Exploitation and Cooperation in Networks*

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Abstract

We consider a game where agents form a network of collaborations to exchange non-rival information. Payoffs correspond to the discounted outcomes of an infinitely repeated game played in the network. In each bilateral interaction, a player decides whether to give his neighbor access to his information. This determines the information flow in the network and therefore the gross benefits to each player. Each player also chooses between cooperating or defecting in a prisoner's dilemma game with his neighbor. The play of this game determines the costs of that interaction and how players share these costs. Our results point toward the reconciling of efficient and stable networks. Players reward efficient plays by transmitting valuable information, while punishing deviations by withholding information. Bilateral punishments propagate over the network and create social punishments that sustain efficient outcomes.

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1 Introduction

Free-riding problems and network externalities characterize many interesting economic and social interactions. Consider the simplest case of a relationship between two agents, who may be friends, firms involved in R&D collaboration, co-authors, doctors, farmers. On the one hand, the relationship bring benefits to the agents involved. Friends exchange information about products and job opportunities, farmers share information about different technologies they have experimented. Similarly, doctors exchange information about different medicines. Co-authors exchange knowledge. In all these situations part of the information exchanged is intrinsic to the agent and another part is obtained from interactions with other agents in their network.[1] On the other hand, maintaining a relationship is costly in terms of effort and time and in many instances an agent has incentives to free-ride on his acquaintance in order to reduce his costs in that relationship. In this paper, we study the trade-off between efficient outcomes and equilibria in environments where social interactions are repeated over time and generate positive externalities and free-riding problems.

We consider a network formation model where network payoffs are endogenously determined by an infinitely repeated game. Each player is endowed with some non-rival and valuable information, which is renewed at every period. Players form connections to exchange information. At the beginning of the game a network forms as a result of a simultaneous game of link announcements. Mutual consensus is required for link creation. Once the network is in place linked players interact for an infinite number of periods. Specifically, at each period each pair of connected players, say Brian and Rob, play simultaneously two games.

We call the first of these two games the *accessibility game*. Here, Brian (Rob) decides either to withhold or to provide the information he accesses in the network to Rob (Brian). If Brian gives Rob access to his information, then information flows from Brian to Rob and from Rob to every other player to whom Rob gives access. Thus, the play of this game determines how information flows in the network and it defines the (gross) benefit that each player obtains from the network. The second game is a standard *prisoner's dilemma game*, where Brian (Rob) decides whether to cooperate or defect with Rob (Brian). The play of this game determines the total costs of the interaction between Brian and Rob and how they share these costs. When Brian and Rob play symmetrically, they share evenly the costs (at the defection level or at the cooperative level); otherwise, the cooperator bears completely the costs (at the exploitative level).

[1] There is a persuasive body of empirical work which illustrates the importance of information exchange in social and economic newtorks. For example, Coleman (1966) presented evidence on how a doctor's prescription of new drugs was influenced by his location in communication networks. Conley and Udry (2004) and Foster and Rosenzweig (1995) present evidence that farmers are influenced by their neighbors in the choice of crops and agricultural inputs. In the context of brand and product choice, Feick and Price (1987), Godes and Mayzlin (2004) present evidence for word of mouth communication.

Thus, at each period, the utility of a player is the total amount of information he accesses, net of the costs he bears in all his interactions. We consider the aggregate discounted utilities for the whole game. For an equilibrium, we require that the network formed in the first stage is a pairwise equilibrium and that the strategy profile is a sequential equilibrium. An outcome is efficient if it maximizes the sum of the utilities of all players.

We first analyze a setting where information flows along network paths without decay. Our findings point towards reconciling efficiency and stability. The reason is the following. On the one hand, efficiency is obtained in minimally connected networks with full flow of information.[2] In fact, a minimally connected network minimizes the costs of connectivity, yet allows for maximal network externalities. On the other hand, players may reward efficient play by giving access to information, while punishing deviations by refusing access. In minimally connected networks, these bilateral punishments spread over the network, inducing *social punishments* that sustain efficient outcomes.

The effectiveness of such punishments depends on the architecture of the network. We then characterize the efficient equilibrium that exists for the widest range of parameters, which we call *best equilibrium*. We distinguish between two cases. In the first case efficiency requires that, for each link, one player free-rides and the other cooperates. We name this an *exploitative efficient outcome*. In this situation, the best equilibrium has the following features: the star network forms, the center of the star always free-rides and the peripheral players always cooperate.[3] The reason behind this result is that the central player of the star implements social punishments immediately. Indeed, if a player linked with the center deviates at some period t, at $t+1$ the center will withhold the information to him so that he will not access any information in the network. In contrast, in any other minimally connected network, social punishments are implemented with some delay.

In the second case, efficiency requires that, for each link, both players mutually cooperate. We name this a *cooperative efficient outcome*. Here, the line network is the unique network which is part of a best equilibrium.[4] The reason being that the incentives of players to play inefficiently is increasing in the number of links a player has. Therefore, a"symmetric" allocation of links is crucial to internalize network externalities, which leads to the line network.

Finally, we extend the analysis to imperfections in the information flow. Our results show that, for a wide range of the decay in the information flow, the exploitative efficient outcome is easier to sustain at equilibrium than the cooperative efficient

[2] A minimally connected network is a network in which each pair of players is connected via a unique sequence of links.

[3] A star network is a minimally connected network in which a player, named the center, has direct links with all the other players, named peripheral players, who are only linked to the center.

[4] A line network is a minimally connected network in which there are exactly two players having only one link.

outcome. When decay in the information flow is very high, the tension between individual and social incentives is severe in both cases.

Section 2 discusses the related literature. Section 3 introduces the model. Section 4 analyses efficient outcomes and efficient equilibria when information flow is perfect. Section 5 studies the case of imperfect information flow. Section 6 concludes. All proofs are provided in the Appendix.

2 Related Literature

Our paper relates to the economic literature of network formation and repeated games. The tension between stable and efficient networks is, since the seminal work by Jackson and Wolinsky (1996), a classical topic in the network literature, e.g. Bala and Goyal (2000), Bloch and Jackson (2004), Currarini and Morelli (2000), Dutta and Mutuswami (1997), and Jackson (2004, 2005). In our model network payoffs are endogenously determined as the discounted outcome of an infinitely repeated game, and therefore allows for a richer class of payoff functions. In this setting, our findings point towards reconciling efficiency with stability.

The economic literature on repeated games is well studied, e.g. Abreu (1988), Abreu and Rubinstein (1988), Ellison (1994), Kandori (1992). This literature considers that players are randomly matched. In contrast, we consider that players' interactions take place in a fixed and endogenous pattern of play.

We now discuss three related papers which study infinitely repeated games in networks. Haag and Lagunoff (2004) analyze repeated prisoners' dilemma games played in given networks. There, agents have different discount factors and the authors investigate the network architectures which support maximum degree of cooperation. Vega-Redondo (2004) and Lippert and Spagnolo (2005) study models where connected players interact according to a collection of infinitely repeated prisoner's dilemma games. In these two papers, the strategic effects of networks result from the fact that players may communicate, through their links, information about the behavior of their neighbors: today Rob may tell to Alice that yesterday Brian deviated. Clearly, the possibility of exchanging information about behavior of other players does not help to sustain cooperation if Alice is neither linked to Brian nor linked to a friend (or a friend of a friend, etc.) of Brian. That is, in order that this mechanism helps to sustain cooperation, the social network cannot be a minimal network.

A novel feature of our work is the presence of endogenous network externalities. We show that players may use network externalities to implement social punishments and therefore to sustain efficient outcomes. This is a different mechanism from the one proposed by Vega-Redondo (2004) and Lippert and Spagnolo (2005).

3 Model

Let $N = \{1, 2, ..., n\}$ be the set of players. To avoid trivialities, we shall assume throughout that $n \geq 3$. Time is considered discrete, $t \in \{0, 1, 2, ...\}$. At every period t, each player is endowed with a non-rival good of value v, to which we refer as information hereafter. We consider the following infinitely repeated game.

t=0: Network Formation

At $t = 0$, each $i \in N$ chooses $\gamma_{i,j} \in \{0, 1\}$, $\forall j \in N\backslash\{i\}$. $\gamma_{i,j} = 1$ means that i proposes a link to j, while $\gamma_{i,j} = 0$ signifies that i does not propose a link to j. Players propose links to create the possibility of exchanging information. A link between two players, i and j, is formed if and only if $\min\{\gamma_{i,j}, \gamma_{j,i}\} = 1$. We denote by g the $n \times n$ symmetric matrix such that $g_{i,i} = 0$ for all $i \in N$, and $g_{i,j} = \min\{\gamma_{i,j}, \gamma_{j,i}\}$ $\forall i, j \in N$, $i \neq j$. Thus, g represents the undirected network induced by $\gamma = \{\gamma_{i,j}\}_{i,j \in N}$. We denote by $N_i(g) = \{j \in N\backslash\{i\} : g_{i,j} = 1\}$ the set of players linked with i and $\mu_i(g) = |N_i(g)|$.

t>0: Infinitely Repeated Game

- *Prisoner's Dilemma Game (PDG).* At each $t > 0$, each pair of linked players, say i and j, plays the PDG described in Table 1, where we assume that $e = 0 > c > d > f$, $2d < f$. We denote by $\alpha_{i,j}^t \in \{C, D\}$ the action for the PDG that player i plays against player j. As usual, C means cooperation and D defection.

${}_i\backslash^j$	C	D
C	c, c	f, e
D	e, f	d, d

Table 1

Thus, if i and j play the same action, i.e. they play symmetrically, the total cost of their link is either $2c$, if they cooperate, or $2d$, if they defect. In both cases, they share evenly the total cost of the link. If i and j play asymmetrically, then the total cost of the link is f and the cooperator has to meet such cost. We denote by $\phi_{i,j}(\alpha_{i,j}^t, \alpha_{j,i}^t)$ the cost of player i associated to the link $g_{i,j} = 1$.

- *Accessibility Game (AG).* At each $t > 0$, simultaneously to the PDG, each $i \in N$ chooses $\lambda_i^t = (\lambda_{i,1}^t, ..., \lambda_{i,n}^t)$, $\lambda_{i,j}^t \in \{0, 1\}$ for all $j \in N$. $\lambda_{i,j}^t = 1$ signifies that i gives j access to information. Of course, the information can effectively flow directly from i to j only if $g_{i,j} = 1$. Formally, we construct the directed flow network $\tilde{g}^t$, where, for every $i, j \in N$, $\tilde{g}_{i,j}^t = \min\{g_{i,j}, \lambda_{j,i}^t\}$. That is, i accesses directly the information from j if and only if there exists a link between i and j and j gives information access to i.

A *flow path* from j to i is denoted by $j \xrightarrow{\tilde{g}^t} i$. There is a flow path from j to i if there exists $\{j_1, ..., j_m\} \subseteq N$ such that $\tilde{g}_{j_1,j_2}^t = ... = \tilde{g}_{j_{m-1},j_m}^t = 1$, where $j_1 = i$ and $j_m = j$. Given $\tilde{g}^t$, the set of players that i accesses at period t is $\tilde{N}_i\left(\tilde{g}^t\right) = \{j \in N\backslash\left\{i\right\} : j \xrightarrow{\tilde{g}^t} i\}$.

Let $\tilde{g}^t_{i,j} = 1$, then the set of players that agent i accesses exclusively via a path containing j is $\mathcal{I}_{i,j}(\tilde{g}^t) = \tilde{N}_i(\tilde{g}^t)\backslash\tilde{N}_i(\tilde{g}^t_{-j})$, where $\tilde{g}^t_{-j}$ is the directed network induced by $\tilde{g}^t$ on $N\backslash\{j\}$. Let $I_{i,j}(\tilde{g}^t) = |\mathcal{I}_{i,j}(\tilde{g}^t)|$, then $I_{i,j}(\tilde{g}^t)$ is the amount of information that i would not obtain once j is removed from the network.[5]

Strategy Profiles. At each period $t \geq 1$, we denote by $A_i = (\{C, D\} \times \{0, 1\})^{n-1}$ the set of actions of a player $i \in N$. An arbitrary $a^t_i \in A_i$ consists of a choice $\alpha^t_{i,j} \in \{C, D\}$ and a choice $\lambda^t_{i,j} \in \{0, 1\}$ for all $j \in N\backslash\{i\}$.[6] We assume that, at each $t \geq 1$, each player $i \in N$ observes:

(i) the network, i.e. g;

(ii) his own past actions, i.e. a^τ_i, $\forall\tau < t$;

(iii) the past actions of all his neighbors in their interactions with i, i.e. $(\alpha^\tau_{j,i}, \lambda^\tau_{j,i})$, $\forall j \in N_i(g)$, $\forall\tau < t$;

(iv) the amount of exclusive information that i has obtained from each neighbor and vice-versa, i.e. $I_{i,j}(\tilde{g}^t)$ and $I_{j,i}(\tilde{g}^t)$, $\forall j \in N_i(g)$, $\forall\tau < t$.

The set of histories for player i at period t, $\mathcal{H}^t(i)$, is formed by all feasible observations of agent i at period t. We focus on the set of pure strategy profiles, that we denote by S. A pure strategy of player i, $s_i = \{\gamma_i, \omega^1_i, ..., \omega^t_i, ...\}$, consists on a set of link proposals, $\gamma_i \equiv \{\gamma_{i,j}\}_{j\in N\backslash\{i\}}$, and a sequence of functions, $\omega^1_i, ..., \omega^t_i, ...$, each of them mapping from the set of histories of player i to his set of actions, i.e. $\omega^t_i : \mathcal{H}^t(i) \to A_i$, $\forall t \geq 1$. We denote by $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$ a pure strategy profile, where $\omega^t = \{\omega^t_i\}_{i\in N}$.

Payoffs, Equilibrium and Efficiency. Generally, information flow is subject to decay. We measure the level of decay of the information flow by a parameter $\rho \in (0, 1]$. Given a flow-network $\tilde{g}^t$, the distance between i and j, denoted by $d(i, j; \tilde{g}^t)$, is the number of links in the shortest flow path from j to i. We assume that the value of $j's$ information to i is $v\rho^{d(i,j;\tilde{g}^t)}$. Given $s \in S$, the utility of player i at period t is:

$$u^t_i(s) = \sum_{j\in\tilde{N}_i(\tilde{g}^t)} v\rho^{d(i,j;\tilde{g}^t)} + \sum_{j\in N_i(g)} \phi_{i,j}(\alpha^t_{i,j}, \alpha^t_{j,i}).$$

Summing up across periods, we have that:

$$u_i(s) = \sum_{t=1}^{\infty} \delta^{t-1}u^t_i(s),$$

[5] For example in a star network, the center provides $n - 1$ pieces of information to each of his neighbors. Note that the same definitions apply at $t = 0$ once we impose that $\tilde{g}^0 = g$.

[6] Clearly, given a network g, only $(\alpha^t_{i,j}, \lambda^t_{i,j}), \forall j \in N_i(g)$ is in fact relevant. However, our notation allows to make the action space A_i invariant to the set of feasible networks.

where $\delta \in (0,1)$ is the common discount factor. The social welfare generated by $s \in S$ is:

$$V(s) = \sum_{i \in N} u_i(s).$$

Note that, when $\rho = 1$, information flows along links without decay. We focus on this case in the first part of our analysis.

In this model, for an equilibrium we require two conditions. The first condition is related to the network formation stage: the network $g(\gamma)$ must be pairwise equilibrium.[7] A network is a pairwise equilibrium whenever no pair of players has incentives to form an additional link and no individual player has incentives to remove any subset of his links. Second, we require that $\{\omega^1, ..., \omega^t, ...\}$ is a sequential equilibrium. Formally,

Definition 3.1 *Consider a strategy profile $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$. The network $g(\gamma)$ is a pairwise equilibrium if:*

(i) γ is a Nash equilibrium.

(ii) For each $i, j \in N$, if $g_{i,j}(\gamma) = 0$ then $u_i(\tilde{s}) > u_i(s) \Rightarrow u_j(\tilde{s}) < u_j(s)$, where $\tilde{s}$ results from s by setting $\gamma_{i,j} = \gamma_{j,i} = 1$ (i.e. $g_{i,j}(\gamma) = 1$).

Definition 3.2 *A strategy profile $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$ is an equilibrium if:*

(i) $g(\gamma)$ is a pairwise equilibrium.

(ii) $\{\omega^1, ..., \omega^t, ...\}$ is a sequential equilibrium.

We note that the above definition of equilibrium is equivalent to require that a strategy s is a sequential equilibrium and every pair of disconnected players do not have an incentive to create a link between them. To conclude the specification of the model, we introduce the notion of efficiency. There are different ways of measuring efficiency; we follow the convention in this literature and focus on the sum of payoffs of all players. Formally,

Definition 3.3 *A strategy profile s is efficient if $V(s) \geq V(\hat{s})$ for any other strategy profile $\hat{s} \in S$.*

Our notion of efficiency is equivalent to the concept of strong efficiency in Jackson and Wolinsky (1996).[8]

[7] Goyal and Joshi (2005) introduce the notion of pairwise equilibrium. Pairwise equilibrium is an extension of the notion of pairwise stability introduced by Jackson and Wolinsky (1996).

[8] An alternative definition would be in term of Pareto dominance. In settings where utility is not transferable, efficient networks are always Pareto-efficient, but the converse is generally not true. However, if payoffs are transferable across players, then clearly the two notions are equivalent.

4 Perfect Information Flow

As a benchmark case we study the trade-off between efficient outcomes and equilibrium outcomes when information in the network flows without decay, i.e. $\rho = 1$. The following definitions are useful to state our results. Given a network g, we say there is a *path of links* between j and i if, either $g_{i,j} = 1$, or there exists a sequence of players $\{j_1, ..., j_m\} \subseteq N \backslash \{i, j\}$, such that $g_{i,j_1} = g_{j_1,j_2} = ... = g_{j_m,j} = 1$. A network g is connected if there exists a path between every pair of players. A network g is minimally connected if, between every pair of players, there is a unique path. The empty network is a network g with no links. A network g has a star architecture and player i is the central player if $g_{i,j} = 1$, $\forall j \in N \backslash \{i\}$, and there are no other links. A line is a minimally connected network where there are only two players which have only one link.

4.1 Efficient Outcomes

We start by characterizing efficient outcomes.

Proposition 4.1 *Assume $\rho = 1$.*

I. If $nv + \max\{2c, f\} > 0$, then s is efficient if and only if g is minimally connected and, for each $t \geq 1$, each player gives his neighbors access to information and the total cost for each link is $\max\{2c, f\}$.

II. If $nv + \max\{2c, f\} \leq 0$, then s is efficient if and only if g is the empty network.

Part I of Proposition 4.1 shows that, for sufficiently low costs of linking, the efficient outcome is obtained when a minimally connected network forms and every player gives his neighbors access to his information. It is useful to distinguish between two situations.

The first is a situation in which efficiency requires that, for each link, one of the players cooperates while the other defects, i.e. $f > 2c$. For example, if the maintenance of a link requires players to exert a total amount of effort and the costs of effort for each player are concave, it is more efficient to let one player to provide the necessary amount of effort.[9] We refer to this case as the *exploitative efficient outcome*. The corresponding social welfare is $(n-1)(nv + f)/(1-\delta)$.

The second situation is one where efficiency requires players to mutually cooperate in every link, i.e. $2c > f$. This would occur when players must exert effort to maintain links and the costs of effort for a player are convex. In this case splitting the maintenance tasks between the two parties is more efficient than leaving one

[9] This will be the case when the maintenance of the link requires coordination across players or the opening of different bureaucratic procedures.

player to bear all the cost. We refer to this case as the *cooperative efficient outcome*. The corresponding social welfare is $(n-1)(nv+2c)/(1-\delta)$.

Part II of Proposition 4.1 tells us that when costs of linking are very high, then the efficient outcome is obtained in the empty network. Note that in this case an efficient equilibrium always exists: players form the empty network and, for every other network, they defect and they do not transmit information. We then focus on the case where $\max\{2c, f\} + nv > 0$.

4.2 Efficient Equilibria

We now study when efficient equilibria exist. In this setting, we focus on maximal punishment strategies and we characterize the best equilibria. A best equilibrium is an efficient equilibrium which exists for the widest range of parameters.

4.2.1 Exploitative Efficient Equilibria

We start by analyzing exploitative efficient equilibria.

Proposition 4.2 *Assume $\rho = 1$ and $2c < f$.*

I. *An efficient equilibrium exists if $f + (n-1)v \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$.*

II. *For sufficiently large n, an efficient equilibrium exists if and only if $f+(n-1)v \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$, and the best equilibrium has the following features: the star network forms and, in the equilibrium path, all players provide information, the center of the star free-rides and the other players cooperate.*

We now illustrate Proposition 4.2. We construct a strategy profile, called *exploitative strategy*, which characterizes the best equilibrium. The formal definition is provided in the appendix. In words, this strategy prescribes that, between two connected players, the agent who values more the link cooperates, while the other agent free-rides.[10] For example, in the star network the center provides $n-1$ units of information to each of his neighbors, while each other player provides one unit of information to the center. Thus, the exploitative strategy prescribes that the center free-rides, while the other players cooperate. A deviation today triggers information withholding tomorrow, and forever, from the part of directly interacting players, in addition to turning to play defection in the PDG. When the underlying network is minimally connected, in terms of information accessibility, it is as if the underlying network were split in at least two different components. Hence, every player in the network can infer, the day after tomorrow, that there has been some deviation. The bilateral punishment then becomes a *social punishment*.

[10] In a different setting, Meléndez-Jiménez (2002) obtains that when two agents bargain on the cost sharing of a link, the agent who values more the link bears a higher part of the cost.

We show that the threat of such social punishment sustains each minimally connected network as an efficient equilibrium. However, not each minimally connected network is equally good in the sense that there are some architectures where social punishments are more effective. This leads us to ask: in which network architectures are players and social incentives best aligned?

We then characterize the best equilibrium. In the best equilibrium, the star network is formed, and the center of the star free-rides, while the peripheral players cooperate. Two observations are important. First, the fact that peripheral players bear entirely the costs of the link with the center of the star is crucial to "fully" internalize network externalities. Second, in the star network the center never wish to deviate and he is in the position to punish directly each possible inefficient deviation in the continuation game. This implies that social punishments are implemented just one period after a deviation has occurred. In contrast, in every other minimally connected network social punishments are implemented only with some delay. That is, if a player deviates today, tomorrow he has still access to some of the information flowing in the network.

We finally remark that punishments are more effective the larger is the size of the population. Indeed, a corollary of Proposition 4.2 is that for every discount factor $\delta \in (0, 1]$, there exists $\tilde{n}$, such that for every $n > \tilde{n}$ there exists an exploitative efficient equilibrium.

4.2.2 Cooperative Efficient Equilibria

We now study cooperative efficient equilibria. Denote $\hat{\delta} = \frac{\sqrt{2cd+10cv-2dv-4cnv+c^2+d^2+v^2}-(v+c-d)}{2((n-2)v-d)}$.

Proposition 4.3 *Assume $\rho = 1$ and $2c > f$. An efficient equilibrium exists if and only if $(n-1)v+2c \geq 0$ and $\delta \geq \hat{\delta}$, and the best equilibrium has the following features: the line network forms and in the equilibrium path all players provide information and cooperate.*

Similarly to the previous subsection, also in this case the efficient outcome can be sustained via a strategy profile which generates social punishments by punishing deviations with information withholding. However, the best equilibrium is obtained when the network is a line. The architectural properties of the line network contrast with the ones of the star. The line architecture is very symmetric. Symmetry here is important because it reduces the number of links each player has. Since, for a cooperative efficient outcome, players must mutually cooperate, networks where a few players have many links are less stable than networks where links are distributed "fairly" across agents. As an illustration, note that the star network is part of a cooperative efficient equilibrium only if $v \geq -c$. In fact, when this condition is violated the center has an incentive to remove all the links he has in the network, regardless of the plays in the continuation game. Finally, note that, also in this case, for every discount factor $\delta \in (0,1]$, there exists n high enough such that a cooperative efficient equilibrium exists.

In this Section we have shown that efficient outcomes can be sustained as equilibria for a wide range of parameters and that, for every given discount factor, there exists a sufficiently large n which assures the existence of efficient equilibrium. We have also shown that the architecture of the network is crucial in shaping the incentives of players. In particular, efficient exploitative outcomes are easily sustained in networks in which links are distributed unequally across players. In contrast, efficient cooperative outcomes are easily sustained in symmetric networks.

A crucial aspect behind these findings is that efficient outcomes are obtained in minimally connected networks. When these networks form, players may implement social punishments by breaking the information flow. The next section takes the case where information flows in the network with decay, which implies that in some situations efficient outcomes are obtained in over-connected networks.

5 Decay in the Information Flow

So far, we have analysed the extreme case where information flows without decay. In what follows, we relax this assumption. We say that g is the complete network if $g_{i,j} = 1, \forall i,j \in N, i \neq j$. We start by characterizing efficient outcomes.

Proposition 5.1 *Assume $\rho \in (0,1)$.*

I. If $-\max\{c, f/2\} < v\rho(1-\rho)$, then s is efficient if and only if g is the complete network and, for all $t \geq 1$, each player gives all his neighbors access to information and the total cost of each link is $\max\{2c, f\}$.

II. If $v\rho(1-\rho) < -\max\{c, f/2\} < v\rho(1+\frac{n-2}{2}\rho)$, then s is efficient if and only if g is the star network and, for all $t \geq 1$, each player gives all his neighbors access to information and the total cost of each link is $\max\{2c, f\}$.

III. If $-\max\{c, f/2\} > v\rho(1+\frac{n-2}{2}\rho)$, then s is efficient if and only if g is the empty network.

The characterization of Proposition 5.1 is similar to the one provided by Jackson and Wolinsky (1996) for the symmetric connections model. Note that imperfections in the information flow narrow the set of efficient network architectures. Specifically, if we fix the costs of a link we can identify three situations. First, for a low level of decay in information flow, each player should observe every other player at distance one, so that the complete network is uniquely efficient. Second, when the decay in information flow is intermediate, it is efficient that a player accesses every other player at distance two, instead of being directly connected and paying the costs of an additional link. In this case, the star network is uniquely efficient. Third, when information flow has high decay, it is efficient that players are disconnected.

We now study efficient equilibria. We are interested in understanding how sensitive the results obtained in the previous section are with respect to the decay factor in the information flow. Since the empty network can always be sustained as equilibrium, we assume that $-\max\{c, 2f\} < v\rho\left(1 + \frac{n-2}{2}\rho\right)$, hereafter.

We first consider exploitative efficient equilibria. Let $\delta^* = \frac{\sqrt{(\rho v - f)^2 + 4(\rho v - d)(d-f)} - (\rho v - f)}{2(\rho v - d)}$.

Proposition 5.2 *Assume $\rho \in (0,1)$ and $2c < f$.*

I. Suppose $-f/2 < v\rho(1-\rho)$. If $\delta < \delta^*$ there is no efficient equilibrium.[11]

II Suppose $v\rho(1-\rho) < -f/2 < v\rho(1 + \frac{n-2}{2}\rho)$. An efficient equilibrium exists if $\delta \geq \frac{d-f}{v\rho(1+(n-2)\rho)}$ and $-f < v\rho(1 + (n-1)\rho)$. Moreover, for sufficiently large n, and efficient equilibrium exists if and only if $\delta \geq \frac{d-f}{v\rho(1+(n-2)\rho)}$ and $-f < v\rho(1 + (n-1)\rho)$.

Part II of Proposition 5.2 shows that the results presented in Proposition 4.2 for the case of perfect information flow are robust to the presence of a substantial amount of decay in the information flow. Indeed, in this case we can adapt to the case of imperfect information flow the exploitative strategy used in Proposition 4.2 to establish the new result.

In contrast, Part I shows that when ρ is very high so that the complete network is efficient (i.e. when $-f/2 < v\rho(1-\rho)$) then the trade-off between efficient and stable networks re-emerges. That is to say, the range of parameters for which efficient equilibria do not exist does not vanish as n increases. It is important to note that even in the complete network social punishments may be implemented. Indeed, a player may credible threat a deviation at stage t, by defecting at $t+1$ with all connected players. Since in the complete network each player is connected with all other players, if at t a deviation occurs, at $t+1$ all players would realize it. However, even if social punishments are feasible, they are not very effective. The reason behind this is that in the complete network there is always a player that must pay for at least half of the links at the exploitative level. Since this is very costly, when the discount factor is low, such player has incentives to play inefficiently to save connections costs, even if social punishments will be immediately implemented.

We conclude with an analysis of cooperative efficient equilibria.

[11] It is easy to show that for high δ an efficient equilibrium exists. However, we were unable to pin down the best equilibrium in this situation. The main reason for this is that in a complete network the strategy which sustains the best equilibrium may require to alternate over time the players who pay for the costs of each link. This makes the analysis very complex.

Proposition 5.3 *Assume $\rho \in (0,1)$ and $2c > f$. If $-c < v\rho(1 + \frac{(n-2)}{2}\rho)$ then an efficient equilibrium exists if and only if $\delta \geq \frac{-c}{v\rho - d}$ and $-c \leq v\rho$.*

There are two remarks we would like to make. The first is that when efficiency requires cooperation even small imperfections in the information flow make it difficult to sustain efficient equilibria. Therefore, the results presented in Proposition 4.3 are specific to the extreme assumption of perfect information flow. The second remark is that the conditions for a cooperative efficient equilibrium are the same in the complete network and in the star network. The reason behind this is that in both these networks the player who has the highest incentives to deviate is the player with the highest number of links. This player is the central player in the star network, while it is an arbitrary player in the complete network. Furthermore, the utilities these players obtain by playing efficiently is the same in both networks. Thus, in these two architectures the critical players have the same incentives to deviate.

6 Discussion

We have studied a network formation game, where payoffs are endogenously determined as the discounted outcome of an infinitely repeated game. The model is well suited to study environments where interactions across players are characterized by free-riding problems as well as network externalities. We have shown that, when efficiency requires asymmetric play in each bilateral interaction, individual and social incentives are not in conflict for a wide range of players' discount factor and decay in the information flow. In contrast, cooperative efficient outcomes are more difficult to be equilibrium outcomes.

We would like to elaborate on some assumptions of our model. We first remark on the linearity of the players' payoffs. It is easy to see that the main insights of our analysis hold, as far as payoffs are increasing in the amount of information a player accesses from others, regardless of concavity and convexity considerations.

The second remark is about the timing of the game. We assume that the network is formed at the beginning of the game, while it is fixed afterward. The strategies which sustain efficient equilibria, can be easily re-defined for a setting where the network is formed in each period (simultaneously with the play of the AG and PDG). Therefore, the results will be qualitatively unaltered.

The third remark is about the information structure of the game. We assume that players observe the information they access from each of their social contacts, but they only posses local information about the behavior of their social contacts. It may seem that there is a discrepancy between the fact that players observe the information flow and that they have only limited information about the play of the game. We note that the possibility for players to communicate via links the information about the play of their neighbors will not change our results. The reason is that, as we have shown, the presence of network externalities is enough to induce social punishments in efficient networks. As we have already pointed out in Section 2, mechanisms of social controls based on communication of players' actions via social contacts have already been investigated by Vega-Redondo (2004) and Lippert and Spagnolo (2005). Our paper presents a different complementary mechanism of "social control", which relies on network externalities and it is proved to be very effective when efficiency requires that players are connected in minimal networks.

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Appendix

Efficient Outcomes

Proof of Proposition 4.1.

Part I. First note that, for each network, social welfare is maximized if and only if players always provide information and the cost of each link in each period is $\max\{2c, f\}$. Moreover, since $\rho = 1$, a network must be minimal. It is readily seen that, if the network is minimal but not connected, then efficiency can be improved by adding links between different components and making the network minimally connected. The prove of Part I. then follows.

The proof of Part II. is trivial and omitted. ■

Exploitative Efficient Equilibria

We first introduce the exploitative strategy profile.

Definition 6.1 *For any two players $i,j \in N$, the action taken by i against j at period t, $a^t_{i,j} \equiv (\alpha^t_{i,j}, \lambda^t_{i,j})$, is well-behaved, WB, with respect to their relative flow of information if*

$$a^t_{i,j} = \begin{cases} (C,1) \text{ if } I_{i,j}(\tilde{g}^t) = I_{i,j}(g) \leq I_{j,i}(g) = I_{j,i}(\tilde{g}^t) \\ (D,1) \text{ otherwise.} \end{cases}$$

The exploitative strategy profile $s^E = \{\gamma_i, \omega_i^{E,1}, ..., \omega_i^{E,t}, ...\}_{i\in N}$, where $\omega_i^{E,t} = \{\omega_{i,1}^{E,t}, ..., \omega_{i,n}^{E,t}\}$ is such that, for every network $\hat{g}$ and each $j \in N$,

$$\begin{aligned} \omega_{i,j}^{E,1} &= \begin{cases} (D,1) & \text{if } \hat{g} = g \text{ and } I_{i,j}(g) > I_{j,i}(g) \\ (C,1) & \text{if } \hat{g} = g \text{ and } I_{i,j}(g) \leq I_{j,i}(g) \\ (D,0) & \text{otherwise} \end{cases} \\ \text{and, } \forall t &\geq 2, \\ \omega_{i,j}^{E,t} &= \begin{cases} (D,1) & \text{if } a_{k,i}^{t-1} \text{ and } a_{i,k}^{t-1} \text{ are } WB,\ \forall k \in N_i(g) \text{, and } I_{i,j}(g) > I_{j,i}(g) \\ (C,1) & \text{if } a_{k,i}^{t-1} \text{ and } a_{i,k}^{t-1} \text{ are } WB,\ \forall k \in N_i(g) \text{, and } I_{i,j}(g) \leq I_{j,i}(g) \\ (D,0) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof of Proposition 4.2. The proof follows from the next three lemmas.

Lemma 1 *Assume $\rho = 1$ and $2c < f$. The strategy profile s^E is an efficient equilibrium if and only if $\delta \geq \frac{d-f}{(n-1)v}$ and $(n-1)v + f \geq 0$.*

Proof. Consider the strategy $s^E = \left\{\gamma, \omega^{E,1}, ..., \omega^{E,t}, ...\right\}$, where g is minimally connected. We first observe that, given s^E, in order to obtain an asymmetric efficient outcome, we need to focus on minimally connected networks where $I_{i,j}(g) \neq I_{j,i}(g)$ $\forall i,j \in N : g_{i,j} = 1$. Next, we observe that, given s^E, in every minimally connected network, each player pays at most the cost of one link at the exploitative level. It is now readily seen that g is a pairwise equilibrium iff $(n-1)v + f \geq 0$.

We now derive the conditions for a sequential equilibrium. Note that, if any deviation occurs, the strategy s^E prescribes optimal behavior, regardless of the players' beliefs. Thus, to derive the conditions for sequential equilibrium, we focus on players' incentives on the path induced by s^E.

Assume first that g is a star network. The central player, say j, does not deviate because he gets the maximum achievable payoff in this game, i.e. $u_j\left(s^E\right) = \frac{(n-1)v}{1-\delta}$. Note that the incentives of every $i \in N\backslash\{j\}$ are equivalent. Select a player, say i, in this set. We show that i does not deviate iff $\delta \geq \frac{d-f}{(n-1)v}$. Given s^E, i's utility is $u_i\left(s^E\right) = \frac{(n-1)v+f}{1-\delta}$. If he deviates, he obtains $u_i\left(\hat{s}_i, s^E_{-i}\right) = (n-1)v + \frac{d}{1-\delta}$. For an equilibrium $u_i\left(s^E\right) \geq u_i\left(\hat{s}_i, s_{-i}^E\right)$, which holds iff $\delta \geq \frac{d-f}{(n-1)v}$. Hence, if $(n-1)v+f \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$, the strategy s^E where g is a star network is an equilibrium.

Suppose that g is not a star network, then at least two end-agents exist. Again, an end-agent does not deviate from s^E only if $\delta \geq \frac{d-f}{(n-1)v}$.

The proof of Lemma 1 is complete.■

Note that Lemma 1 proves Part I of the proposition.

Let $\bar{n} = \frac{(v-2d)(d-f)}{v^2} + 1$.

Lemma 2 *Assume $\rho = 1$ and $2c < f$. If $n > \bar{n}$, then s^E is an efficient equilibrium for the widest range of parameters when g is a star network.*

Proof.

Suppose not; then it must be the case that when $n > \bar{n}$, $(n-1)v + f \geq 0$ and $\delta = \frac{d-f}{(n-1)v}$, s^E is an equilibrium for some minimally connected network g different from the star. Since g is not a star there exists a player j, who has $k \geq 1$ links with k end-agents and one additional link with a non end-agent, i.e. $\mu_j(g) = k+1$. Given s^E, j obtains $u_j\left(s^E\right) = \frac{(n-1)v+f}{1-\delta}$. Assume player j deviates with a non end-agent. Then he gets $u_j\left(\hat{s}_j, s_{-j}^E\right) = (n-1)v + \frac{d}{1-\delta} + kv\delta + \frac{kd\delta^2}{1-\delta}$. Since g is part of equilibrium, the incentives to deviate of an arbitrary end-agent i must be weakly higher than the incentives of player j, i.e. $u_i\left(\hat{s}_i, s_{-i}^E\right) \geq u_j\left(\hat{s}_j, s_{-j}^E\right)$.[12] This is satisfied if and only if $(n-1)v + \frac{d}{1-\delta} \geq (n-1)v + \frac{d}{1-\delta} + kv\delta + \frac{kd\delta^2}{1-\delta}$, which can be rewritten as $v - \delta(v-d) \leq 0$. Note that, when $\delta = \frac{d-f}{(n-1)v}$, the condition $v - \delta(v-d) = v - \frac{(d-f)(v-d)}{(n-1)v} \leq 0$ holds iff $n \leq \frac{(d-f)(v-d)}{v^2} + 1$. Since $\frac{(d-f)(v-d)}{v^2} + 1 < \bar{n}$, this contradicts our hypothesis that $n > \bar{n}$. Hence, if $n > \bar{n}$, the star network uniquely allows the strategy s^E to be an equilibrium for the widest range of parameters.

The proof of Lemma 2 is complete.■

Lemma 3 *Assume $\rho = 1$ and $2c < f$. For sufficiently large n, an efficient equilibrium exists if and only if $f+(n-1)v \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$. Further, if $f+(n-1)v = 0$, $\delta = \frac{d-f}{(n-1)v}$ and n is sufficiently large, then every strategy $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$ different from s^E is not an efficient equilibrium.*

Proof. Let $f > 2c$.Assume that the outcome of $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$ is efficient. We denote by $\tilde{S}$ the set of pure strategy profiles whose induced paths result in the same outcome for every $t > 0$. The proof of the Lemma is divided in two parts. In the first part we prove the Lemma for every $s \in \tilde{S}$, in the second part we prove the Lemma for every $s \notin \tilde{S}$.

First Part: Suppose $s \in \tilde{S}$. We start by showing that s is an equilibrium only if $f + (n-1)v \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$. It is readily seen that s is a pairwise equilibrium

[12] This is true because $u_i(s^E) = u_j(s^E)$.

only if $f + (n-1)v \geq 0$. Next, we show that s is a sequential equilibrium only if $\delta \geq \frac{d-f}{(n-1)v}$. To get a lower bound on δ, it must be the case that s prescribes maximal punishments out of the equilibrium path. Thus, s must prescribe that, if at some period t an agent $i \in N$ deviates with some $j \in N_i(g)$, then j plays $a_{i,j}^{\tau} = (D, 0)$ $\forall \tau > t$. Given s, two possibilities may occur: (1) there is at least one end agent, say j who pays for his link and (2) no end-agent pays for his link.

Consider case (1); the arguments in Lemma 1 imply that s constitutes an equilibrium only if $\delta \geq \frac{d-f}{(n-1)v}$.

Consider case (2), i.e. no end-agent pays for his link. We claim that there exists some player who is not an end-agent and is paying all his direct links. We prove this by construction. For a minimally connected network g^t defined on the set $N(g^t) \subseteq N$, let $E(g^t) \in N(g^t)$ be the set of end-agents in g^t, and $M(g^t) \equiv N(g^t) \backslash E(g^t)$. Construct the following finite sequence of networks $(g, g^1, ..., g^T)$, where g^t is the sub-network of g^{t-1}defined on $M(g^{t-1})$. Let T be such that g^T is either a star network or a network composed of a link between two players. It is easy to see that, starting from any minimally connected network g, such T exists.

For a contradiction, assume that in g no end-agent pays for his link and no agent $i \in M(g)$ pays all his direct links. By construction, it must be the case that each agent $i \in M(g^t)$ linked to a player $j \in E(g^t)$ is paying for the link. Indeed, note that $j \in E(g^t)$ implies $j \in M(g^{t-1})$ and $N_j(g^{t-1}) = N_j(g)$. Thus, if j pays for the link with i, then j would be paying all his direct links in g, a contradiction. Suppose then that g^T is a star network and let $i \in N(g^T)$ be the central player. Then i must be paying for all the links he has in g^T, which actually coincides with the links he has in g, a contradiction. By the same argument, if g^T is a connected network of two nodes, the player who pays for the link is indeed paying for all his direct links in g, a contradiction. This proves the claim.

This claim implies that there exists a player i who pays for all his links, i.e. $\mu_i^p(g, s) = \mu_i(g) > 1$, where $\mu_i^p(g, s)$ is the number of links that agent i pays for. Using this fact, note that $u_i(s) = \frac{(n-1)v + \mu_i(g)f}{1-\delta}$. The utility from deviating in all his interactions is $u_i(\hat{s}_i, s_{-i}) = (n-1)v + \frac{\mu_i(g)d}{1-\delta}$. An equilibrium requires $u_i(s) \geq u_i(\hat{s}_i, s_{-i})$; solving we obtain $\delta \geq \frac{\mu_i(g)(d-f)}{(n-1)v}$. Since $\mu_i(g) \geq 2$, then $\frac{\mu_i(g)(d-f)}{(n-1)v} > \frac{d-f}{(n-1)v}$. Hence, we have shown that $(n-1)\,v + f \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$ are necessary conditions for s to be an efficient equilibrium.

We now prove that, if these two conditions bind and $n > \bar{n}$, then every strategy s different from s^E cannot be an efficient equilibrium. Assume that the conditions bind and that $s = \{\gamma, \omega^1, ..., \omega^t, ...\}$ is an efficient equilibrium. Since $(n-1)v + f = 0$ and s is an efficient equilibrium, each $i \in N$ pays at most for one link. Select an arbitrary pair of linked players, i and j. Without loss of generality, assume that $I_{i,j}(g) > I_{j,i}(g)$. There are two possibilities, which we analyze in turn.

A.) Suppose that the paths induced by s and s^E are equivalent. Since s^E prescribes maximal punishments for every possible deviation, then the incentive of player j (i) to follow s with i (j) cannot be higher than to follow s^E. In this case, we can use Lemma 1 to prove the claim.

B.) Suppose that, in the path induced by s, player i cooperates and player j defects. The utility of player i to follow s is $u_i(s) = \frac{(n-1)v+f}{1-\delta}$. If player i deviates (using his best deviation) against player j at some period t, the utility he obtains in the continuation game is $u_i(\hat{s}_i, s_{-i}) = (n-1)v + (n-1-I_{j,i}(g))v\delta + \frac{d}{1-\delta} + \frac{(\mu_i(g)-1)d\delta^2}{1-\delta}$. Now we claim that player i has incentives to deviate for $\delta = \frac{d-f}{(n-1)v}$. Indeed, the case where player i has the lowest incentives to deviate is when $I_{j,i}(g) = n/2$ and $\mu_i(g) = n-2$. In this case, $\bar{u}_i(\hat{s}_i, s_{-i}) = (n-1)v + (\frac{n-2}{2})v\delta + \frac{d}{1-\delta} + \frac{(n-2)d\delta^2}{1-\delta}$. Note that $u_i(s) \geq \bar{u}_i(\hat{s}_i, s_{-i})$ iff $(n-1)v\delta - (\frac{n-2}{2})v\delta(1-\delta) - (n-2)d\delta^2 > d-f$. Using the fact that $\delta = \frac{d-f}{(n-1)v}$, we can rewrite this condition as $-v + \delta(v-2d) = -v + \frac{d-f}{(n-1)v}(v-2d) > 0$, which is satisfied iff $n < \bar{n}$. This contradicts the fact that $n > \bar{n}$.

Second Part: Suppose $s \notin \tilde{S}$. We now show that for sufficiently large n and when conditions $(i)-(ii)$ do not hold, then s is not an efficient equilibrium.

We first claim that if $\delta \leq \frac{d-f}{(n-1)v}$ and n is large enough, s cannot be an efficient equilibrium. The condition $\delta \geq \frac{d-f}{(n-1)v}$ has been obtained by analyzing the incentives of an end agent to deviate, given that he paid for the link in every period. Thus, to improve the condition, s must require that an agent i linked to an end-agent pays for that link, say, each W periods, where W can be arbitrarily large. A lower bound to the incentives of player i to deviate is obtained when he only pays the link with the end agent each W periods, while he does not pay for any other link. Without loss of generality suppose i has $k+1$ links. Then if i follows the strategy s he gets $u_i(s) = \frac{(n-1)v}{1-\delta} + \frac{f}{1-\delta^W}$, while if he deviates he gets at most $u_i(\bar{s}_i, s_{-i}) = (n-1)v + \delta(n-2)v + \frac{d}{1-\delta} + \frac{\delta^2 kd}{1-\delta}$. Now note that $u_i(s) > u_i(\bar{s}_i, s_{-i})$ iff $\frac{1-\delta}{1-\delta^W}f > -\delta v - \delta^2(n-2)v + d + \delta^2 kd$. Substituting $\delta = \frac{d-f}{(n-1)v}$, it follows that $u_i(s) > u_i(\bar{s}_i, s_{-i})$ if and only if

$$\frac{1-\frac{d-f}{(n-1)v}}{1-\left(\frac{d-f}{(n-1)v}\right)^W} f > -\frac{d-f}{(n-1)v}v - \left(\frac{d-f}{(n-1)v}\right)^2 (n-2)v + d + \left(\frac{d-f}{(n-1)v}\right)^2 kd.$$

Taking the limit as $n \to \infty$ we get that $u_i(s) > u_i(\bar{s}_i, s_{-i})$ if and only if $f > d$, a contradiction.

Finally, as we have shown that (for high n) the end agent must (always) pays for his link, the condition $f + (n-1)v \geq 0$ must hold. This completes the proof.■

Part II of the proposition follows from Lemmas 1-3. The proof of the Proposition is complete.■

Cooperative Efficient Equilibria

The cooperative strategy profile is $s^C = \{\gamma_i, \omega_i^{C,1}, ..., \omega_i^{C,t}, ...\}_{i\in N}$, where $\omega_i^{C,t} = \{\omega_{i,1}^{C,t}, ..., \omega_{i,n}^{C,t}\}$ is such that, for every $\hat{g}$ and for each $j \in N$,

$$\omega_{i,j}^{C,1} = \begin{cases} (C,1) & \text{if } \hat{g} = g \\ (D,0) & \text{otherwise} \end{cases}$$

and, $\forall t \geq 2$,

$$\omega_{i,j}^{C,t} = \begin{cases} (C,1) & \text{if } a_{k,i}^{t-1} = a_{i,k}^{t-1} = (C,1) \text{ and } (I_{k,i}(\tilde{g}^{t-1}), I_{i,k}(\tilde{g}^{t-1})) = (I_{k,i}(g), I_{i,k}(g)),\ \forall k \in N_i(g) \\ (D,0) & \text{otherwise.} \end{cases}$$

Proof of Proposition 4.3. First observe that s^C prescribes players to play the Nash equilibrium $(D,0)$ after any possible deviation. Thus, to derive the conditions for sequential equilibrium, we need to focus only on individuals' incentives in the path induced by s^C. The proof follows from the next two Lemmas which are stated and proved below.

Lemma 4. *Assume $\rho = 1$ and $2c > f$. The cooperative strategy profile, $s^C = \{\gamma, \omega^{C,1}, ..., \omega^{C,t}, ...\}$ is an efficient equilibrium if and only if $(n-1)v + 2c \geq 0$ and $\delta \geq \frac{\sqrt{2cd+10cv-2dv-4cnv+c^2+d^2+v^2}-(v+c-d)}{2((n-2)v-d)}$. Furthermore, s^C is an efficient equilibrium for the widest range of parameters if g is a line network.*

Proof. Consider $s^C = \{\gamma, \omega^{C,1}, ..., \omega^{C,t}, ...\}$, where g is a minimally connected network.

We first claim that the line is the network which is a pairwise equilibrium for the widest range of parameters. Given s^C, the utility of a player, say i, is $u_i(s^C) = \frac{(n-1)v+\mu_i(g)c}{1-\delta}$, where $\mu_i(g) \in \{1, 2, ..., n-1\}$. Suppose i deviates at $t = 0$. It is clear that the best deviation of i is to delete every link he has. Then i obtains zero utility. Hence, player i follows s^C iff $u_i(s^C) \geq 0$, i.e. $(n-1)v + \mu_i(g)c \geq 0$. Thus, no player deviates at $t = 0$ iff $(n-1)v + \mu^*(g)c \geq 0$, where $\mu^*(g) = \max_{i\in N} \mu_i(g)$. Observe now that $\mu^*(g^{line}) = 2 < \mu^*(g)$ for any minimally connected network g different from the line. This proves the claim.

We now analyze the conditions for sequential equilibrium. Assume that g is the line network. We claim that the player who has the highest incentive to deviate in the line network at any $t > 0$ is a player j linked to an end agent.

Note that such a player, say j, he has two links: one with an end-agent, say i, and one with a non-end agent, say j'. Thus, at any $t > 0$, player j has two relevant possible deviations. One, j deviates only with i at some t and deviate with j' at $t+1$. Denote this deviation strategy by $\hat{s}_j$; then $u_j(\hat{s}_j, s_{-j}^C) = (n-1)v + (n-2)v\delta + c + \frac{d\delta}{1-\delta} + \frac{d\delta^2}{1-\delta}$. Two, j may deviate both with i and j' at some t. Denote this deviation strategy by

$\bar{s}_j$; then $u_j(\bar{s}_j, s^C_{-j}) = (n-1)v + \frac{2d\delta}{1-\delta}$.[13]

Select now a player, say j', who is not an end-agent and who is not linked with an end-agent. Then he is linked with two non end-agents. The essential deviations of j' are: to deviate only with the agent who is closer to an end-agent of the line, say $\hat{s}_{j'}$, and to deviate with both of his social contacts, say $\bar{s}_{j'}$.

It is readily seen that $u_j(\hat{s}_j, s^C_{-j}) > u_{j'}(\hat{s}_{j'}, s^C_{-j'})$ and $u_j(\bar{s}_j, s^C_{-j}) = u_{j'}(\bar{s}_{j'}, s^C_{-j'})$. Since $u_j(s^C) = u_{j'}(s^C) = \frac{(n-1)v+2c}{1-\delta}$, to prove our claim, it suffices to show that j has a higher incentive to deviate than the end-agent i. Note that $u_i(s^C) = \frac{(n-1)v+c}{1-\delta}$ while, if i deviates, he gets $(n-1)v + \frac{d\delta}{1-\delta}$. Thus, i follows s^C iff $\delta \geq \frac{-c}{(n-1)v-d}$. A necessary condition for player j not to deviate is $u_j(s^C) \geq u_j(\bar{s}_j, s^C_{-j})$. Solving we obtain that $\delta \geq \bar{\delta} \equiv \frac{-2c}{(n-1)v-2d}$. However, note that $\bar{\delta} > \frac{-c}{(n-1)v-d}$. These observations prove the claim, i.e. player j has the highest incentive to deviate.

We now claim that the condition for j not to deviate is

$$\delta \geq \frac{\sqrt{2cd + 10cv - 2dv - 4cnv + c^2 + d^2 + v^2} - (v + c - d)}{2((n-2)v - d)}.$$

First, note that $u_j(\hat{s}_j, s^C_{-j}) \geq u_j(\bar{s}_j, s^C_{-j})$ iff $\delta \geq \tilde{\delta} \equiv \frac{-c}{(n-2)v-d}$. Again note that $\bar{\delta} > \tilde{\delta}$. Therefore, since when $\delta > \tilde{\delta}$ player j's best deviation is $\hat{s}_j$, it follows that, s^C with the line network is a sequential equilibrium iff $u_j(s^C) \geq u_j(\hat{s}_j, s^C_{-j})$. This is equivalent to solve $\delta[v + (n-2)\,v\delta + c - d - d\delta] + c \geq 0$. Define the following function: $\Upsilon(\delta) = \delta[v + (n-2)\,v\delta + c - d - d\delta] + c$. Note that $\frac{\partial \Upsilon(\delta)}{\partial \delta} > 0$. Since $\hat{\delta} = \frac{\sqrt{2cd+10cv-2dv-4cnv+c^2+d^2+v^2}-(v+c-d)}{2((n-2)v-d)}$ solves $\Upsilon(\hat{\delta}) = 0$, the claim follows.[14]

Finally, let g be a minimally connected network and let it be different from the line. Assume $\delta < \hat{\delta}$. We claim that, if s^C prescribes g to form, then s^C is not an equilibrium. We first observe that in g it must exist a player j'' who has k links with k end-agents ($k \geq 1$) and one additional link, which may be either with a non end-agent, or with an end-agent (this last case would only be possible with the star network).

If $k = 1$, then j'' has the same incentive to deviate that a player linked with an end-agent in a line network (player j above). Hence, in this case, the claim follows.

Assume next that $k \geq 2$. By construction, $\mu_{j''}(g) = k + 1$ and $u_{j''}(s^C) = \frac{(n-1)v+(k+1)c}{1-\delta}$. Assume j'' deviates with the k end-agents at some t and with the

[13] Note that the deviation in which player j deviates only with player i at period t is strictly dominated by $\hat{s}_j$. Also note that the deviations in which player j deviates with player j' at period t and either deviate with i at period $t+1$ or not are strictly dominated by $\bar{s}_j$.

[14] Some algebra shows that $\bar{\delta} < \hat{\delta} < 1$ for any $n \geq 3$.

remaining player at $t+1$, $\hat{s}_{j''}$. Then

$$u_{j''}(\hat{s}_{j''}, s^C_{-j''}) = (n-1)v + (n-1-k)v\delta + c + \frac{kd\delta + d\delta^2}{1-\delta}.$$

Thus, j'' follows s^C only if $u_{j''}(s^C) \geq u_{j''}(\hat{s}_{j''}, s^C_{-j''})$. After some algebra this condition is equivalent to: $\delta[kv + (n-1-k)v\delta + c - kd - d\delta] \geq -kc$.

We now show that, when $\delta = \hat{\delta}$, player j'' deviates. Indeed, note that $\Upsilon(\hat{\delta}) = 0$ implies that $-c = \hat{\delta}[v + (n-2)v\hat{\delta} + c - d - d\hat{\delta}]$. For a contradiction, suppose j'' does not want to deviate at $\hat{\delta}$; that is,

$$\hat{\delta}[kv + (n-1-k)v\hat{\delta} + c - kd - d\hat{\delta}] \geq -kc$$

If, in the RHS, we substitute $-c$ from the equation $\Upsilon(\hat{\delta}) = 0$, we obtain

$$\hat{\delta}[kv + (n-1-k)v\hat{\delta} + c - kd - d\hat{\delta}] \geq -k\hat{\delta}[v + (n-2)v\hat{\delta} + c - d - d\hat{\delta}],$$

which becomes $\hat{\delta} \leq \frac{-c}{(n-1)v-d}$. Since we have shown that $\hat{\delta} > \bar{\delta} > \frac{-c}{(n-1)v-d}$, this constitutes a contradiction. This proves the claim and it completes the proof of the Lemma.■

Lemma 5. *Assume $\rho = 1$ and $2c > f$. If a strategy s is an efficient equilibrium, then s^C (with the network g coinciding with the network prescribed by s) is also an efficient equilibrium.*

Proof. Given $2c > f$, to prove the result, it is enough to note that s^C is a maximal punishment strategy profile.■

Lemmas 4 and 5 prove the Proposition.■

Decay

Proof of Proposition 5.1. The network characterization follows from Jackson and Wolinsky (1996). The arguments used in Proposition 4.1 apply to prove the additional conditions required in Part I and Part II of the Proposition. ■

Proof of Proposition 5.2.

Part I. Assume $-f/2 < v\rho(1-\rho)$ and $\delta < \frac{\sqrt{(\rho v-f)^2+4(\rho v-d)(d-f)}-(\rho v-f)}{2(\rho v-d)}$. Furthermore, assume, for the sake of contradiction, that there exist a strategy $\hat{s}$ that sustains an efficient equilibrium and that $\hat{s}$ is a best equilibrium. Since $\hat{s}$ is an efficient equilibrium, it must prescribe that the complete network forms and that at any period, each player provides accessibility to all his neighbors and each link is unilaterally paid by one agent at the exploitative level. It is then clear that, at any period, there must exist at least one agent who is paying (at least) half of his links.

Since $\hat{s}$ is the best equilibrium it must involve maximal punishment. That is, $\hat{s}$ prescribes that, when any player observes a deviation, he plays $(D,0)$ in all his interactions from the following period onwards. Note that such behavior in the out-of-equilibrium path is optimal, since if a player, say j deviates with player l at period t, the fact that both j and l will play $(D,0)$ in all their interactions at period $t+1$ results in the fact that all the population realizes a deviation after period $t+1$. Then $((D,0),(D,0))$ is played in all the interactions from period $t+2$ onwards, which is an equilibrium of the stage game.

Consider an agent i that at period t is paying for at least half of his links. Given $\hat{s}$, let $\hat{N}_i^t \subseteq N\backslash\{i\}$ be the agents with whom i is cooperating at period t. Also, let $\hat{k}_i^t = \left|\hat{N}_i^t \cap \hat{N}_i^{t+1}\right|$, $k_i^t = \left|\hat{N}_i^t\right| - \hat{k}_i^t$ and $\tilde{k}_i^{t+1} = \left|\hat{N}_i^{t+1}\right| - \hat{k}_i^t$. Note that $\hat{k}_i^t \geq 0$, $k_i^t \geq 0$, $\tilde{k}_i^{t+1} \geq 0$ and $\hat{k}_i^t + k_i^t \geq \frac{n-1}{2}$.

Consider s_i' consisting in player i deviating in all the $\hat{k}_i^t + k_i^t$ links he is paying for at period t. Then:

$$u_i(s_i', \hat{s}_{-i}) = (n-1)\rho v + \delta(n-1-(\hat{k}_i^t + k_i^t))\rho v + (\hat{k}_i^t + k_i^t)d + \delta(\hat{k}_i^t + k_i^t + \tilde{k}_i^{t+1})d + \frac{\delta^2(n-1)d}{1-\delta}.$$

Whereas, if he follows the strategy $\hat{s}_i$, then:

$$u_i(\hat{s}) = (n-1)\rho v + \delta(n-1)\rho v + (\hat{k}_i^t + k_i^t)f + \delta(\hat{k}_i^t + \tilde{k}_i^{t+1})f + U_i^{t+2}(\hat{s}),$$

where $U_i^{t+2}(\hat{s})$ is the discounted sum of utilities that i obtains from period $t+2$ onwards given $\hat{s}$. Note that $U_i^{t+2}(\hat{s}) \leq \frac{\delta^2(n-1)\rho v}{1-\delta}$.

Since $\hat{s}$ is an equilibrium it must be the case that $u_i(\hat{s}) - u_i(s_i', \hat{s}_{-i}) \geq 0$. Define the function $\Phi(\hat{k}_i^t, k_i^t, \tilde{k}_i^{t+1}, U_i^{t+2}) = u_i(\hat{s}) - u_i(s_i', \hat{s}_{-i})$. Then

$$\begin{aligned}\Phi(\hat{k}_i^t, k_i^t, \tilde{k}_i^{t+1}, U_i^{t+2}) &= \hat{k}_i^t(\delta\rho v - (1+\delta)(d-f)) + k_i^t(\delta\rho v - (1+\delta)d + f) \\ &\quad + \tilde{k}_i^{t+1}\delta(f-d) + U_i^{t+2} - \frac{\delta^2(n-1)d}{1-\delta}.\end{aligned}$$

Note that $\partial\Phi/\partial\tilde{k}_i^{t+1} < 0$, $\partial\Phi/\partial U_i^{t+2} > 0$, $\partial\Phi/\partial k_i^t < 0$ and $\partial\Phi/\partial\hat{k}_i^t < \partial\Phi/\partial k_i^t$.[15] Hence, $\Phi(0, \frac{n-1}{2}, 0, \frac{\delta^2(n-1)\rho v}{1-\delta})$ is a higher bound of $u_i(\hat{s}) - u_i(s_i', \hat{s}_{-i})$. That is,

$$\Phi(0, \frac{n-1}{2}, 0, \frac{\delta^2(n-1)\rho v}{1-\delta}) \geq u_i(\hat{s}) - u_i(s_i', \hat{s}_{-i}).$$

Next, note that $\Phi(0, \frac{n-1}{2}, 0, \frac{\delta^2(n-1)\rho v}{1-\delta}) \geq 0$ if and only if $\delta \geq \frac{\sqrt{(\rho v-f)^2+4(\rho v-d)(d-f)}-(\rho v-f)}{2(\rho v-d)}$, a contradiction with the fact that $\hat{s}$ is an equilibrium for $\delta < \frac{\sqrt{(\rho v-f)^2+4(\rho v-d)(d-f)}-(\rho v-f)}{2(\rho v-d)}$.

[15] To see that $\partial\Phi/\partial k_i^t < 0$, note that $\delta < \frac{\sqrt{(\rho v-f)^2+4(\rho v-d)(d-f)}-(\rho v-f)}{2(\rho v-d)}$ implies $\delta < \frac{d-f}{\rho v-d}$.

Finally, after some algebra it is easy to show that $0 < \frac{\sqrt{(\rho v-f)^2+4(\rho v-d)(d-f)}-(\rho v-f)}{2(\rho v-d)} < 1$. This proves part I.

Part II. The proof of the fact that an efficient equilibrium exists if $\delta \geq \frac{d-f}{v\rho(1+(n-2)\rho)}$ and $-f < v\rho(1+(n-1)\rho)$ follows from the same arguments used in Proposition 4.2. We can indeed adapt the exploitative strategy profile, s^E, defined for the no-decay case, for the case in which $\rho \in (0,1)$. The proof of the fact that if n is sufficiently large then an efficient equilibrium exists only if $\delta \geq \frac{d-f}{v\rho(1+(n-2)\rho)}$ and $-f < v\rho(1+(n-1)\rho)$ also follows adapting Lemma 3 to the case where $\rho \in (0,1)$.

This completes the proof of the Proposition.■

Proof of Proposition 5.3.

First suppose that $-c < v\rho(1-\rho)$ so that the complete network is efficient. Consider the strategy $\hat{s}$ which prescribes that players form the complete network. If another network is observed, then each player plays $(D,0)$ forever with every neighbor. Otherwise, each player cooperates and provides information. If i deviates with j at t, then from $t+1$ onwards i and j play $(D,0)$ in all their interactions and whenever an agent detects that there has been a deviation, he plays $(D,0)$ in all his links. This is clearly a maximal punishment strategy, so that the conditions for an equilibrium are necessary and sufficient.

We first observe that out of the path induced by $\hat{s}$, no agent would have any incentives to deviate from $\hat{s}$. Thus, to analyze the condition for sequential equilibrium we can focus on the agents incentives in the path induced by $\hat{s}$.

Second, we observe that $u_i(\hat{s}) = \frac{(n-1)(\rho v+c)}{1-\delta}$. Hence, pairwise stability requires that $-c < v\rho$.

Third, note that the utility of an agent if he deviates with k partners is

$$u_i(s_i^k, \hat{s}_{-i}) = (n-1)v\rho + (n-1-k)v\rho\delta + (n-1-k)c + \delta dk + \frac{\delta^2(n-1)d}{1-\delta},$$

and $\frac{\partial u_i(s_i^k,\hat{s}_{-i})}{\partial k} > 0$ iff $\delta < \frac{-c}{v\rho-d}$. Therefore if $\delta < \frac{-c}{v\rho-d}$ then the best deviation for i is s_i^{n-1}, otherwise it is s_i^1. Consider that $\delta < \frac{-c}{v\rho-d}$, then $u_i(\hat{s}_i,\hat{s}_{-i}) \geq u_i(s_i^{n-1},\hat{s}_{-i})$ iff $\delta \geq \frac{-c}{v\rho-d}$. Thus, if $\delta < \frac{-c}{v\rho-d}$ an efficient equilibrium can not be sustained. Now consider $\delta \geq \frac{-c}{v\rho-d}$. In this case the best deviation is s_i^1. $u_i(\hat{s}_i,\hat{s}_{-i}) \geq u_i(s_i^1,\hat{s}_{-i})$ iff $\delta \geq \frac{-c}{v\rho-d}$. This proves the Proposition for the case in which $-c < v\rho(1-\rho)$.

Assume now that $v\rho(1-\rho) < -c < v\rho(1+\frac{n-2}{2}\rho)$ so that efficiency requires a star network. Consider the cooperative strategy profile s^C where g is star network, properly adapted to the decay case. Note that this is a maximal punishment strategy. Also observe that the center of the star faces the same problem analyzed in the previous case. Therefore, the center does not deviate iff $\delta \geq \frac{-c}{v\rho-d}$ and $-c \leq v\rho$.

Next, if a spoke player, say j, deviates, he gets

$$u_j(\hat{s}_j, s_{-j}^C) = \rho v + (n-2)\rho^2 v + \frac{d\delta}{1-\delta},$$

while, $u_j(s_j^C, s_{-j}^C) = \frac{\rho v + (n-2)\rho^2 v + c}{1-\delta}$. Thus, $u_j(s^C) \geq u_j(\hat{s}_j, s_{-j}^C)$ iff $\delta \geq \frac{-c}{\rho v + (n-2)\rho^2 v - d}$. It is readily seen that $\frac{-c}{\rho v - d} > \frac{-c}{\rho v + (n-2)\rho^2 v - d}$. This concludes the proof. ■