Trade with Heterogeneous Beliefs[1]

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Preliminary and incomplete !

Abstract:

The paper analyzes an economy with asymmetric information in which agents trade in contingent assets. The new feature in the model is that each agent may have any prior belief on the states of nature and thus the posterior belief of an agent maybe any probability distribution that is consistent with his private information. The main result is that the set of equilibrium prices of the assets at a given state s can be characterized by the concept of the core applied to a cooperative game where the states of nature are players and the value of a coalition E (which is an event) is the amount of money in the hands of agents who know at the state s that E has occurred. Thus, the way in which the knowledge of agents restricts the set of trades can be captured by the concept of the core. Furthermore, the characterization that is obtained applies to a broad class of preferences which includes all preferences that can be represented by the expectation of a state dependent monotone utility function.

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1 Introduction

The paper analyzes an economy with asymmetric information in which agents trade in contingent assets. Let $S = \{1, .., n\}$ be the set of states of nature. For $s \in S$, A_s is an asset that pays \$1 in the state s and zero at any other state. The assets are commitments to make contingent payments which are issued by some agents and bought by others. An agent who buys (sells) one unit of the asset A_s gets(pays) \$1 if the state is s and zero otherwise. Different non-trade theorems establish that if risk averse agents have a common prior on the set of states then there will be no trade. In particular, the only rational expectations equilibrium is the fully-revealing equilibrium where in the state s the price of the asset A_s is 1 and the price of every other asset is zero.

In this paper we relax the assumption of a common prior and ask what is the set of equilibrium outcomes when agents may have different priors on S ?

We study two solution concepts. The first concept, which we simply call equilibrium, assumes rationality and market clearing. The second concept, common knowledge equilibrium (CKE), makes the stronger assumption that the rationality of the agents (R), market clearing (MC), and the parameters that define the economy ($\mathcal{E}$) are all common knowledge among the agents. Our two main results, theorem 1 and theorem 2, characterize the set of equilibrium prices and the set of CKE prices respectively. To get a better sense for the concepts and the results we now describe some of the elements in the model. The information of an agent i is represented by an information partition of S, ψ_{i} . For $s \in S$ $\psi_i(s) \subseteq S$ is the event that agent i knows at the state s. The concept of equilibrium that we study puts no restriction on the posterior belief of an agent i at a state s, $\gamma^i (= \gamma^i_s)$, except the obvious restriction that γ^i be consistent with the private information of i at s (that is, $\gamma^i \in \Delta(\psi_i(s))$.) Let I denote the set of agents and let $p = (p_s)_{s \in S}$ be a price vector where p_s is the price of the asset A_s. A tuple $((\gamma^i, z^i)_{i \in I}, p)$, where γ^i and z^i are, respectively, a belief and a bundle of assets for agent i and p is a price vector, is an <u>equilibrium at a state s</u> if the following two conditions are satisfied: (1) <u>Rationality,</u> $\gamma^i \in \Delta(\psi_i(s))$ and z^i is optimal w.r.t γ^i in the budget set of i (defined by p and the initial endowment of i.) (2) <u>Market Clearing:</u> The demand for each asset equals its supply. For $s \in S$ we let $\overline{P}^s$ denote the set of equilibrium prices at the state s. Formally, $\overline{P}^s \equiv \{p \,|\text{There exists an equilibrium } ((\gamma^i, z^i)_{i \in I}, p) \text{ at } s\}$. Thus, $\overline{P}^s$ is the set of equilibrium prices that is generated by all the profiles of subjective beliefs $(\gamma^i)_{i\in I}$ such that $\gamma^i \in \Delta(\psi_i(s))$. Similarly P^s is the set of CKE prices at the state s. A precise definition of P^s is more involved and therefore we delay its presentation to section 2. The idea that underlies the definition is that the beliefs that support a given price p are further restricted by the requirement that each agent i assigns a positive probability only to states in which p is an equilibrium, furthermore, in each state to which i assigns a positive probability p is supported by beliefs which assign a positive probability only to sates where p is an equilibrium price and so forth.

Theorem 1 characterizes the set $\overline{P}^s$ and theorem 2 characterizes the set P^s. To state theorem 1 we need just two additional definitions. For $E \subseteq S$ and $\widehat{s} \in S$ $I^E_{\widehat{s}}$ denotes the set of agents who know the event E at the state $\widehat{s}$. We let $m(I^E_{\widehat{s}})$ denote the aggregate amount of money in the hands of the agents in $I^E_{\widehat{s}}$. Finally, we normalize the aggregate amount of money in the economy to be 1. Theorem 1 states that $p = (p_s)_{s\in S}$ is an equilibrium price vector at a state $\widehat{s}$ iff

(1) $\sum\limits_{s\in S} p_s = 1$

(2) For every $E \subseteq S$ $\; m(I^E_{\widehat{s}}) \leq \sum\limits_{s\in E} p_s$

We observe that for $E \subseteq S$ $\sum\limits_{s\in E} p_s$ is the price of the composite asset $A_E \equiv \sum\limits_{s\in E} A_s$ which pays \$1 in the event E and zero otherwise. Since A_S is equivalent to money condition 1 is just a non-arbitrage constraint. Condition 2 states that for any event $E \subseteq S$ the price of the asset A_E is greater or equal to the aggregate amount of money in the hands of agents who know E at the state $\widehat{s}$.

Theorems 1 and 2 have three notable features:

(a) The sets $\overline{P}^s$ and P^s are characterized in terms of the parameters $m(I^E_s)$, $E \subseteq S$,which specify for each event E the aggregate amount of money in the hands of agents who know the event E at the state s.

(b) The characterizations are related to the concept of the core. In particular conditions (1) and (2) in theorem 1 define the core of the cooperative game, $G_{\widehat{s}}$, where the set of players is S and the value of a coalition $E \subseteq S$ is $m(I^E_s)$.

(c) Finally, and perhaps most surprising, the characterizations in theorems 1 and 2 apply for a very broad class of preferences of agents over outcomes. This class, which we denote by $\mathcal{M}$ (for monotonicity), includes all preferences that can be represented by an expectation of a monotone state dependent utility from money[3]. The characterizations in theorems 1 and 2 apply to <u>every</u> profile of preferences in $\mathcal{M}$. This implies, in particular, that the set of equilibrium prices (CKE prices) is the <u>same</u> set for every profile of preferences in $\mathcal{M}$. In particular, the set of equilibrium prices (CKE prices) does not depend on whether agents are risk averse or on their degree of risk aversion.

2 The model and the results.

In this section I define an economy with asymmetric information in which agents trade in contingent assets. I then present the main results, theorem 1 and theorem 2, and demonstrate them by means of a simple example.

The economy is defined as follows:

The set of agents is $I \equiv [0,1]$. The set of states of nature is $S \equiv \{1,..,n\}$. For $s \in S$ we let A_s denote the <u>asset</u> which pays \$1 in the state s and zero otherwise. The asset A_{n+1} is money, i.e., A_{n+1} pays \$1 in every state. For $E \subseteq S$ define $A_E \equiv \sum_{s \in E} A_s$. A_E is the composite asset which pays \$1 in the event E and zero otherwise. The assets are commitments to make contingent payments which are issued by some agents and bought by others. An agent who buys (sales) one unit of the asset A_s, $s \in S$, gets (pays) \$1 if the state is s and zero otherwise. A <u>bundle of assets</u> is a vector $z = (z_1, ..., z_{n+1}) \in R^{n+1}$ where z_k, $k = 1,..,n+1$, is the number of units of the asset A_k in the bundle. For $s \in S$ $z_s < 0$ means that $|z_s|$ units of the asset A_s have been sold. A bundle $z = (z_1, ..., z_{n+1})$ defines an <u>outcome</u> $x \in R^n$, $x = x(z)$, as follows:

For $s \in S$ $x_s \equiv z_s + z_{n+1}$.

x_s is the number of \$ that an agent who holds the bundle z will have if the true state is s.

We assume that each agent is restricted to the choice of bundles that generate outcomes in $X \equiv R^n_+$. (The reason for this restriction will become

[3] A bundle of assets z defines an outcome $x(z) \in R^n$ that specifies the amount of money that z generates at each state $s \in S$. To say that the preference of an agent i can be represented by the expectation of a state dependent utility from money $u(\cdot, s)$ means that i evaluates a bundle z by the expectation of $u(x(z)_s, s)$ w.r.t his subjective probability distribution γ^i.

clear later on). Thus, each agent $i \in I$ is characterized by:

(1) $m_i -$ an initial amount of money.

(2) $\psi_i -$ an information partition of S. For $s \in S$ $\psi_i(s)$ is the event that i knows at the state s.

(3) $\succsim_i \equiv \{\succeq_i^\gamma\}_{\gamma \in \Delta(S)}$ where $\succsim_i^\gamma$ is the preference relation of agent i on X w.r.t. the subjective probability distribution γ.

We make only two assumptions on $\succsim_i^\gamma$: For $x, y \in X$,

(1) Monotonicity, (M): If $x \geqq y$ then $x \succsim_i^\gamma y$ and if for some $s \in S$ s.t. $\gamma(s) > 0$ $x_s > y_s$ then $x \succ_i^\gamma y$.

(2) Null events don't count (N): If $x_s \neq y_s \Rightarrow \gamma(s) = 0$ then $x \sim_i^\gamma y$.

Remarks:

(1) The class of preferences that satisfy (M) and (N) include all the preferences that can be represented by an expectation of a monotone state-dependent utility function.

(2) Assumptions (M) and (N) allow for an incomplete preference. We say that a bundle z is an optimal choice for an agent i w.r.t $\succsim_i^\gamma$and a choice (budget) set B if there is no bundle $z' \in B$ such that $x(z') \succ_i^\gamma x(z)$.

(3) We assume that the function $m(i) = m_i$ is integrable and we normalize the aggregate amount of money to 1. Thus, $\int\limits_{i \in I} m_i = 1$.

We let $p = (p_s)_{s \in S}$ denote a vector of prices of the assets.

Since $A_S \equiv \sum\limits_{s \in S} A_s$ is equivalent to money non-arbitrage implies $\sum\limits_{s \in S} p_s = 1$. (Non-arbitrage is implied by the definition of an equilibrium which will be given later on.)

It is convenient to think of the economy as operating in two periods. In period 1 nature selects a state $\widehat{s}$. Each agent i gets his private signal $\psi_i(\widehat{s})$ and then as a function of the vector of prices p and his subjective belief γ^i on $\psi_i(\widehat{s})$ agent i chooses a bundle of assets z in his budget set $B(p, m_i)$ ($B(p, m_i)$ is defined in the next paragraph.) In period 2 the state $\widehat{s}$ becomes common knowledge and the transactions which the assets define are implemented.

The budget set $B(p, m)$ is defined as follows:

A vector $z \in R^n$ belongs to $B(p, m)$ iff:

(1) Income constraint (IC): $\sum_{s \in S} p_s \cdot z_s + z_{n+1} \leqq m$.

(2) No borrowing (NB): $z_{n+1} \geqq 0$.

(3) Complete Coverage (CC): $\forall s \in S \; -z_s \leqq z_{n+1}$.

The constraints (IC) and (NB) are standard. The constraint (CC) requires that an agent will be able to pay back at every state. In particular, if an agent sold $|z_s|$ units of A_s then (CC) requires that the amount of money that is available for him at the state $s, \$z_{n+1}$, is sufficient to cover the payment that he has to make which is $\$ - z_s$. To further motivate (CC) and get a better sense of the model we present lemma 1. Lemma 1 states that the purchase of an asset A_E, $E \subseteq S$, is equivalent to the sale of the complementary asset $A_{S \setminus E}$ in that both transactions generate the same outcome. Furthermore, the purchase of A_E satisfies (NB) iff the sale of $A_{S \setminus E}$ satisfies (CC). Put differently, If (CC) is relaxed then the agent is in effect in a situation where he can borrow money because any outcome that can be generated by borrowing money to buy some asset $A_E, E \subseteq S$, can also be generated by selling short the complementary asset $A_{S \setminus E}$.

Lemma 1:

Let p be a price vector s.t. $\sum_{s \in S} p_s = 1$ and let $m \geqq 0$ be an initial endowment of money. Let $\overline{z}_E^y$ be the bundle where the agent buys y units of the asset E and let $z_{S \setminus E}^y$ be the bundle where the agent sells y units of the asset $S \setminus E$, that is

$$(\overline{z}_E^y)_k \equiv \begin{cases} y & k \in E \\ 0 & k \in S \setminus E \\ m - y \cdot (\sum_{s \in E} p_s) & k = n+1 \end{cases}$$

$$(z_{S \setminus E}^y)_k \equiv \begin{cases} 0 & k \in E \\ -y & k \in S \setminus E \\ m + y \cdot (\sum_{s \in S \setminus E} p_s) & k = n+1 \end{cases}$$

then $x(\overline{z}_E^y) = x(z_{S \setminus E}^y)$ and $\overline{z}_E^y$ satisfies the constraint (NB) iff z_E^y satisfies (CC).

Proof:

It is easy to see that the definition of the outcome function $x(\cdot)$ and the equation $\sum_{s\in S} p_s = 1$ imply that $x(\overline{z}^y_E)_s = x(z^y_E)_s = \begin{cases} m + y\cdot(\sum_{s\in S\setminus E} p_s) & s\in E \\ m - y\cdot(\sum_{s\in E} p_s) & s\in S\setminus E \end{cases}$

Also, $\overline{z}^y_E$ satisfies the constraint (NB) iff $m \geq y\cdot(\sum_{s\in E} p_s) \Leftrightarrow y \leq \frac{m}{\sum_{s\in E} p_s} \Leftrightarrow$ $y \leq \frac{m}{1-\sum_{s\in S\setminus E} p_s} \Leftrightarrow y \leq m + y\cdot(\sum_{s\in S\setminus E} p_s)$. Now, we observe that $z^y_{S\setminus E}$ satisfies the constraint (CC) iff $y \leq m + y\cdot(\sum_{s\in S\setminus E} p_s)$. (The LHS is the payment that the agent will have to make at a state $s\in S\setminus E$ and the RHS is the amount of money that he has.)

Definition: A belief and demand realization (BDR) is a profile $(\gamma^i, z^i)_{i\in I}$ which specifies a belief $\gamma^i \in \Delta(S)$ and a bundle $z^i \in R^{n+1}$ for each agent $i \in I$.

We are now ready to give a definition of an equilibrium in the economy.

Definition: Let $(\gamma^i, z^i)_{i\in I}$ be a BDR and let $p = (p_s)_{s\in S}$ be a price vector. We say that $((\gamma^i, z^i)_{i\in I}, p)$ is an equilibrium at the state $\widehat{s}$ if

(1) Rationality: $\gamma^i \in \Delta(\psi_i(\widehat{s}))$ and z^i is optimal w.r.t $\succsim_i^{\gamma^i}$ in the budget set $B(p, m_i)$.

(2) Market Clearing: $\int_{i\in I} z^i = (0, ..., 0, 1)$.

Definition: A vector of prices p is an equilibrium at a state $\widehat{s}$ if there exists a BDR $(\gamma^i, z^i)_{i\in I}$ such that $((\gamma^i, z^i)_{i\in I}, p)$ is an equilibrium at $\widehat{s}$.

To state our main result we need two additional definitions. Let $\widehat{s} \in S$ and let $E \subseteq S$ be an event. Define $I^E_{\widehat{s}} \equiv \{i \,|\, i\in I \ \psi_i(\widehat{s}) \subseteq E\}$. $I^E_{\widehat{s}}$ is the set of agents who know the event E at the state $\widehat{s}$. Let $J \subseteq I$ be a measurable set of agents. Define $m(J) = \int_{i\in J} m_i$. $m(J)$ is the aggregate amount of money in the hands of agents in J. In particular, $m(I^E_{\widehat{s}})$ is the aggregate amount of money in the hands of agents who know E at the state $\widehat{s}$[4].

[4]We assume that for every $E \subseteq S$ and $\widehat{s} \in S$ $I^E_{\widehat{s}}$ is measurable.

Theorem 1:
The price vector $p = (p_s)_{s \in S}$ is an equilibrium price at a state $\widehat{s}$ iff:
1. $\sum\limits_{s \in S} p_s = 1$.
2. For every $E \subseteq S$ $m(I_{\widehat{s}}^E) \leq \sum\limits_{s \in E} p_s$.

Remarks:

1. The set of price vectors that satisfy conditions 1 and 2 is independent of the profile of preferences $\{\succsim^i\}_{i \in I}$ of the agents. Thus, theorem 1 implies in particular that for any profile of preferences that satisfy (M) and (N) the set of equilibrium prices is the same set.

2. For $E \subseteq S$ $\sum\limits_{s \in E} p_s$ is the price of the asset A_E. Since A_S is equivalent to money condition 1 is just a non-arbitrage constraint. Condition 2 states that for any event $E \subseteq S$ the price of the asset A_E is greater or equal to the aggregate amount of money in the hands of agents who know E at the state $\widehat{s}$.

3. To get some immediate sense for the result we consider two extreme cases:
(a) Every agent knows the true state $\widehat{s}$. In this case $m(I_{\widehat{s}}^{\{\widehat{s}\}}) = 1$ and therefore conditions 1 and 2 imply that $p_{\widehat{s}} = 1$ and for $s \neq \widehat{s}$ $p_s = 0$ (which is of course what we would expect.)
(b) No one knows anything, i.e., $\psi_i(\widehat{s}) = S$ for every $i \in I$. In this case $m(I_{\widehat{s}}^E) = 0$ for every $E \subsetneq S$ and therefore theorem 1 implies that a price vector p is an equilibrium price iff $\sum\limits_{s \in S} p_s = 1$.

4. To see the relationship of the result to the concept of the core define a cooperative game $G_{\widehat{s}}$ as follows: The set of players is S and the value of a coalition E, $E \subseteq S$, is $m(I_{\widehat{s}}^E)$. It is easy to see that a payoff vector $p = (p_s)_{s \in S}$ is in the core of the game $G_{\widehat{s}}$ iff it satisfies conditions 1 and 2.

We now present a simple example which on one hand demonstrates theorem 1 and on the other hand motivates the definition of common knowledge equilibrium (CKE)[5]

[5]Examples of a similar nature in the context of exchange economies are studied in Desgranges and Guesnerie (2002), Desgranges(2004), and Ben-Porath and Heifetz(2006). I present the example here because it nicely demonstrates theorem 1.

Example 1

$S = \{1,2\}$. $I = I_1 \cup I_2$ where $I_1 = [0,\delta]$ and $I_2 = (\delta,1]$. Every agent in I_1 knows the true state ($\psi_i(s) = s$) while every agent in I_2 does not know anything ($\psi_j(s) = S$.) All the agents have an initial endowment of \$1 and all of them evaluate an outcome by its expectation. That is, for every $i \in I$ $\gamma \in \Delta(S)$ and $x, y \in R^2$ $x \succsim_i^\gamma y$ iff $\gamma(1) \cdot x_1 + \gamma(2) \cdot x_2 \geq \gamma(1) \cdot y_1 + \gamma(2) \cdot y_2$[6]. Let $\overline{P}^s$ denote the set of equilibrium prices at the state s. We will now apply theorem 1 to solve $\overline{P}^1$. Table 1 summarizes the different constraints that are imposed by condition 2

Event E	$m(I_1^E) \leq \sum_{s \in E} p_s$
$\{1\}$	$\delta \leq p_1$
$\{2\}$	$0 \leq p_2$
$\{1,2\}$	$1 \leq p_1 + p_2$

Table 1

Adding the constraint $p_1 + p_2 = 1$ which is implied by condition 1 we obtain that

$\overline{P}^1 = \{p = (p_1, p_2) \mid \delta \leq p_1,\ p_1 + p_2 = 1\}$

We observe that the set $\overline{P}^1$ depends on δ, the fraction of agents who know the true state, in an intuitive way, as δ increases $\overline{P}^1$ shrinks. In a similar way we obtain that $\overline{P}^2$, the set of equilibrium prices at the state 2, is given by

$\overline{P}^2 = \{p = (p_1, p_2) \mid \delta \leq p_2,\ p_1 + p_2 = 1\}$.

We now use example 1 to motivate the introduction of CKE. Consider a price $p \in \overline{P}^1$ such that $0 < p_2 < \delta$ $(1 > p_1 > 1 - \delta)$. The price p does not belong to the set $\overline{P}^2$. This means that at the state 2 p is not consistent with the assumption of rationality and market clearing. Thus, an agent in I_2 who knows the parameters that define the economy, knows that every agent made a rational choice, and knows that the markets clear at p should conclude that the true state must be 1, but if all agents reach this conclusion then p cannot be a clearing price (the clearing price is $(1,0)$). Thus, p is not consistent with common knowledge (CK) of the parameters that define the economy ($\mathcal{E}$), the rationality of the agents (R), and market clearing (MC). We now present the concept of common knowledge equilibrium, CKE, and then use the result in theorem 1 to characterize the set of CKE prices[7].

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[6] Since an agent in I_1 knows the true state she assigns it a probability 1.

[7] The concept of CKE was first defined by Desgranges (2004). It is also studied in

Definition:

1. A price vector $p \in R^n$ is a common knowledge equilibrium (CKE) price w.r.t a set $\widehat{S} \subseteq S$ if for every $s \in \widehat{S}$ there exists an equilibrium $((\gamma_s^i, z_s^i), p)$ in s such that for every $i \in I$ $\gamma_s^i \in \Delta(\psi_i(s) \cap \widehat{S})$.

2. A price vector p is CKE at a state $\widehat{s}$ if there exists a set $\widehat{S}$ such that $\widehat{s} \in \widehat{S}$ and p is a CKE w.r.t. $\widehat{S}$.

The idea that underlies the definition of CKE is that if p can be supported at every state $s \in \widehat{S}$ by some profile of beliefs $\{\gamma_s^i\}_{i \in I}$ with support in $\widehat{S}$ (and which respect the private information of the agents) then p is consistent with CK of $\mathcal{E}$, R, and MC at any state $s \in \widehat{S}$ because each belief γ_s^i assigns a positive probability to some set of states $\widehat{S}_{i,s} \subseteq \widehat{S}$ in which p is an equilibrium price. Furthermore, in each state $s' \in \widehat{S}_{i,s}$ p is supported by beliefs $\left\{\gamma_{s'}^j\right\}_{j \in I}$ such that $\gamma_{s'}^j$ assigns a positive probability only to some set of states $\widehat{S}_{j,s'} \subseteq \widehat{S}$ in which p is an equilibrium price, and so forth.[8]

Define:

For $\widehat{s} \in S$ $P^{\widehat{s}} \equiv \{p \,|p$ is a CKE price at $\widehat{s}\}$

For $\overline{S} \subseteq S$ $P_{\overline{S}} \equiv \left\{p \,\middle|\, p \text{ is a } CKE \text{ price w.r.t } \overline{S}\right\}$

For $\overline{S} \subseteq S$ and $s \in \overline{S}$

$P_{\overline{S}}^s \equiv \left\{p \,\middle|\, \text{there exists an equilibrium } ((\gamma_s^i, z_s^i)_{i \in I}, p) \text{ at } s \text{ s.t. } \gamma_s^i \in \Delta(\psi_i(s) \cap \overline{S})\right\}$

The definition of CKE implies that

(1.1) $P^{\widehat{s}} = \cup_{\overline{S}, \widehat{s} \in \overline{S}} P_{\overline{S}} = \cup_{\overline{S}, \widehat{s} \in \overline{S}} (\cap_{s \in \overline{S}} P_{\overline{S}}^s)$

Consider example 1. We will compute P^1. From the left equation of (1.1) we have

(1.2) $P^1 = \cup_{\overline{S}, 1 \in \overline{S}} P_{\overline{S}} = P_{\{1\}} \cup P_{\{1,2\}}$.

$P_{\{1\}}$ is the set of equilibrium prices when the state is 1 and every agent assigns probability 1 to the state 1. Clearly, $P_{\{1\}} = \{(1, 0)\}$.

From the right equation in (1.1) we have

(1.3) $P_{\{1,2\}} = P_{\{1,2\}}^1 \cap P_{\{1,2\}}^2$.

Now,

(1.4) $P_{\{1,2\}}^1 = \overline{P}^1 = \{p \,|p_1 + p_2 = 1 \text{ and } \delta \leq p_1\}$ and similarly

(1.5) $P_{\{1,2\}}^2 = \overline{P}^2 = \{p \,|p_1 + p_2 = 1 \text{ and } \delta \leq p_2\}$

Ben-Porath and Heifetz (2006).

[8]The next draft will include a formal framework in which the argument that p is consisitent with CK of $\mathcal{E}, R$, and MC at a state $\widehat{s}$ iff p is a CKE at $\widehat{s}$ can be made precise.

(we remind that $\overline{P}^1$ and $\overline{P}^2$ are, respectively, the set of equilibrium prices at the state 1 and the set of equilibrium prices at the state 2.)

Putting (1.3), (1.4) and (1.5) together we obtain that

If $\delta \leq \frac{1}{2}$ $P_{\{1,2\}} = \{p\,|p_1 + p_2 = 1\ \delta \leq p_1 \leq 1-\delta\}$ and if $\delta > \frac{1}{2}$ then $P_{\{1,2\}} = \varnothing$.

Now from (1.2) we get

If $\delta \leq \frac{1}{2}$ then $P^1 = \{(1,0)\} \cup \{p\,|p_1 + p_2 = 1\ \delta \leq p_1 \leq 1-\delta\}$

If $\delta > \frac{1}{2}$ then $P^1 = \{(1,0)\}$.

Similarly,

If $\delta \leq \frac{1}{2}$ then $P^2 = \{(1,0)\} \cup \{p\,|p_1 + p_2 = 1\ \delta \leq p_1 \leq 1-\delta\}$

If $\delta > \frac{1}{2}$ then $P^2 = \{(1,0)\}$.

In particular, when $\delta > \frac{1}{2}$ then CK of $\mathcal{E}, R$, and MC select the fully revealing rational expectations equilibrium even when players have heterogeneous beliefs.

For $\overline{S} \subseteq S$ and $s \in \overline{S}$ the result in theorem 1 provides a characterization of the set $P^s_{\overline{S}}$. Specifically, define a cooperative game $G_{\overline{S},s}$ as follows: The set of players is $\overline{S}$. The value of a coalition (event) $E \subseteq \overline{S}$ is $m(I_s^{E\cup(S\setminus\overline{S})})$. Let $C(G_{\overline{S},s})$ denote the core of the game $G_{\overline{S},s}$. We claim that theorem 1 implies that $P^s_{\overline{S}} = C(G_{\overline{S},s})$. To see this we remind that $p \in P^s_{\overline{S}}$ iff there is a profile of beliefs $(\gamma_s^i)_{i\in I}$ and a profile of bundles $(z_s^i)_{i\in I}$ such that $((\gamma_s^i, z_s^i)_{i\in I}, p)$ is an equilibrium at s and

(1.6) $\gamma_s^i \in \Delta(\psi_i(s) \cap \overline{S})$.

Define now an economy $\mathcal{E}_{\overline{S}}$ that is obtained from the original economy $\mathcal{E}$ by restricting the set of states to $\overline{S}$ and defining the information partition of an agent i, $\overline{\psi}_i$, as follows:

For $s' \in \overline{S}$ $\overline{\psi}_i(s') \equiv \psi_i(s') \cap \overline{S}$.

Clearly, it follows from (1.6) that $p \in P^s_{\overline{S}}$ iff p is an equilibrium price in the economy $\mathcal{E}_{\overline{S}}$ at the state s. It is easy to see that in the economy $\mathcal{E}_{\overline{S}}$ the set of agents who know the event $E \subseteq S$ at the state s is $I_s^{E\cup(S\setminus\overline{S})}$. The claim now follows.

Putting this together with (1.1) we obtain theorem 2 which characterizes the set of prices, $P^{\widehat{s}}$ that are CKE at a state $\widehat{s}$.

<u>Theorem 2:</u> $P^{\widehat{s}} = \cup_{\overline{S},\widehat{s}\in\overline{S}} \cap_{s\in\overline{S}} C(G_{\overline{S},s})$.

3 The proof of theorem 1.

We show, first, that if p is an equilibrium price at a state $\widehat{s}$ then p satisfies conditions (1) and (2). Start with condition (1). Since the asset $A_S \equiv \sum\limits_{s\in S} A_s$ is equivalent to money the argument that the price of A_S, $\sum\limits_{s\in S} p_s$, must equal 1 is an argument that points to the possibility of arbitrage if this equation is not satisfied. Specifically, assume by contradiction that $\sum\limits_{s\in S} p_s > 1$. For every number $x > 0$ an agent i can obtain an outcome which gives $\$x \cdot (\sum\limits_{s\in S} p_s - 1)$ in every state s by selling x units of A_S. (Condition (2) in the definition of the budget set is satisfied because the sale commits agent i to a payment of $\$x$ in each state while the amount of money in his hands is $\$1 + x \cdot (\sum\limits_{s\in S} p_s)$.) It follows that each agent can obtain an unbounded amount of money in every state. Since the aggregate amount of money in the economy is 1 p cannot be a clearing price. Similarly, assume by contradiction that $\sum\limits_{s\in S} p_s < 1$. An optimal bundle z^* must satisfy $z^*_{n+1} = 0$. Otherwise, the agent could obtain an outcome that pays better in every state by using his $\$z^*_{n+1}$ to buy the asset A_S. Now, if every agent does not want to hold money then there is an excess supply of money and therefore p is not an equilibrium price.

We turn now to condition (2).

Lemma 2: Let $p \in R^n$ be a price vector such that $\sum\limits_{s\in S} p_s = 1$ and let $z \in R^{n+1}$ be a bundle for agent i in the budget set $B(p, m_i)$. There exists a bundle $\overline{z} \in B(p, m_i)$ such that:

(1) $\overline{z}_k \geq 0$ for every $k = 1, .., n+1$.

(2) $x(\overline{z}) = x(z)$.

In words, every outcome that can be generated by a bundle of assets in $B(p, m_i)$ can also be generated by a bundle in which agent i does not sell any asset.

The proof of the lemma is similar to the proof of lemma 1 and is given in the appendix. We now show how it implies condition (2). Let p be an equilibrium price in $\widehat{s}$ and assume by contradiction that there exists an event E such that $m(I^E_{\widehat{s}}) > \sum\limits_{s\in E} p_s$. Let $((\gamma^i, z^i)_{i\in I}, p)$ be an equilibrium at $\widehat{s}$. Lemma 2 implies that for any $i \in I$ there exists a bundle $\overline{z}^i$ such that $\overline{z}^i_k \geq 0$ for every

$k = 1, .., n+1$ and $x(\overline{z}^i) = x(z^i)$. Since z^i is optimal for agent i in $B(p, m_i)$ so is $\overline{z}^i$. We claim that $\overline{z}^i_k = 0$ for $k \notin E$. To see that we observe that for $s \notin E$ agent i will not buy the asset A_s because he assigns the state s probability zero. Also, $\overline{z}^i_{n+1} > 0$ is impossible because agent i assigns probability 1 to the event E and therefore could obtain a better outcome by spending the $\$\overline{z}^i_{n+1}$ on buying the asset A_E. For $s \in E$ define $\overline{m}^i_s \equiv p_s \cdot \overline{z}^i_s$ and $\overline{m}_s \equiv \int\limits_{i \in I^E_{\widehat{s}}} \overline{m}^i_s$.

Thus, $\overline{m}^i_s$ is the number of \$ that agent i spends on buying the asset A_s and $\overline{m}_s$ is the aggregate amount of money that agents in $I^E_{\widehat{s}}$ spend on A_s. Since $\overline{z}^i_k = 0$ for every $i \in I^E_{\widehat{s}}$ and $k \notin E$ we have $\sum\limits_{s \in E} \overline{m}_s = m(I^E_{\widehat{s}})$. It follows that $\sum\limits_{s \in E} \overline{m}_s > \sum\limits_{s \in E} p_s$ and therefore there exists a state $\widetilde{s}$ such that $m_{\widetilde{s}} > p_{\widetilde{s}}$. This last inequality implies that

$$\int\limits_{i \in I^E_{\widehat{s}}} x(\overline{z}^i)_{\widetilde{s}} > 1. \tag{3.1}$$

That is, the aggregate amount of money that agents in $I^E_{\widehat{s}}$ obtain in the state $\widetilde{s}$ is larger than 1. Since the aggregate amount of money in the economy is 1 p cannot be a clearing price. Thus, we have obtained a contradiction to the assumption that there exists an event E that does not satisfy condition (2).

We have shown that if p is an equilibrium price at a state $\widehat{s}$ then conditions (1) and (2) must be satisfied. We now show that if p satisfies conditions (1) and (2) w.r.t. the state $\widehat{s}$ then p is an equilibrium price at $\widehat{s}$. Lemma 3 below plays a central role in this part.

Lemma 3: Let $p = (p_1, ..., p_n)$ be a price vector that satisfies conditions (1) and (2) w.r.t. the state $\widehat{s}$. There exists a partition of I, $\widehat{I}_1, ..., \widehat{I}_n$, such that for $s = 1, ..., n$:

(a) If $i \in \widehat{I}_s$ then $s \in \psi_i(\widehat{s})$.

(b) $m(\widehat{I}_s) = p_s$.

We first prove the theorem from the lemma. Define a profile of subjective beliefs as follows: For $i \in \widehat{I}_s$ $\gamma^i(s) = 1$ (and $\gamma^i(s') = 0$ for $s' \neq s$). Define $\widetilde{S} \equiv \{s \, | p_s > 0\}$. We can ignore the agents in $\cup_{s \in S \setminus \widetilde{S}} I_s$ because $m(\cup_{s \in S \setminus \widetilde{S}} I_s) = 0$ and thus their behavior does not influence the equilibrium price. Let $\widetilde{s}$ be some arbitrary state in $\widetilde{S}$. For every $s \in \widetilde{S} \setminus \{\widetilde{s}\}$ an agent i that belongs to the set I_s chooses a bundle z^i in which he spends all his money on the purchase of the asset A_s. Since $m(\widehat{I}_s) = p_s$ the aggregate demand for each asset $A_s, s \in \widetilde{S} \diagdown \{\widetilde{s}\}$,is 1. Consider now the agents in $\widehat{I}_{\widetilde{s}}$. Define $\overline{S} \equiv \widetilde{S} \diagdown \{\widetilde{s}\}$. Each agent $i \in I_{\widetilde{s}}$ chooses a bundle z^i where he uses all his income as a cover for the sale of the asset $A_{\overline{S}}$. Let y^i denote the number of units of $A_{\overline{S}}$ that i sells ($y^i = |z^i_s|$ for $s \in \overline{S}$). Since i uses all his income as a cover for the sale of $A_{\overline{S}}$ y^i is defined by the equation $m_i + y^i \cdot (\sum_{s \in \overline{S}} p_s) = y^i$ (The RHS is the payment that i will have to make at a state $s \in \overline{S}$ while the LHS is the amount of money that he holds.) It follows that

(3.2) $y^i = \frac{m_i}{1 - \sum_{s \in \overline{S}} p_s} = \frac{m_i}{p_{\widetilde{s}}}$.

It is easy to see that since $\gamma^i(\widetilde{s}) = 1$ the outcome that z^i generates is equivalent to the outcome that is generated by a bundle where agent i spends all his income on the purchase of the asset $A_{\widetilde{s}}$ (In both cases agent i gets $\$\frac{m_i}{p_{\widetilde{s}}}$ in the state $\widetilde{s}$.) It follows that z^i is an optimal choice for agent i w.r.t γ^i. Also, since $m(\widehat{I}_{\widetilde{s}}) = p_{\widetilde{s}}$, (3.2) implies that the aggregate supply of $A_{\overline{S}}$ by the agents in $I_{\widetilde{s}}$ is 1. It follows that the markets for all the assets clear and thus $((\gamma^i, z^i)_{i \in I}, p)$ is an equilibrium at $\widehat{s}$.

Proof of lemma 3:

The proof of lemma 3 is based on a continuous version of the famous marriage lemma (Hall 1935) which is due to Hart and Kohlberg (1974)[9].

Lemma 4 (Hart and Kohlberg):

Let $(\Omega, \mathcal{B}, \mu)$ be a non-atomic measure space and let $\{F_i\}_{i=1}^n \subset \Omega$ and $\{\alpha_i\}_{i=1}^n \in R_+$ such that for all $L \subseteq \{1, .., n\}$:

(1) $\mu(\cup_{i \in L} F_i) \geq \sum_{i \in L} \alpha_i$

(2) $\mu(\cup_{i=1}^n F_i) = \sum_{i=1}^n \alpha_i$.

Then there exist disjoint sets $\{T_i\}_{i=1}^n$ such that $T_i \subseteq F_i$ and $\mu(T_i) = \alpha_i$.

Lemma 3 is now proved as follows. Let $(\Omega, \mathcal{B}, \mu)$ be the measure space where $\Omega = I = [0, 1]$, $\mathcal{B}$ is the set of Borel sets and for $J \in \mathcal{B}$ μ is defined by $\mu(J) \equiv \int_{i \in J} m_i$. (That is, $\mu(J)$ is the aggregate amount of money in the

[9]I thank Sergiu Hart for pointing out that lemma 4 can be used to prove lemma 3.

hands of agents in J.) For $s \in S$ define $J_s \equiv \{i \,|s \in \psi_i(\widehat{s})\}$. Condition (2) states that for every $E \subseteq S$ $\sum\limits_{s \in E} p_s \geqq \mu(I_{\widehat{s}}^E)$. Since $\sum\limits_{s \in S} p_s = 1$ and $\mu(I) = 1$ we obtain that $\sum\limits_{s \in E^c} p_s \leqq \mu((I_{\widehat{s}}^E)^C)$ where $(I_{\widehat{s}}^E)^C$ is the complement of $I_{\widehat{s}}^E$. Now since $(I_{\widehat{s}}^E)^C = \cup_{s \in E^C} J_s$ we obtain that for every $E \subseteq S$ $\sum\limits_{s \in E^C} p_s \leqq \mu(\cup_{s \in E^C} J_s)$. This, of course, means that for every $E \subseteq S$ $\sum\limits_{s \in E} p_s \leqq \mu(\cup_{s \in E} J_s)$.In addition, we have $\sum\limits_{s \in S} p_s = \mu(\cup_{s \in S} J_s) = 1$. Applying lemma 4 by setting $p_s \equiv \alpha_s$ and $J_s \equiv F_s, s \in S$, we obtain that there exist disjoint sets $\widehat{I}_s, s \in S$, such that $\widehat{I}_s \subseteq J_s$ and $m(\widehat{I}_s) = \mu(\widehat{I}_s) = p_s$. The proof of lemma 3 is now complete.

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Appendix

Proof of lemma 2:

Let $z \in B(p, m_i)$.

Define $\overline{S} \equiv \{s \,|s \in S \; z_s < 0\}$. $\overline{S}$ is the set of assets that are sold in the bundle z. Define $y \equiv \min_{s\in\overline{S}} |z_s|$. We will now construct a bundle $\overline{z}$ such that $\overline{z} \in B(p, m_i), x(\overline{z}) = x(z)$, and such that the set of assets that are sold in $\overline{z}$ is strictly contained in $\overline{S}$. the bundle $\overline{z}$ is obtained from z by reducing the sale of each asset $A_s, s \in \overline{S}$, by y units and then using $\$y \cdot (\sum_{s\in S\setminus\overline{S}} p_s)$ to buy y units of the asset $A_{S\setminus\overline{S}}$. Formally, the bundle $\overline{z}$ is defined as follows:

$$\overline{z}_k \equiv \begin{cases} z_k + y & k \in S \\ z_{n+1} - y & k = n+1 \end{cases}$$

It is easy to see that $x(\overline{z}) = x(z)$. To see that $\overline{z}$ does indeed belong to the budget set $B(p, m_i)$ we observe that since the sales of the asset $A_{\overline{S}}$ in the bundle $\overline{z}$ is lower by y units in comparison to the sale in the bundle z the commitment of agent i to pay back is lower by $\$y$. On the other hand the reduction in the sales of $A_{\overline{S}}$ decreases the amount of money in the hands of the agent by $\$y \cdot (\sum_{s\in\overline{S}} p_s)$. Putting this together we see that the reduction of y units in the sale of the asset $A_{\overline{S}}$ releases $\$(y - y \cdot (\sum_{s\in\overline{S}} p_s)) = \$y \cdot (\sum_{s\in S\setminus\overline{S}} p_s)$ that can be used for the purchase of the asset $A_{S\setminus\overline{S}}$. It follows that $\overline{z}$ belongs to $B(p, m_i)$. Also, the set of assets that are sold in $\overline{z}$ is strictly contained in $\overline{S}$. If $\overline{z} \geqq 0$ we are done. Otherwise, we repeat the procedure and obtain, after at most $\left|\overline{S}\right|$ steps, a bundle $\widetilde{z} \in B(p, m_i)$ such that $\widetilde{z} \geqq 0$ and $x(\widetilde{z}) = x(z)$.