Subjective Beliefs and Ex-Ante Agreeable Trade

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Abstract

We find a necessary and sufficient condition for ex-ante trade when agents are non-expected utility maximizers. The condition is that they share subjective beliefs. Our result holds for a class of *convex preferences* that contains many functional forms used in applications. In a special case of expected utility, the condition becomes exactly the common prior assumption. It can also be articulated in the language of other functional forms, confirming results existing in the literature, generating new results, and providing a useful tool for applications.

Another contribution of this paper is a characterization of a general definition of beliefs for convex preferences. We show that this definition can be characterized in terms of market behavior. Moreover, it coincides with the usual one for an important class of convex invariant biseparable preferences.

1 Introduction

Betting should occur very often in a world where people are subjective expected utility maximizers. Every time two or more people disagree about the probability of some event, there is a possibility of a mutually beneficial exchange. In fact, every time we observe such an exchange, it must have been caused by differing subjective probabilities. More formally, the absence of ex-ante agreeable trade is equivalent to a common prior. Even though this result has been known for many years, it is still surprising how little betting takes place in the real world, where the possible sources of disagreement abound.

A very interesting perspective has been recently offered by Billot, Chateauneuf, Gilboa and Tallon (2000), who relax the expected utility hypothesis and allow each agent to hold multiple probability distributions *à la* Gilboa and Schmeidler (1989). Within this maxmin expected utility (MEU) framework, they show that ex-ante agreeable trade is absent if and

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only if there is at least one probability distribution common to all agents. This offers a very appealing explanation for the rarity of betting. Among subjective expected utility (SEU) agents, *any* disagreement causes trade, however among MEU agents this happens only in the case of *complete* disagreement. If the agents have different, but not disjoint, sets of priors, i.e. if they disagree only mildly, they will not be willing to bet against each other.

A similar explanation follows from Bewley's (1986) model of incomplete preferences. In a recent paper, applying his model and building on Bewley (1989), Rigotti and Shannon (2005) come to a similar conclusion as Billot et al. (2000). If initially fully insured, the agents will not bet against each other if and only if they share at least one probability distribution.

Our paper departs from the SEU hypothesis in a more radical way than Billot et al. (2000) or Rigotti and Shannon (2005). We characterize the necessary and sufficient conditions for ex-ante trade assuming only *convexity* of preferences. This includes as special cases the risk averse MEU and Bewley preferences, therefore our result is an extension of the previous ones. Indeed, it builds on the observation that convexity is what essentially drives the argument. Because of this, the result holds for many other preferences, provided they are convex, such as the invariant biseparable preferences of Ghirardato, Maccheroni and Marinacci (2004), the smooth model of decision making proposed by Klibanoff, Marinacci and Mukerji (2004) and the model of variational preferences proposed by Maccheroni, Marinacci and Rustichini (2004), of which Hansen and Sargent's (2001) multiplier preferences is a special case.

An important problem that we have to address before being able to make this extension is that beliefs are not obviously defined without assuming a specific functional form for preferences. The first step of our analysis is devoted to studying a general definition of beliefs for convex preferences, which builds on Yaari (1969). We provide a characterization of this definition in terms of market behavior in the spirit of Segal and Spivak (1990) and Dow and Werlang (1992). In addition, we show that for invariant biseparable preferences this definition coincides with the one given by Ghirardato et al. (2004). We also provide results for the other special cases mentioned above.

Using this definition we show that the results of Billot et al. (2000) and Rigotti and Shannon (2005) generalize to the class of convex preferences. Our result is important for two reasons: First, it makes it possible to study economic interactions of decision makers with alternative preferences. Second, by nesting most models, it allows for cross-model comparisons. This enables the study of situations in which agents with different violations of SEU interact. It could not have been achieved by studying the implications of each model in isolation, since the conditions resulting from such study would be model-specific, for example expressed in terms of the functional form representation.

The paper is organized as follows. Sections 2-3 deal with complete preferences. Section 2 presents the class of convex preferences. Section 3 studies the definition of beliefs and its behavioral properties. It also compares these beliefs to beliefs in known representations. Section 4 studies trade between agents with convex, complete preferences. The main result, Theorem 2, shows that the agents will trade if and only if they completely disagree. Section 5 extends this result to the class of incomplete preferences. Section 6 discusses related literature and Section 7 concludes. All proofs appear in the Appendix.

2 Convex Preferences

Let S be the set of states of the world, which for reasons of simplicity is assumed to be finite. The set of consequences is $\mathbb{R}_+$, which we interpret as monetary payoffs. The set of acts is $\mathcal{F} = \mathbb{R}_+^S$ with the standard topology. Its elements will be denoted by f, g, h, etc., and for any $c \in \mathbb{R}_+$ we will abuse the notation by writing $c \in \mathcal{F}$, which stands for a constant act with payoff c in each state of the world. Let $\succ$ be a binary relation on $\mathcal{F}$. We will say that $\succ$ is a *convex preference* if and only if it satisfies the following axioms:

Axiom 1 (Preference). *The relation $\succ$ is asymmetric and negatively transitive.*

Axiom 2 (Continuity). *For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} | g \succ f\}$ and $\{g \in \mathcal{F} | f \succ g\}$ are open.*

Axiom 3 (Monotonicity). *For all $f, g \in \mathcal{F}$, if $f(s) > g(s)$ for all $s \in S$, then $f \succ g$.*

Axiom 4 (Convexity). *For all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} | g \succ f\}$ is convex.*

Axioms 1 and 2 are standard in the literature.[1] Axiom 3 means that money is desirable.[2] Axiom 4 means that $\succ$ exhibits preference for portfolio diversification (see for example Chateauneuf and Tallon (2002)). It implies both risk aversion and ambiguity aversion, because our setup is too general to make an explicit difference between the two. The word that we will henceforth use to refer to any kind of unpredictability of outcomes is *uncertainty*. In fact, this axiom is stronger than any known definition of uncertainty aversion, including Yaari's (1969), as well as Ghirardato and Marinacci's (2002) and Grant and Quiggin's (2004). The relationship between convexity and risk aversion has been studied by Dekel (1989) for probabilistically sophisticated agents. The relationship between convexity and uncertainty aversion has been studied by Chateauneuf and Tallon (2002) for Choquet expected utility (CEU) agents.

The class of convex preferences is too large to have any specific functional form representation. The following proposition asserts that without imposing further axioms we cannot say anything about the functional form.

Proposition 1. *$\succ$ is convex if and only if there exists a continuous, increasing and quasi-concave function $V : \mathcal{F} \to \mathbb{R}$, such that for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if $V(f) > V(g)$.*

This generality allows to include most of the models of choice under uncertainty, however it makes studying beliefs difficult. The difficulty is that we cannot derive beliefs from the functional form representation and have to rely on another definition of beliefs. The next section is studying a general definition of beliefs for convex preferences and its characterizations in terms of market behavior.

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[1] As usual, we write $f \succsim g$ if $g \not\succ f$ and $f \sim g$ if $f \succsim g$ and $g \succsim f$. Axiom 1 is equivalent to completeness and transitivity of $\succsim$. Axiom 2 is equivalent to closedness of upper and lower contour sets.

[2] Axiom 3 is equivalent to the conjunction of the following two axioms. **1**: For all $x, y \in \mathbb{R}$, $x > y$ implies $x \succ y$. **2**: For all $f, g \in \mathcal{F}$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

3 A Definition of Beliefs

3.1 Supporting Hyperplanes and Beliefs

The traditional decision theoretic approach of Ramsey and Savage is to identify the subjective probability of a decision maker with the odds at which he is willing to make small bets. Following this route, Yaari (1969) identifies the subjective probability with a hyperplane that supports the upper contour set. The following simple example explains this idea.[3]

Example 1. *Suppose that $S = \{s_1, s_2\}$ and the agent maximizes his subjective expected utility $V(f) = p_1 u(f(s_1)) + p_2 u(f(s_2))$, for some strictly increasing, concave and differentiable u. At the act c on the 45^o line (certainty line) the slope of his indifference curve can be calculated as the marginal rate of substitution between states:* $\frac{\mathrm{d}s_1}{\mathrm{d}s_2}|_{V=V(c)} = -\frac{p_1}{p_2}\frac{u'(c)}{u'(c)} = -\frac{p_1}{p_2}$

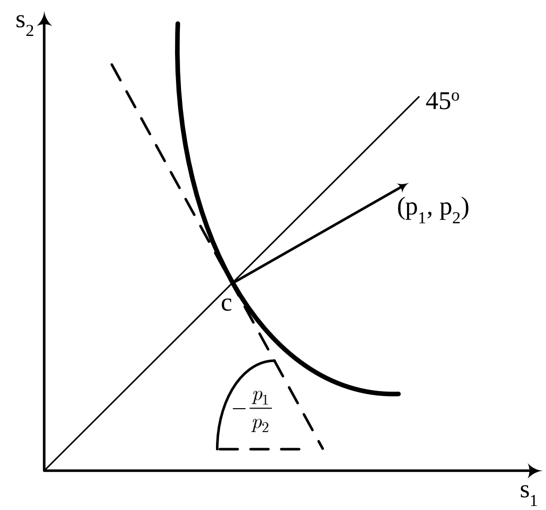

Figure 1. Deriving probabilities from indifference curves.

In his paper Yaari defines subjective beliefs only for decision makers whose indifference curve are smooth. We view this as a serious restriction, since it excludes many important preferences, such as maxmin expected utility. The way to define subjective beliefs for an arbitrary convex preference is to consider the set of *all* supporting hyperplanes.

Definition 1 (Subjective Beliefs). *The set of subjective beliefs at a constant act c is:*

$$\boldsymbol{\pi}^s(c) = \{p \in \Delta S |\ E_p f \geq c \text{ for all } f \succsim c\}.$$

The following proposition asserts that this set is well defined and has desirable properties.

Proposition 2. *If preferences satisfy Axioms 1-4, then for all constant acts c, $\boldsymbol{\pi}^s(c)$ is a nonempty, convex and compact set.*

[3] It is common in the finance literature to call give the name of risk-neutral, or risk-adjusted probabilities, to the probabilities derived in this way. We will avoid this terminology for reasons of first-order risk aversion, discussed later.

An equivalent definition of beliefs was introduced by Chambers and Quiggin (2002). It relies on the natural duality between supporting hyperplanes and the superdifferential of V. For that reason we refer to it as *differential beliefs*.[4] Because concavity is not guaranteed in general, Chambers and Quiggin (2002) rely on the usage of benefit function $b\colon \mathcal{F} \times \mathbb{R} \to \mathbb{R}$ which is defined by: $b(f, v) = \max\{\beta \in \mathbb{R} | V(f - \beta \mathbf{1}) \geq v\}$. One of its most important properties is that it can be used to locally represent preferences, i.e. $f \succ g$ if and only if $b(f, V(g)) > b(g, V(g))$. Moreover, b is concave if V is quasiconcave, which motivates:

Definition 2 (Differential Beliefs). *The set of differential beliefs at a constant act c is:*

$$\boldsymbol{\pi}^d(c) = \partial b(c, V(c)).$$

A natural question arises about the meaning of these definitions. In what sense does the slope of indifference curves represent agent's beliefs if he does not think in a probabilistic manner? What is the behavioral meaning of this slope: what exactly is its relationship to the Ramsey-Savage approach of small bets? The next section answers those questions by studying market behavior and in particular willingness to trade small amounts of assets.

3.2 Market Behavior and Beliefs

In this section we examine the behavioral content of the definition of subjective beliefs $\boldsymbol{\pi}^s$. We study two properties of beliefs, which are familiar from the MEU model. We show that both of them are satisfied by $\boldsymbol{\pi}^s$. More importantly, each of them exactly characterizes subjective beliefs, i.e. beliefs defined by each property coincide with $\boldsymbol{\pi}^s$.

Example 2. *Consider a MEU agent with a set of priors P and a concave, differentiable u. First, consider an act f with $E_p f = c$ for some $p \in P$, drawn on the left panel of Figure 2. The agent has zero demand for f. Second, consider an act f such that $E_p f > c$ for all $p \in P$, drawn on the right panel of Figure 2. The agent can find a small ε, s.t. $\varepsilon f + (1 - \varepsilon)c \succ c$.*

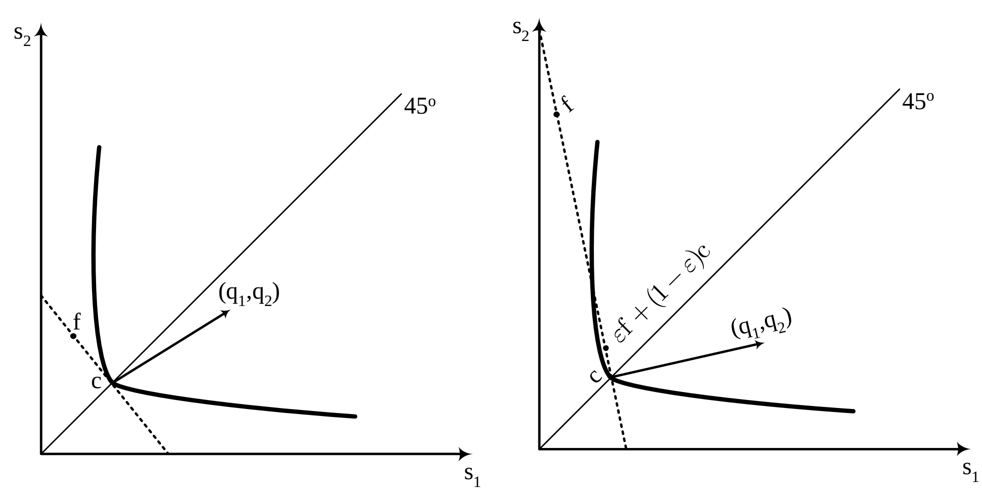

Figure 2. Behavioral properties of beliefs in the MEU model.

[4]There are two kinds of derivatives. In Chambers and Quiggin (2002) and Ghirardato et al. (2004), differentiation is with respect to acts. A different approach of Epstein (1999) and Machina (2004*a*; 2004*b*) is to differentiate with respect to events, which requires an assumption of a rich state space.

The first property — unwillingness to trade — holds for all elements of $\boldsymbol{\pi}^s$. More precisely, if $E_p f = c$ for some $p \in \boldsymbol{\pi}^s(c)$, then it follows that $c \succsim f$. In fact, this property *defines* $\boldsymbol{\pi}^s$. Theorem 1 establishes that the following definition of unwillingness-to-trade revealed beliefs is equivalent to $\boldsymbol{\pi}^s$.

Definition 3 (Unwillingness-to-trade Revealed Beliefs). *The set of beliefs revealed by unwillingness to trade at a constant act c is:*

$$\boldsymbol{\pi}^u(c) = \{q \in \Delta S |\ c \succsim f \text{ for all } f \text{ such that } E_q f = c\}.$$

This set gathers all beliefs for which the agent is unwilling to trade assets with zero expected net returns. It can also be interpreted as a set of Arrow-Debreu prices for which the agent endowed with c will have zero net demand. At any price $q \in \boldsymbol{\pi}^s(c)$ he prefers c to all elements of his budget set.

The second property — willingness to trade — holds for the set of all beliefs $\boldsymbol{\pi}^s$. More precisely, if $E_p f > c$ for all $p \in \boldsymbol{\pi}^s(c)$ then $\varepsilon f + (1-\varepsilon)c \succ c$ for sufficiently small ε. This property also can be used to define $\boldsymbol{\pi}^s$. Consider a set of probability distributions $P \in \Delta S$ with the property that if $E_p f > c$ for all $p \in P$ then $\varepsilon f + (1-\varepsilon)c \succ c$ for sufficiently small ε. Let $\mathcal{P}(c)$ denote the collection of all compact, convex sets P with this property.[5] We define the willingness-to-trade revealed beliefs as the smallest such set.

Definition 4 (Willingness-to-trade Revealed Beliefs). *The set of beliefs revealed by willingness to trade at a constant act c is:*

$$\boldsymbol{\pi}^w(c) = \bigcap \mathcal{P}(c).$$

Remark 1. *The relationship between beliefs and willingness to trade has been studied by Segal and Spivak (1990) and Dow and Werlang (1992). The approach of Segal and Spivak (1990) differs from ours in that unwillingness to trade is attributed to probabilistic first-order risk aversion, whereas we emphasize the role of non-probabilistic uncertainty aversion. For a similar point, see Ghirardato and Marinacci (2002). Dow and Werlang (1992) study the Choquet expected utility model with convex capacity. Example 2 with the MEU model and Theorem 1 extend their work. However, both the analysis of Dow and Werlang (1992), as well as Segal and Spivak's (1990) are restricted to studying how beliefs influence behavior. Our Theorem 1 additionally shows the other direction: how to derive beliefs from behavior.*

The following theorem establishes equivalence between all four definitions. This gives behavioral content to — rather abstract — Definitions 1 and 2. Subjective beliefs are related to observable behavior in terms of willingness to make small bets or trade small amounts of assets. In this way our definition fulfills the requirements of the Ramsey-Savage program. Moreover, for any convex preference the relationship between beliefs and market behavior is the same as in the MEU model. This result means that — locally — every convex preference behaves as MEU.

Theorem 1 (Equivalence). *If preferences satisfy Axioms 1-4, then for all constant acts c,* $\boldsymbol{\pi}^s(c) = \boldsymbol{\pi}^d(c) = \boldsymbol{\pi}^u(c) = \boldsymbol{\pi}^w(c)$.

[5] Note that $\mathcal{P}(c)$ is always nonempty, because $\Delta S \in \mathcal{P}(c)$ by Axiom 3. Also $\mathcal{P}(c)$ is closed under intersections, which follows from the proof of Theorem 1.

3.3 Examples

3.3.1 Invariant Biseparable Preferences

Studied by Ghirardato et al. (2004, 2005), invariant biseparable preferences are the largest class that achieves separation of utility and beliefs. This class contains as special cases the MEU model axiomatized by Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff and Ozdenoren (2000) and the Choquet expected utility (CEU) model axiomatized by Schmeidler (1989) and Gilboa (1987).

They are represented by $V(f) = I(u \circ f)$, where I is a unique, monotonic and constant linear functional, and u is unique up to positive affine transformations. Ghirardato et al. (2004) develop a definition of beliefs for such preferences and study its differential characterization. They show that beliefs can be represented by $\partial I(0)$, the Clarke differential of I. The following proposition shows the relationship with the definition studied in our paper.[6]

Proposition 3. *Let $\succsim$ be an invariant biseparable preference represented by I and a continuous, strictly increasing u. It satisfies Axiom 4 if and only if I and u are concave. Moreover, if u is differentiable, then Axiom 4 implies that $\boldsymbol{\pi}^s(c) = \partial I(0)$ for all $c > 0$.*

Corollary 1. *This proposition implies that the set of priors of a MEU maximizer coincides with $\boldsymbol{\pi}^s$, provided that his utility function is concave, strictly increasing and differentiable. Also for a CEU maximizer with such u the core of his capacity coincides with $\boldsymbol{\pi}^s$, provided that the capacity is convex. More generally, for convex preferences, the Ghirardato et al.'s (2004) definition and the one studied here are the same modulo kinks in u.*[7]

3.3.2 Smooth Model

An important model of non-biseparable preferences was developed by Klibanoff et al. (2004). Their *smooth model*, has the representation: $V(f) = E_\mu \phi(E_p u(f))$, where μ is a probability distribution on the set of probabilities, that the decision maker considers to be possible descriptions of the world.[8] This model allows preferences to display non-neutral attitudes towards ambiguity, but avoids kinks in the indifference curves. The set of subjective beliefs of an agent whose preferences are described by the smooth model with u and ϕ concave, differentiable functions is a singleton consisting of the mixture of all probabilities in the support of his measure μ.

Proposition 4. *Let $\succsim$ be a smooth model preference with u and ϕ concave and differentiable. Then it satisfies Axioms 1-4 and $\boldsymbol{\pi}^s(c) = E_\mu p$ for all $c > 0$.*

[6] We are not relying on any axiomatization of invariant biseparable preferences in the Savage setup, but just studying the functional form. As mentioned in Ghirardato, Maccheroni and Marinacci (2005), such an axiomatization exists and is employing the subjective mixtures of Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003).

[7] Segal and Spivak (1990) show that even in the case of SEU, the behavior of an agent with a non-differentiable u will be first-order risk averse. Because our method attributes all such behavior to beliefs, we will not be able to recover the original prior. On the other hand, the method of Ghirardato et al. (2004) will be able to isolate the prior correctly. Nevertheless, neither method would be correct in the case of a RDEU agent with differentiable u. See also section B.4 of the Appendix.

[8] For similar models, see Segal (1990), Nau (2003) and Ergin and Gul (2004).

3.3.3 Variational Preferences

Introduced and axiomatized by Maccheroni et al. (2004), variational preferences have the following representation $V(f) = \min_{p\in\Delta S}(E_p u(f) + c^\star(p))$, where $c^\star\colon \Delta S \to [0,\infty]$ is a convex, lower semicontinuous function, such that $c^\star(p) = 0$ for at least one $p \in \Delta S$. As special cases, it includes the MEU preferences (where $c^\star$ is the indicator function of the set of priors), the multiplier preferences of Hansen and Sargent (2001) (where $c^\star(p) = R(p \parallel q)$ is the relative entropy between p and some reference distribution q), and the mean-variance preference of Markovitz and Tobin (where $c^\star(p) = G(p \parallel q)$ is the relative Gini concentration index between p and some reference distribution q).

Proposition 5. *Let $\succsim$ be a variational preference with u concave and differentiable. Then it satisfies Axioms 1-4 and $\boldsymbol{\pi}^s(c) = \{p \in \Delta S | c^\star(p) = 0\}$ for all $c > 0$.*

The set of subjective beliefs $\boldsymbol{\pi}^s(c)$ is equal to the set of probabilities for which the index of ambiguity aversion is zero. An interesting special case of this result is that the subjective beliefs of an agent with Hansen and Sargent (2001) multiplier preferences are equal to the singleton $\{q\}$ consisting of the reference probability, since $R(p \parallel q) = 0$ if and only if $p = q$. For the same reason this is also true for the mean-variance preferences.

4 Ex-Ante Trade

Following the literature, we are studying an exchange economy with one good and the space of uncertainty S. There are n agents in the economy, indexed by i. We assume that there is no aggregate uncertainty in the economy, *i.e.* the aggregate endowment w is constant across states. Each agent's consumption set is the set of acts $\mathcal{F}$. An allocation $f = (f_1, \dots, f_n) \in \mathcal{F}^n$ is feasible if $\sum_{i=1}^n f_i(s) = w$ for each $s \in S$. An allocation f is interior, if $f_i(s) > 0$ for all s and for all i. An allocation f, for which f_i is constant across states for all i, will be referred to as a full insurance allocation and any other allocation will be interpreted as betting.

The theorem that we are aiming at characterizes full insurance allocations. Recall that in the case of SEU, full insurance is guaranteed only if people are strictly risk averse and their utilities are strictly increasing in money. Otherwise there may exist allocations to which the agents are indifferent, but which do not provide full insurance. The following two axioms guarantee that this cannot happen.

Axiom 5 (Strong Monotonicity). *For all $f, g \in \mathcal{F}$, if $f \geq g$ and $f \neq g$, then $f \succ g$.*

Axiom 6 (Strict Convexity). *For all $f, g \in \mathcal{F}$, $\alpha \in (0,1)$, if $f \neq g$ and $f \sim g$, then $\alpha f + (1-\alpha) g \succ g$.*

In order to make sure that there is no uncertainty about the fundamentals of the economy, we have made the constant aggregate endowment assumption, which rules out uncertainty about *wealth*. Another feature that we need to rule out is uncertainty about *tastes*, i.e. state dependence. Although formally this is a restriction on preferences, it is more appropriate to view it as a restriction on the nature of uncertainty. We propose the following axiom, which is satisfied by all important special cases of convex preferences.

Axiom 7 (State Independent Utility). *For all constant acts c and c', $\boldsymbol{\pi}^s(c) = \boldsymbol{\pi}^s(c')$.*

The intuition for this axiom comes from the SEU model with state-dependent utility, where the slope of the indifference curves changes at the certainty line. This slope is constant, if utility is state independent. The axiom allows to write $\boldsymbol{\pi}^s$ instead of $\boldsymbol{\pi}^s(c)$.

Before turning to the main theorem, it is useful to observe that our economy can be studied using the standard general equilibrium techniques. The following proposition summarizes the most important results.

Proposition 6. *If $\{\succ_i\}_{i=1}^n$ satisfy Axioms 1-6, then the economy has a Pareto optimum and a Walrasian equilibrium exists. Moreover, all Walrasian equilibrium allocations are Pareto efficient and all interior Pareto efficient allocations are Walrasian equilibrium allocations.*

This proposition asserts that the two, potentially different standards of agreement to trade, i.e. Pareto efficiency and Walrasian equilibrium, are in fact the same. Our results are therefore invariant with respect to the concept used to predict aggregate behavior. In what follows, we restrict attention to Pareto optima. The following theorem is the main result of our paper and is a generalization of Billot et al.'s (2000) theorem (for a finite space).

Theorem 2. *If preferences satisfy Axioms 1-7, then the following statements are equivalent:*

(i) There exists a full insurance Pareto optimal allocation.

(ii) Any interior Pareto optimal allocation is a full insurance allocation.

(iii) Every interior full insurance allocation is Pareto optimal.

(iv) $\bigcap_{i=1}^n \boldsymbol{\pi}_i \neq \emptyset$.

This result means that beliefs are the only information that we need to extract from preferences in order to predict the occurrence of betting. Regardless of the details of the functional form of their preferences, agents are going to trade among themselves, only if their beliefs are drastically different. If they differ mildly, or agree, they will not engage in trade.

5 Incomplete Preferences

In this section we are studying convex, incomplete preferences, a generalization of Bewley's (1986) model (published in Bewley (2002)). In his model a preference relation $\succ$ is represented by a convex, compact set of probability distributions $P \subseteq \Delta S$ and a continuous, concave, strictly increasing utility index u. For all $f, g \in \mathcal{F}$, $f \succ g$ if and only if $E_p u(f) > E_p u(g)$ for all $p \in P$. To generalize this model, consider the following weakening of Axiom 1.

Axiom 1'. The relation $\succ$ is asymmetric and transitive.

We will say that $\succ$ is a *convex, incomplete* preference if it satisfies Axioms 1' and 2-4. Some readers might disagree with our definition, because asymmetry of $\succ$ is equivalent to completeness of $\succsim$. We offer two arguments. First, this relationship is purely formal because $\succsim$, derived from $\succ$ as in footnote 2, looses its meaning of weak preference when $\succ$ does not satisfy Axiom 1. Its true meaning is "preferred, indifferent, or incomparable". For this reason we will avoid using $\succsim$ in what follows. Second, the original model of Bewley (1986), which is expressed in terms of $\succ$, satisfies Axiom 1' (which is equivalent to Bewley's Assumption 1.3.). Bewley's definition of completeness is that for all $f \in \mathcal{F}$, $\mathrm{cl}\,\{y \in \mathcal{F} | x \succ y \text{ or } y \succ x\} = \mathcal{F}$.

In Bewley's model subjective beliefs are associated with the set of priors P. We define subjective beliefs for convex, incomplete preferences as follows. For Bewley's model, our definition coincides with the set of priors P (by Corollary 1 in Rigotti and Shannon (2005)).

Definition 5. *The set of subjective beliefs at a constant act c is*

$$\boldsymbol{\pi}^b(c) = \{p \in \Delta S |\ E_p f > c \ \text{ for all } f \succ c\}$$

The following proposition establishes that for complete preferences it is equivalent to the definition(s) used before.

Proposition 7. *If $\succ$ satisfies Axioms 1-4, then $\boldsymbol{\pi}^b(c) = \boldsymbol{\pi}^s(c)$.*

The foundation for our approach of deriving beliefs from preferences is weaker in the presence of incompleteness. This is also reflected by the fact that (the analogue of) Theorem 1 does not hold in general. However it is possible to show the following useful inclusion.

Proposition 8. *If $\succ$ satisfies Axioms 1' and 2-4, then for all constant acts c, $\boldsymbol{\pi}^w(c) \subseteq \boldsymbol{\pi}^b(c)$.*

The following theorem is an analogue of Theorem 2 for incomplete preferences and is a generalization of Rigotti and Shannon's (2005) result.

Theorem 3. *If preferences satisfy Axioms 1' and 2-7, then the following statements are equivalent:*

(i) There exists a full insurance Pareto optimal allocation.

(iii) Every interior full insurance allocation is Pareto optimal.

(iv) $\bigcap_{i=1}^n \boldsymbol{\pi}_i \neq \emptyset$.

Because of incompleteness, this theorem is necessarily weaker than Theorem 2. This is also the case with Rigotti and Shannon's (2005) theorem. Condition *(ii)* is not equivalent to the other ones, because — even if not fully insured — agents might not want to purchase insurance, unless they are certain about its value. This is one of the main differences between the incomplete preferences model of Bewley (1986) and the MEU model of Gilboa and Schmeidler (1989), where agents buy insurance much more willingly. Our result suggests that the reason behind this difference is deeper than just the functional forms of these preferences and stems from incompleteness.

6 Related Literature

The result for SEU preferences was formally proved by Cass and Shell (1983). Balasko (1983) analyzed symmetric smooth preferences and Goenka and Shell (1997) symmetric convex preferences. Billot et al. (2000) studied the case of MEU, and Tallon (1998) and Billot, Chateauneuf, Gilboa and Tallon (2002) analyzed the case of CEU. Kajii and Ui (2005) found necessary and sufficient conditions for agreeable binary trades within the model of Billot et al. (2000). Rigotti and Shannon (2005) studied the Bewley's (1986) model.

The only assumption that we make is convexity. As observed by Tallon (1998) for the case of CEU, lack of convexity makes sunspots relevant for the agents. In fact, as Billot et al. (2002) show, a continuum of agents is needed for the equivalence. As Chateauneuf, Dana and Tallon (2000) show, shared belief implies no trade, but the converse is not true.

Milgrom and Stokey's (1982) no-trade theorem asserts that when people have asymmetric information there will be no interim agreeable trade, provided that their posteriors are Bayesian updates of a common prior. In fact, depending on the environment studied, the prior does not have to be exactly the same (Morris, 1994). Billot et al.'s (2000) approach, as well as ours, does not permit the disagreement to be created by differential information.

Yaari (1969) was first to define beliefs for convex preferences. His definition was restricted only to smooth preferences. Chambers and Quiggin (2002) proposed an equivalent definition, allowing for kinks. Our contribution is to provide a behavioral characterization. The relationship with earlier work of Segal and Spivak (1990) and Dow and Werlang (1992) is clarified in Remark 1. Ghirardato et al. (2004) study beliefs for invariant biseparable preferences. Imposing convexity makes their definition of beliefs equivalent to the one studied here. Machina and Schmeidler (1992) introduced yet another definition of subjective beliefs by imposing consistency of belief over the whole domain. However, it is not equivalent to Ghirardato et al.'s (2004) nor to the one studied here, for example in the case of RDEU, see also Chambers and Quiggin (2002) and Ghirardato and Marinacci (2002).

7 Conclusions

This paper has accomplished two things. First, using a general definition of beliefs for convex preferences we studied the relationship between beliefs and market behavior. We discovered that, locally, any convex preference behaves as a maxmin expected utility preference. Second, using this definition we showed that the condition for no ex-ante trade discovered by Billot et al. (2000) applies far more generally. We also showed how to use our approach in the case of incomplete preferences, generalizing the result of Rigotti and Shannon (2005). Some special cases of our analysis are the (convex restrictions of) invariant biseparable model of Ghirardato et al. (2004), the smooth model of Klibanoff et al. (2004) and the variational preferences of Maccheroni et al. (2004), which nest the multiplier preferences of Hansen and Sargent (2001). An additional value of our result is that it allows to analyze situations in which agents with different models of preferences interact. The traditional approach of studying each model in isolation does not provide any meaningful results; however, our analysis which nests those models as special cases, does.

Appendix: Proofs

A Preferences and Beliefs

We will be using the notation $\mathcal{U}(f) = \{g \in \mathcal{F} | g \succsim f\}$, $\mathcal{L}(f) = \{g \in \mathcal{F} | f \succsim g\}$ and $\mathcal{W}(f) = \{g \in \mathcal{F} | g \succ f\}$.

Proof of Proposition 1. The proposition follows from Debreu's theorem on existence of utility functions (see Debreu, 1959, page 56). Monotonicity and quasi-concavity of the utility function are standard. $\square$

Proof of Proposition 2. The set $\boldsymbol{\pi}^s(c)$ is nonempty by the supporting hyperplane theorem (see for example Theorem M.G.3 of Mas-Colell, Whinston and Green, 1995). It is a closed and convex set, because it is described by a certain (infinite) system of weak linear inequalities. Being a subset of ΔS, it is compact. $\square$

Lemma 1. *Assume that Axioms 1-4 are satisfied and let c be a constant act. If f is such that $c \geq E_p f$ for some $p \in \boldsymbol{\pi}^s(c)$, then $c \succsim f$.*

Proof. We will show that $f \succ c$, implies $E_p f > c$ for all $p \in \boldsymbol{\pi}^s(c)$. Take some $p \in \boldsymbol{\pi}^s(c)$. By definition, for all acts $g \succsim c$ we have $E_p g \geq c$. In particular f is such an act, therefore $E_p f \geq c$. It remains to show that $E_p f = c$ is impossible. Because the set $\mathcal{L}(c)$ is closed, it's complement is open. Therefore, there exists an open ball around f, such that for all acts g in this ball $E_p g \geq c$. By monotonicity, it follows that $E_p g > c$, so in particular $E_p f > c$. $\square$

Lemma 2. *Assume that Axioms 1-4 and 6 are satisfied and let c be a constant act. If $f \neq c$ is such that $c \geq E_p f$ for some $p \in \boldsymbol{\pi}^s(c)$, then $c \succ f$.*

Proof. We will show that $f \succsim c$, $f \neq c$ implies $E_p f > c$ for all $p \in \boldsymbol{\pi}^s(c)$. Take some $p \in \boldsymbol{\pi}^s(c)$. Two cases are possible. In Case 1, $f \succ c$ and by Lemma 1 we conclude that $E_p f > c$. In Case 2, $f \sim c$ and by definition of $\boldsymbol{\pi}^s(c)$ we conclude that $E_p f \geq c$. It remains to show that $E_p f = c$ is impossible in this case. Let $g = \frac{1}{2} f + \frac{1}{2} c$. Note that $g \neq c$, because $f \neq c$, and that $E_p g = c$. Because of strict convexity, $f \sim c$ implies that $g \succ c$. But from Case 1, we know that $g \succ c$ implies $E_p g > c$. Contradiction. $\square$

Lemma 3. *Assume that Axioms 1-4 are satisfied and let c be a constant act. Then $\mathcal{W}(c) = \operatorname{int} \mathcal{U}(c)$ and $\mathcal{U}(c) = \operatorname{cl} \mathcal{W}(c)$.*

Proof. By continuity $\mathcal{W}(c)$ is open, so by definition of interior $\mathcal{W}(c) \subseteq \operatorname{int} \mathcal{U}(c)$. Conversely, if $y \in \operatorname{int} \mathcal{U}(c)$, then $B(y, \varepsilon) \subseteq \mathcal{U}(c)$ for some $\varepsilon > 0$. Therefore $y - 0.5\varepsilon \succsim c$, hence $y \succ c$ by monotonicity. The second equality follows from the first one and Theorem 6.3 in Rockafellar (1970), because $\operatorname{aff} \mathcal{U}(c) = \mathcal{U}(c)$, by monotonicity. $\square$

Proof of Theorem 1. We show the sequence of inclusions:

$\boldsymbol{\pi}^s(c) \subseteq \boldsymbol{\pi}^u(c)$: Follows from Lemma 1.

$\boldsymbol{\pi}^u(c) \subseteq \boldsymbol{\pi}^s(c)$: Let $p \in \boldsymbol{\pi}^u(c)$ and let $f \succsim c$. We need to show that $E_p f \geq c$. Assume not, and let $a = c - E_p f > 0$. If we let $g = f + a$, then $E_p g = c$, so $c \succsim g$, because $p \in \boldsymbol{\pi}^u(c)$. Axiom 3 implies that $g \succ f$, because $a > 0$. Therefore $c \succ f$, by Axiom 1. Contradiction.

$\boldsymbol{\pi}^w(c) \subseteq \boldsymbol{\pi}^s(c)$: We need to show that $\boldsymbol{\pi}^s(c) \in \mathcal{P}(c)$. Assume that f is such that $E_p f > c$ for all $p \in \boldsymbol{\pi}^s(c)$. It suffices to show that there exists $\delta > 0$ such that $h = \delta f + (1-\delta)c \succ c$.[9] Let $G = \{g \in \mathcal{F} | E_p g > c \text{ for all } p \in \boldsymbol{\pi}^s(c)\}$. We will show that for all $f \in G$, there exists $\alpha > 0$ and $g \succ c$, such that $f = c + \alpha(g - c)$, i.e. $G = c + \bigcup_{\alpha>0} \alpha(\mathcal{W}(c) - c)$.

First, let $N(c)$ be the normal cone to $\mathcal{U}(c)$ at c, and $T(c)$ be the tangent cone to $\mathcal{U}(c)$ at c, i.e. $N(c) = \{h \in \mathbb{R}^S | f \in \mathcal{U}(c) \Rightarrow h.(f-c) \leq 0\}$ and $T(c) = \{g \in \mathbb{R}^S | h \in N(c) \Rightarrow g.h \leq 0\}$. By monotonicity, $-N(c) \subseteq \mathbb{R}^S_+ - \{0\}$, hence $G = c + \operatorname{int} T(c)$. By Corollary 6.3.7 in Borwein and Lewis (2000), $T(c) = \operatorname{cl}(\bigcup_{\alpha \geq 0} \alpha(\mathcal{U}(c) - c))$, hence $G = c + \operatorname{int}(\operatorname{cl}(\bigcup_{\alpha\geq 0} \alpha(\mathcal{U}(c) - c)))$. Second, observe that $\operatorname{aff} \mathcal{U}(c) = \mathbb{R}^S = \operatorname{aff} G$, so their interior and relative interiors coincide. By Theorem 6.3 in Rockafellar (1970) $\operatorname{int}(\operatorname{cl}(\bigcup_{\alpha\geq 0} \alpha(\mathcal{U}(c) - c))) = \operatorname{int}(\bigcup_{\alpha \geq 0} \alpha(\mathcal{U}(c) - c)$. Finally we have that $\operatorname{int}(\bigcup_{\alpha\geq 0} \alpha(\mathcal{U}(c) - c) = \bigcup_{\alpha>0} \alpha \operatorname{int}(\mathcal{U}(c) - c)$ (see Rockafellar (1970), p.50). Hence $G = c + \bigcup_{\alpha>0} \alpha(\operatorname{int} \mathcal{U}(c) - c) = c + \bigcup_{\alpha>0} \alpha(\mathcal{W}(c) - c)$, by Lemma 3.

$\boldsymbol{\pi}^s(c) \subseteq \boldsymbol{\pi}^w(c)$: We need to show that $\boldsymbol{\pi}^s(c) \subseteq P$ for all $P \in \mathcal{P}(c)$. Suppose not, then there exists a set of priors $P \in \mathcal{P}(c)$ and a probability $q \in \boldsymbol{\pi}^s(c)$ such that $q \notin P$. Because P is a closed, convex set and $q \notin P$, by the separating hyperplane theorem, we can find an act $g \in \mathbb{R}^S$ such that $E_p g > c$ for all $p \in P$, and $E_q g < c$. It remains to show that $E_q g < c$ implies that the agent will not want to bet even a little on g. Assume that there exists an ε such that $f = \varepsilon g + (1-\varepsilon)c \succ c$. Lemma 1 implies that $E_q f > c$. But $E_q f = \varepsilon E_q g + (1-\varepsilon)c < c$. Contradiction.

$\boldsymbol{\pi}^d(c) \subseteq \boldsymbol{\pi}^s(c)$: Let $p \in \boldsymbol{\pi}^d(c)$ and let f be an act such that $f \succsim c$. This is implies that $b(c, V(c)) \leq b(f, V(c))$. By definition, p is such that $b(f, V(c)) \leq b(c, V(c)) + E_p(f - c)$, therefore we conclude that $b(c, V(c)) \leq b(c, V(c)) + E_p(f - c)$. This implies that $E_p f \geq c$.

$\boldsymbol{\pi}^u(c) \subseteq \boldsymbol{\pi}^d(c)$: Let $p \in \boldsymbol{\pi}^u(c)$. We need to show that for all acts f, $b(f, V(c)) \leq b(c, V(c)) + E_p(f - c)$. By the translation property of benefit functions, this is equivalent to showing that $b(f - E_p(f-c), V(c)) \leq b(c, V(c))$, which means that $g = f - E_p(f-c) \precsim c$. But this is true, because $E_p g = c$ and $p \in \boldsymbol{\pi}^u(c)$. □

Lemma 4. *$\boldsymbol{\pi}^s(c) = \{p \in \Delta S |\ E_p f \geq c$ for all $f \in \operatorname{cl} \mathcal{W}(c)\} = \{p \in \Delta S |\ E_p f > c$ for all $f \in \mathcal{W}(c)\}$.*

Proof. Let p be such that $E_p f \geq c$ for all $f \in \operatorname{cl} \mathcal{W}(c)$ and let $g \in \mathcal{W}(c)$. Then $g \in \operatorname{cl} \mathcal{W}(c)$, so $E_p g \geq c$. Suppose $E_p g = c$, then $g - \varepsilon \in \mathcal{W}(c)$ by continuity, so also $E_p(g - \varepsilon) \geq c$. But $E_p(g - \varepsilon) = c - \varepsilon < c$. Contradiction.

Let p be such that $E_p f > c$ for all $f \in \mathcal{W}(c)$ and let $g \in \operatorname{cl} \mathcal{W}(c)$. There is $\{g_n\} \subseteq \mathcal{W}(c)$ such that $g_n \to g$. Then $E_p g_n > c$, so $E_p g \geq c$. □

[9] All points $\lambda h + (1-\lambda)c$ for $\lambda \in (0,1)$ will be strictly preferred to c. By continuity, there exists $\varepsilon > 0$ such that $h - \varepsilon \succ c$. Therefore by convexity $\lambda(h-\varepsilon) + (1-\lambda)c \succsim c$. But $\lambda(h-\varepsilon) + (1-\lambda)c < \lambda h + (1-\lambda)c \precsim c - \lambda\varepsilon$. Hence the result follows by monotonicity.

B Examples

Lemma 5. *Assume that $\succsim$ satisfies Axioms 1-4 and V, which represents $\succsim$ is concave. Denote $\boldsymbol{\pi}^\partial(c) = \{\frac{f}{\|f\|} | f \in \partial V(c)\}$. Then $\boldsymbol{\pi}^s(c) = \boldsymbol{\pi}^\partial(c)$.*

Proof. By monotonicity, $\boldsymbol{\pi}^\partial \subseteq \Delta S$. First, we show that $\boldsymbol{\pi}^\partial \subseteq \boldsymbol{\pi}^s$. Let $p \in \boldsymbol{\pi}^\partial$, i.e. $p = \frac{g}{\|g\|}$ for some $g \in \partial V(c)$. Let $f \succsim c$, which implies that $V(f) \geq V(c)$. We have $0 \leq V(f) - V(c) \leq \langle g, f - c\rangle$, hence $\langle g, c\rangle \geq \langle g, f\rangle$, so $E_p f \geq c$. To prove the converse, we will show that $\boldsymbol{\pi}^w \subseteq \boldsymbol{\pi}^\partial$. We need to $\boldsymbol{\pi}^\partial(c) \in \mathcal{P}(c)$. Assume that f is such that $E_p f > c$ for all $p \in \boldsymbol{\pi}^\partial(c)$. We need to show that the agent will want to buy a little bit of f, i.e. that $V(\varepsilon f + (1-\varepsilon)c) > V(c)$ for sufficiently small ε. V is a proper concave function, therefore the one sided directional derivatives $V'(c; h)$ exist in all directions $h \in \mathbb{R}^S$, and $V'(c; h) = \min\{\langle l, h\rangle | l \in \partial V(c)\}$.[10] Hence we can write:

$$\begin{aligned}
V(\varepsilon f + (1-\varepsilon)c) &= V(c + \varepsilon(f - c)) \\
&= V(c) + \varepsilon V'(c; f - c) + o(\varepsilon) \\
&= V(c) + \varepsilon \min\{\langle l, (f - c)\rangle |\, l \in \partial V(c)\} + o(\varepsilon) \\
&= V(c) + \varepsilon \langle g, (f - c)\rangle + o(\varepsilon) \\
&= V(c) + \varepsilon[\langle g, (f - c)\rangle + o(1)],
\end{aligned}$$

where g is some element of $\partial V(c)$. Hence $g = \|g\| p$, for some $p \in \boldsymbol{\pi}^\partial$ and by assumption, $\langle g, (f - c)\rangle = \|g\| E_p(f - c) > 0$. Therefore, there exists a $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$, $[E_p(f - c) + o(\varepsilon)] > 0$, hence $V(\varepsilon f + (1 - \varepsilon)c) - v(c) > 0$. $\square$

B.1 Invariant Biseparable Preferences

Lemma 6. *Let $\succsim$ be an invariant biseparable preference. Then I is quasiconcave if and only if I is concave.*

Proof. The proposition follows from the fact that I is positive homogenous and the fact that all quasiconcave positive homogenous functions are concave. $\square$

Lemma 7. *Let $\succsim$ be an invariant biseparable preference represented by I and a continuous, strictly increasing u. Then $\succsim$ satisfies Axiom 4 if and only if u is concave and I is quasiconcave.*

Proof. If u is concave and I is quasiconcave, then $V = I(u)$ is quasiconcave (by monotonicity of I), so Axiom 4 is satisfied. Conversely, if Axiom 4 holds, then $\succsim$ exhibits preference for comonotone diversification and therefore by Theorem 17 of Ghirardato and Marinacci (2001) u is concave. The rest of the proof shows that I is quasiconcave. Assume not. Then there exist $\phi, \psi \in \mathbb{R}^S$, such that $I(\phi) = I(\psi) = a$, $\lambda \in (0, 1)$ and $e > 0$, such that that $I(\lambda\phi + (1 - \lambda)\psi) = a - e$. By constant linearity of I, we have that for any $d \in \mathbb{R}^S$

[10] Theorem 23.4 of Rockafellar (1970) implies that $V'(c; h) = \inf\{\langle l, h\rangle | l \in \partial V(c)\}$ for all h. Because V is a proper concave function, $\partial V(c)$ is a compact set, hence the infimum is achieved.

and $t > 0$, $I(\lambda(d + t\phi) + (1 - \lambda)(d + t\psi)) = d + t(a - e)$. Choose d and t, such that $d = u(c)$ for some $c > 0$ such that u is differentiable at c (this can be done, since u being a concave function is differentiable almost everywhere) and there exist acts f^t, g^t, such that $f^t(s) = u^{-1} \circ (d + t\phi)$ and $g^t = u^{-1} \circ (d + t\psi)$ (this can be done by choosing sufficiently small t). It follows that $V(f^t) = V(g^t) = d + ta$. Axiom 4 implies that V is quasiconcave, hence $I(u(\lambda f^t + (1 - \lambda)g^t)) = V(\lambda f^t + (1 - \lambda)g^t) \geq d + ta$. Hence $0 < e < \frac{1}{t}\left[I(u \circ (\lambda f^t + (1 - \lambda)g^t)) - I(\lambda u \circ (f^t) + (1 - \lambda)u \circ (g^t))\right]$. By Lipschitz continuity of I, we have that $0 < e < \frac{1}{t}\left\|u \circ (\lambda f^t + (1 - \lambda)g^t) - \lambda u \circ (f^t) - (1 - \lambda)u \circ (g^t)\right\|$. We will now show that the right hand side converges to zero, which is a contradiction. Fix a state s and observe that $u(\lambda f^t(s) + (1 - \lambda)g^t(s)) = u(c) + u'(c)[\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c)] + o(\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c))$. Moreover, u is a differentiable, strictly increasing function, hence u^{-1} is continuous, and the inverse function theorem applies, so $f^t(s) = u^{-1}(d + t\phi(s)) = c + t\frac{\phi(s)}{u'(c)} + o(t)$ and similarly for $g^t(s)$. Hence

$$
\begin{aligned}
&\frac{1}{t}\left[u(\lambda f^t(s) + (1 - \lambda)g^t(s)) - \lambda u(f^t(s)) - (1 - \lambda)u(g^t(s))\right] \\
&= \frac{o(t)}{t} + \frac{1}{t}o(\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c) \\
&= \frac{o(t)}{t} + \frac{o(\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c))}{\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c)}\frac{\lambda(f^t(s) - c) + (1 - \lambda)(g^t(s) - c)}{t} \\
&= o(1) + o(1)\frac{\lambda t\frac{\phi(s)}{u'(c)} + (1 - \lambda)t\frac{\psi(s)}{u'(c)} + o(t)}{t} = o(1)
\end{aligned}
$$
□

Lemma 8. *Let $I\colon \mathbb{R}^S \to \mathbb{R}$ be a constant linear, concave functional, $u\colon \mathbb{R}_+ \to \mathbb{R}$ a differentiable, concave function and let $V(f) = I(u \circ f)$. Then for all $c > 0$, $\partial V(c) = u'(c)\partial I(0)$.*

Proof. It suffices to show that $V'(c; f) = u'(c)I'(0; f)$ for all f. Observe first that because u is differentiable, we have $V(c+tf) = I(u\circ(c+tf)) = I(u(c)+tu'(c)f+o(t))$. It follows that $V'(c; f) - u'(c)I'(0; f) = \lim_{t\downarrow 0}\frac{1}{t}[V(c + tf) - V(c)] - \lim_{t\downarrow 0}\frac{1}{t}[I(u(c) + tu'(c)f) - I(u(c))] = \lim_{t\downarrow 0}\frac{1}{t}[I(u(c) + tu'(c)f + o(t)) - I(u(c) + tu'(c)f)]$. By constant-linearity if I, we have that $\|V'(c; f) - u'(c)I'(0; f)\| \leq u'(c)\lim_{t\downarrow 0}\|I(f + \frac{o(t)}{tu'(c)}) - I(f)\| \leq \lim_{t\downarrow 0}\|\frac{o(t)}{t}\| = 0$ by Lipschitz continuity of I. □

Proof of Proposition 3. Let $V(f) = I(u\circ f)$. The first part of the proposition corresponds to Lemmas 7 and 6. For the second part, note that if u and I are concave, so is V. Lemma 8 implies that for all $c > 0$, $\partial V(c) = u'(c)\partial I(0)$. Therefore, by Lemma 5, for all $c > 0$ $\boldsymbol{\pi}^s(c) = \boldsymbol{\pi}^\partial(c) = \partial I(0)$, because $\partial V(0) \subseteq \Delta S$. □

Proof of Corollary 1. Convexity: $V(\lambda f+(1-\lambda)g) = \min_{p\in P} E_p u(\lambda f+(1-\lambda)g) = E_q u(\lambda f+(1 - \lambda)g)$ for some probability $q \in P$. Also, $E_q u(\lambda f + (1 - \lambda)g) \geq E_q[\lambda u(f) + (1 - \lambda)u(g)] = \lambda E_q u(f) + (1-\lambda)E_q u(g) \geq \lambda \min_{p\in P} E_p u(f) + (1-\lambda)\min_{p\in P} E_p u(g) = \lambda V(f) + (1-\lambda)V(g)$. Monotonicity is trivial and continuity follows from Berge's (1963) theorem of maximum. Therefore MEU preferences are convex and the set of priors P coincides with $\boldsymbol{\pi}^s$. For a CEU maximizer Chateauneuf and Tallon (2002, Theorem 1, Proposition 1) show that preferences are convex if and only if u is concave and ν is convex. In this case they become a special case of MEU preferences, where the set of priors is equal to the core of the capacity. □

B.2 Smooth Model

Proof of Proposition 4. Convexity: $V(\lambda f + (1-\lambda)g) = E_\mu \phi(E_p u(\lambda f + (1-\lambda)g)) \geq E_\mu \phi(E_p[\lambda u(f) + (1-\lambda)u(g)]) = E_\mu \phi(\lambda E_p u(f) + (1-\lambda)E_p u(g)) \geq E_\mu[\lambda \phi(E_p u(f)) + (1-\lambda)\phi(E_p u(g))] = \lambda E_\mu \phi(E_p u(f)) + (1-\lambda)E_\mu \phi(E_p u(g)) = \lambda V(f) + (1-\lambda)V(g)$. Continuity and monotonicity are routine. To calculate $\boldsymbol{\pi}^s(c)$, we compute directional derivatives at f:

$$\begin{aligned}
V(f+tg) &= E_\mu \phi(E_p u(f+tg)) \\
&= E_\mu \phi(E_p[u(f) + tu'(f)g + o(t)]) \\
&= E_\mu \phi(E_p u(f) + t[E_p u'(f)g + o(1)]) \\
&= E_\mu\{\phi(E_p u(f)) + t\phi'(E_p u(f))[E_p u'(f)g + o(1)]) + o(t)\} \\
&= E_\mu\{\phi(E_p u(f)) + tE_\mu \phi'(E_p u(f))E_p u'(f)g + o(t)\} \\
&= V(c) + tE_\mu\{\phi'(E_p u(f))E_p u'(f)g\} + o(t).
\end{aligned}$$

From this, it follows, that $V'(f;g) = E_\mu \phi'(E_p u(f))E_p u'(f)g$, which is a linear function of g. Also, for each g, $V'(\cdot;g)$ is a continuous function of f, therefore V is differentiable at each point. In particular, it is differentiable at a constant act c, where $V'(c;g) = \phi'(u(c))u'(c)E_q g$ for a probability distribution $q(s) = E_\mu p(s)$. Therefore the (super)differential $\partial V(c)$ is equal to $\{u'(c)q\}$, so by Lemma 5, $\boldsymbol{\pi}^s(c) = \{q\}$. □

B.3 Variational Preferences

Proof of Proposition 5. Convexity: $V(\lambda f+(1-\lambda)g) = \min_{p\in\Delta S}(E_p u(\lambda f+(1-\lambda)g) + c^\star(p))$ $= E_q u(\lambda f+(1-\lambda)g)+c^\star(q)$ for some $q \in \Delta S$. Also, $E_q u(\lambda f+(1-\lambda)g)+c^\star(q) \geq E_q(\lambda u(f)+(1-\lambda)u(g)) + c^\star(q) = \lambda E_q(u(f) + c^\star(q)) + (1-\lambda)E_q(u(g) + c^\star(q)) \geq \lambda \min_{p\in\Delta S} E_p(u(f) + c^\star(p)) + (1-\lambda)\min_{p\in\Delta S} E_p(u(g) + c^\star(p)) = \lambda V(f) + (1-\lambda)V(g)$. To prove continuity and monotonicity, observe that $I(\varphi) = \min_{p\in\Delta S}(E_p\varphi + c^\star(p))$ is a monotonic, vertically invariant functional, hence Lipschitz continuous. Hence $I \circ u$ is continuous (and monotone).

Let $P^\star = \{p \in \Delta S | c^\star(p) = 0\}$. We first show the inclusion $P^\star \subseteq \boldsymbol{\pi}^s(c)$. Let V be the preference functional, i.e. $V(f) = \min_{p\in\Delta S} E_p u(f) + c^\star(p)$. Let p be such that $c^\star(p) = 0$ and and let f be an act, such that $E_p f = c$. We need to show that $c \succsim f$. By definition, $V(f) \leq E_p u(f) + c^\star(p) = E_p u(f)$. Also, because $E_p f = c$, we know that by concavity of u, $E_p u(f) \leq u(c)$. Therefore $V(f) \leq V(c)$, so $p \in \boldsymbol{\pi}^s(c)$. Now we show the inclusion $\boldsymbol{\pi}^w(c) \subseteq P^\star$. We need to prove that $P^\star \in \mathcal{P}$, i.e. that $E_p > c$ for all $p \in P^\star$, implies that there exists a $\delta > 0$ such that for all $\varepsilon \in (0,\delta)$, $\varepsilon f + (1-\varepsilon)c \succ c$. Because $c^\star$ is lower semicontinuous, $P^\star$ is compact, so we have that $e = \min_{p\in P^\star} E_p f - c > 0$. Recall that $\frac{o(\varepsilon)}{\varepsilon} = o(1) \to 0$, as $\varepsilon \to 0$ and observe that by linear expansion of u around c we have:

$$\begin{aligned}
V(\varepsilon f+(1-\varepsilon)c) &= \min_{p\in\Delta S} E_p u(\varepsilon f+(1-\varepsilon)c)+c^\star(p)\\
&= \min_{p\in\Delta S} E_p\left[u(c)+\varepsilon u'(c)(f-c)+o(\varepsilon)+c^\star(p)\right]\\
&= u(c)+\varepsilon u'(c)\left[\min_{p\in\Delta S} E_p f-c\right]+c^\star(p)+o(\varepsilon)\\
&= V(c)+\varepsilon u'(c)\left[\min_{p\in\Delta S} E_p f-c+\frac{1}{\varepsilon}c^\star(p)+o(1)\right].
\end{aligned}$$

Hence

$$\frac{V(\varepsilon f+(1-\varepsilon)c)-V(c)}{\varepsilon} = u'(c)\left[\min_{p\in\Delta S} E_p f-c+\frac{1}{\varepsilon}c^\star(p)+o(1)\right].$$

We need to show that the right hand side of this equation eventually becomes positive. We will show that eventually $\min_{p\in\Delta S} E_p f-c+\frac{1}{\varepsilon}c^\star(p)>\frac{e}{2}$. Assume to the contrary, that

($\star$) for all $\delta>0$ there exists an $\varepsilon\in(0,\delta)$, such that $\min_{p\in\Delta S} E_p f-c+\frac{1}{\varepsilon}c^\star(p)\le\frac{e}{2}$.

This means that there exists a sequence $\varepsilon^N\to 0$, and a sequence $p^N\in\Delta S$, such that $E_{p^N}f+\frac{1}{\varepsilon^N}c^\star(p^N)\le c+\frac{e}{2}$. Hence, for each N, $0\le c^\star(p^N)\le\varepsilon^N\left[c+\frac{e}{2}-E_{p^N}f\right]\le\varepsilon^N\left[c+\frac{e}{2}-m\right]$, where $m=\min_{p\in\Delta S}E_p f$ and therefore $c^\star(p^N)\to 0$ as $N\to\infty$. Now take a convergent subsequence $\{p^{N_k}\}\to p$ and observe that necessarily $p\in P^\star$, as a consequence of lower semicontinuity of $c^\star$. This implies that $E_{p^{N_k}}f\to E_p f\ge e$. Therefore $\liminf_{k\to\infty}E_{p^{N_k}}f+\frac{1}{\varepsilon^{N_k}}c^\star(p^{N_k})\ge e$, which violates ($\star$). □

B.4 Probabilistically sophisticated preferences

As observed by Chambers and Quiggin (2002), if the decision maker is a probabilistically sophisticated non-expected utility maximizer, the subjective beliefs derived from supporting hyperplanes do not have to coincide with his true subjective probability. For example consider a RDEU maximizer with subjective probability p, distortion function g and differentiable u. In this case $\boldsymbol{\pi}^s(c)=\{q\in\Delta S|q(E)\le g(p(E))\text{ for all }E\subseteq S\}$ for all constant acts c. Therefore the set of subjective probabilities derived from the supporting hyperplanes is different (in this case strictly larger) than the true subjective probability p.

This problem is not specific to our analysis, since the beliefs of Ghirardato et al. (2004) have a similar drawback, as discussed in Ghirardato and Marinacci (2002). (In fact, for the example above, Theorem 1 holds). This is because all deviations from SEU are attributed to non-probabilistic uncertainty aversion, rather than probabilistic first-order risk aversion. Also, compare the definitions of Epstein (1999) and Ghirardato and Marinacci (2002).

There exists another problem with both methods, as well as with the standard SEU theory. The probability identified from the SEU representation does not have to be the true subjective probability if *utility* is state dependent (Karni, Schmeidler and Vind, 1983; Karni and Mongin, 2000). The identified probability can only have a behavioral interpretation but does not have any cognitive meaning. In the same way, the beliefs derived from supporting hyperplanes do not have any cognitive meaning but are just a description of agent's behavior.

C Ex-Ante Trade

Proof of Proposition 6. The proof amounts to checking that assumptions of standard theorems are satisfied:
(i) follows from Debreu's theorem on existence of Pareto optima (see Debreu, 1959, page 92). This is because $\mathcal{F}$ is a closed, connected subset of $\mathbb{R}^S$, with 0 as a lower bound, and $\succsim_i$ is continuous.
(ii) follows from Propositions 17.B.2 and 17.C.1 of Mas-Colell et al. (1995).
(iii) follows from Proposition 16.C.1 of Mas-Colell et al. (1995).
(iv) follows from Propositions 16.D.1, 16.D.2 and 16.D.3 of Mas-Colell et al. (1995). $\square$

Proof of Theorem 2. First, we will prove that (iv) implies (ii), which trivially implies (i), because by Proposition 6 Pareto efficient allocations always exist. Next we show that (iv) implies (iii), which also trivially implies (i). Finally, we show that (i) implies (iv).

(iv) $\Rightarrow$ (ii): Let's assume without loss of generality that f_1 is not constant. We will construct a new allocation c which Pareto dominates f. Let $p \in \bigcap_{i=1}^n \boldsymbol{\pi}_i^s$ and define $c_i = E_p f_i$, for $i = 1, \ldots, n$. By Theorem 1 $p \in \boldsymbol{\pi}_i^u(c)$, which implies that $c_i \succsim_i f_i$ for $i = 1, \ldots, n$. Also, by assumption f_1 is not constant, so by Lemma 2, $c_1 \succ_1 f_1$, because $p \in \boldsymbol{\pi}_i^s$ and $\succ_1$ is strictly convex. It remains to check that the allocation c is aggregate feasible. $\sum_{i=1}^n c_i = \sum_{i=1}^n E_p f_i = E_p \sum_{i=1}^n f_i = w$, because $\sum_{i=1}^n f_i = w$ by aggregate feasibility of f.

(iv) $\Rightarrow$ (iii): Suppose to the contrary, that d is a full insurance allocation, which is dominated by some other allocation f. By continuity and monotonicity we can assume that the dominance is strict. Let $p \in \bigcap \boldsymbol{\pi}_i^s$. By Lemma 4 $E_p f_i > d_i$ for all i. Hence $w = \sum_{i=1}^n f_i = E_p \sum_{i=1}^n f_i = \sum_{i=1}^n E_p f_i > \sum_{i=1}^n d_i = w$. Contradiction.

(i) $\Rightarrow$ (iv): Suppose that $\bigcap \boldsymbol{\pi}_i^s = \emptyset$ and let c be an interior full insurance allocation. We will construct a new allocation h, which Pareto dominates c. By the separation theorem of Billot et al. (2000) applied to $X = \mathbb{R}^S$, there exists a nonempty set $I \subseteq \{1, \ldots, n\}$, a point $p \in co(\bigcup_{i \in I} \boldsymbol{\pi}_i^s) \subseteq \Delta S$ and for each $i \in I$, a continuous linear functional $\Upsilon_i \colon X \to \mathbb{R}$. We identify Υ_i with an act $g_i \in \mathbb{R}^S$ such that for all $p \in X$, $\Upsilon_i(p) = E_p g_i$. Moreover, the theorem imposes the following restrictions:

(a) $\forall_{i \in I}, E_q g_i - E_p g_i > 0$ for all $q \in \boldsymbol{\pi}_i^s$,

(b) $\sum_{i \in I} g_i = 0$.

For $i \in I$, define $h_i^\varepsilon = c_i + \varepsilon(g_i - E_p g_i) = \varepsilon[c_i + (g_i - E_p g_i)] + (1-\varepsilon)c_i$. Condition (a) implies that $E_q[c_i + (g_i - E_p g_i)] > c_i$, for all $q \in \boldsymbol{\pi}_i$. By Theorem, 1 $\boldsymbol{\pi}_i^w(c) \subseteq \boldsymbol{\pi}_i^s$, hence there is a sufficiently small ε, such that $h_i^\varepsilon \succ_i c_i$, for all $i \in I$. Let $h_i = h_i^\varepsilon$ for $i \in I$ and $h_i = c_i$ for $i \notin I$. To conclude that h Pareto dominates c, it remains to check that the allocation h is feasible. $\sum_{i=1}^n h_i = \sum_{i=1}^n c_i + \varepsilon[\sum_{i \in I}(g_i - E_p g_i)] = w + \varepsilon[\sum_{i \in I} g_i - E_p \sum_{i \in I} g_i] = w$, because of condition (b). Choosing ε small enough ensures individual feasibility. $\square$

D Incomplete Preferences

Proof of Proposition 7. Follows from Lemmas 3 and 4. □

Proof of Proposition 8. Mimics the proof of the appropriate part of Theorem 1, which does not rely on negative transitivity. □

Proof of Theorem 3. Mimics the proof of appropriate parts of Theorem 2, which do not rely on negative transitivity. □

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