\sigma)$ vote for $\mathcal{P}$. This proves the ordered nature of the cut-off strategies, and establishes the lemma, under the assumption that.$\beta(S|piv,\pi,\sigma)$ is well-defined. ■

Denote $x(p_l)$ as x_l and $x(p_r)$ as x_r. The the cut-off strategies are given by (6) and (7):

$$\left\{\begin{array}{c} \pi(x,l) = \left\{\begin{array}{c} 1\ if\ x \geq x_l \\ 0\ otherwise \end{array}\right., \\ \pi(x,r) = \left\{\begin{array}{c} 1\ if\ x \geq x_r \\ 0\ otherwise \end{array}\right. \end{array}\right\} \text{when } R \quad > \quad L > 0 \tag{6}$$

$$\left\{\begin{array}{c} \pi(x,l) = \left\{\begin{array}{c} 1\ if\ x \leq x_l \\ 0\ otherwise \end{array}\right., \\ \pi(x,r) = \left\{\begin{array}{c} 1\ if\ x \leq x_r \\ 0\ otherwise \end{array}\right. \end{array}\right\} \text{when } L \quad < \quad R < 0 \tag{7}$$

Remark 2 *Note that for* $\beta_L = 1,\ x_r = x_l = \frac{L}{2}$, *and likewise for* $\beta_L = 0,\ x_r = x_l = \frac{R}{2}$.*Since* $\frac{dx(p)}{dp} = -\frac{1}{2}\left(\frac{(R-L)LR}{(Lp+R(1-p))^2}\right) < 0$, *and since* $p_l > p_r$ *for* $\beta_L \in (0,1)$, $x_r > x_l$ *for these values of* β_L.

Thus, for any induced prior, the strategies in the benchmark case are characterised by cutpoints x_l and x_r, with $x_l \leq x_r$. If $R > L > 0$, types $x < x_l$ always vote for $\mathcal{Q}$, types $x \in [x_l, x_r]$ vote for $\mathcal{P}$ if they get signal l and $\mathcal{Q}$ if they get signal r, and the types $x > x_r$ vote for $\mathcal{P}$ regardless of the signal. If $L < R < 0$, types left of x_l always vote for $\mathcal{P}$ and those right of x_r vote for $\mathcal{Q}$ while types in $[x_l, x_r]$ vote informatively. In either case, $[x_l, x_r]$ is the responsive set, while the other types vote their bias. Henceforward, we shall deal only with the case $L < R < 0$, noting that the other case is completely symmetric.

Note that the ordered cutoff nature of the strategies ensures that there will always be one and only one responsive interval. Also, irrespective of the location of the cutoffs, the responsive set is always *aligned* with the society. This means that whenever the responsive set is influential, information will be aggregated. Thus, for consequential rules, all we need to show for information aggregation is that in any limiting equilibrium, the responsive set is indeed influential. For this, we need monotonicity of the vote shares under both states, which is again ensured by the ordered nature of the cut off strategies. We define the probability of an individual voting for the alternative $\mathcal{P}$ given σ as z_σ, *i.e.* $z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma)dF$. For $L < R < 0$, we have from (7),

$$z_\sigma = F(x_\sigma),\ \sigma = \{l, r\}$$

Therefore, using (2) we write[11]:

$$\left.\begin{array}{l} t(L,\pi) = qz_l + (1-q)z_r = qF(x_l) + (1-q)F(x_r) \\ t(R,\pi) = (1-q)z_l + qz_r = (1-q)F(x_l) + qF(x_r) \end{array}\right\} \tag{8}$$

Note that since the cut-offs x_l and x_r are functions of the induced prior, the vote shares $t(L,\pi)$ and $t(R,\pi)$ are also functions of β_L. The following lemma examines how the vote share in each state changes as a function of the induced prior.

Lemma 2 *The expected share of votes $t(S,\pi)$ in state S decreases strictly with the induced prior β_L from $F(\frac{R}{2})$ at $\beta_L = 0$ to $F(\frac{L}{2})$ at $\beta_L = 1$. Also, for all interior values of the induced prior, i.e. for all $\beta_L \in (0,1)$, $t(L,\pi) < t(R,\pi)$[12].*

Proof. By Remark 2, at $\beta_L = 0$, $z_l = z_r = F(\frac{R}{2}) \Rightarrow t(S,\pi) = F(\frac{R}{2})$ for $S \in \{L,R\}$. Similarly, at $\beta_L = 1$, $t(S,\pi) = F(\frac{L}{2})$ for $S \in \{L,R\}$. Also, since p_σ is a strictly increasing function of β_L, x_σ is decreasing in β_L by Remark 2. The full support assumption guarantees that $F(\cdot)$ is strictly increasing. Hence, $t(S,\pi)$ is strictly decreasing in β_L. For the second part of the lemma, note that

$$t(L,\pi) - t(R,\pi) = (2q-1)\left(F(x_l) - F(x_r)\right)$$

By remark 2 again, for $\beta \in (0,1)$, $F(x_l) - F(x_r) < 0$, and since $q > \frac{1}{2}$, we have $t(L,\pi) < t(R,\pi)$. $\blacksquare$

The above lemma states that as the induced prior probability of the state being L (conditional of being pivotal) increases, the expected share of votes for the alternative policy decreases under either state because the state L is deemed to be more "extreme". Informative voting by the responsive set ensures that the policy receives more votes in the "moderate" state (R) unless the prior is degenerate. Note also that at any induced prior, the difference in expected vote shares is increasing in the informativeness of the signal. The expected vote shares in the two states are plotted against the induced prior in figure 3.

[11] For $0 < L < R$, we have $z_\sigma = G(x_\sigma)$, where $G(y) \equiv 1 - F(y)$, $y \in [-1,1]$

[12] If $0 < L < R$, then both $t(L,\pi)$ and $t(R,\pi)$ increase strictly with the induced prior β_L from $F(\frac{R}{2})$ at $\beta_L = 0$ to $F(\frac{L}{2})$ at $\beta_L = 1$. Also, for all $\beta_L \in (0,1)$, $t(L,\pi) > t(R,\pi)$.

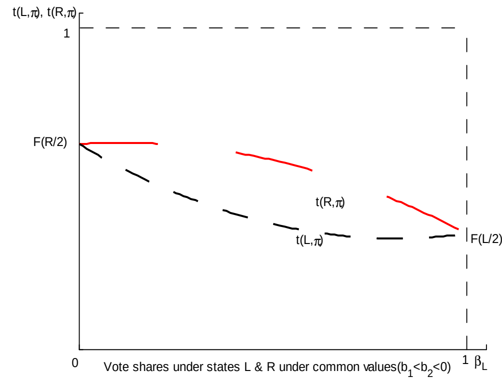

Figure 3

Lemma 2 also ensures that since $t(S,\pi)$ lies strictly between 0 and 1, and $\beta(S|piv,\pi,\sigma)$ is always well-defined. Intuitively, since the types left of $\frac{L}{2}$ are $\mathcal{P}$-partisans and those to the right of $\frac{R}{2}$ are $\mathcal{Q}$-partisans, there is always a positive probability for any type to be pivotal. This finally proves our Lemma 1.

The following proposition guarantees the existence of an equilibrium of the common values voting game $(F(\cdot),q,L,R,n,\theta)$.

Proposition 1 *In the common values case, there exists a voting equilibrium π^* for every population size n and every voting rule $\theta \in (0,1)$ characterized by ordered cut-off strategies x_σ given by the solution of the equation $E(v(x_\sigma,s)|piv,\pi^*,\sigma) = 0$ for $\sigma = (l,r)$.*

Proof. For the proof of this proposition, we first note that the strategy for each voter can be denoted by two numbers x_l and x_r, both lying between $\frac{L}{2}$ and $\frac{R}{2}$. Thus the strategy space is a compact, convex and non-empty set $\left[\frac{L}{2},\frac{R}{2}\right] \times \left[\frac{L}{2},\frac{R}{2}\right]$. The rest of the proof follows from the proof of Proposition 1 in F-P. ■

To find the equilibrium of the model, what we essentially do is find a fixed point on the belief space $\beta_L \in [0,1]$. In other words, suppose everyone else holds some belief β_L, which determines two distributions $p_\sigma(\beta_L)$ according to (5) and correspondingly, the cut-off strategies $x_\sigma(\beta_L)$ according to (7). From the cutoff stratefies, the expected shares of votes for the policies $t(S,\pi)$ in the two states $S = L,R$ is determined by (8). Given these shares, the number of players n and the voting rule θ, a player forms $\Pr(piv|\pi,S)$: probabilities of being pivotal in each state according to the pivot equations (3). These probabilities define belief $\widetilde{\beta_L}$ by (4), which in turn yields the cutoffs $x_\sigma\left(\widetilde{\beta_L}\right)$ that are the best response to the strategy $x_\sigma(\beta_L)$. In equilibrium, the cutoffs $x_\sigma(\beta_L)$ and $x_\sigma\left(\widetilde{\beta_L}\right)$ must be the same, i.e. the pivot equations using shares with belief β_L should deliver pivotal probabilities that lead to the same belief β_L. Note that

$$\frac{\Pr(piv|\pi, L)}{\Pr(piv|\pi, R)} = \frac{\beta\left(L|piv, \pi\right)}{\beta\left(R|piv, \pi\right)} = \frac{\beta_L}{1-\beta_L}$$

Thus, using the above and the pivot equations, the *equilibrium condition* can be simply stated as:

$$\frac{\beta_L}{1-\beta_L} = \frac{\Pr(piv|\pi, L)}{\Pr(piv|\pi, R)} = \left[\frac{\left(t(L,\pi^n)\right)^{\theta}\left(1-t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^{\theta}\left(1-t(R,\pi^n)\right)^{1-\theta}}\right]^n \tag{9}$$

4.2 Limiting Equilibria under Common Values

In this section, we consider the properties of the voting equilibria as the electorate grows in size arbitrarily, keeping all other parameters of the model constant.Therefore, we superscript everything by the number of voters n. At times we will suppress the superscript n when there is no ambiguity Suppose, given L, R and θ for some n, the equilibrium is π^n, and the cutoffs are x^n_σ. As long as common values assumption is satisfied, existence of equilibrium for any n implies the existence of a convergent subsequence with an accumulation point as $n \to \infty$. If a limit of this sequence exists, we call it π^0. By continuity arguments , as $x^n_\sigma \to x^0_\sigma$, $t(S,\pi^n)$, β^n_L, p^n_l,and p^n_r all converge to finite limits $t(S,\pi^0)$, β^0_L, p^0_l, and p^0_r respectively along the sequence.

Rewriting the equilibrium condition:

$$\frac{\beta^n_L}{1-\beta^n_L} = \left[\frac{\left(t(L,\pi^n)\right)^{\theta}\left(1-t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^{\theta}\left(1-t(R,\pi^n)\right)^{1-\theta}}\right]^n \quad for\ all\ n \tag{10}$$

By Proposition 1, a solution to (10) exists for every n. From continuity, if a limit exists, we can also say that the above relation has to hold in the limit; call this the *limiting equilibrium condition.*

$$\frac{\beta^0_L}{1-\beta^0_L} = \lim_{n\to\infty}\left[\frac{\left(t(L,\pi^0)\right)^{\theta}\left(1-t(L,\pi^0)\right)^{1-\theta}}{\left(t(R,\pi^0)\right)^{\theta}\left(1-t(R,\pi^0)\right)^{1-\theta}}\right]^n \tag{11}$$

To avoid writing complicated expressions, we define:

$$\alpha_n = \frac{\left(t(L,\pi^n)\right)^{\theta}\left(1-t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^{\theta}\left(1-t(R,\pi^n)\right)^{1-\theta}} \text{ and } \alpha_0 = \frac{\left(t(L,\pi^0)\right)^{\theta}\left(1-t(L,\pi^0)\right)^{1-\theta}}{\left(t(R,\pi^0)\right)^{\theta}\left(1-t(R,\pi^0)\right)^{1-\theta}}$$

Note that the vote shares $t(S,\pi^n)$ are functions of β^n_L. Next, we look at the properties of the limit, assuming existence for the time being. We later show that in the commmon values game, for any voting rule, there is only one accumulation point of π^n which must be the limit.

Lemma 3 *If* $\beta^0_L \in (0,1)$, $\alpha_0 = \lim_{n\to\infty} \alpha_n = 1$

Proof. See Appendix. Note that this lemma does not use the common values condition, so it is true of non-common values too. ■

Lemma 4 *If $\beta_L^0 = 1$, then $x_\sigma^n \to \frac{R}{2}$ from the left for $\sigma = l, r$. Similarly, if $\beta_L^0 = 0$, then $x_\sigma^n \to \frac{L}{2}$ from the right for $\sigma = l, r$*

Proof. Follows from continuity of x_σ^n in p_σ^n and of p_σ^n is β_L^n, along with Remark 2. ■

Note, as an aside to Lemma 4, that although under both signals the cutoffs converge to $\frac{R}{2}$ or $\frac{L}{2}$ if the induced prior converges to 1 or 0, by remark 2, we always have $x_l^n < x_r^n$. Thus, in the responsive set, the voters always vote for $\mathcal{Q}$ if they get moderate signal r and $\mathcal{P}$ if they get the extreme signal l. In the limit, the responsive interval is vanishingly small as the induced prior distribution converges to state R , grows for intermediate values of the prior, and again shrinks to a vanishing size as the distribution converges to a degenerate distribution at state L. Thus, given q, a level of precision of the signals, the difference between expected shares in the two states is low for extreme values of the induced prior and high for intermediate values.

Lemma 3 and Lemma 4 together imply that for any limiting induced prior, given a voting rule under any equilibrium the vote shares in each state must be related in a certain way, which is stated in Proposition 2 below. According to Lemma 3, if α_n is bounded away from 1, then β_L^0 must be either 0 or1. And under conditions of Lemma 4, if β_L^n is indeed 0 (or 1), then the voters are almost sure of the state in which they are pivotal and vote as if under (almost) full information. Every type except those in a vanishing set votes uninformatively, and the vote shares under either state are the same in the limit. Thus, in equilibrium, we have $\alpha_0 = 1$ for all values of the induced prior.

Proposition 2 *In all limiting equilibria, we must have $\alpha_0 = 1$, i.e.*

$$\left(t(L,\pi^0)\right)^\theta \left(1 - t(L,\pi^0)\right)^{1-\theta} = \left(t(R,\pi^0)\right)^\theta \left(1 - t(R,\pi^0)\right)^{1-\theta}, i.e.\ \alpha_0 = 1$$

Proof. For any equilibrium with $\beta_L^0 \in (0,1)$, the proposition follows straightforwardly from Lemma 3. If $\beta_L^0 = 1$, the first part of Lemma 4 implies that

$$\begin{aligned} t(L,\pi^n) &= qF(x_l^n) + (1-q)F(x_r^n) \to qF(\frac{R}{2}) + (1-q)\,F(\frac{R}{2}) \to F(\frac{R}{2}) \\ t(R,\pi^n) &= qF(x_r^n) + (1-q)F(x_l^n) \to qF(\frac{R}{2}) + (1-q)\,F(\frac{R}{2}) \to F(\frac{R}{2}) \end{aligned}$$

$$\therefore \alpha_0 = \lim_{n\to\infty} \frac{\left(t(L,\pi^n)\right)^\theta \left(1 - t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^\theta \left(1 - t(R,\pi^n)\right)^{1-\theta}} = \frac{\left(F(\frac{R}{2})\right)^\theta \left(1 - F(\frac{R}{2})\right)^{1-\theta}}{\left(F(\frac{R}{2})\right)^\theta \left(1 - F(\frac{R}{2})\right)^{1-\theta}} = 1 \left(\because F(\frac{R}{2}) \in (0,1)\right)$$

If $\beta_L^0 = 0$, the proof follows in exactly the same way since the second part of Lemma 4 implies that then $t(S,\pi^n) \to F(\frac{L}{2})$ for $S \in \{L, R\}$. ■

Note that the above proposition is based on a necessary condition that must be true for a β_L^0 to which induced belief converges in the limiting equilibrium.

It helps exclude certain voting rules that cannot support a given value of β_L in the limit. To do that formally, we define $\Theta(\beta_L)$ as the set of voting rules that can support β_L as an induced belief in the *limiting equilibrium condition* (11) for *some* distribution of preferences in the cut-off equilibrium. To emphasize that $t(S, \pi)$ is a function of β_L, we write $t(S, \pi)$ as $t_S(\beta_L)$ for $S \in \{L, R\}$.

Lemma 5 *Under common values, (i) If $\beta_L \in (0,1)$, then $\Theta(\beta_L)$ is a strictly increasing function $\theta^*(\beta_L)$, with $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$. (ii) Otherwise, $\Theta(1) = \{\theta : \theta < F(\frac{L}{2})\}$, and $\Theta(0) = \{\theta : \theta > F(\frac{R}{2})\}$*

Proof. In Appendix. ■

The first part of the lemma is almost a corollary of Proposition 2. For each interior value β_L of the induced prior, it identifies a unique θ as the only possible voting rule to support β_L in the limiting equilibrium. As long as the expected vote shares in the two states are different, the only voting rule that can satisfy Proposition 2 is one that lies strictly between the two shares. This has the implication that under one state the status quo wins, while in the other, the policy wins. If there are any equilibria with beliefs that place positive probability on both states, then the responsive set of types for these equilibria are always influential. The lemma also notes that such equilibria are possible only for consequential rules. The second part of the lemma says that the extreme beliefs can be supported only by extreme values of the voting rules.

Note that since $\theta^*(\beta_L)$ is strictly increasing, its inverse function $\beta_L^{-1}(\theta)$ exists for $\theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right)$ and is strictly increasing. Thus, according to Lemma 5, for every θ, there is a unique β_L that is supportable as an induced prior in the limit, *for any distribution of types.* Call it $\beta(\theta)$. We can write:

$$\beta(\theta) = \begin{cases} 1 \; if \; \theta < F(\frac{L}{2}) \\ \beta_L^{-1}(\theta) \; \; if \; \theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right) \\ 0 \; if \; \theta > F(\frac{R}{2}) \end{cases} \tag{12}$$

We plot the correspondence $\Theta(\beta_L)$ along with the expected vote shares in each state against the induced prior in figure 4.

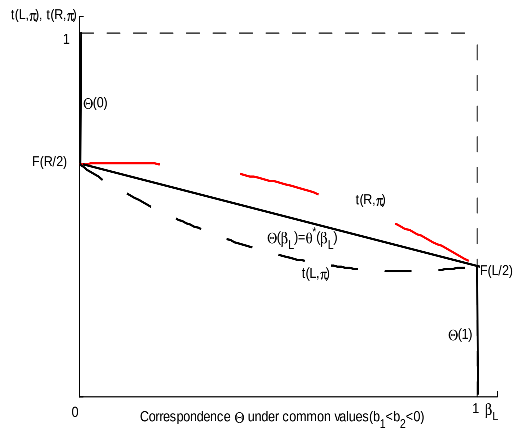

Figure 4

The next theorem gives a characterization of cut-off equilibria in large populations for different voting rules under common values.

Theorem 1 *Assume L, R satisfy the common values condition, $F(\cdot)$ satisfies full support, and $q \in (\frac{1}{2}, 1)$. Fix a voting rule $\theta \in (0,1)$. Then there is a unique limiting equilibrium π^0 with ordered cut-off strategies and with the induced prior converging to β_L if and only if $\theta \in \Theta(\beta_L)$, or alternatively, if and only if $\beta_L = \beta(\theta)$.*

Proof. According Proposition 2, a voting equilibrium π^n with ordered cutoff strategies exists for a given θ for any n. Since β_L lies in a compact set, there is an accumulation point π^a given θ. We show in the appendix that this π^a is the limiting equilibrium π^0 given θ. Lemma 5 states that for any distribution of types, if a limit exists, there is a unique number $\beta(\theta)$ to which the induced prior converges in the limit along the sequence of equilibria under voting rule θ. ■

Note that once the limiting value of the induced prior β_L is established, the limiting posterior distributions p_σ, the limit cut-offs x_σ etc. are all determined from β_L. Thus this theorem describes all relevant information about strategies, vote shares and statewise outcomes in equilibria with a voting rule when the population size becomes large. Note that, by the Law of Large numbers, the actual vote shares are arbitrarily close to the expected vote shares[13]. From here onwards, we do not make a distinction between the expected and actual, and just call it "vote share". We postpone the discussion of the actual outcomes for each voting rule till the next section.

[13] More specifically, given any $\epsilon > 0$ and $\delta > 0$, we can find some number N such that as long as the polupation size is larger than N, the actual vote share is within ϵ of the exoected share with a probability higher than $1 - \delta$.

4.3 Outcomes and Information Aggregation

In the introduction we have informally discussed a classification of voting rules according to the outcomes produced under full information. Here we formalise the discussion, and then examine the information aggregation properties of each class of voting rules.

For purposes of this paper define a social choice rule H as a function that maps a state to an outcome, i.e.

$$H : \{L, R\} \to \{\mathcal{P}, \mathcal{Q}\}$$

When $H(\cdot)$ is a constant function, i.e. when the planner wants the same outcome in both states, we call it a *trivial rule*. There are two trivial rules - one where the planner always wants the status quo to prevail ($H(L) = H(R) = \mathcal{Q}$), and the one that maps both states to the policy ($H(L) = H(R) = \mathcal{P}$). We call the first one $\mathcal{Q}$-*trivial* and the second one $\mathcal{P}$-*trivial rule*. If the function maps different states to different outcomes ($H(L) \neq H(R)$), we call it a *consequential rule*.

A voting rule is said to *correspond* to a particular social choice rule if *under full information* of the state, the voting outcome is the same as the outcome determined by the social choice rule for that state. A voting rule is said to *implement* the corresponding social choice rule $H(\cdot)$ if, for any $\epsilon > 0$, we can find a number N such that when the population size is larger than N, in either state the outcome of the voting game *under incomplete information* of the state is the same as the outcome determined by the social choice rule with a probability larger than $1-\epsilon$. When a voting rule implements the corresponding social choice rule, then voting rule is said to satisfy *full information equivalence*[14]. In other words, the voting game under incomplete information gives the same outcome that would have occurred if there were common knowledge of the state.

With full information, under state L, the policy would get $F(\frac{L}{2})$ share of votes; and similarly under state R, the policy would get $F(\frac{R}{2})$ share of votes. Therefore:

- Any voting rule $\theta < F(\frac{L}{2})$ corresponds to the $\mathcal{P}$-trivial rule, i.e. $\mathcal{P}$ wins under both states.
- Any voting rule $F(\frac{L}{2}) < \theta < F\left(\frac{R}{2}\right)$ corresponds to a consequential rule, i.e. $\mathcal{P}$ wins in state R and $\mathcal{Q}$ in state L[15].
- Any voting rule $\theta > F\left(\frac{R}{2}\right)$ corresponds to the $\mathcal{Q}$-trivial rule, i.e. outcome is status quo under both states.

[14] The concept of full information equivalence was formalised by F-P, and we use the same definition adapted for our setting.

[15] Note that the other consequential rule, i.e. $\{G(L) = \mathcal{P}, G(R) = \mathcal{Q}\}$ cannot be implemented under full information by the plurality rule with the common values case we are considering, i.e. $L < R < 0$.

We classify the respective voting rules by the name of the social choice rule they correspond to.

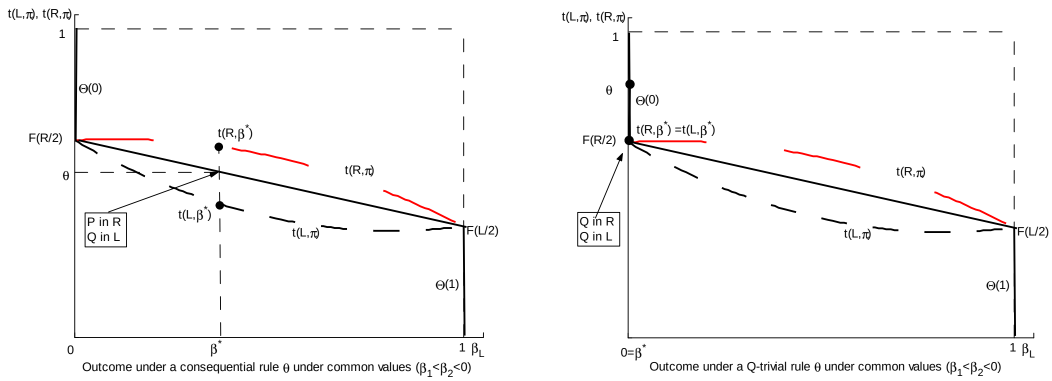

Figure 5(a), 5(b)

Theorem 2 *Under common values, any plurality rule $\theta \in (0,1)$ satisfies full information equivalence for any distribution of types.*

Proof. In appendix. ■

According to the theorem, under common values, any voting rule aggregates information. Since the vote shares in each state is between $F(\frac{L}{2})$ and $F\left(\frac{R}{2}\right)$, any trivial rule aggregates information. Essentially, the responsive types lying between $\frac{L}{2}$ and $\frac{R}{2}$ can never be influential with trivial rules. With $\mathcal{P}$-trivial rules, everyone is virtually sure that conditional on being pivotal, the state is L. In other words, under such a rule, being pivotal at state L (when $\mathcal{P}$ receives least votes) is infinitely more probable than being pivotal at state R. Similarly, with any $\mathcal{Q}$-trivial rule, one has far higher chance of being pivotal in state R (when $\mathcal{P}$ receives most votes) than in state L. We depict the outcome in the limiting equilibrium with a $\mathcal{Q}$-trivial rule in figure 5(a). On the other hand, for any consequential rule, the induced prior places positive probability on both states in the limit, and the responsive set is influential. Since the responsive types are aligned too, we have outcome $\mathcal{P}$ in state R and $\mathcal{Q}$ in state L almost surely, and hence we have information aggregation. The limiting equilibrium outcome with a consequential rule is depicted in figure 5(b).

5 Non-common values

Recall that a non-common values situation occurs if $L < 0 < R$. We now look at the strategies and equilibria in this situation and compare and contrast its properties with that of the benchmark common value model. Specifically, we show how voting can fail to aggregate information in the presence of heterogeneous groups with competing interests.

We shall simplify the model a bit and consider a slightly special case with $L = -b$ and $R = b > 0$. Note that this is not too strong an assumption as we are considering all possible distributions of voter ideal points. However, we need to make an additional assumption on the informativeness of the signals. Assume that :

$$\Pr(l|L) = \Pr(r|R) = q > \frac{1}{2} + \frac{b}{4}$$

Call it Assumption I. The full support assumption is heneceforth referred to as Assumption F. We denote a non-common value setting by the collection $(F(\cdot), q, b)$. To be able to compare and contrast this with the common value setting, we denote similar lemmata, propositions and theorems in this sections with numbers analogous to those assigned in Section 4.

5.1 Strategies and equilibria

A voter with signal $\sigma, (\sigma \in \{l, r\})$ evaluates the state using the distribution $\beta(S|piv, \pi, \sigma)$ and votes for $\mathcal{P}$ if and only if the expected value is non-negative. So, the condition for voting for the policy after having received σ is:

$$Ev(x, \sigma) \geq 0 \Rightarrow 2x(1 - 2p_\sigma) \geq b$$

Hence, the voter votes for $\mathcal{P}$ iff

$$1 \geq |x| \ \geq \frac{b}{2(1 - 2p_\sigma)} \tag{13}$$

Using (13), we can determine the cut-offs:

$$x_\sigma = \begin{cases} \min(1, \frac{b}{2(1-2p_\sigma)}), & 0 \leq p_\sigma < \frac{1}{2} \\ \max(-1, \frac{b}{2(1-2p_\sigma)}), & \frac{1}{2} \leq p_\sigma \leq 1 \end{cases} \tag{14}$$

Now, according to the above definitions of the cutoff, we get:

$$\pi(x, \sigma) = \begin{cases} \left.\begin{array}{c} 1 \ for \ x \leq x_\sigma \\ 0 \ for \ x > x_\sigma \end{array}\right\} if \ \frac{1}{2} \leq p_\sigma \leq 1 \\ \left.\begin{array}{c} 1 \ for \ x \geq x_\sigma \\ 0 \ for \ x < x_\sigma \end{array}\right\} if \ \ 0 \leq p_\sigma < \frac{1}{2} \end{cases} \tag{15}$$

Or alternatively, combining (14) and (15), we define the strategies in terms of p_σ as follows:

$$\pi(x, \sigma) = \left\{ \begin{array}{c} \left.\begin{array}{c} 1 \ for \ x \leq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x > \frac{b}{2(1-2p_\sigma)} \end{array}\right\} \ if \ p_\sigma \geq \frac{1}{2} + \frac{b}{4} \\ 0 \ for \ all \ x \quad if \ p_\sigma \in \left(\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}\right) \\ \left.\begin{array}{c} 1 \ for \ x \geq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x < \frac{b}{2(1-2p_\sigma)} \end{array}\right\} \ if \ p_\sigma \leq \frac{1}{2} - \frac{b}{4} \end{array} \right\}$$

Any equilibria must have strategies of the above form. Note, first, that $p_\sigma \in [0,1] \Rightarrow -1 \leq 1-2p_\sigma \leq 1$ and so $x_\sigma \in [-1,-\frac{b}{2}] \cup [\frac{b}{2},1]$, and second, that for all values of p_σ, $\pi(x,\sigma)=0$ in the range $(-\frac{b}{2},\frac{b}{2})$. Thus a voter with her bliss point in this range always votes for the status quo irrespective of the signal.

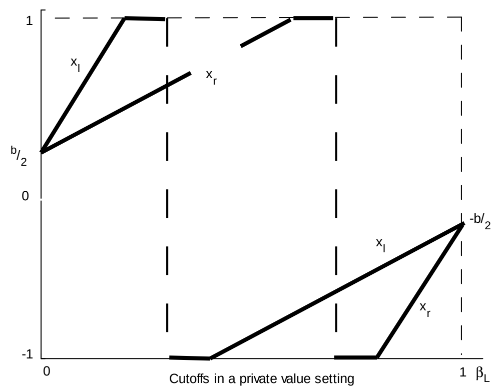

Figure 6

Thus, although all equilibria must have cut-off strategies, the cut-offs are not ordered. The cutoffs as functions of the induced prior are plotted in figure 6. From the figure we see the cutoff functions have a discontinuity that leads to a nonconvexity in the strategy space. When a cut-off is in $[-1,-\frac{b}{2}]$ (the L-group), the types to the left of the cut-off vote for $\mathcal{P}$, and when the cut-off lies in $[\frac{b}{2},1]$ (the R-group), types to the right of the cut-off vote for $\mathcal{P}$. This has several implications. First, the responsive types lying in these two groups would vote in opposite ways based on the same information. Thus. one of the groups is aligned with the society and the other is not. Second, the vote shares in either state are not monotonic functions of the induced belief. Note that the monotonicity in vote shares was crucial for information aggregation with consequential rules in the common values case. Third, with unordered cut-offs, the existence of a well-defined induced prior is no longer trivial, and we need the informativeness assumption I on signals to guarantee that. Lastly, with a loss of the ordering property, uniqueness of the responsive set is no longer assured. This can give rise to a certain kind of equilibria that is not seen in the common values case, as we shall see in Proposition 3.

Recall that the probability of an individual voting for the alternative $\mathcal{P}$ given σ is $z_\sigma, i.e.\ z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma)dF$. In any equilibrium, we have:

$$z_\sigma = \begin{cases} F(x_\sigma)\ if\ x_\sigma \leq -\frac{b}{2} \\ 1-F(x_\sigma)\ if\ x_\sigma \geq \frac{b}{2} \\ 0\ otherwise \end{cases} \tag{16}$$

Although the definition of z_σ is different in the non-common values case, the vote shares in the two states in terms of z_σ are still given by equation (8):

$$\begin{aligned} t(L,\pi) &= qz_l + (1-q)z_r \\ t(R,\pi) &= (1-q)z_l + qz_r \end{aligned}$$

Lemma 6 *In any equilibrium in the non-common values setting, the expected share of votes in any state lies strictly between* 0 *and* 1*, i.e.* $t(S,\pi) \in (0,1)$ *for* $S \in \{L,R\}$.

Proof. See Appendix. ■

Lemma 6 guarantees that the induced prior is indeed always well-defined. The expected share of people voting is less than unity because there is always a set of types close enough to 0 (between $-\frac{b}{2}$ and $\frac{b}{2}$) who vote for the $\mathcal{Q}$. On the other hand, the signal being informative enough (Assumption I) guarantees that the cut-offs are sufficiently distant when the induced priors are not overwhelmingly strong. In other words, if for one signal, no type votes for $\mathcal{P}$, there is an interior cut-off for the other signal. This ensures positive expected share for intermediate induced priors in each state. To see that from figure 2, note that the range of β_L for which p_l lies between $\frac{1}{2}-\frac{b}{4}$ and $\frac{1}{2}+\frac{b}{4}$ lies entirely to the left of $\frac{1}{2}$, while the range of β_L for which p_r lies between $\frac{1}{2}-\frac{b}{4}$ and $\frac{1}{2}+\frac{b}{4}$ lies entirely to the right of $\frac{1}{2}$. This guarantees that, for any induced prior, at least one signal always leads to an interior cut-off.

Next, we show the existence of an equilibrium for the non-common value game $(F(\cdot),q,b,n,\theta)$. This is the analogous result to Proposition 1. Although the strategy set is non-convex and we cannot use a fixed point theorem to prove existence the way we did in the common values setting, we can still show the existence of a solution to equation (10), which is the *equilibrium condition*.

Proposition 1a In the non-common values case, there exists a voting equilibrium π^* for every population size n and every voting rule $\theta \in (0,1)$. The equilibrium is characterized by cut-off strategies x_σ given by the solution of $E(v(x_\sigma,s)|piv,\pi^*,\sigma)=0$ for $\sigma=(l,r)$.

Proof. From Lemma 5, we know that $t(S,\pi)$, is bounded by positive numbers both above and below. This implies that for any n, the right hand side of equation (10) is bounded above and below. However, as β_L goes from 0 to 1, the left hand side continuously increases from 0 to ∞. This guarantees the existence of a solution β_L^n to the equation, and hence existence. ■

We can immediately identify one particular equilibrium for the case with a distribution of types with pdf $f(\cdot)$ that is symmetric about 0.

Proposition 3 *For any* $F(\cdot)$ *for which the pdf* $f(\cdot)$ *is symmetric about* 0*, there is an equilibrium with* $x_l^* = -\frac{b}{2(2q-1)}$ *and* $x_r^* = -x_l^*$*. This is an equilibrium for all values of* $\theta \in (0,1)$ *and* n.

Proof. Consider the situation where everyone else plays $x_\sigma = x_\sigma^*$, and $\sigma \in \{l,r\}$. Note that $x_l^* < -\frac{b}{2}$ and $x_r^* > \frac{b}{2}$. So, $z_l^* = F(x_l^*)$ and $z_r^* = 1-F(x_r^*) =$

$1-F\left(-x_l^*\right)=F\left(x_l^*\right)=z_l^*$, by symmetry of $f(\cdot)$. Therefore, $t(L,\pi)=t(R,\pi)=F\left(x_l^*\right)$ for each n, which implies that $\beta_L=\frac{1}{2}$ for every θ and n. Thus, the signals are fully informative, and we have $p_l=q$ and $p_r=1-q$. These, coupled with the assumption I, imply that the best response to x_σ^* is indeed x_σ^*, which establishes the claim. ■

The proposition says that if the commonly held induced priors are uninformative, then sufficiently extreme types vote for the alternative $\mathcal{P}$ if and only if they get favourable signals, and everyone else votes "uninformatively". There are a few things to be noted about the above equilibrium. First, this is the only "stable" equilibrium sequence in the sense that the strategies do not change with the number of players. Second, in this equilibrium, the expected vote share does not change with the state or the voting rule. If the required plurality for the policy to pass is higher than $F\left(x_\sigma^*\right)$, then the status quo always passes, and if the required share is lower than $F\left(x_\sigma^*\right)$, then the status quo always loses. If $\theta=F\left(x_\sigma^*\right)$, then we get either alternative (policy or status quo) with equal probability. As we shall see later in section 5.3, this constitutes a failure of information aggregation. We note here that we do not even require the full force of symmetry of $f(\cdot)$ here. As long as we have $F\left(-\frac{b}{2(2q-1)}\right)=1-F\left(\frac{b}{2(2q-1)}\right)$, we shall have this equilibrium. Next, we examine how the vote shares will behave when we do not necessarily have this symmetry.

Lemma 7 *There exists some numbers β_1 and β_2 satisfying $0<\beta_1<\beta_L^*<\beta_2<1$ such that the expected vote share in both states $t(S,\pi)$ strictly decreases with the induced prior β_L for $\beta_L<\beta_1$ and strictly increases with β_L for $\beta_L>\beta_2$. Also, for $\beta_L<\beta_L^*$, $t(R,\pi)>t(L,\pi)$, for $\beta_L>\beta_L^*$, $t(R,\pi)>t(L,\pi)$ and for $\beta_L=\beta_L^*$, $t(R,\pi)=t(L,\pi)$.*

Proof. See Appendix. ■

This Lemma says that if the commonly held induced prior probability that one is pivotal at state L falls below a critical value β_L^*, .then the expected vote share in favour of the policy in state L is higher than that in state R. If, on the contrary, the belief is higher than β_L^*, then the alternative $\mathcal{P}$ is expected to get a higher vote share in state R. However, given a state, the expected share of the votes in favour of the policy alternative increases as one gets more and more extreme beliefs, i.e. as one is surer and surer of the state in which one is pivotal. As the voters get more unsure about the state, only the very extreme types vote for the policy. Note that at β_L^*, we have $F(x_l)+F(x_r)=1$, and under a symmetric distribution of types, $\beta_L^*=\frac{1}{2}$, and we have an equilibrium at $\beta_L=\frac{1}{2}$ according to Proposition 3. The expected share of votes under the two states in the non-common value situation (according to Lemma 7) as functions of the induced prior are shown in figure 7. To illustrate how the shares are constructed according to (8), we also show the functions z_l and z_r (i.e. the probability of voting for $\mathcal{P}$ on getting the signal l and r respectively) in the figure.

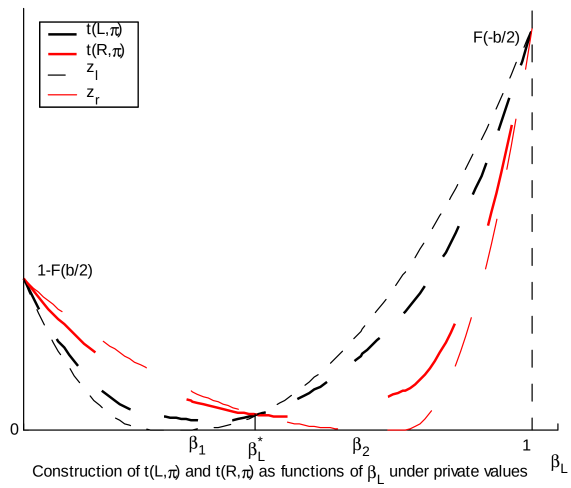

Figure 7

5.2 Limiting equilibria in large elections

Given Proposition 1a, equilibrium exists for every n. Therefore, we use the same notation as in Section 4.2. Since the cutoffs are bounded within a compact set, any sequence of x_σ^n will have a convergent subsequence. We look at such convergent subsequences x_σ^n as $n \to \infty$. We call an accumulation point of such a sequence of cutoffs as x_σ^0, and the resulting equilibrium as π^0. By the continuity arguments , as $x_\sigma^n \to x_\sigma^0$, $t(S, \pi^n)$, β_L^n, p_l^n,and p_r^n all converge to $t(S, \pi^0)$, β_L^0, p_l^0, and p_r^0 respectively along the subsequence. In this section we examine which outcomes can be supported as the limiting values.

The necessary conditions for the limit, the *limiting equilibrium condition* as identified in equation (11) remains exactly the same. Lemma 3 goes through, without any change. Lemma 4 goes through too, with the slight modification that now it is no longer true of all n, but it holds for large enough n. We state this in Lemma 4a. If the induced prior converges to 0 or 1, both the cut-offs are either in the L-group or in the R-group respectively, if the population size is large enough. Locally, the structure of the equilibrium is no different from that in the common values case.

Lemma 4a If $\beta_L^0 = 1$, (i) $\exists$ some m such that $x_l^n > x_r^n$ for all $n > m$; and (ii) $x_\sigma^n \to -\frac{b}{2}$ from the left for $\sigma = l, r$. Similarly, if $\beta_L^0 = 0$, (i) $\exists$ some m_1 such that $x_l^n > x_r^n$ for all $n > m_1$; and (ii) $x_\sigma^n \to \frac{b}{2}$ from the right for $\sigma = l, r$

Proof. See Appendix. ■

Proposition 2 now goes through in exactly the same form. The proof follows from lemma 3 and Lemma 4a analogously. Thus, in the limiting equilibrium, we must have the same relationship between the shares and the voting rule in the common value case and the non-common value case. In other words, the local properties of the limiting equilibria are the same. Next, we examine which voting rules are supportable by a given value of the induced prior in the limit. We look for an equivalent of Lemma 5.

Lemma 5a Under non-common values, $(i)$for $\beta_L \in (0, \beta_L^*) \cup (\beta_L^*, 1)$, $\Theta(\beta_L)$ is a continous function $\theta^*(\beta_L)$, with $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$ for $\beta_L < \beta_L^*$, and $t_L(\beta_L) > \theta^*(\beta_L) > t_R(\beta_L)$ for $\beta_L > \beta_L^*$, (ii)Otherwise, $\Theta(1) = \{\theta : \theta > F(-\frac{b}{2})\}$, $\Theta(0) = \{\theta : \theta > 1 - F(\frac{b}{2})\}$ and $\Theta(\beta_L^*) = \{\theta : \theta \in (0, 1)\}$.

Proof. In Appendix. ■

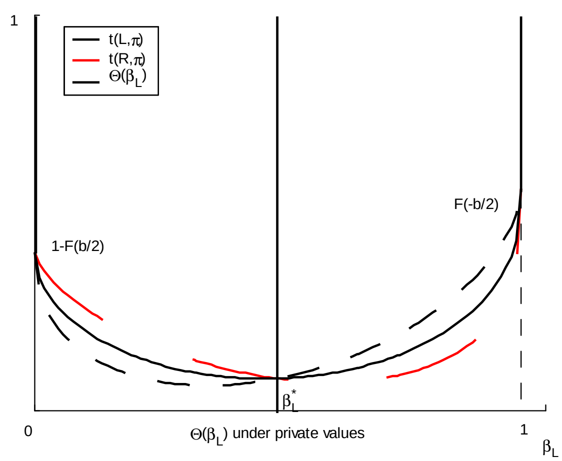

Figure 8

The correspondence $\Theta(\beta_L)$ for the non-common values system, as inferred in Lemma 5a, is depicted in figure 8. Note that in this case, if we invert the correspondence to get the supporting induced belief β_L for each voting rule θ, we no longer get a function $\beta(\theta)$ as defined in (12) in the common values case, but rather a correspondence.

Figure 8 illustrates the relationship between the non-common values and the common values models. If, in any equilibrium, the induced prior converges to a value above β_L^* it must be that conditional on being pivotal, everyone considers the state L to have a likelihood large enough that the R-group votes for $\mathcal{Q}$ in each state en masse and although the extreme left types in the L-group vote for $\mathcal{P}$ uninformatively, some moderate members in the L-group vote informatively. The reason that these two groups behave differently is quadratic preference: in state L, the policy is perceived to be located so far to the extreme left for the

R-group that the members of the group feel very strongly against it, while the members of the L-group have a closer choice between the policy and the status quo. This makes a difference in the voting behaviour of the types in each group when there is uncertainty about the state. Thus, if there is an equilibrium with $\beta_L^0 > \beta_L^*$, locally it will look like an equilibrium in a common value setting where the L-group forms the set of independent types (with the more extreme types within the group having a stronger preference for $\mathcal{P}$ while the more moderate types have a bias towards $\mathcal{Q}$) and the R-group is committed to $\mathcal{Q}$. By extension of this analogy, if there is an equilibrium with $\beta_L^0 = 1$, it can be supported by any voting rule above $F(-\frac{b}{2})$. The case for $\beta_L^0 < \beta_L^*$ is symmetric. Note that we cannot rule out any voting rule at β_L^*, as evidenced from Proposition 3.

For a large set of voting rules, we can construct equilibria in two ways. Based on whether the equilibrium limiting belief is above or below the cut-off β_L^*, the responsive set is expected to be in the L-group or R-group, and we have two different equilibria. Since these two groups have opposite rankings in either state, if the responsive sets are influential in both equilibria, we get opposite outcomes in these two equilibria given a voting rule. If the responsive set is influential in one and not in the other equilibrium, we still get different outcomes in one state and the same outcome in the other.

Next we specify the outcomes for each voting rule. Note that the characterization result in the common values set up depends on the fact that for each θ, there is a unique induced prior $\beta(\theta)$ that can support the voting rule in a limiting equilibrium. This in turn hinges the monotonicity of expected vote shares in the induced prior. In the non-common values setting, we no longer have uniqueness of the supporting induced belief for a given voting rule, although we have an existence result for every θ. Thus, for any given θ, it is difficult to say for certain which beliefs in the set of possible supporting beliefs can support an equilibrium for *all possible distributions of preferences.*

However, we have already noted that, locally, these equilibria should have the same properties as those in the common value setting. More specifically, for $\beta_L > \beta_2$ both the cut-offs are to the left of $-\frac{b}{2}$. For this interval of beliefs, the cut-off strategies are ordered, and therefore, the vote shares are monotonic with beliefs in both states. The same is true for $\beta_L < \beta_1$. Exploiting this similarity with the common values situation, we use Theorem 1 to get a partial characterization theorem for equilibria under non-common values.

Theorem 3 *For any $b > 0$ satisfying the non-common values condition, and for any q satisfying assumption I, identify two numbers $0 < \beta_1 = p_l^{-1}(\frac{1}{2}+\frac{b}{4}) < \beta_2 = p_r^{-1}(\frac{1}{2}-\frac{b}{4}) < 1$. For any distribution of preferences $F(\cdot)$ satisfying assumption F, given a voting rule θ, there is a limiting equilibrium π^0 with cut-off strategies and with the induced prior converging to β_L if $\theta \in \Theta(\beta_L)$, and $\beta_L \in [0, \beta_1) \cup (\beta_2, 1]$.*

Proof. In appendix. ■

We utilise the local nature of the limiting equilibrium to prove this theorem. Given a non-common values set up $(b, q, F(\cdot))$, considering any $\beta_L = \beta$, we look at the vote shares in both states in a neighbourhood of β. All we need for the existence of a limiting equilibrium at voting rule $\Theta(\beta)$ is the existence of a sequence β_L^n that gives rise to the same shares, and converges to β. This only needs to be true in a small neighbourhood of β. We construct an appropriate common values set up such that for the same neighbourhood of β, the vote share functions are exactly the same. We can always do that as long as the vote shares are monotonic in the non-common values set-up. From Theorem 1, we claim that β is supported in limiting equilibrium by $\Theta(\beta)$, which implies that the equilibrium sequence we are looking for indeed exists.

Note that since the shares are not necessarily increasing or decreasing together in both states in the range $[\beta_1, \beta_2]$, we will not find an appropriate common value set up in this range of beiefs. As a result, we cannot guarantee the existence of an equilibrium for these beliefs for every distribution of voter ideal points. The reason for this failure is the non-convexity of the strategy set. Note that in this range of beliefs, while x_l lies in $[-1, -\frac{b}{2}]$, x_r lies in $[\frac{b}{2}, 1]$. This, in turn, translates to a break-up of the responsive set into two disjoint intervals. So we cannot have the usual limiting equilibria. That does not mean, however, that there cannot be any equilibria in this range. Proposition 3, for example, demonstrates the existence of an equilibrium at β_L^* in this range for any voting rule, for any symmetric distribution of preferences.

5.3 Voting rules and Information Aggregation

From Lemma 5a, we can deduce possible outcomes for each value of the induced prior. All these outcomes occur almost surely, in the same way as in the common.values case. Define two intervals $(0, \beta_L^*)$ and $(\beta_L^*, 1)$ of β_L as B_R and B_L respectively.

- For $\beta_L^0 = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in R-group but is not influential.
- For $\beta_L^0 \in B_R$, the only possible outcome is $\mathcal{Q}$ under state L and $\mathcal{P}$ under state R. Here, the responsive set is in R-group and is influential.
- For $\beta_L^0 = \beta_L^*$, the vote shares in each state is fixed, say at z, and the outcome depends on whether the voting rule is greater or less than z.
- For $\beta_L^0 \in B_L$, the only possible outcome is $\mathcal{P}$ under state L and $\mathcal{Q}$ under state R. Here, the responsive set is in L-group and is influential.
- For $\beta_L^0 = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in L-group but is not influential.

From here onwards, we assume with a slight loss of generality that $F(-\frac{b}{2}) > 1 - F(\frac{b}{2})$[16]. In other words, we assume that the L-group is the larger interest group, and hence the group that is aligned with the society. Therefore,

[16] If $F(-\frac{b}{2}) = 1 - F(\frac{b}{2})$,then there are no consequential rules. $\theta = F(-\frac{b}{2})$ would implement a random social choice rule under full information if the L-group is the larger interest group.

- Any voting rule $\theta < 1 - F\left(\frac{b}{2}\right)$ is $\mathcal{P}$-trivial
- Any voting rule $1 - F\left(-\frac{b}{2}\right) \leq \theta < F\left(\frac{b}{2}\right)$ is a consequential rule[17], i.e. the policy wins in state L and the status quo in state R.
- Any voting rule $\theta \geq F\left(-\frac{b}{2}\right)$ is a $\mathcal{Q}$-trivial rule.

For all $\mathcal{Q}$-trivial rules, there are at least two limiting equilibria, one with the induced prior at 0 and one with that at 1. Any possible equilibrium at β_L^* too aggregates information. Figure 9(a) depicts the limiting equilibria for a $\mathcal{Q}$-trivial rule. For consequential rules, we need the responsive set to be influential and in the L-group for information aggregation. For these rules however, there is always one equilibrium with $\beta_L^0 = 0$, the responsive set in the R-group, and is not influential. Hence we get $\mathcal{Q}$ in both states. If for the consequential rule θ is larger than $\theta^*(\beta_2)$, there is an equilibrium with $\beta_L^0 \in B_L$; information is aggregated in this equilibrium. If $\theta < \theta^*(\beta_2)$, this aggregating equilibrium may or may not exist. There *may* be another equilibrium with $\beta_L^0 = \beta_L^*$ for some distributions of preference (e.g. when $f(\cdot)$ is symmetric about 0). Here too, we get $\mathcal{Q}$ in both states with a very high probability. Figure 9(b) depicts all the possible limiting equilibria for a consequential rule.

For $\mathcal{P}$-trivial rules greater than $\phi = \max(\theta^*(\beta_1), \theta^*(\beta_2))$ we have exactly two equilibria with opposite outcomes in the different states. The responsive sets are influential here when information aggregation requires that they not be so. So, for these voting rules we have no information-aggregating equilibrium. Figure 9(c) shows the possible equilibria for one such rule. However, information is aggregated almost surely by the very low $\mathcal{P}$-trivial rules[18].

We summarise the inferences about information aggregation for different voting rules in a non-common value setting in the next proposition. We use the same definition of full information equivalence as in Section 4.3. We define an equilibrium as *non-information aggregating* when in at least one state, voting under incomplete information delivers an outcome different from the full-information outcome with a probability arbitrarily close to 1.

Theorem 3a All (limiting) voting equilibria with $\mathcal{Q}$-trivial voting rules satisfy full information equivalence property. For consequential rules, there is one or more equilibria that are non-information aggregating. For consequential rules that are sufficiently large, there is always one equilibrium that is satisfies full information equivalence. There is some $\mathcal{P}$-trivial rule ϕ such that voting equilibria with all $\mathcal{P}$-trivial rules larger than ϕ are always non-information aggregating.

[17] Note that the other consequential rule, i.e. $\{G(L) = Q, G(R) = P\}$ cannot be implemented under full information by the plurality rule

[18] More speficically, the $\mathcal{P}$-trivial voting rules that aggregate information for sure for any distribution of preferences are those that are below the minimum share of votes received by $\mathcal{P}$ for any belief, i.e. those rules that satisfy $\theta < \min\{\min_{\beta_L} t(L,\pi), \min_{\beta_L} t(L,\pi)\}$. Equilibrium induced prior is β_L^* and equilibrium shares in both states are $z > \theta$ in the limit.

Proof. In appendix. ■

The above theorem establishes the bias in favour of the status quo. Unless the required vote share for the policy to win is very low, competition between two groups along with risk aversion ensures that the status quoi wins in at least one state. Note that the only voting rules for which information ius aggregated in any equilibrium are all $\mathcal{Q}$-trivial rules and the very low $\mathcal{P}$-trivial rules.

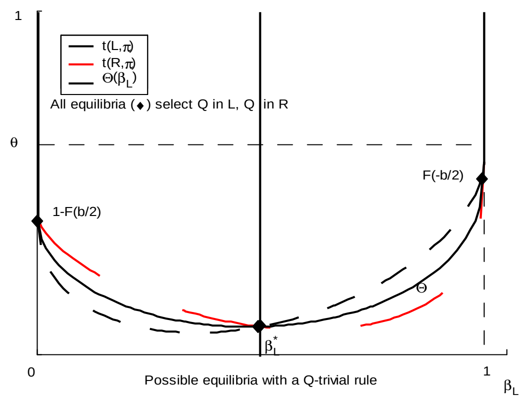

Figure 9(a)

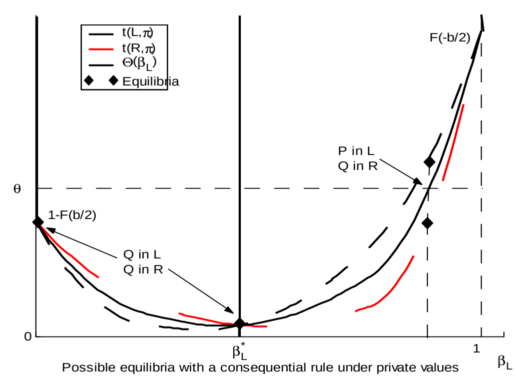

Figure 9(b)

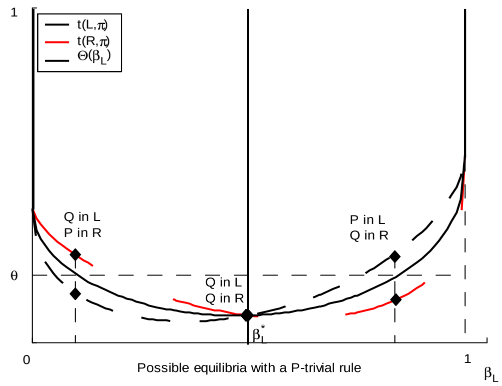

Figure 9(c)

6 Discussion and extension

The unidimensional spatial model brings into sharp relief the role of the common value assumption. It shows how the common value condition ensures that the responsive set voters always swing the election in the direction of the majority preference, which is the full-information outcome. Under non-common values, we have two competing groups with opposed preferences, and it is possible that swing voters can belong to the minority group. Then, depending on the equilibrium, we may have a sure outcome different from the full information outcome. We also show that this competition effect dominates the efficiency-enhancing effect of increased precision of signals. In other words, the same results prevail even if the signals are almost fully informative.

Another issue that the non-common values framework demonstrates is how exactly information aggregation can fail in presence of competing interests. We show that such failure can happen in several ways: the misaligned set of types can be influential, the responsive set of types may be influential when they should not be, or there can be two disjoint sets of responsive voters under any rule, making the swing voters behave in two different ways.

The non-common values model also presents the problem of multiple equilibria, with different equilibria giving completely different results. The multiplicity issue makes the role of beliefs crucial. The model endogenises the process of formation of beliefs about which types are going to be responsive to information. We show that information aggregation can fail because of "wrong" beliefs. For example, while a consequential rule needs the responsive set to be in the larger interest group, voters can believe that almost everyone is voting uninformatively, pretending that the state is known. However, a more serious failure is possible with everyone believing that the responsive set is in the minority interest group and is influential, leading to "wrong" outcome in both states. This requires us to look at a multidimensional policy space.

The framework discussed in the paper also throws light on a problem that is slightly different from the information aggregation issue. From the point of view of implementation one could look for a voting rule which would deliver two pre-specified outcomes in two states with a very high probability in all equilibria. For example, we might look for a voting rule that delivers the majority preferred outcome in both states. We show here that such a rule does not exist unless the pre-specified outcome is the same in both states.

Lastly, one might wonder how empirically relevant the non-common values condition is. In the unidimensional model, unless the uncertainty is somewhat extreme, we do not encounter the non-common value situation. For example, in most elections, it is known whether the challenger is to the left or right of the incumbent. However, in a multidimensional policy space, the common value assumption is much harder to justify. We claim that our framework can readily handle the extension to a multidimensional policy space, and the main conclusions carry over. In this paper, we only provide the intuition for why this is so.

6.1 Multidimensional extension

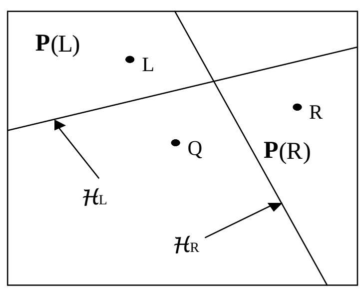

Figure 10: Multidimensional Policy space

Think of the policy space as a many-dimensional cube, with each dimension being $[-1, 1]$. Suppose that the status quo $\mathcal{Q}$ is located at the origin, and the policy alternative $\mathcal{P}$ is located at two points L and R under states L and R respectively. Given a state S, a hyperplane $\mathcal{H}_S$ separates the cube into two parts, one containing the origin that supports $\mathcal{Q}$ and the other that supports $\mathcal{P}$ under full information. Just as described in Section 3, we can define as $\mathbf{P}(S)$ the set of types that support $\mathcal{P}$ in state S. The common value condition is exactly the same - that $\mathbf{P}(L)$ be included in $\mathbf{P}(R)$ or vice versa. Note that this is harder to satisfy. In particular, for a given location L, as the size of the cube increases, the set of locations R for which $\mathbf{P}(S)$ exhibits common values keeps shrinking and approches a ray connecting L with the origin.

If the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ are parallel, we are either in a common value situation or in a situation where there are two disjoint, completely opposed interest groups, much like the unidimensional non-common value situation. Otherwise, we typically have four sets: two of opposed independent types, one type committed to $\mathcal{P}$ under both states and one type committed to $\mathcal{Q}$ under both states. Suppose the hyperplanes meet at a straight line $\mathcal{L}$. Under uncertainty, given a signal σ, the "cutoffs" that separate those who vote for $\mathcal{P}$ from those who vote $\mathcal{Q}$ are hyperplanes $\mathcal{X}_\sigma$. As the induced prior changes from 0 to 1, $\mathcal{X}_\sigma$ rotates about $\mathcal{L}$, starting at $\mathcal{H}_R$, and ending at $\mathcal{H}_L$. The strategy of a voter can also be described by the angle that each of the cutoff hyperplanes makes with the line $\mathcal{L}$. This is a compact set, and therefore, an equilibrium exists. If the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ are parallel, then the cutoffs $\mathcal{X}_\sigma$ do not rotate, but translate from $\mathcal{H}_R$ to $\mathcal{H}_L$.Thus we can trace vote shares t_L and t_R in the two states as a function of the induced prior β_L. Once we have done that, the rest of the analysis is exactly like the way we did in the unidimensional model.

In a common value setting, the vote shares in both states are monotonic-functions of the induced prior, with the derivative having the same sign in both states. Thus all our results for this case hold. However, in the non-common value case, we do not necessarily have U-shaped share functions. The equilibria depend on the particular shape of the distribution of preferences. This makes generalised equilibrium characteristics and aggregation (or non-aggregation) results difficult to get in a multidimensional set-up. However, given a distribution of preferences we can use the limiting equilibirium conditions developed in this paper to identify all the possible voting equilibria for that particular case and make judgements about information aggregation properties of each voting rule.

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8 Appendix

8.1 Proof of Remark 1

Proof. Let us first look at the situation with $0 < L < R$. Here, $\mathbb{P}(R) = \{x : x \geq \frac{R}{2}\} \subset \{x : x \geq \frac{L}{2}\} = \mathbb{P}(L)$. Similarly, if we have $L < R < 0$, $\mathbb{P}(L) = \{x : x \leq \frac{L}{2}\} \subset \{x : x \leq \frac{R}{2}\} = \mathbb{P}(R)$. On the other hand, if $L < 0 < R$, $\mathbb{P}(L) = \{x : x \leq \frac{L}{2}\}$ and $\mathbb{P}(R) = \{x : x \geq \frac{R}{2}\}$, thus $\mathbb{P}(L) \cap \mathbb{P}(R) = \phi$. ■

8.2 Proof of Lemma 3

By hypothesis of the lemma, $\lim\limits_{n\to\infty} \frac{\beta_L^n}{1-\beta_L^n} = \frac{\beta_L^0}{1-\beta_L^0}$ is a finite, positive number. Now suppose $\exists$ some $\varepsilon > 0$ such that $\alpha_n > 1 + \varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n > (1+\varepsilon)^n \to \infty \ as\ n \to \infty$ which is a contradiction. On the other hand, suppose $\exists$ some $\varepsilon \in (0,1)$ such that $\alpha_n < 1 - \varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n < (1-\varepsilon)^n \to 0 \ as\ n \to \infty,$ which is again a contradiction.

8.3 Proof of Lemma 5

Proof. For part (i) of the lemma, since $\beta_L \in (0,1)$, Proposition 2 holds. Suppose $0 < y < x < 1$, and $f(z,\theta) = z^{\theta}(1-z)^{1-\theta}$, with both z and θ lying in $(0,1)$. We show that $\exists!\ \theta^* s.t.\ f(x,\theta^*) = f(y,\theta^*)$, θ^* is increasing in x,and $x < \theta^* < y$. Note that $f(x,0) = 1-x < 1-y = f(y,0)$ and $f(x,1) = x > y = f(y,1)$. Continuity of f in θ establishes the existence of θ^*. For uniqueness, note that $\theta^* = \frac{\log\frac{1-y}{1-x}}{\log\frac{x(1-y)}{y(1-x)}}$. Also, $\frac{d\theta^*}{dx} = \left(\frac{1}{1-x}\right)\left[\frac{\log\frac{x}{y}}{\left(\log\frac{x(1-y)}{y(1-x)}\right)^2}\right]$, which implies that $\frac{d\theta^*}{dx} > 0$ ($\because x < 1$ and $\frac{x}{y} > 1$). For the second part of the lemma, note that $\frac{\partial h}{\partial z} = \frac{\theta}{z} - \frac{1-\theta}{1-z}$, and $\frac{\partial^2 h}{\partial z^2} = -\frac{\theta}{z^2} - \frac{1-\theta}{(1-z)^2} < 0$. So $f(x, \theta^0)$ is single-peaked with the peak at $x = \theta^0$. Since $x > y$, now $f(x, \theta^*) = f(y, \theta^*) \Rightarrow x > \theta^* > y$. Since $0 < F(\frac{L}{2}) < t_L(\beta_L) < t_R(\beta_L) < F(\frac{R}{2}) < 1$, taking $t_R(\beta_L) = x$ and $t_L(\beta_L) = y$ and noting that $t_R(\beta_L)$ is a strictly increasing function of β_L is sufficient for the proof of (i).

For part (ii), note that for any n, by Remark 2, we have $x_l^n < x_r^n$. Since $z_\sigma^n = F(x_\sigma^n)$, we have $z_r^n > z_l^n > 0$. Define, for any n, $h^n = z_r^n - z_l^n > 0$. Substituting, we have: $t(R, \pi^n) = z_l^n + qh^n$, and $t(L, \pi^n) = z_l^n + (1-q)h^n$. Therefore:

$$\frac{1-\beta_L^n}{\beta_L^n} = \left[\frac{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_l^n + qh^n)^\theta (1 - z_l^n - qh^n)^{1-\theta}}{(z_l^n + (1-q)h^n)^\theta (1 - z_l^n - (1-q)h^n)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 0$ (or 1), the left hand side of the above equation goes to infinity (or 0). This requires the term in the bracket large enough n to be greater (or less) than unity, or its logarithm to be positive (or negative). We can write,

$$\log\frac{(z_l^n + qh^n)^\theta (1 - z_l^n - qh^n)^{1-\theta}}{(z_l^n + (1-q)h^n)^\theta (1 - z_l^n - (1-q)h^n)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_l^n, h^n)\ \forall n$$

where the function $\zeta(z_l^n, h^n)$ is defined as:

$$\zeta(z_l^n, h^n) \equiv \frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]}$$

By Lemma 4, we know that for any sequence, with $\beta_L^0 \in \{0,1\}$, $h^n \to 0^+$. Hence,

$$\lim_{h^n\to 0^+,\ z_l^n=t} \zeta(z_l^n, h^n) = \lim_{h^n\to 0^+,\ z_l^n=t}\left(\frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]}\right) = \lim_{z_l^n=t} z_\sigma^n = t$$

By Lemma 4, if $\beta_L^0 = 0, t = F(\frac{R}{2})$, and $\theta > \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta > \lim_{h^n\to 0^+,\ z_l^n=t}\zeta(z_l^n, h^n) = F(\frac{R}{2})$. Similarly, if $\beta_L^0 = 1, t = F(\frac{L}{2})$, and $\theta < \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta < \lim_{h^n\to 0^+,\ z_l^n=t}\zeta(z_l^n, h^n) = F(\frac{L}{2})$. ■

8.4 Proof of Theorem 1

Here we only show that the only accumulation point is also the limit. For this, it is enough to show that given $\theta \in \Theta\left(\beta_L^0\right)$, for any neighbourhood ϵ of β_L^0, there is some large enough N, such that β_L^n in the equilibrium sequence must lie within the nighbourhood for all values of $n > N$.

First consider $\beta_L^0 \in (0,1)$.Suppose the accumulation point is not the limit, and there is an infinite equilibrium subsequence β_L^m of the sequence β_L^n, such that for any $\epsilon > 0$, there is some M so that for all values of m larger than M, β_L^m lies outside $\left(\beta_L^0 - \epsilon, \beta_L^0 + \epsilon\right)$. Since even this subsequence must have an accumulation point, it must be either 0 or 1. But, by the second part of Lemma 5, since the limiting equilibrium condition must hold for accumulation points too, there cannot be an accumulation point for θ in $\Theta\left(\beta_L^0\right)$ at 0 or 1. Hence there is no such infinite subsequence.

The proof for $\beta_L^0 \in \{0,1\}$ is similar.

8.5 Proof of Theorem 2

Proof. Theorem 1 guarantees existence of limiting equilibrium for all θ.

Consider $\theta < F\left(\frac{L}{2}\right)$. By Lemma 2, $t(S,\pi^n) > F\left(\frac{L}{2}\right)$ $\forall n$ for $S = L, R$. Let $\delta = F\left(\frac{L}{2}\right) - \theta$. By Law of large numbers, given ϵ we can find N such that actual share of votes $\tau(S,\pi^n,\theta)$ under rule θ in any state S is greater than $F\left(\frac{L}{2}\right) - \delta > \theta$ for any $n > N$ with a probability larger than $1-\epsilon$. Thus, under both states, $\mathcal{P}$ wins with a probability larger than $1-\epsilon$.

Since $t(S,\pi^n) < F\left(\frac{R}{2}\right)$ $\forall n \forall S$, by the same logic as above, any $\mathcal{Q}$-trivial rule aggregates information too.

Consider a consequential rule θ, for which the only equilibrium induced prior in the limit is $\beta_L^{-1}(\theta)$. By Lemma 5, $t_L\left(\beta_L^{-1}(\theta)\right) < \theta < t_R\left(\beta_L^{-1}(\theta)\right)$.

Also, for any consequential rule θ, we can find a positive number η such that $F\left(\frac{L}{2}\right) + \eta < \theta < F\left(\frac{R}{2}\right) - \eta$. By Lemma 5, we can find a similar number $\kappa > 0$ such that $\kappa < \beta_L^{-1}(\theta) < 1-\kappa$. Now, by Lemma 1 and Lemma 2, we can find some $\lambda > 0$ such that $t_R\left(\beta_L^{-1}(\theta)\right) - t_L\left(\beta_L^{-1}(\theta)\right) > \lambda$. Now, from Proposition 2, we can derive θ from $t_R\left(\beta_L^{-1}(\theta)\right)$ and $t_L\left(\beta_L^{-1}(\theta)\right)$ and can find another number $\mu > 0$ such that $t_L\left(\beta_L^{-1}(\theta)\right) + \mu < \theta < t_R\left(\beta_L^{-1}(\theta)\right) - \mu$. Since $t_R,\ t_L$ and θ^* are all continuous functions of β_L, we can find a number $\xi > 0$ such that for a range $\left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ around $\beta_L^{-1}(\theta)$, $t_L - \frac{\mu}{2} < \theta < t_R + \frac{\mu}{2}$. Given ξ,we can find M_1 such that $\beta_L^n \in \left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ in any π^n whenever $n > M_1$. Now consider $\delta = \min\left((t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \theta,\ \theta - t_L(\beta_L^{-1}(\theta) + \xi) - \frac{\mu}{2}\right)$. By Law of large numbers, given ϵ we can find M_2 such that actual share of votes under rule θ under state R, $\tau(R,\pi^n,\theta)$ is less than $t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \delta < \theta$ for any $n > M_2$ and the actual share under state L, $\tau(L,\pi^n,\theta)$ is greater than $t_L\left(\beta_L^{-1}(\theta) + \xi\right) + \frac{\mu}{2} - \delta > \theta$ for any $n > M_2$ with a probability larger than $1-\epsilon$. Set $N = \max(M_1, M_2)$ and we are done. ■

8.6 Proof of Lemma 6

Proof. If $x_\sigma \leq -\frac{b}{2},\ z_\sigma = F(x_\sigma) \leq F(-\frac{b}{2})$ since $F(\cdot)$ is nondecreasing. If on the other hand, $x_\sigma \geq \frac{b}{2},\ z_\sigma = 1 - F(x_\sigma) \leq 1 - F(\frac{b}{2})$. Thus, for $\sigma \in \{l, r\},\ z_\sigma \leq$

$\max\left(F(-\frac{b}{2}), 1-F(\frac{b}{2})\right)$ .Therefore,

$$t(S,\pi) \leq q\max(z_l, z_r) + (1-q)\max(z_l, z_r) = \max(z_l, z_r) \leq \max\left(F(-\frac{b}{2}), 1-F(\frac{b}{2})\right) < 1$$

The last inequality in the chain is guaranteed by assumption F. To show $t(S,\pi) > 0$, it is sufficient to show that both z_l and z_r cannot be 0 simultaneously. From assumption F and the definition of x_σ, $z_\sigma = 0 \Rightarrow p_\sigma \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}]$. We show that both p_l and p_r cannot be simultaneouly in this range. We start by noting that p_l and p_r increase in tandem, since both increase with β_L. When $p_l = q$, $\beta_L = \frac{1}{2}$. So, $p_r = 1-q$. By the above positive relationship, $p_l < q \Rightarrow p_r < 1-q$ and $p_r > 1-q \Rightarrow p_l > q$. Note that by Assumption I, $q > \frac{1}{2} + \frac{b}{4}$ and $1-q < \frac{1}{2} - \frac{b}{4}$.Hence,

$$p_l \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}] \Rightarrow p_r < 1-q < \frac{1}{2} - \frac{b}{4} \; and \; p_r \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}] \Rightarrow p_l > q > \frac{1}{2} + \frac{b}{4}$$

■

8.7 Proof of Lemma 7

Proof. At $\beta_L = 0$, $x_l = x_r = \frac{b}{2} \Rightarrow z_l = z_r = 1 - F\left(\frac{b}{2}\right)$. Now, consider the interval of β_L such that p_l lies in $(0, \frac{1}{2} + \frac{b}{4}]$. In this interval, $x_l \in (\frac{b}{2}, 1] \cup \{-1\} \Rightarrow z_l = 1 - F(x_l)$. Also, in this interval of β_L, $p_r < \frac{1}{2} - \frac{b}{4} \Rightarrow x_r \in (\frac{b}{2}, 1) \Rightarrow z_r = 1 - F(x_r) > 0$, by assumptions F and I. For values of β_L such that $x_l \leq 1$, $x_r < x_l \Rightarrow z_l = 1 - F(x_l) < 1 - F(x_r) = z_r$, again by assumption F. For values of β_L such that $x_l = -1$, $z_l = 1 - F(-1) = 0 < z_r$. Thus, over this entire interval $z_r > z_l$. Note also that over this set of values of β_L, z_r is strictly decreasing, while z_l first strictly decreases and then stays at 0. For β_L such that $p_l = \frac{1}{2} + \frac{b}{4}$, $z_r = \overline{z_r}$, say. In the same way, consider the interval of β_L such that p_r lies in $[\frac{1}{2} - \frac{b}{4}, 1]$. Here, by the same token, $z_r < z_l$ except for $\beta_L = 1$ where $z_l = z_r = F\left(-\frac{b}{2}\right)$. z_l increases strictly from $\overline{z_l} > 0$ to $F\left(-\frac{b}{2}\right)$ over this interval, while z_r is initially 0 and then strictly increases. Noting that $\beta_1 = p_l^{-1}(\frac{1}{2} + \frac{b}{4})$ and $\beta_2 = p_r^{-1}(\frac{1}{2} - \frac{b}{4})$ suffices for the first part of the proof.

Now, consider the remaining interval of β_L which is $\left(p_l^{-1}(\frac{1}{2} + \frac{b}{4}), p_r^{-1}(\frac{1}{2} - \frac{b}{4})\right)$. That this is a valid nonempty interval is guaranteed by assumption I. In this interval, $x_r \in (\frac{b}{2}, 1]$, and x_r increases with β_L. Thus, $z_r = 1 - F(x_r)$ is a strictly falling continuous function, going from $\overline{z_r} > 0$ to 0 over this interval. Similarly, z_l strictly and continuously increases from 0 to $\overline{z_l} > 0$. Therefore, there exists a unique β_L^* in this interval where $z_l = z_r$. This implies that at β_L^*, $t(L,\pi) = t(R,\pi)$. For all $\beta_L < \beta_L^*$, $z_l < z_r \Rightarrow t(L,\pi) = qz_l + (1-q)z_r < qz_r + (1-q)z_l = t(R,\pi)$. Similarly, for $\beta_L > \beta_L^*$, where $z_l > z_r$, we have $t(L,\pi) > t(R,\pi)$. ■

8.8 Proof of Lemma 4a

Proof. We prove the result for the case $\beta_L^0 = 1$, the other one follows symmetrically. First we look at how $\frac{p_l}{p_r}$ changes with β_L.

$$\frac{p_l}{p_r} = \left(\frac{q}{1-q}\right)\left(\frac{q\beta_R + (1-q)\beta_L}{q\beta_L + (1-q)\beta_R}\right) = \left(\frac{q}{1-q}\right)\left(\frac{q + (1-q)\alpha}{q\alpha + (1-q)}\right),$$

where $\alpha = \frac{\beta_L}{\beta_R}$. Therefore, we have:

$$\frac{d}{d\beta_L}\left(\frac{p_l}{p_r}\right) = \frac{d\alpha}{d\beta_L}\cdot\frac{d}{d\alpha}\left(\frac{p_l}{p_r}\right) = \frac{1}{(1-\beta_L)^2}\left(\frac{q}{1-q}\right)\frac{(1-q)^2 - q^2}{(q\alpha + (1-q))^2} < 0$$

At $\beta_L = 1$, we have $p_l = p_r = 1$. Thus, for $\beta_L \in [0,1)$, we always have $p_l > p_r$ by the above strictly monotonic relationship. Since $\beta_L^0 = 1 \Rightarrow p_r^n \to 1$, by continuity we can find some m large enough such that for all $n > m$, we have $p_r^n > \frac{1}{2} + \frac{b}{4}$. Since $p_l^n > p_r^n$, for all $n > m$, $p_l^n > \frac{1}{2} + \frac{b}{4}$ too. Since we always have $\beta_L^n < 1$, $p_\sigma^n < 1$. Therefore, for all $n > m$, both x_l^n and x_r^n lie in the open interval $(-1, -\frac{b}{2})$. Also, $p_l^n > p_r^n \Rightarrow x_l^n > x_r^n$ for all $n > m$. This proves part (i). Part (ii) follows trivially from $p_\sigma^n \to 1$. ■

8.9 Proof of Lemma 5a

Proof. Part (i) follows from Lemma 5 and Lemma 7.

For part (ii), we first consider the case with $\beta_L^0 = 1$. By Lemma 4a, we know that for any such sequence, $x_\sigma^n \to \left(-\frac{b}{2}\right)^-$ for $\sigma = \{l, r\}$, and $x_l^n > x_r^n$ for all large enough n. For large enough n, $p_\sigma^n > \frac{1}{2} + \frac{b}{4} \Rightarrow z_\sigma^n = F(x_\sigma^n) \Rightarrow z_l^n > z_r^n > 0$ and $z_\sigma^n \to F(-\frac{b}{2})$. Define $h^n = z_l^n - z_r^n \to 0^+$. Substituting, we have: $t(L, \pi^n) = z_r^n + qh^n$, and $t(R, \pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 1$, the left hand side of the above equation goes to infinity. This requires the term in the bracket large enough n to be greater than unity, or its logarithm to be positive .

For the case with $\beta_L^0 = 0$, we again use Lemma 4a which tells us that $x_\sigma^n \to \left(\frac{b}{2}\right)^+$ for $\sigma = \{l, r\}$, and $x_l^n > x_r^n$ for all large enough n.We also know that for large enough n, $p_\sigma^n > \frac{1}{2} - \frac{b}{4} \Rightarrow z_\sigma^n = 1 - F(x_\sigma^n) \Rightarrow z_r^n > z_l^n > 0$ and $z_\sigma^n \to 1 - F(-\frac{b}{2})$. Define $h^n = z_r^n - z_l^n \to 0^+$. Substituting, we have: $t(R, \pi^n) = z_r^n + qh^n$, and $t(L, \pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^{-n}$$

Since the LHS goes to 0 in the limit, the term within the bracket in the RHS has to be greater than 1. Thus we have the exact same situation as in the proof of Lemma 5, and therefore, we need.

$$\log \frac{\left(z_r^n + qh^n\right)^\theta \left(1 - z_r^n - qh^n\right)^{1-\theta}}{\left(z_r^n + (1-q)h^n\right)^\theta \left(1 - z_r^n - (1-q)h^n\right)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_r^n, h^n)\ \forall n$$

where the function $\zeta(z_l^n, h^n)$ is defined as in the proof of lemma 5.

By Lemma 4, if $\beta_L^0 = 0$, $t = 1 - F(\frac{b}{2})$, and $\theta > \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta > \lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = 1 - F(\frac{b}{2})$. Similarly, if $\beta_L^0 = 1$, $t = F(-\frac{b}{2})$, and $\theta > F(-\frac{b}{2})$.

For $\beta_L^0 = \beta_L^*$, from Proposition 3, no value of θ can be ruled out. ■

8.10 Proof of Theorem 3

This is a proof by construction. Consider any non-common value setting $(F(\cdot), q, b)$. We show that every $\beta \in [0, \beta_1) \cup (\beta_2, 1]$ can be supported by any $\theta \in \Theta(\beta)$ for any $F(\cdot)$ satisfying full support. The proof strategy is to find a non-common value setting $(F^c(\cdot), q^c, L, R)$ with the same $\Theta(\beta)$, $t_L(\beta)$, and $t_R(\beta)$ in the neighbourhood of β and use Theorem 1 to guarantee the existence of a limiting sequence of equilibria converging to β.

Consider first some $\beta^0 \in (\beta_2, 1)$. By Lemma 4a, $t_L(\beta)$ and $t_R(\beta)$ are increasing functions, and $t_L(\beta) > .t_R(\beta)$. By assumption F, they are differentiable too. By Proposition 2,

$$\theta^*(\beta^0) = \frac{\log \frac{1 - t_R(\beta^0)}{1 - t_L(\beta^0)}}{\log \frac{t_L(\beta^0)(1 - t_R(\beta^0))}{t_R(\beta^0)(1 - t_L(\beta^0))}}$$

Call $\theta^*(\beta^0)$ simply θ^p. Consider some ϵ such that $\left(\beta^0 - \epsilon,\ \beta^0 + \epsilon\right) \in (\beta_2, 1)$. Consider any $q^c \in (\frac{1}{2}, 1)$ and define the functions $G_L(\beta)$ and $G_R(\beta)$ over the domain $\left[\beta^0 - \epsilon,\ \beta^0 + \epsilon\right]$:

$$\begin{aligned} G_L(\beta) &\equiv \frac{q^c t_L(\beta) + (1 - q^c) t_R(\beta)}{2q^c - 1} \\ G_R(\beta) &\equiv \frac{q^c t_R(\beta) + (1 - q^c) t_L(\beta)}{2q^c - 1} \end{aligned}$$

We need $G_S(\beta)$, $S = L, R$ to satisfy several properties over the domain, and choose q^c accordingly.

1. $G_S(\beta) > 0 \Rightarrow q^c > \max\limits_{[\beta^0 - \epsilon,\ \beta^0 + \epsilon]} \left(\frac{t_L}{t_L + t_R}, \frac{t_R}{t_L + t_R} \right) = M_1$.

2. $G_S(\beta) < 1 \Rightarrow q^c > \max\limits_{[\beta^0 - \epsilon,\ \beta^0 + \epsilon]} \left(\frac{1 - t_L}{(1 - t_L) + (1 - t_R)}, \frac{1 - t_R}{(1 - t_L) + (1 - t_R)} \right) = M_2 \in (\frac{1}{2}, 1)$

3. $\frac{dG_S(\beta)}{d\beta} > 0 \Rightarrow q^c > \max\limits_{[\beta^0 - \epsilon,\ \beta^0 + \epsilon]} \left(\frac{\mathring{t}_L}{\mathring{t}_L + \mathring{t}_R}, \frac{\mathring{t}_R}{\mathring{t}_L + \mathring{t}_R} \right) = M_3 \in (\frac{1}{2}, 1)$, where $\mathring{t}_S = \frac{dt_S(\beta)}{d\beta} > 0$.

4. Also, define $M_4 = \frac{1}{1+\sqrt{\frac{(\beta^0-\epsilon)(1-\beta^0-\epsilon)}{(\beta^0+\epsilon)(1-\beta^0+\epsilon)}}} < 1.$

Choose $q^c \in (\max\{M_1, M_2, M_3, M_4\}, 1)$. Consider any common value setting $0 < L < R < 1$. From Lemma 1, given $\beta,\ L,\ R$ and q^c we can find the cut-offs $x(p_l(\beta))$ and $x(p_r(\beta))$. We claim that $q^c > M_4 \Rightarrow p_l(\beta^0 - \epsilon) > p_r(\beta^0 + \epsilon)$.The proof follows from simple algebra, given that

$$p_l(\beta^0-\epsilon) = \frac{q^c(\beta^0-\epsilon)}{q^c(\beta^0-\epsilon)+(1-q^c)(1-\beta^0+\epsilon)} > \frac{(1-q^c)(\beta^0+\epsilon)}{q^c(1-\beta^0-\epsilon)+(1-q^c)(\beta^0+\epsilon)} = p_r(\beta^0+\epsilon)$$

This implies that $\left[x(p_l(\beta^0-\epsilon)), x(p_l(\beta^0+\epsilon))\right]$ and $\left[x(p_r(\beta^0-\epsilon)), x(p_r(\beta^0+\epsilon))\right]$ are disjoint sets. Now define functions $G_S(\beta)$ over the domain $\beta \in \left[\beta^0-\epsilon,\ \beta^0+\epsilon\right]$ as:

$$G_L(\beta) \equiv G(x(p_l(\beta)) \text{ and } G_R(\beta) \equiv G(x(p_r(\beta))$$

Finally, define the preference distribution $F^c(\cdot)$ as any distribution with $F^c(\cdot) = 1 - G(\cdot)$ for $x \in \left[x(p_l(\beta^0-\epsilon)), x(p_l(\beta^0+\epsilon))\right] \cup \left[x(p_r(\beta^0-\epsilon)), x(p_r(\beta^0+\epsilon))\right]$, and any other legitimate value elsewhere. By choice , $q^c > M_1$ and $q^c > M_2$ guarantee that $F^c(\cdot)$ is a legitimate cumulative distribution function satisfying full support over the range. Since p_σ increases with β, x_σ decreases with p_σ , and $F^c(\cdot)$ increases with $x_\sigma, G(\cdot)$ must be increasing with β, which is satisfied by $q^c > M_3$.

The vote shares $t_L^C(\beta)$ and $t_R^C(\beta)$ in favour of $\mathcal{P}$ in the common value setting are given by:

$$\begin{aligned} t_L^C(\beta) &= q^c(1-F^c(x(p_l(\beta)) + (1-q^c)(1-F^c(x(p_r(\beta)) \\ t_R^C(\beta) &= q^c(1-F^c(x(p_r(\beta)) + (1-q^c)(1-F^c(x(p_l(\beta)) \end{aligned}$$

In the domain $\beta \in \left[\beta^0-\epsilon,\ \beta^0+\epsilon\right]$, we have:

$$\begin{aligned} t_L^C(\beta) &= qG_L(\beta) + (1-q^c)\, G_R(\beta) = t_L(\beta) \\ t_R^C(\beta) &= qG_R(\beta) + (1-q^c)\, G_L(\beta) = t_R(\beta) \end{aligned}$$

By Theorem 1, for a voting rule $\theta^*(\beta^0)$, there exists a sequence of equilibria π^n that converges to π^0, and the induced prior beliefs converge to β^0. Since $(F^c(\cdot), q^c, L, R)$ is a common values setting that has the same values of the functions $t_L(\beta)$ and $t_R(\beta)$ in the chosen domain as our non-common value setting, $\theta^*(\beta^0) = \theta^c$. Thus, for θ^c, there exists a series β^n which satisfies the equilibrium condition lying entirely in the domain for all values of n greater than some N. Hence, in the non-common value setting too, for $\theta^*(\beta^0)$, we can claim the existence of a sequence that satisfies the equilibrium condition (10) for all values larger of n than N, that converges to β^0.

Next, consider the case $\beta^0 = 1$.

Consider any $\theta \in \Theta(1)$ and the non-common value setting $(F(\cdot), b, q)$.

We claim that there exists a sequence of equilibria π^n with β^n satisfying the equilibrium condition with $\beta^n \to 1$. We prove our claim by contradiction.

Suppose the claim is not true and there is some $\overline{\beta} < 1$ and some N such that for all $n > N$, there is no β^n satisfying the equilibrium condition in the non-common value setting $(F(\cdot), b, q)$ with voting rule $\theta \in \Theta(1)$.

Consider the range $\beta \in (\delta, \epsilon)$ with $\delta > \max(\overline{\beta}, \beta_2)$, and $\delta < \epsilon < 1$. Construct a common value setting $(F^c(\cdot), q^c, L, R)$ that has the same values of the functions $t_L(\beta_L)$ and $t_R(\beta_L)$ in this range, and has $G(\frac{L}{2}) \leq F(-\frac{b}{2})$. Since $\theta > F(-\frac{b}{2})$ in the non-common value setting $\theta \in \Theta(1)$ in the common value setting too. By Theorem 1, there is an equilibrium sequence π_c^n with $\beta_c^n \to 1$ in $(F^c(\cdot), q^c, L, R)$. Therefore, the equilibrium condition is satisfied in the non-common value setting $(F(\cdot), b, q)$ for all those values of n for which β_c^n lies in (δ, ϵ). Now, define:

$$K(\epsilon) = \max_n \{\beta_c^n : \delta < \beta_c^n < \epsilon\}$$

Note that since $\beta_c^n \to 1$, $K(\epsilon)$ increases arbitrarily as ϵ gets closer and closer to 1.If $K(\epsilon) > N$, we are done showing the contradiction in our supposition. If $K(\epsilon) \leq N$, let ϵ increase, and we can always find. some ϵ_0 such that $K(\epsilon_0) > N$.

If $\beta^0 < \beta_1$, consider a common value case with $-1 < L < R < 0$, and the construction proceeds the same way.

8.11 Proof of Theorem 3a

Proof. From Lemma 5a, the only possible equilibria with $\mathcal{Q}$-trivial voting rules, i.e. $\theta \geq F\left(-\frac{b}{2}\right)$ involve $\beta_L^0 \in \{0, \beta_L^*, 1\}$. If $\beta_L^0 = 0$, then $t(L, \pi^0) = t(R, \pi^0) = F\left(-\frac{b}{2}\right) \leq \theta$. Similarly, if $\beta_L^0 = 1$, then $t(L, \pi^0) = t(R, \pi^0) = 1 - F\left(\frac{b}{2}\right) < F\left(-\frac{b}{2}\right) \leq \theta$. If on the other hand, $\beta_L^0 = \beta_L^*$,then again, $t(L, \pi^0) = t(R, \pi^0) = z < 1 - F\left(\frac{b}{2}\right) < F\left(-\frac{b}{2}\right) \leq \theta$.In all cases therefore, the outcome is $\mathcal{Q}$ in both states with an arbitrarily high probability. This proves the first part of the proposition.

For the second part, note that for a consequential rule, there is always one equilibrium at $\beta_L^0 = 0$ and for those greater than $\theta^*(\beta_2)$,, there is another one at $\beta_L^0 \in (0, \beta_1)$.The second one aggregates information while the first leads to $\mathcal{Q}$ in either state almost surely. The third possibility, $\beta_L^0 = \beta_L^*$ also leads to $\mathcal{Q}$ in either state almost surely.

For the third part, consider any $\theta \in \left(\phi, 1 - F\left(\frac{b}{2}\right)\right)$. It can support exactly three values of beliefs, one lying in the range $(0, \beta_L^*)$, one in $(\beta_L^*, 1)$ and of course, the third value being β_L^*. In the first two cases. we get different outcomes in different states (namely, for $\beta_L^0 < \beta_L^*$, the policy wins in state R and state R alone, while for $\beta_L^0 > \beta_L^*$, the policy wins only in state L). If the equilibrium is at β_L^*, then in either state, $t(L, \pi^0) = t(R, \pi^0) = z < \theta$. Thus, the status quo wins in both states. This proves the third part. ■