Buyer Substitutability with Allocative Externalities*

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Abstract

We consider the assignment model of homogeneous units of a good in a general framework where preferences may involve both multi-unit demand and allocative externalities: individual payoffs are additive according to the various usages of the good and the payoff for a given usage is given as the sum of an intrinsic valuation term and an externality term that depends negatively on the number of units assigned to the same usage across all buyers. We characterize the externality structures that generate preferences satisfying buyer substitutability, i.e. such that the contribution of an agent to a coalition is nonincreasing in the size of the coalition. As an application, we introduce contingent auctions, a superset of combinatorial auctions where bidders submit bids that are contingent on the whole final assignment, and analyze the ability of two contingent formats to implement the Vickrey payoff point in a robust way.

Keywords: Assignment model, Allocative externalities, Identity-dependent externalities, Buyer substitutability, Buyer submodularity, Multi-unit auctions, Contingent auctions, Combinatorial auctions

JEL classification: D44, D51, D62

1 Introduction

The notion of substitutability is a cornerstone in the economic theory of resource allocation. Two kinds of notions have appeared in the literature: 'item substitutability' and 'buyer substitutability'. The latter, which is related to the joint preferences of the agents in the economy, corresponds that saying that the marginal contribution of an agent to the welfare in the efficient assignment shrinks with the set of agents in the economy. On the contrary, the former notion is related directly to individual preferences and there are a vast number of definitions capturing it, the best known saying roughly that the demand for one

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good does not shrink when the prices for the alternative goods rise. The underlying research program is to go beyond the Arrow-Debreu theory of general equilibrium where the analog of substitutability is captured by some convexity assumptions. The seminal contribution in this literature is Kelso and Crawford [34] who consider indivisibilities (discrete goods) and non-atomic agents (a finite set of agents) and introduce the *gross substitute* condition. This program still receives considerable interest recently including the works of Bikhchandani and Mamer [7], Gul and Stacchetti [25, 26], Bikhchandani and Ostroy [8, 9], Ausubel and Milgrom [3], Ausubel [2], Sun and Yang [44, 45], de Vries, Schummer and Vohra [16] and Milgrom and Strulovici [40].[1,2] All this literature is restricted to pure private goods. In particular, it excludes any allocative externalities which are a crucial element in many real-life assignment problems as investigated, for the auction literature, by numerous papers of Philippe Jehiel and Benny Moldovanu and co-authors [32, 29, 33, 30, 28] where bidders do not care solely on their own assignment but also on the number and identities of the final purchasers. E.g., prior to the German UMTS auction, where the design has left some flexibility with respect to the final number of licensees, Hoppe et al. [28] report that "a major investment bank estimated per-licenses values of Euro 14.75, 15.88, and 17.6 billion for a German UMTS market with 6, 5, and 4 firms, respectively". While the European UMTS auctions were devoted to a specific usage of the spectrum, US spectrum auctions are more flexible meaning that some competing bidders may interact in the downstream markets while others do not. In other words, allocative externalities may be identity specific or more precisely usage specific. More generally, when the assignment problem involves a scarce good that is a main input in some separate downstream markets, the value of the good for a given usage crucially depends on the quantity assigned to the same usage. At a first order level, we can consider that this value depends solely on this quantity and in a negative way reflecting that the competitiveness of a downstream market increases with the quantity of the scarce input assigned to this market.[3] Those are the critical ingredients of our assignment model with allocative externalities which is labeled as the Negative Usage-Dependent Externalities structure (henceforth NUDE) where private values for a given usage are the sum of two elements: an intrinsic valuation structure and an externality structure which depends on the number of units assigned to the given usage.

[1] Sun and Yang [44, 45] deserve a special credit since they consider generalized substitutability notions when buyers are valuing some goods explicitly as complements.

[2] Another departure from the Arrow-Debreu framework has received considerable attention: non-transferabilities as in pure matching models. See Echenique and Oviedo [18] and Hatfield and Milgrom [27].

[3] Such allocative externalities are the cornerstone of the IO literature on entry. See Berry [6] for an empirical analysis in the airline industry. The disassociation of the various submarkets may be questionable for airport slots where a submarket corresponds to a given route, i.e. an origin-destination pair. On the contrary, passenger and freight transportation correspond to two fully disconnected usages of railroad tracks.

When the latter vanishes, the model reduces to the standard pure private value model with nonincreasing marginal valuations inside a given usage and with additive valuations across usages.

The aim of the paper is to investigate the notion of 'buyer substitutability' in terms of private preferences in our NUDE model which involves both multi-unit demand and above all allocative externalities. In other words, the issue is to extend substitutability notions for resource allocation problems in a direction that has not been considered previously by the literature. Without allocative externalities, the link between the 'buyer substitutability' condition and the primitives of the model is well-known: Ausubel and Milgrom [3] show that buyers valuing the goods as 'substitutes' is a sufficient condition for buyer substitutability and provide a partial converse. Unit demand is a special case analyzed previously by Crawford and Knoer [13] and Demange et al. [17]. With multi-unit of an homogenous good, 'item substitutability' reduces to buyers having nonincreasing marginal utilities. Our model imposes explicitly nonincreasing marginal utilities since the failure of buyer substitutability is well known without this assumption to focus only on the new elements, i.e. the externality terms. Our main contribution is to characterize the externality structures that are generating preferences satisfying buyer substitutability. The conditions are directly and easily interpretable in terms of individual preferences.

Without allocative externalities, the aforementioned literature on the theory of resource allocation shows that the substitutability concepts are key elements to obtain various desirable properties including the existence of Walrasian equilibria, the convergence of some simple tâtonnement processes to the efficient assignment, the stability of the Vickrey payoff point (also known as the Vickrey-Clarke-Groves outcome) where stability is captured by the inclusion in the Core, strategic proofness of combinatorial auction designs according to individual deviations but also to some kind of coalitional deviations. As an example, Bikhchandani and Ostroy [8] show that buyer-substitutability is equivalent to the Vickrey payoff point being implementable by a pricing equilibrium (with may involve non-linear and non-anonymous pricing) which is also equivalent to the Vickrey payoff point being in the Core.[4] Under buyer-substitutability, the Core and the pricing equilibria (which coincide with a unique seller) have the lattice property with respect to buyers. In particular, there is a unique buyer-optimal element in this set which corresponds to the Vickrey payoff point. With allocative externalities, the 'buyer substitutability' property is still linked with some desirable properties for the noncooperative analysis of contingent auctions, a superset of

[4] The precise definition of buyer-substitutability in Bikhchandani and Ostroy [8] is slightly weaker than the one mentioned above and considered throughout the paper that corresponds to the one in Ausubel and Milgrom [3] who derived similar results.

combinatorial auctions where bidders are able to submit bids that are contingent on the whole final assignment, as argued in section 6 where our theoretical results are translated in term of auction design insights.

This paper is organized as follows: Section 2 introduces the Negative Usage-Dependent Externalities model. Section 3 considers a more primitive notion of substitutability which is limited to pairs of buyers: the notion of 'substitute buyers' says that the marginal contribution of one of those buyers in any coalition rises when the other one is removed from the coalition. Two kinds of conditions are introduced to obtain sufficient conditions that guarantee that two unit demand buyers l, m are 'substitute buyers': the first is a convexity condition on the total externality that is imposed on the buyers for a given usage as a function of the number of units assigned to this usage; the second is related to the sign of the differences between the slopes of the externality terms of the different potential buyers. Under an appropriate convexity property, proposition 3.1 establishes that if those slopes are greater [smaller] for less efficient buyers according to any usage and if two unit demand buyers l and m consider distinct usages [the same usage] of the good, then l and m are 'substitute buyers'. General sufficient conditions for unit demand buyers can be easily extended to multi-unit demand buyers from the bundling argument if we view those buyers as 'bundles' of unit demand buyers. The key intuition that drives our sufficient conditions is the following: we have to identify the impact of removing unit demand buyer l from the coalition on the chance of unit demand buyer m to obtain a unit of the good. If it increases then buyer m is 'intuitively' a substitute to buyer l. Consider the removal of a winning unit demand buyer l corresponding to usage g. If the externality term is steeper for more efficient buyers, then the subsequent most efficient buyers for usage g will suffer from a smaller externality if additional units are assigned to this usage. On the whole, the number of units assigned to usage g will not shrink from more than one unit, i.e. the one previously assigned to l, such that the chance of obtaining a unit for some unit demand buyer m according to usage g can only increase. On the contrary, the chance of obtaining a unit of the good for a unit demand buyer according to a different usage $g' \neq g$ may decrease after buyer l's removal as established later when we will investigate converses. Similarly, when the externality term is steeper for less efficient buyers, it is the reverse comparative statics that holds: the number of units assigned to the usages $g' \neq g$ will not shrink while the number of units assigned to usage g may shrink.

Theorem 1 in section 4 characterizes the externality structures that 'generate' joint preferences that satisfy buyer substitutability: the result is 'negative'[5] since it reduces to a

[5]'Negative' reflects the disappointing flavor of the result with respect to the subsequent applications.

set of non-generic structures that are called uniform and convex. The 'uniform' condition imposes in particular that for a given usage all valuations for an additional unit are equal up to a constant, i.e. the externality term is the same for all marginal valuations. In other words, it excludes any private information with respect to the externality terms. We then move to a slightly tighter concept for how to 'generate' preferences for some externality structure, in a way such that more efficient marginal buyers correspond to steeper externality terms. We then obtain a 'positive' result in environments with a single usage: Theorem 2 provides a general sufficient condition for buyer substitutability under a convexity condition and when the externality terms can be fully ordered according to their steepness. Contrary to the 'uniform' condition, we emphasize that the 'steepness' condition does not impose any constraint with respect to the dimensionality of the private information of the agents. In a nutshell, this is a condition that is satisfied for a generic set of preferences in the NUDE framework. Theorem 2 contains also a partial converse. In particular the result does not hold anymore with multiple usages.

Additionally to a brief discussion with respect to previous models with allocative externalities in the IO literature, section 5 is devoted to the illustration of the NUDE framework and of our key conditions through simple economic examples. We apply our theoretical results in section 6 where we compare two kinds of contingent auctions that generalize their combinatorial counterparts: the Vickrey auction versus one that picks a buyer-optimal Core outcome given the reported preferences. Without allocative externalities, both appear to be robust if and only if buyers have substitute preferences (see Ausubel and Milgrom [3]).[6] The main insight to have in mind and which explains the links between our characterizations of buyer substitutability and our non-cooperative analysis of contingent auctions is that the equivalence between buyer substitutability and the Vickrey payoff point being in the buyer-optimal frontier of the Core still holds under our general framework with allocative externalities. However, with allocative externalities, the comparison between auction formats becomes more subtle: there is a difference between having true preferences that satisfy buyer substitutability and having an auction design that can restrict bidders to report private preferences such that the joint preferences satisfy buyer substitutability. Without allocative externalities, this difference was innocuous since the characterization of the preferences satisfying buyer substitutability leads solely to constraints on individual preferences. On the contrary, our characterizations lead to constraints between the exter-

[6] Ausubel and Milgrom [3] consider also another dimension in the comparison since they compare a dynamic version of a Core selecting auction to a static Vikrey auction. In the present paper, we leave aside any dynamic consideration which can be another source of benefit either from an informational linkage perspective when valuations are interdependent (Ausubel [1]) or from an information acquisition perspective (Compte and Jehiel [11]).

nality terms of the potential buyers: the set of preferences satisfying buyer substitutability has not a cartesian product nature anymore, which will provide a new argument in favor of Core selecting auctions. Section 7 concludes with an auction design perspective. Our main proofs are relegated in Appendices A-D.

2 The Negative Usage-Dependent Externalities Model

We consider the assignment of M homogeneous indivisible units of a good to a set N of potential buyers. The units are originally owned by a single seller indexed by $l = 0$. While being homogeneous, the good can be used in various ways by its purchasers: let G be the set of potential usages. With a slight abuse of notation, N $[G]$ will represent the set as well as the number of potential buyers [usages] indexed by $l = 1, \dots, N$ $[g = 1, \dots, G]$. In the special case with a single usage ($G = 1$) which deserves some interest on its own, we drop the dependence on g in the notation. We assume that the usage of any unit can be contracted upon. A final assignment is then given by $\mathcal{Y} = \{\mathcal{Y}_1, \dots, \mathcal{Y}_N\}$ with $\mathcal{Y}_l = (\mathcal{Y}_l^1, \dots, \mathcal{Y}_l^G)$ where $\mathcal{Y}_l^g$ denotes the number of units assigned to buyer l according to usage g. The feasibility constraints require that $\mathcal{Y}_l^g \geq 0$ and $\sum_{g=1}^{G} \sum_{l=1}^{N} \mathcal{Y}_l^g \leq M$. Let $\mathcal{Y}_0 = M - \sum_{g=1}^{G} \sum_{l=1}^{N} \mathcal{Y}_l^g$ denote the number of units remaining in the seller's hands and $\mathbf{A}$ the set of feasible assignments. For any function $F(.)$, let $\Delta F(x) := F(x) - F(x-1)$. If the function F contains multiple variables then we will use the notation Δ_x to explicit the variable that is used in the difference. Each agent $l = 0, \dots, N$ has a valuation vector $\theta_l = (\theta_l(\mathcal{Y}) : \mathcal{Y} \in \mathbf{A})$, where $\theta_l(\mathcal{Y})$ specifies the value of the final assignment $\mathcal{Y}$ to agent l. We emphasize that we make a significant departure from the standard assignment literature since buyer l's valuations $\theta_l(.)$ may not solely depend on $\mathcal{Y}_l$ the set of items he has acquired but also on the assignment of his opponent buyers: we allow allocative externalities. We impose the following structure on the set of preferences which is labeled as the Negative Usage-Dependent Externality (NUDE) framework.

The NUDE framework

1. *Private Values*: each buyer l knows his own value vector θ_l.
2. *Quasilinear preferences*: if the final assignment is $\mathcal{Y}$ and the final vector of monetary transfers is $(b_l)_{l=1,\cdots,N}$, then buyer l's net payoff is $\theta_l(\mathcal{Y}) - b_l$.
3. *Zero seller value*: $\theta_0(.) \equiv 0$.
4. *No externality on non-purchasers*: if a buyer does not acquire any item, his utility is normalized to zero, $\theta_l(\mathcal{Y}) = 0$ if $\mathcal{Y}_l = 0$.

5. *The Negative Usage-Dependent Externalities structure*:

$$\theta_l(\mathcal{Y}) = \sum_{g=1}^{G} \sum_{y=1}^{\mathcal{Y}_l^g} [V_l^g(y) - E_l^g(\sum_{l=1}^{N} \mathcal{Y}_l^g, y)], \quad \text{for any } l \in N, \tag{1}$$

where $V_l^g(.) \geq 0$ and $E_l^g(., y)$ is nondecreasing for any $y \in [1, M]$. We impose the normalization $E_l^g(1, y) = 0$ for any $y \in [1, M]$.

6. *Intrinsic substitutability*: $V_l^g(.)$ is nondecreasing for any g and l.

The private value assumption is crucial in our intrinsically multi-dimensional environment due to the allocative externalities: with interdependent values and multidimensional signals, the efficient assignment is generically not implementable as argued by Jehiel and Moldovanu [31]. The second assumption excludes any financial externalities[7]: while buyers may care about the items assigned to their opponent buyers, they do not care about the price they have paid. The third assumption will be relaxed later to allow convex production costs. The normalization in the fourth assumption is far from being innocuous: it imposes that a non-purchaser is indifferent to the final assignment. Thus we restrict our analysis to allocative externalities that are somehow orthogonal to the externalities in Jehiel et al. papers ([29], [30], [32], [33]) where a single item is for sale but where the losers in the auction care about the identity of the winner. The fifth assumption imposes some structure on the allocative externalities: on the one hand the valuation of a buyer l is additive across usages. On the other hand the value of the allocation for a given usage g is decomposed into two terms: the first, $\sum_{y=1}^{\mathcal{Y}_l^g} V_l^g(y)$, corresponds to his intrinsic valuation for the items, the second, $-\sum_{y=1}^{\mathcal{Y}_l^g} E_l^g(., y)$, corresponds to the externality term that depends on $\sum_{l=1}^{N} \mathcal{Y}_l^g$ the total number of items assigned for this usage g. The terminology *usage-dependent* comes from this latter aspect of the externality whereas the terminology *negative* reflects that $E_l^g(., y)$ is nondecreasing. The subsequent normalization is then innocuous: given that $V_l^g(y) \geq 0$, it only assumes implicitly free disposal for the first unit for one usage. Finally the last assumption is the standard 'nonincreasing marginal utilities' assumption which corresponds to the 'item substitutability' property for the assignment of an homogenous item without any allocative externalities, i.e. if $E_l^g(., .) \equiv 0$ in our framework. In a nutshell, potential buyers' preferences are characterized by the intrinsic valuation terms $(V_l^g(.))_{g \in G, l \in N}$ and the externality terms $(E_l^g(.))_{g \in G, l \in N}$. Note that the NUDE framework imposes constraints that are solely related to individual preferences.

[7] With budget constraints that may alter future interactions or with crossholdings between buyers, such externalities may be of primary importance. See Ettinger [21] and Massland and Onderstal [38].

Let $\Theta_l^{M,G}$ be the set of buyer l's individual preferences compatible with the NUDE framework with M units and the usages in the set G. Formally, $\Theta_l^{M,G}$ is thus given by:

$$\Theta_l^{M,G} = \{\theta_l | \text{ there exists } (V^g(y), E^g(x,y)), g \in G, x \in [1,M], y \in [1,M], V^g(.) \text{ and } E^g(.,y) \text{ are nondecreasing}, V^g(.) \geq 0, E^g(1,y) = 0 \text{ and such that } \theta_l(\mathcal{Y}) \text{ is given by equation (1)}\}.$$

which does not depend on l and is also denoted by $\Theta^{M,G}$. Let $\overline{\Theta}^{M,G}$ be the cartesian product $\prod_{l=1}^{N} \Theta_l^{M,G}$ be the set of joint preferences compatible with the NUDE framework among the N potential buyers. It can be easily checked that the inequalities involving the definition of $\Theta^{M,G}$ are such that $\Theta^{M,G}$ is a closed subset of $\mathbb{R}^{M^2 \cdot G}$ whose interior is an open set. On the whole, our model deals with joint preferences that have $N \cdot M^2 \cdot G$ dimensions while the corresponding model without allocative externalities is limited to $N \cdot M$ dimensions.

Important remark We do not impose any global monotonicity or free disposal assumptions: a buyer may be worse off if more items are assigned to him while the set of items assigned to his opponents remains fixed. It could be even up to a point where the buyer would have preferred to obtain no unit at all. This comes from the loss in the externality term that may overcome the gain in the intrinsic utility. At first glance, it seems at odd with many applications we have in mind as the ones suggested in the introduction where a buyer is able not to use some units he has bought. In other words, the externality term should correspond to the number of items that are effectively used and not to the number that are bought. However, for the analysis of the related coalitional form of this assignment problem, such a specification does not matter: if the value of a coalition is reached by an assignment involving some kind of preemption, it can be replaced by the assignment where the preempted units are remaining in the seller's hands.[8] More generally, all our analysis remains unchanged if we modify the part of the individual preferences where $V_l^g(y) - E_l^g(\sum_{l=1}^{N} \mathcal{Y}_l^g, y) \leq 0$ provided that such marginal valuations remain negative: at the end, it is never optimal to assign units to a buyer up to a negative marginal valuation. E.g. our analysis is unchanged if we replace the expression of $\theta_l(\mathcal{Y})$ in (1) by $\sum_{g=1}^{G} \sum_{y=1}^{\mathcal{Y}_l^g} \max\{V_l^g(y) - E_l^g(\sum_{l=1}^{N} \mathcal{Y}_l^g, y), 0\}$.

The (selected) efficient assignment For a given set of joint preferences $\theta = (\theta_1, \ldots, \theta_n)$

[8] In the analysis of competition for scarce capacity before a Cournot market, a particular case of the NUDE framework, Eso, Nocke and White [20] focus on preemption though they consider efficient assignments, possibly implemented by a Vickrey-Clarke-Groves auction. Preemption matters in their model while not in ours because they impose a constraint on the final assignment: the final number of units assigned to the potential buyers is fixed ex ante: the seller is unable to keep some units.

and a given coalition of buyers $S \subset N$, we are interested in assigning the units among the buyers in S in an efficient way, which consists in solving the maximization program $\max_{\mathcal{Y} \in \mathrm{A}} \sum_{l \in S} \theta_l(\mathcal{Y})$. Let $\mathcal{Y}_S^*(\theta)$ be the efficient assignment selected in the following way[9]: among the efficient assignment select first the ones which assign the greatest number of units to the usages with the lower indices (the seller being identified as the index 0) according to the lexicographic order. Second among this pre-selection of efficient assignments, pick the one which assigns the greatest number of units to the buyers with the lower indices. Let $\mathcal{X}_S^g(\theta)$ denote the number of units assigned to usage g in this assignment: $\mathcal{X}_S^g(\theta) = \sum_{l \in N} \mathcal{Y}_l^g$ with $\mathcal{Y} = \mathcal{Y}_S^*(\theta)$. Let $\mathcal{X}_S^*(\theta) = (\mathcal{X}_S^g(\theta))_{g \in G}$.

The coalitional form A useful tool in the analysis of assignment models is a related coalitional form game $(N \cup \{0\}, w_\theta)$, where w_θ is the coalitional value function associated to buyers preferences θ. When there is no ambiguity with respect to the buyers' preferences, we drop the dependence on θ in our subsequent notation. For any coalition of bidders $S \subset N \cup \{0\}$, w_θ is defined by:

$$w_\theta(S) = \sum_{l \in S} \theta_l(\mathcal{Y}_S^*(\theta)), \quad \text{if } \ 0 \in S, \qquad w_\theta(S) = 0, \quad \text{if } \ 0 \notin S.^{10} \tag{2}$$

In the analysis of assignment models, a crucial condition on the coalitional value function has emerged in the literature: buyer submodularity.

Definition 1 (Buyer substitutability) *The buyer substitutability condition, which corresponds also to saying that the coalitional value function w is buyer submodular, is satisfied if for any $l \in N$ and any coalitions S and S' satisfying $0 \in S \subset S'$, we have*

$$w(S \cup \{l\}) - w(S) \geq w(S' \cup \{l\}) - w(S'). \tag{3}$$

For $l \notin S$, the term $w(S \cup \{l\}) - w(S)$ represents the surplus associated to adding buyer l to the coalition S. Buyer substitutability is a kind of 'substitute' condition insofar as it says that such surplus are nonincreasing with the size of the coalition. In other words, equation (3) says that buyer l is a substitute to the buyers in $S' \setminus S$. A more primitive notion of substitutability is one between only two buyers as defined below.

[9] Generically, efficient assignments are unique. This innocuous selection is useful for some definitions and to alleviate some arguments in our proofs.

[10] If the seller is not a member of the coalition, the coalition obtains no items. Due to our negative externality assumption, this point is independent of any 'dumping' assumption, i.e. whether the auctioneer can dump objects on bidders who do not belong to the seller's coalition, if the seller is supposed to maximize the welfare of the buyers inside her coalition.

Definition 2 (Substitute buyers) *A pair of buyers $l, m \in N$ is called 'substitute buyers' if for any $S \subset N$, we have*

$$w(S \cup \{l, m\}) - w(S \cup \{m\}) \geq w(S \cup \{l\}) - w(S).$$

As shown in Milgrom [39] [Theorem 8.2], w is buyer submodular if and only if any pair of buyers $l, m \in N$ is a pair of substitute buyers.

Restricted classes of buyers In our general NUDE framework a buyer may be interested in different usages of the good and may also have multi-unit demand for a given usage. The following definition introduces unit demand buyers that are interested in only one unit of the good and only for a given specific usage. This subclass of buyers will play a crucial role in order to establish buyer substitutability.

Definition 3 (Unit demand buyers) *A buyer l is a unit demand buyer if there exists $g^* \in G$ such that $V_l^g(.) \equiv 0$ for $g \neq g^*$ and $V_l^{g^*}(y) = 0$ for $y > 1$.*

For a unit demand buyer l, we can focus only on the preferences with respect to the first unit and drop any dependence with respect to the variable y and the usage he is interested in: let V_l and $E_l(x)$ denote respectively the intrinsic valuation and the externality term. We say that a potential buyer has null preferences if $V_l^g(.) \equiv 0$, for any $g \in G$ or equivalently if l is a unit demand buyer with $V_l = 0$. Next definition introduces the less restrictive class of single usage buyers that are interested in some units only for one specific usage.

Definition 4 (Single usage buyers) *A buyer l is a single usage buyer if there exists $g^* \in G$ such that $V_l^g(.) \equiv 0$ for $g \neq g^*$.*

The non-crossing property The non-crossing property imposes that for any given usage g, the marginal valuation functions $x \to V_l^g(y) - E_l^g(x, y)$ can be ranked unambiguously across all l and y. Note that the ranking between $x \to V_l^g(y) - E_l^g(x, y)$ for two different y and the same buyer l is w.l.o.g. if we have in mind that the buyers have different opportunities to use one unit of capacity for a given usage: he chooses the most profitable opportunities.

Assumption 2.1 (The non-crossing property) *For any $g \in G$, $l, l' \in N$ and $y, y' > 0$, we have either [$V_{l'}^g(y') - E_{l'}^g(x, y') \geq 0 \Rightarrow V_l^g(y) - E_l^g(x, y) \geq V_{l'}^g(y') - E_{l'}^g(x, y')$] for any $x \geq 1$ or [$V_l^g(y) - E_l^g(x, y) \geq 0 \Rightarrow V_{l'}^g(y') - E_{l'}^g(x, y') \geq V_l^g(y) - E_l^g(x, y)$] for any $x \geq 1$. In the former [latter] case, we say that the marginal buyer (or marginal valuation) (l, y) is more [less] efficient than (l', y').*

We do not incorporate this assumption directly in the NUDE framework since it imposes global constraints on the joint preferences on the contrary to the constraints of the NUDE framework which are constraints on individual preferences. In particular, in our contingent auctions application, we force bidders to report individual preferences that fit the NUDE framework while the joint preferences may not satisfy the non-crossing assumption. However, we assume all over the paper that buyers' true preferences satisfy the non-crossing property.

Under the non-crossing assumption and conditional on assigning a given number of units to usage g, it is always optimal to assign those items to the more efficient marginal valuations according to usage g. When we are looking for efficient assignments for some coalition, we can thus restrict ourselves to the search of the number of items assigned to each usage which is given by the vector $\mathcal{X} = (\mathcal{X}^g)_{g\in G}$, where $\mathcal{X}^g$ denotes the number of items assigned to usage $g \in G$. For a given coalition S and a given $\mathcal{X}^g = x$, let $\mathcal{Y}_l^g(x, S)$ denote the number of units assigned to buyer l according to usage g in the assignment that allocates exactly x units to usage g to the most efficient marginal valuations among the buyers in S and which assigns the greatest number of unit to the buyers with the lowest indices in presence of ties. Let $H_S^g(.)$ denote the function mapping the number of units assigned to usage g among the buyers in S to the contribution of this partial (efficient) assignment to the welfare:

$$H_S^g(x) = \sum_{l\in S} \sum_{y=1}^{\mathcal{Y}_l^g(x,S)} [V_l^g(y) - E_l^g(x, y)] = [\sum_{l\in S} \sum_{y=1}^{\mathcal{Y}_l^g(x,S)} V_l^g(y)] - E_S^g(x), \tag{4}$$

where $E_S^g(x)$ denotes the sum of the externality terms for usage g when the number of units assigned among the buyers in S and to this usage is x, i.e.

$$E_S^g(x) = \sum_{l\in S} \sum_{y=1}^{\mathcal{Y}_l^g(x,S)} E_l^g(x, y). \tag{5}$$

The notation above covers the case $x = 0$, where $H_S^g(0) = E_S^g(0) = 0$. For $S \subset N$ and if $0 \in S$, $w(S)$ is then the solution of the following maximization program:

$$w(S) = \max_{\mathcal{X}:\sum_{g\in G} \mathcal{X}^g \leq M} \sum_{g\in G} H_S^g(\mathcal{X}^g). \tag{6}$$

A simple example The following example is motivated by the German UMTS auction mentioned in the introduction where unit demand buyers are competing for licences and where the value of a licence depends solely on the total number of licensees. Consider three identical potential unit demand buyers $N = \{1, 2, 3\}$ and three licences intended for a unique usage. The intrinsic valuation is set to 11 while the externality term is given by $E(1) = 0$, $E(2) = 4$ and $E(3) = 5$. The numerical values are chosen such that the externality term $x \to x \cdot E(x)$ fails to be convex, exactly as in the figures reported by Hoppe et al. [28], a property which is shown later to be related to buyer substitutability. Externalities are mild enough such that for any coalition it is an efficient allocation to assign the maximum number of licences, i.e. one to each member of the coalition. The coalitional value function is then given by $w(N) = 18$, $w(N \setminus \{l\}) = 14$ and $w(N \setminus \{l, m\}) = 11$ for $l, m \in 1, 2, 3$. It is not buyer-submodular because $w(N) - w(N \setminus \{l\}) = 4 > 3 = w(N \setminus \{m\}) - w(N \setminus \{l, m\})$ for $l, m \in 1, 2, 3$. Any pair of buyers l and m can be viewed as *complement buyers* instead of substitute buyers since the additional value to the total surplus provided by buyer l grows with the mere presence of buyer m.

3 Sufficient conditions for buyers to be substitute

We introduce below several properties on the preferences and additional notation that will play a central role in our analysis. We introduce first the notion of an ordered list of marginal buyers that is required to define then some crucial convexity properties.

Definition 5

- *An ordered list of marginal buyers for a given usage $g \in G$ is a list $L = \{k_1, \ldots, k_{t_L}\}$ such that any k_j corresponds to a marginal buyer with respect to usage g, i.e. has the form $k_j = (\sigma_L^g(j), \gamma_L^g(j))$ with $\sigma_L^g(j) \in N$ and $\gamma_L^g(j) \in [1, M]$ and such that the marginal buyer k_i is more efficient than k_j if $i \leq j$. The integer t_L corresponds to the length of the list.*
- *Let $\mathcal{L}^g$ be the set of nonempty ordered lists of marginal buyers for the usage g that contain at most one duplicate, i.e. $\sigma_L^g(i) = \sigma_L^g(j)$ and $\gamma_L^g(i) = \gamma_L^g(j)$ for at most one pair (i, j) with $i \neq j$.*

We define $H_L^g(x)$ and $E_L^g(x)$ for a given ordered list L, a given usage $g \in G$ and on the range $[0, \min\{t_L, M\}]$ as the generalization to ordered lists of marginal buyers of the equations (4) and (5) that were devoted to subset of buyers:

$$H_L^g(x) = \sum_{j=1}^{x} [V_{\sigma_L^g(j)}^g(\gamma_L^g(j)) - E_{\sigma_L^g(j)}^g(x, \gamma_L^g(j))], \tag{7}$$

$$E_L^g(x) = \sum_{j=1}^{x} E_{\sigma_L^g(j)}^g(x, \gamma_L^g(j)). \tag{8}$$

In a similar way as before we define $\mathcal{Y}_L^*$ as the selected efficient assignment among the marginal buyers in the list $L = (L^1, \ldots, L^G) \in \prod_{g \in G} \mathcal{L}^g$ and $\mathcal{X}_L^g$ as the number of units assigned to usage g in the selected efficient assignment. The vector $\mathcal{X}_L^* = (\mathcal{X}_L^g)_{g \in G}$ is thus a solution of the maximization program:

$$\max_{\mathcal{X}: \sum_{g \in G} \mathcal{X}^g \leq M, \mathcal{X}^g \leq t_{L^g}} \sum_{g \in G} H_{L^g}^g(\mathcal{X}^g). \tag{9}$$

Definition 6 • *The* ***convexity property A*** *holds if $\Delta E_L^g(.)$ is nondecreasing for any $g \in G$ and any $L \subset \mathcal{L}^g$.*

- *The* ***convexity property B*** *holds if $\sum_{j=1}^{x-1} \Delta_x E_{\sigma_L^g(j)}^g(x, \gamma_L^g(j))$ is nondecreasing for any $g \in G$ and any $L \subset \mathcal{L}^g$.*

Definition 7 *Preferences are* ***steeper for more [less] efficient (marginal) buyers*** *if for any $l, l' \in N$, $y, y' \geq 1$ and $g \in G$, $[V_l^g(y) \geq V_{l'}^g(y') \Rightarrow \Delta_x E_l^g(x, y) \geq [\leq] \Delta_x E_{l'}^g(x+1, y')$ for any $x \in [2, M]]$.*

As the non-crossing property, those properties are involving global constraints on the joint preferences. We can check immediately that if one of those properties is satisfied for a set of buyers then it satisfied for any subset of the initial set of buyers. The convexity properties are used in our preliminary propositions to prove our main theorems, where, on the contrary, more directly interpretable convexity conditions on the functions $E_l^g(., y)$ are used. Note that the convexity property always holds in the special case where $M = 2$. The 'steeper for more efficient (marginal) buyers' property is an increasing differences property. Having in mind that the pair $k = (l, y)$ can be ordered according to the corresponding marginal valuations as it is guaranteed by the non-crossing assumption, this property is equivalent to saying that $(x, k) \to E_l^g(x, y)$ has increasing differences, a property that plays a key role for deriving comparative statics results (see e.g. Topkis [46]). In the same way, the 'steeper for more efficient (marginal) buyers' property corresponds to $(x, k) \to E_l^g(x, y)$ having decreasing differences.

The two convexity properties seem to be closely related to each other since $\Delta E_L^g(x) = E_{\sigma_L^g(x)}^g(x, \gamma_L^g(x)) + \sum_{j=1}^{x-1} \Delta_x E_{\sigma_L^g(j)}^g(x, \gamma_L^g(j))$. However, in general, neither implies the other one since the term $E_{\sigma_L^g(x)}^g(x, \gamma_L^g(x))$ has no reason to be either nonincreasing or nondecreasing without any further assumption. Nevertheless, if preferences are steeper for less efficient buyers then $E_{\sigma_L^g(x+1)}^g(x+1, \gamma_L^g(x+1)) \geq E_{\sigma_L^g(x)}^g(x, \gamma_L^g(x))$ while negative externalities imply $E_{\sigma_L^g(x)}^g(x+1, \gamma_L^g(x)) \geq E_{\sigma_L^g(x)}^g(x, \gamma_L^g(x))$ which guarantees finally that $E_{\sigma_L^g(x)}^g(x, \gamma_L^g(x))$ is nondecreasing such that the convexity property B implies the convexity property A.[11]

As a preliminary step for the analysis of buyer substitutability, we provide sufficient conditions that guarantee that a pair of unit demand buyers are substitute buyers: one condition is devoted to the case where the two buyers consider the same usage of the good while the other one is devoted to the alternative case where the two buyers consider distinct usages of the good.

Proposition 3.1 (Substitute buyers with unit demand buyers) *Under the convexity property A or B,*

1. *If preferences are steeper for less efficient buyers, then any pair of unit demand buyers according to distinct usages is a pair of substitute buyers.*

2. *If preferences are steeper for more efficient buyers, then any pair of unit demand buyers according to the same usage is a pair of substitute buyers.*

Proposition 3.1 is the central result that drives all our subsequent results about buyer substitutability. There are three main steps which are sketched below while the formal proof is relegated to Appendix A. While the formal proof involves our generalized concepts for ordered lists of marginal buyers and thus deals with the maximization programs (9), to give the underlying intuitions we restrict ourselves to the simpler optimization programs (6). This presentaiton is illustrated by Figure 1 with $M = 3$.

First the convexity of the externality structure guarantees that the functions $H_S^g(.)$ are concave (lemma A.1). As a corollary, efficient assignments are the ones that put the units to the M highest 'incremental opportunities' among the set of such opportunities $\{\Delta H_S^g(x)\}_{g \in G, x \in [1,M]}$, provided that they are remaining positive. When such opportunities become negative, it is better to leave the additional units into the seller's hands. In Figure 1, the incremental opportunities for usage g when the set of potential buyers is S or $S \setminus \{l\}$ are depicted respectively by the thick black and thick light curves. The dotted curve depicts the highest opportunity among the alternative usages, which is equal to $\max_{g' \neq g, x \in [1,M]} \Delta H_S^g(x)$ for the coalition S and $\max_{g' \neq g, x \in [1,M]} \Delta H_{S \setminus \{l\}}^g(x)$ for the coalition $S \setminus \{l\}$, which are equal since buyer l is assumed to be a unit demand buyer with respect to usage g. For our illustration purposes, such a positive opportunity with an alternative usage is assumed to be unique in Figure 1. In the left panel, when the outside opportunity is low (before the upgrades) the efficient assignment among the coalition S [$S \setminus \{l\}$] is to assign the three units to usage g [two units to usage g and one unit to usage g']. When the outside opportunity

[11] We emphasize that both properties are needed for our analysis. Property A [resp. B] is used for Theorem 1 [resp. Theorem 2].

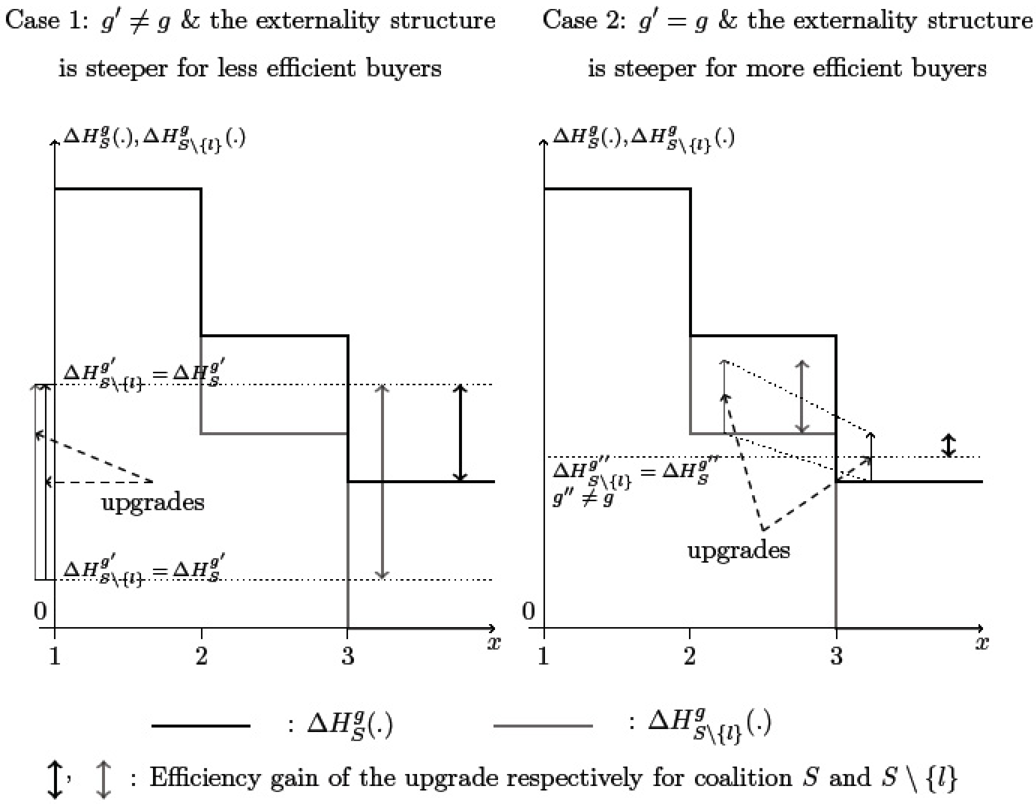

Figure 1: Sketch of Proposition 3.1's proof

is high (after the upgrades) the efficient assignment is two units to usage g and one unit to usage g' for both coalitions S and $S \setminus \{l\}$.

Second, we identify in lemma A.2 how the incremental opportunities change when one unit demand buyer l (with respect to usage g) is removed from a given coalition S. Those incremental opportunities are the sum of two terms: the first (positive) is the intrinsic valuation from the incremental buyer and thus decreases when one buyer is removed; the second (negative) corresponds to the increase in the externality imposed on all buyers. When the externality structure is steeper for less efficient buyers, this externality term shrinks when one buyer is removed from the initial coalition. Consequently buyer l's removal worsens the next opportunities according to usage g as it is illustrated in the left panel of Figure 1 where the thick dark curse is above the thick light curve: $\Delta H^g_S(x) \geq \Delta H^g_{S\setminus\{l\}}(x)$. On the contrary, if the externality structure is steeper for more efficient buyers (right panel of Figure 1), this externality term raises. We obtain finally that $\Delta H^g_{S\setminus\{l\}}(x) \geq \Delta H^g_S(x+1)$. The effect on the first term of buyer l's removal prevents the inequality $\Delta H^g_{S\setminus\{l\}}(x) \geq \Delta H^g_S(x)$ to be satisfied. In both cases, removing buyer l is neutral according to the opportunities with respect to the other usages.

Third, to show that a pair of unit demand buyers (l, m) are substitute buyers, i.e. $w(S) - w(S \setminus \{l\}) \leq w(S \setminus \{m\}) - w(S \setminus \{l, m\})$, we adopt of differential approach: we construct a path which corresponds to varying buyer m's preferences from the lowest (or null) preferences which corresponds to buyer m's removal to buyer m's original full preferences and we show then that the marginal contribution of buyer l to the welfare is nonincreasing along this path (lemma A.3) according to the right combinations of a steepness condition and whether buyer m may or may not consider the same usage as buyer l.

The intuition for this last part of the proof is rather simple. Consider for example the case where the externality structure is steeper for less efficient buyers and a unit demand buyer m with usage $g' \neq g$. Now consider some infinitesimal variation δv with respect to V_m: the corresponding infinitesimal gain in $w(S)$ is then either 0 (if V_m is small such that buyer m does not receive any unit) or δv (if V_m is high enough such that buyer m does receive one unit). From lemma A.2, if the efficient allocation assign a unit to buyer m for $S \subset N$ then it will also assign a unit to buyer m for the set $S \setminus \{l\}$. In other words, we have $\delta w(S \setminus \{l\})/\delta V_m \geq \delta w(S)/\delta V_m$. The integration of the above inequality from 0 to V_m leads thus to the buyer substitutability property between l and m. When the externality structure is steeper for more efficient buyers, the same logic leads to stating that the buyer substitutability property holds between unit demand buyers according to the same usage.

Though the above argument gives the right intuition for how we end the proof of proposition 3.1, it does not formally work since variations in V_m may break the non-crossing and the steepness properties which would prevent the use of lemmata A.1 and A.2. We have thus to build a path (parameterized by the variable u) for buyer m's preferences which goes from the lowest preferences $V_m = 0$ and $E_m(x) = E_{l^*}^{g'}(x, y^*)$ with (l^*, y^*) corresponding to the less efficient marginal buyer with respect to usage g', such that buyer m does not receive any unit under those null preferences to the final preferences that satisfies the non-crossing and the steepness properties and such that: first we are remaining in the NUDE framework along the path; second, the non-crossing, the convexity and the steepness properties never break down along the path and third, buyer m's preferences are increasing which will guarantee that the conditions on the derivatives (that are properly defined almost everywhere) $\delta w(S \setminus \{l\})/\delta u \geq \delta w(S)/\delta u$ hold.

We end this section with a generalization of Proposition 3.1 to single usage buyers. This result is indeed a direct corollary of Proposition 3.1 if we have in mind the *bundling argument* that is presented below and which will be useful later for our general theorems. This argument allows us more generally to generalize substitutability results for unit demand buyers to substitutability results for buyers that can be viewed as 'bundles' of unit demand buyers, as it is the case with the separability structure of the NUDE framework.

Proposition 3.2 (Substitute buyers with single usage buyers) *Under the convexity property A or B,*

1. *If preferences are steeper for less efficient buyers, then any pair of single usage buyers according to distinct usages is a pair of substitute buyers.*

2. *If preferences are steeper for more efficient buyers, then any pair of single usage buyers according to the same usage is a pair of substitute buyers.*

The Bundling argument For a given set of preferences according to the NUDE framework that is characterized by $\{M, G, N, (V_l^g(.))_{g\in G, l\in N}, (E_l^g(., y))_{g\in G, l\in N, y\in[1,M]}\}$, we build another set of preferences with only unit demand buyers and which should be viewed as a break up of all marginal valuations into single pieces. The new framework is then characterized by $\{M, G, \overline{N}, (\overline{V}_k(.))_{k\in\overline{N}}, (\overline{E}_k(.))_{k\in\overline{N}}\}$ where $\overline{N} = G\cdot M\cdot N$ and where $\overline{V}_k(.) = V_{k_1}^{k_3}(.)$ and $\overline{E}_k(.) = V_{k_1}^{k_3}(., k_2)$ for any $k \in \overline{N}$ and where (k_1, k_2, k_3) is the unique element in $N\times M\times G$ such that $k = k_1\cdot M\cdot G + k_2\cdot G + k_3$, this decomposition corresponds to standard euclidian division. Let w and $\overline{w}$ denote the respective coalitional value functions of those two frameworks. From the separability structure of the NUDE framework, we obtain that w can be fully expressed as a function of $\overline{w}$ by:

$$w(S \cup \{0\}) = \overline{w}(\bigcup_{l\in S}\{(l-1)\cdot M\cdot G + 1, \dots, l\cdot M\cdot G\} \cup \{0\}), \text{ for any } S \subset N. \quad (10)$$

The bundling argument says that if any pair of (unit demand) buyers $((l-1)\cdot M\cdot G + k_1, (m-1)\cdot M\cdot G + k_2)$ for any $(k_1, k_2) \in [1, GM]^2$ in the 'disaggregated' framework are substitute buyers, then (l, m) is a pair of substitute buyers in the original framework.

For $S \subset N$, let $\overline{S} = \bigcup_{l\in S}\{(l-1)\cdot M\cdot G + 1, \dots, l\cdot M\cdot G\}$. For any $l \in N$ and $k \in [1, M\cdot G]$, let $J_l^k = \{(l-1)\cdot M\cdot G + 1, \dots, (l-1)\cdot M\cdot G + k\}$. In particular $J_l^{M\cdot G} = \overline{\{l\}}$. Let $J_l^0 = \emptyset$ for any $l \in N$. The bundling argument comes from the decomposition:

$$\begin{aligned}
&(w(S\cup\{0\}) - w(S\cup\{0\}\setminus\{l\})) - (w(S\cup\{0\}\setminus\{m\}) - w(S\cup\{0\}\setminus\{l,m\}))\\
&\quad = (\overline{w}(\overline{S}\cup\{0\}) - \overline{w}(\overline{S}\cup\{0\}\setminus\overline{\{l\}})) - (\overline{w}(\overline{S}\cup\{0\}\setminus\overline{\{m\}}) - \overline{w}(\overline{S}\cup\{0\}\setminus\overline{\{l,m\}}))\\
= &\sum_{k_1=0}^{G\cdot M-1}\sum_{k_2=0}^{G\cdot M-1} \underbrace{[(w(\overline{S}\setminus\{J_l^{k_1}, J_m^{k_2}\}) - w(\overline{S}\setminus\{J_l^{k_1+1}, J_m^{k_2}\})) - (w(\overline{S}\setminus\{J_l^{k_1}, J_m^{k_2+1}\}) - w(\overline{S}\setminus\{J_l^{k_1+1}, J_m^{k_2+1}\}))]}_{\leq 0 \ \text{ if buyers } (l-1)\cdot M\cdot G+1+k_1 \text{ and } (m-1)\cdot M\cdot G1+k_2 \text{ are substitute buyers}}
\end{aligned}$$

An important remark to make use of the bundling argument is to note that the non-crossing condition, the convexity property and the steepness properties on the preferences remains true in the 'disaggregated' framework when they are satisfied in the original framework.

4 Characterization of buyer substitutability

As a preliminary, we first define what we mean by the set of preferences that an externality structure generates and then introduce some basic properties on externality structures. Let J be a set of $N \cdot M$ elements.

Definition 8

- *An **externality structure** is a set* $\{\mathcal{E}_j^g(.)\}_{g \in G, j \in J}$*, where* $\mathcal{E}_j^g(.)$ *is a nondecreasing real-valued function on* $[1, M]$ *such that* $\mathcal{E}_j^g(1) = 0$.
- *Preferences are **generated** by an externality structure* $\{\mathcal{E}_j^g(.)\}_{g \in G, j \in J}$ *if for any* $g \in G$, $l \in N$ *and* $y \in [1, M]$*, there exists* $j \in J$ *such that* $E_l^g(., y) = \mathcal{E}_j^g(.)$.[12]
- *An externality structure is **uniform** if for any* $g \in G$*, all the functions in the set* $\{\mathcal{E}_j^g(.)\}_{j \in J}$ *are equal. If so, let* $\mathcal{E}^g(.)$ *denote the unique element in the set* $\{\mathcal{E}_j^g(.)\}_{j \in J}$.[13]
- *An externality structure is **convex** if for any* $g \in G$ *and* $j \in J$, $x \to x \cdot \mathcal{E}_j^g(x)$ *is convex.*

The following theorem provides a general characterization of buyer substitutability. In the special case without allocative externalities, i.e. $E_l^g(., y) \equiv 0$ for any g, l, y, then w is guaranteed to be submodular, a point that is well-known since we assumed substitutability for the intrinsic valuations: the functions $V_l^g(.)$ are nondecreasing. Such an externality structure is uniform and convex and next result guarantees thus buyer substitutability as a special case.

Theorem 1 (Characterization of buyer substitutability: a 'negative' result) *If preferences are generated by a uniform and convex externality structure, then* w *is buyer submodular.*

Conversely, for $N \geq 3$*, if an externality structure is not uniform and convex, then there exists a set of preferences generated by this externality structure such that* w *is not buyer submodular.*

[12] Naturally, we still require that the preferences satisfy the underlying assumption of our model, in particular the non-crossing assumption.

[13] Note that if the externality structure is uniform, then the non-crossing assumption is automatically satisfied for any intrinsic valuation structure.

As already emphasized in the introduction, the uniform externality structure condition imposes constraints not solely on the individual preferences but mainly on the joint preferences of the buyers. This characterization suggests more generally that an individual-related notion of substitutability can not emerge in frameworks with allocative externalities contrary to the literature without allocative externalities.

Proof First note that a uniform and convex externality structure generates preferences that satisfy the convexity property A and that are steeper for both less and more efficient buyers. We can thus apply the two cases in Proposition 3.1 which guarantees that any pair of unit demand buyers are substitute buyers. From the so-called 'bundling argument', we obtain that any pair of buyers are substitute buyers and finally the first part of the theorem. The converse part is a corollary of Propositions 4.1 & 4.2 which are partial converses developed below. **CQFD**

Partial converses

Proposition 4.1 is the partial converse with respect to the convexity of the externality structure, while Proposition 4.2 considers convex externality structures that fail to be uniform.

Proposition 4.1 *For $N \geq 3$, consider an externality structure that is not convex, then there exists a set of preferences generated by this externality structure such that w is not buyer submodular.*

Proposition 4.2 *For $N \geq 3$, consider a convex externality structure that is not uniform, then there exists a set of preferences generated by this externality structure such that w is not buyer submodular.*

Those converses require at least three potential buyers. For $N = 2$, the buyer submodularity condition reduces to $w(\{1\}) + w(\{2\}) \geq w(\{1, 2\})$ which is always true in the NUDE framework independently of any additional assumption on the externality structure: more generally it always true in assignment models with negative externalities, i.e. if putting some goods in the hands of one buyer can only make his competitors worse-off since, for each buyer $i = 1, 2$, $w(\{i\})$ is guaranteed to be as big as buyer i's payoff in the efficient assignment among the full coalition $N = \{1, 2\}$. Those converses do not make any restriction on the total number of available units M. However the failure of the convexity condition requires $M \geq 3$.

We sketch below the intuitions for those converses, which are also illustrated by Figure 2. The formal proofs are relegated to Appendix B.

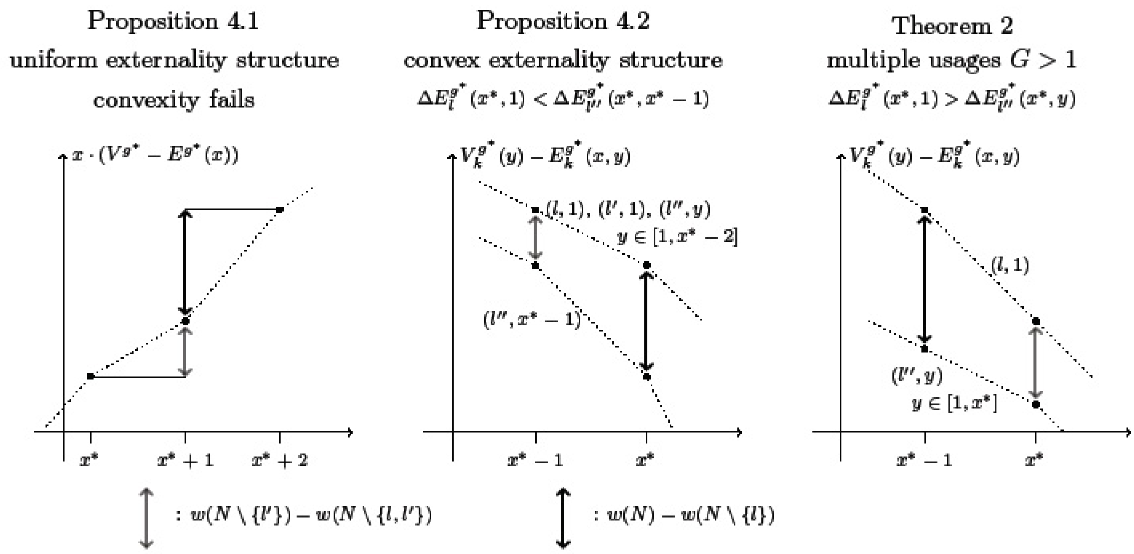

Figure 2: Illustration of the converse parts

Consider an externality structure that is not convex, then take $g^* \in G$ and $j \in J$ such that $x \to x\mathcal{E}_j^{g^*}(x)$ is not convex. The contribution to the welfare of the assignment of x units to usage g^* to a set of symmetric marginal buyers according to the externality term $E^{g^*}(.) = \mathcal{E}_j^{g^*}(.)$ is then given by $x \cdot (V^g - \mathcal{E}_j^g(x))$, which is not convex. In other words, the welfare differential when a additional unit is assigned may strictly increase in x at some point as depicted in the left panel of Figure 2. If the intrinsic valuations are chosen high enough such that the optimal assignment corresponds exactly to the allocation that assigns a unit to all those marginal valuations (the remaining being null), then the marginal contribution of some marginal valuation may strictly increase with the set of marginal valuations. It corresponds exactly to the simple example presented in section 2.

Consider a convex externality structure that is not uniform, then we can build joint preferences including two marginal valuations labeled as a and b, characterized respectively by $(V_a, \mathcal{E}_j^{g^*}(.))$ and $(V_b, \mathcal{E}_{j'}^{g^*}(.))$, such that the unit that is assigned to the more efficient one is then assigned to the less efficient one after his removal and also such that between two points $x^* - 1$ and x^* the externality term for the less efficient marginal valuation is steeper than for the more efficient one, i.e. $\Delta\mathcal{E}_{j'}^{g^*}(x^*) > \Delta\mathcal{E}_j^{g^*}(x^*)$. In the middle panel of Figure 2, those marginal valuations a and b are corresponding respectively to $(l, 1)$ and $(l'', x^* - 1)$ with $\Delta E_{l''}^{g^*}(x^*, 1) > \Delta E_l^{g^*}(x^*, x^* - 1)$. The welfare benefit from switching from the less efficient efficient marginal buyer to the more efficient one if the total number of items assigned to usage g^* is $x^* - 1$ is then given by: $V_a - V_b - (\mathcal{E}_j^{g^*}(x^* - 1) - \mathcal{E}_{j'}^{g^*}(x^* - 1))$ which is thus strictly smaller than the corresponding benefit $V_a - V_b - (\mathcal{E}_j^{g^*}(x^*) - \mathcal{E}_{j'}^{g^*}(x^*))$

if the total number of items assigned to usage g^* is x^*. The preferences can be built with an additional marginal valuation c such that without it [respectively with it] the number of units assigned to usage g^* corresponds exactly to $x^* - 1$ [resp. x^*] in a way such that the aforementioned welfare benefits correspond respectively to the marginal contribution of the marginal valuation a without and with the marginal valuation c.

Part of the negative flavor of Theorem 1's characterization comes from the very general way an externality structure can generate a set of preferences. To obtain a more positive result, we consider below a slightly tighter way an externality structure can generate preferences: when the steepness of two externality terms $\mathcal{E}^g_j(.)$ and $\mathcal{E}^g_{j'}(.)$ according to the same usage are ordered, i.e. e.g. $\Delta\mathcal{E}^g_j(x) \geq \Delta\mathcal{E}^g_{j'}(x)$, then we constraint the generation of preferences such that marginal buyers that are generated from the term $\mathcal{E}^g_j(.)$ are more efficient than those that are generated from the term $\mathcal{E}^g_{j'}(.)$. We use then the terminology that an externality structure 'SFME-generates' some preferences, where "SFME" stands for 'Steeper For More Efficient'. As a preliminary, we define formally what we mean by the set of preferences that an externality structure SFME-generates and introduce additional useful properties on externality structures.

Definition 9

- *For $g \in G$ and $j, j' \in J$, we say that the externality term $\mathcal{E}^g_j(.)$ is* ***steeper*** *than the externality term $\mathcal{E}^g_{j'}(.)$ if $\Delta\mathcal{E}^g_j(x) \geq \Delta\mathcal{E}^g_j(x)$ for any $x \in [2, M]$.*
- *An externality structure $\{\mathcal{E}^g_j(.)\}_{g \in G, j \in J}$ is* ***ordered*** *if for any $g \in G$ and $j, j' \in J$, either $\mathcal{E}^g_j(.)$ is steeper than the externality term $\mathcal{E}^g_{j'}(.)$ or $\mathcal{E}^g_{j'}(.)$ is steeper than the externality term $\mathcal{E}^g_j(.)$.*
- *Preferences are* ***SFME-generated*** *by an externality structure $\{\mathcal{E}^g_j(.)\}_{g \in G, j \in J}$ if for any $g \in G$, $l \in N$ and $y \in [1, M]$, there exists $j \in J$ such that $E^g_l(., y) = \mathcal{E}^g_j(.)$ and if for any $g \in G$, $l, l' \in N$ and $y, y' \in [1, M]$ the following property is satisfied: if the externality term corresponding to (l, y) is steeper than the one corresponding to (l', y'), then (l, y) is more efficient than (l', y').*
- *An externality structure is* ***strongly convex*** *if for any $g \in G$ and $j \in J$, $\mathcal{E}^g_j(.)$ is convex.*[14]

Theorem 2 (Characterization of buyer substitutability: a 'positive' result) *For $G = 1$, if preferences are SFME-generated by an ordered and strongly convex externality structure, then w is buyer submodular.*

[14] A strongly convex externality structure is necessarily convex since externalities are negative, i.e. $\mathcal{E}^g_j(.)$ is nondecreasing.

Conversely, for $N \geq 3$ and for any strongly convex externality structure, if either the externality structure is not uniform and $G > 1$ or if the externality structure is not ordered, then there exists a set of preferences SFME-generated by this externality structure such that w is not buyer submodular.

The difference between the 'impossibility result' in Theorem 1 when there are multiple usages to the 'possibility result' with a single usage in Theorem 2 has the same flavor as Jehiel and Moldavonu [31] impossibility result for efficient Bayesian implementation with interdependent valuations when they move from unidimensional to multidimensional signals while no formal connection is established. In the same way as [31], the conditions for a single usage are based on inequalities which allows the condition to be satisfied generically, i.e. on some open sets of $\overline{\Theta}^{M,G}$ while the conditions for multiple usages are based on equalities that are not satisfied generically.

It is clear from the proof of proposition 4.2 that we can not obtain a similar result with an alternative generation structure that would assign steeper externality terms to less efficient buyers.

Proof The first element is to note that preferences that are SFME-generated by an ordered and strongly convex externality structure are steeper for more efficient buyers and such that $E_l(., y)$ is convex on $[1, M]$ for any $l \in N, y \in [1, M]$.[15] It is then sufficient to prove that the convexity property B holds. Then we can apply proposition 3.2 which guarantees that w is buyer submodular in the case of a single usage. For any ordered list of marginal buyers L, we have: $\sum_{j=1}^{x} \Delta_x E^g_{\sigma^g_L(j)}(x+1, \gamma^g_L(j)) - \sum_{j=1}^{x-1} \Delta_x E^g_{\sigma^g_L(j)}(x, \gamma^g_L(j)) = \Delta_x E^g_{\sigma^g_L(x)}(x+1, \gamma^g_L(x)) + \sum_{j=1}^{x-1} [\Delta_x E_{\sigma^g_L(j)}(x+1, \gamma^g_L(j)) - \Delta_x E_{\sigma^g_L(j)}(x, \gamma^g_L(j))]$. The first term is positive since externalities are negative while each term in the sum is positive from the convexity of the terms $E_l(., y)$. The proof of the converse part is relegated to Appendix B. **CQFD**

Extension with seller preferences All our results extends to the more general case where it is costly to produce the good provided that the production costs are convex and does not depend on the usage of the units. It can be easily verified: the seller can be viewed as an additional potential buyer for a new usage. However, in the perspective of our contingent auctions applications that aim to implement the Vickrey payoff point, the seller should be distinguished from the potential buyers: she does not have the same incentives to report her real preferences as the potential buyers since she captures also the revenue. It is well-known (see e.g. Hatfield and Milgrom [27]) that in both auctions and matching

[15] Indeed, any preferences that are steeper for more efficient buyers and such that $E_l(., y)$ is convex on $[1, M]$ for any $l \in N, y \in [1, M]$ can be viewed as being SMFE-generated by some ordered and strongly convex externality structure.

algorithms that strategic-proofness is obtained only for one side of the market. Thus the strategic robustness of our contingent auctions in this more general environment would be limited to buyers strategic-proofness, i.e. if we force the seller to be non-strategic.

5 Economic examples

Models with allocative externalities in the IO literature The NUDE framework is a combination of two ingredients that appear previously in the literature: externalities that depend on the total 'level of trade' of all competitors as in Segal [43] which introduce a model that covers numerous applications and identity-dependent externalities where the externalities depend on the exact identities of the competitors as in Jehiel and Moldavonu's papers. With a single usage, our model captures only the first ingredient and then covers numerous applications that previously appeared in the literature. Some specifications in Segal [43] correspond to the NUDE model with a single usage, especially if his condition L (for linearity) is satisfied as in the basic specification of the vertical contracting literature. In the literature about the design of market structure, models where firms differ only in terms of some fixed costs (see Dana and Spier [14], Eso et al. [20], Gebhardt and Wambach [24], Ranger [41]) correspond to the NUDE framework with a uniform externality structure. We emphasize that our model does not encompass in models that consider issues related to incumbency (see Krishna [35] and Hoppe et al. [28]): first, the 'No externality on non-purchasers' fails in those models; second, the externality does not depend on the aggregate number of units of capacity assigned to a given usage but on a weighted aggregate where incumbents and entrants receive different weights.[16]

In the two economic applications we discuss below we emphasize that the efficient assignment is limited to buyers' perspective and does not account of final consumers' contribution to the welfare.[17] The first one about advertising slots is a simple illustration of preferences that are steeper for more efficient buyers. The second one about the assignment of capacities of production, which is thus linked to the literature on the design of market structure mentioned previously, is an example of a uniform externality structure in its simplest version while natural generalizations of the model lead to preferences that have few chances to be steeper for more efficient buyers.

[16]The weights are respectively 0 and 1 in the basic specifications. In Fullerton and McAfee [23], who consider the value of entering a contest, the corresponding aggregate is weighted by a research cost that is heterogenous across firms. In the specific case where firms are homogenous, their model fits with ours with some symmetric unit demand buyers such that the externality structure is uniform. Note however that the externality term $x \cdot E(x)$ would fail to be convex in their model.

[17]Consumers' preferences, e.g. increasing in the number of units assigned to capture market power mitigation, is an obvious source of failure of buyer substitutability and thus has not been considered.

5.1 Advertising slots

We consider that each firm $i = 1, \dots, N$ has a set of products $y = 1 \dots, Y_i$ characterized by $v_i^y \geq 0$ the value of being viewed by a consumer.[18] Products are ranked w.l.o.g. according their profitability v_i^y that is thus supposed to be nondecreasing in y. Let $\alpha(x)$ denote the expected number of consumers viewing a given advertising slot which is assumed to depend only on the total number of slots. Naturally, we consider that $\alpha(.)$ is nonincreasing: enlarging the number of slots reduces the probability of each slot to be viewed, i.e. the probability to attract consumers' attention. Finally the payoff of firm i for k slots if a total number of $x \geq k$ slots is assigned is given by:

$$\theta_i(k, x) = \sum_{y=1}^{\min\{k, Y_i\}} v_i^y \cdot \alpha(x).$$

This simple structure fits our NUDE framework. Preferences are always steeper for more efficient buyers: if the value to be viewed is higher for a given product (i, y) then it suffers more from the negative externality. Since the environment is a single usage environment, then we can apply Theorem 2 that guarantees that the associated coalitional value function is buyer submodular if $\alpha(.)$ is concave.

5.2 Capacity of production[19]

Suppose that a set of firms $i = 1, \dots, N$ is competing for some capacity of production prior to a Cournot market interaction for some output with a nonincreasing demand function denoted by $D_{out}(p)$.[20] Each firm has a set of projects $y = 1 \dots, Y_i$ to produce the output. The production costs of those projects are the sum of two terms: one is an idiosyncratic production cost denoted by $c_i^y \geq 0$ (that is thus supposed to be nondecreasing in y w.l.o.g.) while the second, $C(x)$, comes from a primary input whose price is determined by the total number of units of capacity that are assigned. The input price is given by the nondecreasing demand function on the input market: $x = D_{in}(p)$, where $D_{out}(.)$ is a nonincreasing demand function. Finally the payoff of firm i for $k \leq Y_i$ units when a total number of $x \geq k$ units is assigned is given by:

$$\theta_i(k, x) = \sum_{y=1}^{k} [D_{out}^{-1}(x) - D_{in}^{-1}(x) - c_i^y].$$

[18]In the perspective of internet keyword auctions (see Edelman et al. [19]) it would correspond to the value per click.

[19]This covers not only physical capacities of production but also non-physical capacities as pollution permits, an application that is especially relevant in the perspective of auction design by a monopoly seller.

[20]As argued in section 2, we can assume w.l.o.g. that any assigned capacity should be used.

This simple structure fits our NUDE framework. The externality structure is uniform: all project (i, y) are suffering in the same way the negative externalities when more outputs are produced which comes either from a fall of the price of the output or from a rise of the price of the primary input. The corresponding externality structure is then uniform and convex if and only if $x \to x \cdot (D_{out}^{-1}(x) - D_{in}^{-1}(x))$ is concave. If it is the case, then Theorem 1 guarantees that the associated coalitional value function is buyer submodular and the result still holds in more general environments where the capacity of production can be used for different kinds of activities whose output and input prices are fully disconnected.[21]

Let us briefly discuss two possible extensions of the firms' preferences.

First, consider that for each project, each firm has signed an hedging contract on the price of the input for a share of the required input, which is labeled as s_i^y, a contract that is furthermore conditional of the acquisition of the related capacity. Let p^* denote the price specified by the contract. The expression of $\theta_i(k, x)$ is now given by:

$$\theta_i(k, x) = \sum_{y=1}^{k} [D_{out}^{-1}(x) - [(1 - s_i^y) \cdot D_{in}^{-1}(x) + s_i^y \cdot p^*] - c_i^y].$$

This structure fits the NUDE framework. If more efficient projects in terms of production costs have a larger share of their input needs that are hedged[22], i.e. for any $i, j \in N$ and $y, y' \geq 1$, $c_i^y \leq c_j^{y'} \Rightarrow s_i^y \geq s_j^{y'}$,then it opens the door for an externality structure that is steeper for less efficient buyers if the preferences satisfy the non-crossing property. Remark that the same logic applies if the hedging concerns the price of the output instead of the price of the input: it is the most efficient units that have thus the highest probability to obtain units that have the greatest incentives to reduce the externality ex ante.

Second, a more sophisticated model could also involves heterogeneity in terms of the productivity of the projects. Each project (i, y) would be also characterized by the two productivity coefficients β_i^y, γ_i^y such that each unit of capacity requires β_i^y units of inputs to produce γ_i^y units of output. The expression of $\theta_i(k_1, \ldots, k_N)$, where $k_i \leq Y_i$ denotes the number of units assigned to firm i, is now given by:

$$\theta_i(k_1, \ldots, k_N) = \sum_{y=1}^{k} [\gamma_i^y \cdot D_{out}^{-1}(\sum_{i \in N} \sum_{y=1}^{k_i} \gamma_i^y) - \beta_i^y \cdot D_{in}^{-1}(\sum_{i \in N} \sum_{y=1}^{k_i} \beta_i^y)] - c_i^y].$$

Such a structure does not encompass in the NUDE framework: the externalities which

[21]The insight of Ranger [41] is precisely to derive the buyer substitutability property in this uniform convex framework but without the externality coming from the term $D_{in}^{-1}(x)$. Note a minor technical difference: Ranger's analysis covers a scope without indivisibilities.

[22]If the hedge contractual options represent some sunk cost, it is then the natural assumption since those projects have a greater probability to be run.

come from the price of the output and the price of the input do not depend solely of the total number of assigned capacities but on the exact productivities of the buyers. Nevertheless, it does not mean that a similar logic that the one we run could not be used to obtain buyer substitutability for a single usage: if a buyer is removed, then, for any remaining potential buyers, the number of assigned units in the efficient allocation does not shrink. Indeed, it is left to the reader to check that in the case where the costs c_i^y and the productivity coefficients β_i^y are homogenous across all projects and if the function $x \to x \cdot D_{out}^{-1}(x)$ is concave, then the buyer substitutability condition holds. The intuition is the same as in the NUDE framework: more efficient projects, γ_i^y is high, have steeper preferences and their removal will induce an extension of the set of marginal buyers that receive a unit in the efficient allocation. On the contrary, if the heterogeneity reduces to the productivity coefficients β_i^y, then the opposite logic applies: less efficient projects are more sensitive to the input price and have thus steeper preferences which closes the door of buyer substitutability.

We have learnt two elements from those possible extensions. First, efficient buyers having steeper preferences does not seem to be the appropriate property that holds: it is rather the opposite property that seems more natural for the assignment of capacities which leaves thus no room for buyer substitutability. Second, similar intuitions as the ones we develop for the NUDE framework can be useful for alternative models and used in order to derive the buyer substitutability property. Such developments are left for further research.

We should be cautious about the relevance of our analysis for many real-life capacity markets: it implicitly assumes that potential buyers do not own any capacity prior to the market such that non-purchasers are indifferent to the total amount of capacity assigned. Nevertheless, it occurs frequently that the scarce resource is in possession of a monopoly who has then the power to sell it once: application of this sort ranges from spectrum to network infrastructure as airport slots, railroad tracks and electricity or gas transmission systems.

6 Application: Contingent Auctions

Contingent auctions are mechanisms where bidders have the opportunity to report preferences that depend not solely on their own assignment but on the whole final assignment. A critical aspect of the definition of contingent auctions, as the mechanisms generalizing the Vickrey-Clarke-Grove mechanism, is the individual set of reports that the bidders can send to the auctioneer. Without any restriction it can lead to important problems. First the possibility to report preferences involving ‘positive allocative externalities’, i.e. such that putting an item into an opponent buyer's hands instead of the seller's make a given buyer better off, opens the door to positive transfers from the seller to the potential buyers. Second, even if available reports exclude any kind of positive externality, without any additional structure on the possible reports, some pair of potential buyers with low valuations may obtain easily all the units by simple joint deviations where they are reporting high intrinsic valuations for the items, that they are suffering for high negative externalities when the item is put into their opponent potential buyers' hands and such that they are not suffering from any externality between each other.[23] For those reasons, we consider 'NUDE-contingent auctions' where bidders are constrained to report individual preferences according to the NUDE framework. In the following, they are simply labeled as 'contingent auctions' though we emphasize that the constrained set of report is a crucial element.

Definition 10 *A **contingent auction** mechanism* $(A, b) : \overline{\Theta}^{M,G} \to \mathbf{A} \times \mathbb{R}^N$ *is a function mapping a vector of types* $\theta = (\theta_1, \dots, \theta_N) \in \overline{\Theta}^{M,G}$ *into an allocation* $A(\theta)$ *and a vector of transfers such that* $b_l(\theta)$ *represents the transfer paid by bidder* l *to the seller.*[24]

Denote by u_l the net payoff of bidder l. In a contingent auction (A, b), if bidder l true type is θ_l and the vector reported preferences is $\widehat{\theta}$, then $u_l = \theta_l(A(\widehat{\theta})) - b_l(\widehat{\theta})$.

Note that contingent auctions should be viewed as a superset of combinatorial auctions. Since the complexity to compute the efficient assignment in those latter formats is known to be NP-complete in general while the number of possible reports grows exponentially as a function of M, we can wonder whether computational tractability is an issue in the class of contingent auctions we consider. On the contrary, in the NUDE framework, the size of the set of reports is considerably reduced compared to general environment with allocative externality - it grows in $O(M^2)$- while the complexity to compute the efficient assignment remains polynomial (bounded by $O(M^G)$).

In order to introduce two contingent mechanisms that are natural generalizations of familiar mechanisms in purely private setups, we introduce additional definitions and notations.

The Vickrey payoff point Let $u^V(S)$ denote the Vickrey payoff point with respect to coalition $S \subset N$: for the buyers $l \neq 0$, $u^V_l(S) = w(S) - w(S \setminus \{l\})$ while the seller's payoff is given by $u^V_0(S) = w(S) - \sum_{l=1}^{N} u^V_l(S)$. This payoff corresponds to the dominant-strategy equilibrium of the generalized Vickrey auction among the bidders in S. Without further

[23] See the working paper version [36] for details on those issues.

[24] In general environments with allocative externalities, contingent auction designs should also specify a non-participation report as emphasized by Figueroa and Skreta [22]. In the auctions considered below it does not play any role and non participation will be equivalent to the null report such that $V^g(.) \equiv 0$ for any $g \in G$.

indication, the Vickrey payoff point corresponds to $u^V(N)$, the one corresponding to the whole coalition N, and is denoted u^V.

The Core payoff points From the coalitional value function w we have defined in section 2, we can define a related set of Core payoff points for any coalition $S \subset N$, denoted by $Core(S, w)$:

$$Core(S, w) = \left\{ (u_l)_{l \in S \cup \{0\}} \mid (a) : \sum_{l \in S \cup \{0\}} u_l = w(S);\ (b) : \forall S' \subset S \cup \{0\},\ w(S') \leq \sum_{l \in S'} u_l \right\}$$

(a) is the feasibility condition, whereas (b) means that the payoffs are not blocked by any coalition S'. In general environments with allocative externalities, it is well-known that there are several natural definitions for the Core and furthermore that all natural Core concepts may be empty as argued in Jehiel and Moldovanu [29]. However, our assumption that non-purchasers suffer from no externalities guarantees that the Core is never empty in our framework: the payoff vector $(w(S), 0, \cdots, 0)$ is always in the Core. Without further indication, the Core payoff points correspond to the set $Core(N, w)$, the one corresponding to the whole coalition N.

A subset of the Core plays a central role in the analysis of combinatorial auctions and similarly for contingent auctions: those are the Pareto-optima from the perspective of the buyers. This set is qualified as the buyer-optimal frontier of the Core.

Definition 11 *The **buyer-optimal frontier of the Core** is the set containing the elements $(u_l)_{0 \leq l \leq N} \in Core(N, w)$ such that there exists no $(u'_l)_{0 \leq l \leq N} \in Core(N, w)$ where $u'_l \geq u_l$ for all $l = 1 \ldots N$ and such that at least one inequality is strict.*

The following results that appear in Milgrom [39] in pure private value environments immediately extends to our environment with allocative externalities.

Lemma 6.1 (Theorem 8.1 in Milgrom [39]) *The Vickrey payoff point is in the Core if and only if the buyer-optimal frontier of the Core is a singleton. In such a case, it reduces to u^V.*

Lemma 6.2 (Theorem 8.3 in Milgrom [39]) *$u^V(S) \in Core(S, w)$ for any $S \subset N$ if and only if w is buyer submodular.*

Remark In the characterization results of theorems 1 and 2, 'w being [or not] buyer submodular' can be replaced by the weaker property 'the Vickrey payoff point u^V being [or not] in $Core(N, w)$' since those properties coincide in the special cases where only three potential buyers have non null preferences, i.e. for the preferences that are used in our proofs of the converse parts in Appendix B.

A simple example (continuation) The failure of the buyer substitutability property is also reflected by the structure of the Core outcomes $(u_l)_{l\in N\cup\{0\}}$ which are defined by the following constraints:

$$u_0 + u_1 + u_2 + u_3 = 18 \qquad \text{Feasibility Constraint} \tag{11a}$$

$$u_0 + u_l + u_m \geq 14 \ \ , \text{ for any } l, m \in \{1,2,3\} \qquad \{0,l,m\} \text{ do not block} \tag{11b}$$

$$u_0 + u_l \geq 11 \ \ , \text{ for any } l \in \{1,2,3\} \qquad \{0,l\} \text{ do not block} \tag{11c}$$

$$u_l \geq 0 \ \ , \text{ for any } l \in \{0,1,2,3\} \ \text{ Rationality Constraint } \ \{l\} \text{ do not block} \tag{11d}$$

The Vickrey payoff point is the one such that the constraints (11b) are all binding, it is given by $(6,4,4,4)$ which lies notably outside the Core since the constraints (11c) are violated. We can also check that the buyer-optimal frontier of the Core is not a singleton: $(7.5,3.5,3.5,3.5)$ is the symmetric outcome in this set, while $(11,4,3,0)$ is buyer 1's most preferred outcome in this set which notably gives him his Vickrey payoff.[25]

We introduce two kinds of contingent auctions, the Vickrey contingent auction and buyer-optimal Core selecting contingent auctions, that implement respectively the Vickrey payoff point and buyer-optimal Core payoff points with respect to the reported preferences.[26,27]

Definition 12 (Two standard contingent auctions) *In the following contingent auctions, the seller chooses an efficient assignment given the set of reports $\widehat{\theta}$: $A(\theta) = \mathcal{Y}_N^*(\theta)$. Then the transfers for $l \in N$ are given by:*

- ***The Vickrey contingent auction:*** $b_l(\widehat{\theta}) = \sum_{k\neq l} \widehat{\theta}_k(\mathcal{Y}^*_{N\setminus\{l\}}(\widehat{\theta})) - \sum_{k\neq l} \widehat{\theta}_k(\mathcal{Y}^*_N(\widehat{\theta}))$.
- ***Buyer-optimal Core selecting contingent auctions:*** $b_l(\widehat{\theta})$ *is given such that the vector* $(\widehat{\theta}_l(\mathcal{Y}^*_N(\widehat{\theta})) - b_l(\widehat{\theta}))_{0\leq l\leq N}$ *belongs to the buyer-optimal frontier of* $Core(N, w_{\widehat{\theta}})$.

In particular, those mechanisms always select an efficient assignment if bidders are reporting their true preferences. We now discuss the robustness of truthful reporting in

[25] The buyer-optimal frontier of the Core are the payoff vectors $(u_l)_{l\in N\cup\{0\}}$ such that $u_0 \in [7.5, 11]$, $u_1 = 11 - u_0$, $u_2 = \max\{u_1, 14 - u_0 - u_1\}$ and $u_3 = 18 - \sum_{l=0}^2 u_l$ (up to any permutation of buyers indices).

[26] In a pure private value setup, Day and Milgrom [15] introduce Core-selecting combinatorial auctions. They emphasize that the auctions that select a payoff in the buyer-optimal frontier are the ones among this class that minimize the incentives of the bidders to misreport their preferences.

[27] Bernheim and Whinston [5]'s first price 'menu' auction is an alternative contingent auction format. We do not consider this format here since it leaves no room for truthful reporting and is generally inefficient under incomplete information. However, our analysis can be applied for a better understanding of the environments where the set of coalitional-proof Nash equilibrium outcomes (under complete information) is a singleton since this set corresponds to the buyer-optimal frontier of the Core.

those contingent auctions. Robustness is captured by various criteria: first the standard strategy-proofness requiring that truthful reporting is always a best-response or equivalently a dominant strategy, second we consider two forms of 'coalitional flavored' deviations that have been previously investigated by Ausubel and Milgrom [3], Milgrom [39] and Yokoo et al. [47]: shill bidding and losing bidders' joint deviations.

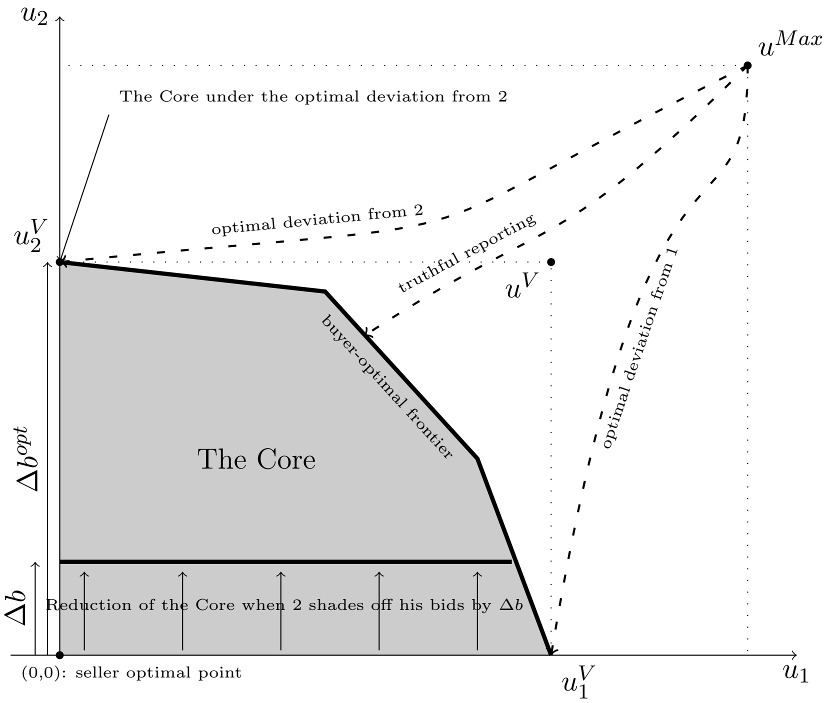

Figure 3: Buyer-optimal Core selecting contingent auctions

6.1 Robustness to individual deviations

Truthful reporting is always an equilibrium in the Vickrey contingent auction. On the contrary, this desirable property is satisfied only on a restricted domain of preferences in buyer-optimal Core selecting contingent auctions. It corresponds to those preferences that satisfy buyer substitutability.

Proposition 6.1 *For any buyer-optimal Core selecting contingent auction, truthful reporting is an equilibrium if and only if the Vickrey payoff point is in the Core.*

The following proof is illustrated by Figure 3.

Proof If $u^V \in Core(N, w)$, then the outcome under truthful reporting is u^V since lemma 6.1 guarantees that the buyer-optimal frontier of the Core reduces to u^V. In a buyer-optimal Core selecting contingent auction, an individual can never obtain more than his Vickrey payoff according to the set of preferences reported by his opponent bidders and his own preferences (Theorem 1 in Day and Milgrom [15] that extends straightforwardly to our environment) and it is thus not profitable to deviate from truthful reporting.

Consider that $u^V \notin Core(N, w)$ such that the final outcome u under truthful reporting is such that there exists $l \in N$ with $u < u_l^V$. It is the case that is depicted in Figure 3. Then from truthful reporting, bidder l has a profitable deviation that guarantees the Vickrey payoff point (Theorem 2 in Day and Milgrom [15]): it consists in a truncation of its preferences, instead of reporting the true preferences $\theta_l(\mathcal{Y})$ he reports $\widehat{\theta}_l(\mathcal{Y}) = \max\{\theta_l(\mathcal{Y}) - \Delta b, 0\}$ which eliminate some of his less favorable payoffs in the Core. The optimal truncation Δb^{opt} is the one where only his Vickrey payoff remains a Core outcome. **CQFD**

If preferences are generated according the conditions in Theorem 1 or 2, then truthful reporting is a dominant strategy in buyer-optimal Core selecting contingent auctions. Given proposition 6.1, a closer look at our proof for the converse part of theorems 1 and 2 shows that we can replace 'w is not buyer submodular' by 'truthful reporting is not an equilibrium in buyer-optimal Core selecting auctions' in those converses.

6.2 Robustness to shill bidding

Shill bidding is the possibility for a buyer to duplicate his identities in the auction. We say that a contingent auction is shill-bidding proof on a domain $\Theta \subset \Theta^{M,G}$ if, in the expanded strategy space where bidders are free to use shills provided that any reports lies in the set Θ and for any bidder l, there exists a best reply for bidder l that does not use shill bids.[28] In the same way as strategy-proofness is the key property of the Vickrey auction, shill-bidding proofness is one of the key properties of buyer-optimal Core selecting auctions (see Day and Milgrom [15]). This property immediately extends in our environment with allocative externalities. On the contrary, this property requires some restrictions on the domain of preferences to be satisfied in the Vickrey contingent auction. Yokoo et al. [47] establishes that buyer substitutability is a sufficient condition to obtain the robustness to shill bidding activity, a result that goes beyond the pure private value case to include allocative externalities.

Proposition 6.2 (Generalization of Proposition 2 & 3 in Yokoo et al. [47]) *On a domain of preferences $\Theta \subset \Theta^{M,G}$ such that w_θ is buyer submodular for any $\theta \in \Theta^k$ and if the reported preferences can only belong to such a domain, then the Vickrey contingent auction is shill-bidding proof on the domain Θ.*

[28] For a formal definition in a combinatorial setup, see Yokoo et al. [47]. In particular, the definition of our contingent auction should be extended to allow any number of bidders. Note that it should not be confused with a shill bidding activity coming from the seller as in Lamy [37].

Contrary to proposition 6.1 where the robustness criterium is obtained if buyer substitutability is satisfied for the true preferences (a condition that holds under reasonable conditions as established by Theorems 1 & 2), proposition 6.2 uses a much stronger condition: the seller is able to force bidders to report preferences that satisfies buyer substitutability. The difference between those conditions does not really matter without allocative externalities since the buyer substitutability condition has been shown to be related to individual preferences: without allocative externalities the set of preferences such that w is buyer submodular is a cartesian product which corresponds to each individual buyer having substitute preferences. On the contrary, we emphasize that this difference is crucial with allocative externalities since this condition imposes constraints on the joint preferences and that it may be difficult if not fully unrealistic to force bidders to report preferences satisfying this condition on the joint preferences.[29] From our characterization of buyer substitutability in Theorem 1, we obtain that only the restriction to preferences that are generated by a given uniform externality structure would work as the following converse part makes clear.

Proposition 6.3 (Robustness to shill bidding: a converse) *For $N \geq 2$, for any externality structure that is not uniform and convex and for the Vickrey contingent auction where bidders are forced to report individual preferences generated by such an externality structure, there exists a set of preferences generated by this externality structure such that truthful reporting is not robust to shill bidding: some bidders' best reply involve necessarily a shill bidding activity.*

Proof If preferences are generated by an externality structure that is not uniform and convex, then propositions 4.1 and 4.2 show that we can build preferences with three agents labeled as A, B and C such that $w(\{A, B, C\}) - w(\{A, B\}) > w(\{A, C\}) - w(\{A\})$. Consider now the environment with $N \geq 2$ where agent 1 has the same preferences as agent A, agent 2 has the same preferences as the 'bundle' of the agents B and C (bundling is the inverse operation to the break up of all marginal valuations into single pieces that has being considered in section 3) and where the remaining agents have null preferences. Under truthful reporting, the payoff of agent 2 in the Vickrey auction is equal to $w(\{A, B, C\}) - w(\{A\})$. If agent 2 use two identifiers: one bidding according to agent B's preferences and one according agent C's preferences then his final payoff in the Vickrey auction corresponds to the sum of agent B and C payoff if they were real different bidders (this comes from

[29] Naturally, from a purely theoretical point of view, we could slightly modify the Vickrey contingent auction such that the seller keeps all units if the reported preferences fails to satisfy buyer substitutability. From a practical auction design perspective, this is an issue since it requires a strong commitment power for the seller. Moreover, if the true preferences may fail to satisfy buyer substitutability, it would involve the risk of implementing a disastrous assignment.

the separability of the preferences), i.e. $[w(\{A, B, C\}) - w(\{A, B\})] + [w(\{A, B, C\}) - w(\{A, C\})]$. Without shill bidding activity, bidder 2 can never obtain a better payoff that the Vickrey payoff $w(\{A, B, C\}) - w(\{A\})$. On the contrary, with two identifiers and reports specified as above, he obtains a strictly higher payoff. **CQFD**

We emphasize that w being buyer submodular at the true preferences, e.g. if they are generated by a uniform and convex externality structure, does not guarantee that the truthful equilibrium is immune to shill bidding without any restriction on the set of preference that bidders can report, e.g. $\overline{\Theta}^{M,G}$.

6.3 Robustness to losing bidders' joint deviations

Losing bidders for a given assignment $\mathcal{Y}$ are the ones that obtain no unit at all, i.e. bidders l such that $\mathcal{Y}_l^g = 0$ for any $g \in G$. Denote by $S^{los}(\theta)$ the set of losing bidders in the allocation $\mathcal{Y}_N^*(\theta)$. The outcome under truthful reporting is robust to losing bidders deviations if the losing bidders can not coordinate their reports such that they jointly benefit from the deviation: formally, for any $\widehat{\theta} \in \overline{\Theta}^{M,G}$ such that $\widehat{\theta}_l = \theta_l$ if $l \notin S^{los}(\theta)$, we have $\sum_{l \in S^{los}(\theta)} \theta_l(A(\widehat{\theta})) - b_l(\widehat{\theta}) \leq 0$. Robustness to losing bidders' joint deviations is another important property of auctions that always implement a Core outcome and it is thus satisfied for buyer-optimal Core selecting auctions. The argument is the following. Since the outcome ends in the Core, the losing bidders on the whole would have to pay at least the externality on the winning bidders, denoted by $S^{win}(\theta) = N \setminus S^{los}(\theta)$, which is equal to $\sum_{l \in S^{win}(\theta)} \theta_l(A(\theta)) - \theta_l(A(\widehat{\theta}))$ which is itself bigger than $\sum_{l \in S^{los}(\theta)} \theta_l(A(\widehat{\theta})) - \theta_l(A(\theta))$ since $A(\theta)$ is the efficient assignment according to the preferences θ, this last expression is equal to $\sum_{l \in S^{los}(\theta)} \theta_l(A(\widehat{\theta}))$ (since the bidders in $S^{los}(\theta)$ are losing bidders such that their payoffs are null under the assignment $A(\theta)$) the joint payoff of the losing bidders under the deviation. On the whole, the losing bidders have to pay more than what they gain.

On the contrary, the robustness to losing bidders' joint deviations property requires some restrictions on the preferences to be satisfied in the Vickrey contingent auction as illustrated by the following example.

A simple example (continuation) We come back to the example in section 2 but now we slightly modify the numerical values such that the externality term $x \to x \cdot E(x)$ is now convex while $E(.)$ remains concave. We consider the uniform externality structure given by $E(0) = 0$, $E(1) = 4$ and $E(2) = 5.5$. We also break the symmetry assumption by considering that there is one strong potential buyer 1 with intrinsic valuation 11 while the two remaining (weak) potential buyers $2, 3$ have an intrinsic valuation set to 8. The externalities are now strong enough such that for any coalition it is an efficient allocation to assign a single licence to its most efficient member. The coalitional value function for any $S \neq \emptyset$ is now given by $w(S) = 11$ if $1 \in S$ and $w(S) = 8$ otherwise. We can check immediately that w is buyer submodular. Note that the true preferences are generated according to the stringent conditions of theorem 1. The Vickrey payoff point is given by $(8, 3, 0, 0)$ which belongs to the Core, which is given by $\{(x, 11 - x, 0, 0)|x \in [8, 11]\}$. However, the Vickrey contingent auction is not robust to a joint deviation of the losing bidders: if buyers 2 and 3 are reporting a private valuation that does not suffer from any externality and that is greater than 4 then the efficient assignment (with respect to those reports) becomes the three-licensees structure while the final net payoffs under the Vickrey contingent auction pricing rule is now given by $(3.5, 5.5, 1, 1)$. On the whole both 'deviant' buyers strictly benefit from the joint coalition which is also beneficial to buyer 1.

Without allocative externalities, Ausubel and Milgrom [3] show that the restriction on the preferences that makes the Vickrey auction robust to losing bidders' joint deviations corresponds to item substitutability. In particular, they obtain exactly the same condition as for the robustness against shill bidding of the same format, which corresponds also to the condition that makes truthful reporting an equilibrium in buyer-optimal Core selecting auctions. However, with allocative externalities such an equivalence will not hold anymore.

First if the reported preferences are restricted to a domain such that the joint preferences satisfy buyer substitutability then it guarantees that the final outcome belongs to the Core which guarantees robustness to losing bidders' joint deviations as argued above. Thus the analog of proposition 6.2 can be derived, but as also argued above such a result is of little use since it would involve unrealistic constraints on the feasible reports of the bidders. Second, in the same vein as proposition 6.3, which should be interpreted as a 'negative' result since any departure from uniform externality structures breaks shill-bidding proofness, we show that with multiple usages if the set of feasible reports allows to report joint preferences that are not generated by a uniform externality structure, then it opens the door to losing bidders' joint deviations and no restriction on the set of available reports.

Proposition 6.4 (Robustness to losing bidders' joint deviations: $G > 1$) *For $N \geq 3$ and $G > 1$, consider a convex externality structure that fails to be uniform and suppose that bidders are forced to report individual preferences that are generated by this externality structure, there exists a set of preferences generated by this externality structure[30], such that truthful reporting in the Vickrey auction is not immune to losing bidders' joint deviation.*

[30] Indeed we show a slightly stronger result since the true preferences that fail to be robust to losing bidders' joint deviation can be generated by a uniform externality structure: the key point is that the set of available reports is not limited to preferences that are generated by a uniform externality structure.

The proof is relegated to Appendix C. On the contrary, next proposition derives a 'positive' result in the single usage case which relies solely on convexity conditions.

Proposition 6.5 (Robustness to losing bidders deviations: $G = 1$**)** *For $G = 1$, if preferences are generated by a strongly convex externality structure, then losing bidders in the truthful reporting equilibrium of the Vickrey auction have no profitable joint deviations.*

Partial converse: *For $N \geq 3$ and $G = 1$, consider an externality structure that fails to be strongly convex, that includes externality terms that involve no externality, i.e. there exists $j \in J$ such that $\mathcal{E}_j(.) \equiv 0$ and suppose that bidders are forced to report individual preferences that are generated by this externality structure, then there exits preferences generated by this externality structure such that truthful reporting in the Vickrey auction is not immune to losing bidders' joint deviation.*

The proof is relegated to Appendix D. The continuation of the simple example above illustrates the partial converse.

Previously, we mentioned that the practical assumption that any assigned units are effectively used by the buyers was w.l.o.g., the argument that buyers can express through their reports their preemption motives does not work anymore with 'coalitional' deviations as shill bidding or losing bidders' deviations, e.g. a shill bidder can not express the preemption motives of its second identifier. To illustrate this point consider $M = 2$, a single usage and preferences such that units are worthless if two units are used. Consider three bidders whose preferences are then fully characterized by the value of a unit if a single unit is used, denoted by V_i, $i = 1, 2, 3$. Suppose that $V_1 > V_2 = V_3 > 0$. Under the assumption that the units that are sold have to be used, then the truthful equilibrium is robust to shill bidding and losing bidders deviations. There is no possible strictly positive gain if both units are sold at the final outcome since the final payoff is null. If a single unit is sold to a buyer that differs from agent 1, then the winner has to pay at least V_1 which leads to negative profits. On the contrary, if buyers can preempt units, then there are strictly profitable 'coalitional' deviations: with two identifiers that report intrinsic valuations that are bigger than V_1 and with preferences that do not suffer from any externality then both units are sold at a null price and the final payoff of the winners is V_2 if they can decide to use effectively only one unit. On the whole preemption motives will thus reinforce our insights in favor of buyer-optimal Core selecting contingent auctions.

7 Conclusion: an auction design perspective

A practical issue for policy makers facing an assignment problem is to choose between a centralized administrative procedure and a decentralized market-based procedure. Banks et al. [4] report experimental evidence supporting that the latter price-based mechanisms outperform the former.[31] Rassenti et al. [42] and Brewer and Plott [10] provide additional experimental evidence supporting that auction outcomes are very close to the efficient ones in complex multi-object assignment problems as the allocation of airport slots or the use of a railroad track.[32] A critical element is that bidders' reservation values may not be additive in the objects for sale but may involve both substitutabilities and/or complementarities. In such environments, combinatorial auctions where bidders can submit bids on packages of objects have received a growing attention.[33] More generally the main insight of the literature on complex assignment problems is the need for non-linear and non-anonymous pricing rules. A critical condition emerges in this literature: the buyer-substitutability condition which says that the marginal contribution of a buyer to a set of potential buyers is nonincreasing in this set. This condition is satisfied for a large set of preferences: those where each individual buyers are valuing the items as substitutes.

Nevertheless, a closer look at real-life assignment problems as the ones that motivated the experimental works mentioned above reveals that allocative externalities may be as important as the non-additive nature of valuations that has been the focus of the previous literature: purchasers do not care solely on their own assignment but on the whole assignment. The buyer-substitutability plays a similar role to analyze the robustness of auction mechanisms in this generalized framework. Our characterization of the externality structure generating preferences that satisfy buyer-substitutability leads to two lessons from an auction design perspective. First, in environments where the good may be used for different usages corresponding to different submarkets, the answer is negative: buyer-substitutability is guaranteed only for non-generic externality structures where the externality term for a given usage is the same for all bidders. Second, in environments where the good can be used only for a unique usage (corresponding to a specific submarket), the answer is mixed:

[31]The modeling of administrative assignment rules is rather ad hoc in Banks et al. [4]: they consider random assignment arguing that it is not an unreasonable modeling of NASA's Space Transportation System pricing and allocation policy. More generally, there is much doubt that complex systems of hierarchical committees and detailed administrative rules are able to allocate efficiently scarce resources. The allocation of airport slots in Europe under the grandfathering rights amended by a use-it-or-it clause is a good illustration of inefficiencies as it is indicated by the so-called 'baby-sitting' of slots, i.e. the use of slots for unprofitable markets to preempt entry.

[32]This last assignment problem is even more complex because the nature of the allocated goods are determined endogenously by the market that has to take into account the multiple constraints resulting from railroad operation systems.

[33]See also the collection of papers in Cramton et al. [12].

in environments where the most efficient bidders are suffering more from the externalities then there is a room for buyer-substitutability.

Naturally, the buyer-substitutability condition and the desirable theoretical predictions it brings are only a limited part of the picture that has to be completed with experimental investigations. The NUDE-framework and our characterization of buyer-substitutability can be very useful for the design of such experiments and to structure the discussion of the results. Our analysis also focus in an environment where the seller can monitor the way a purchased unit is used. If the usage of the units is not contractible, such incompleteness of the market is a source of inefficiency: it may be optimal to assign a unit to a given bidder for a given usage but we can not reach this efficient assignment if that bidder prefer to use this unit for an alternative usage. In a similar way, resale markets may be a source of inefficiency in the multi-usage case. Such investigations are left for further research.

Appendix

A Proof of Proposition 3.1

From equation (7), we derive the expression of the marginal gain from assigning an additional unit to usage g among the list of marginal buyers $L \subset \mathcal{L}^g$ from an initial assignment that allocates $x-1$ units to usage g: for $x \geq 1$,

$$\Delta H^g_L(x) = V^g_{\sigma^g_L(x)}(\gamma^g_L(x)) - \Delta E^g_L(x) = [V^g_{\sigma^g_L(x)}(\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x, \gamma^g_L(x))] - \sum_{j=1}^{x-1} \Delta_x E^g_{\sigma^g_L(j)}(x, \gamma^g_L(j)). \tag{12}$$

Lemma A.1 *For any usage $g \in G$ and any ordered list $L \subset \mathcal{L}^g$, if either the convexity property A or the convexity property B holds, then $H^g_L(.)$ is concave.*

Proof From equation (12), $\Delta H^g_L(x)$ is the sum of two terms: $V^g_{\sigma^g_L(x)}(\gamma^g_L(x))$ and $-\Delta E^g_L(x)$. The first term is nonincreasing in x from the construction of the mapping $\sigma^g_L(.)$. The second term is nonincreasing in x under the convexity property A. The same argument works with convexity property B and the second decomposition in (12) since the non-crossing property guarantees that $V^g_{\sigma^g_L(x)}(\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x, \gamma^g_L(x))$ is nonincreasing in x. **CQFD**

Lemma A.1 guarantees that the optimization program in (9) is concave which means that the selected efficient assignment among a set of marginal buyers $L = (L^1, \ldots, L^G) \in$

$\prod_{g\in G}\mathcal{L}^g$ can be easily characterized. $\mathcal{X}_L^*$ can be easily computed by the following 'greedy' algorithm where $X_L = (X_L^g)_{g\in G}$ is the current state of the algorithm:

- Initialization step (all units are initially in the seller's hands): let $X_L := (0,\ldots,0)$.
- Incremental assignment step (at each step at most one unit is assigned): let $g^* := \min_{g\in G}\{\operatorname{Arg\,max}_{g\in G}\Delta H^g_{L^g}(X^g_L+1)\}$, i.e. g^* is the most promising usage to assign an additional unit. If $\sum_{g\in G}X^g_L < M$ and $\Delta H^{g^*}_{L^g}(X^{g^*}_S+1) > 0$, i.e. if there is still at least one available unit and the best usage is strictly better than the one who assigns this unit to the seller, then let $X_L := (X^1_L,\ldots,X^{g^*-1}_L, X^{g^*}_L+1, X^{g^*+1}_L,\ldots,X^G_L)$ and re-run the incremental step. Otherwise stop the algorithm.

The final state X_L corresponds to the assignment across usages in the selected efficient allocation: $X_L = \mathcal{X}^*_L$.

To investigate the difference between the efficient assignment before and after the removal of one marginal buyer, the key point is how the incremental opportunities evolve. First note that removing the marginal buyer $l = (\sigma^g_L(\overline{x}), \gamma^g_L(\overline{x}))$ that consider usage g does not modify the profit function for the other usages: $\Delta H^{g'}_{L\setminus\{l\}}(.) = \Delta H^{g'}_L(.)$ for $g' \neq g$. We are now left with the comparison between $\Delta H^g_{L\setminus\{l\}}(.)$ and $\Delta H^g_L(.)$ which is the purpose of the following lemma.

Lemma A.2 *For any usage $g \in G$ and any $L \in \mathcal{L}^g$. Suppose that either the convexity property A or the convexity property B holds and consider a marginal buyer $l = (\sigma^g_L(\overline{x}), \gamma^g_L(\overline{x}))$ with respect to usage g.*

- *If preferences are steeper for less efficient buyers, then $\Delta H^g_L(x) \geq \Delta H^g_{L\setminus\{l\}}(x)$ for any $x \in [1, \min\{t_L, M\}]$.*
- *If preferences are steeper for more efficient buyers, then $\Delta H^g_{L\setminus\{l\}}(x) \geq \Delta H^g_L(x+1)$ for any $x \in [1, \min\{t_L, M\}]$.*

Proof For $x < \overline{x}$, we have $\Delta H^g_{L\setminus\{l\}}(x) = \Delta H^g_L(x)$. Rewriting equation (12) for the set $L \setminus \{l\}$, we have:

$$\Delta H^g_{L\setminus\{l\}}(x) = V^g_{\sigma^g_{L\setminus\{l\}}(x)}(\gamma^g_{L\setminus\{l\}}(x)) - \Delta E^g_{L\setminus\{l\}}(x). \tag{13}$$

For $x \geq \overline{x}$, we have: $\sigma^g_{L\setminus\{l\}}(x) = \sigma^g_L(x+1)$ (the x^{th} most efficient marginal buyer among $L \setminus \{l\}$ corresponds to the $(x+1)^{th}$ most efficient marginal buyer among L), $\gamma^g_{L\setminus\{l\}}(x) =$

$\gamma_L^g(x+1)$ and

$$\Delta E^g_{L\setminus\{l\}}(x) = \Delta E^g_L(x) + (E^g_{\sigma^g_L(x+1)}(x,\gamma^g_L(x+1)) - E^g_{\sigma^g_L(x)}(x-1,\gamma^g_L(x))) - (E^g_l(x,1) - E^g_l(x-1,1)).$$

Finally after plugging all those elements, we obtain an expression for $\Delta H^g_{L\setminus\{l\}}(x)$ that is easily comparable to $\Delta H^g_L(x)$. For $x \geq \overline{x}$, we obtain:

$$\begin{aligned}\Delta H^g_L(x) - \Delta H^g_{L\setminus\{l\}}(x) = &\overbrace{(V^g_{\sigma^g_L(x)}(\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x,\gamma^g_L(x))) - (V^g_{\sigma^g_L(x+1)}(\gamma^g_L(x+1)) - E^g_{\sigma^g_L(x+1)}(x,\gamma^g_L(x+1)))}^{\geq 0 \text{ since the marginal valuation } (\sigma^g_L(x),\gamma^g_L(x)) \text{ if more efficient than } (\sigma^g_L(x+1),\gamma^g_L(x+1))} \\ &+ \underbrace{(E^g_{\sigma^g_L(x)}(x,\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x-1,\gamma^g_L(x))) - (E_l(x) - E_l(x-1))}_{\geq 0 \text{ if preferences are steeper for less efficient buyers}}\end{aligned} \quad (14)$$

$$\begin{aligned}\Delta H^g_{L\setminus\{l\}}(x) - \Delta H^g_L(x+1) = &\overbrace{(E_l(x) - E_l(x-1)) - (E^g_{\sigma^g_L(x)}(x,\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x-1,\gamma^g_L(x)))}^{\geq 0 \text{ if preferences are steeper for more efficient buyers}} \\ &+ \underbrace{(\Delta E^g_L(x+1) - \Delta E^g_L(x))}_{\geq 0 \text{ under convexity property A}} + \underbrace{E^g_{\sigma^g_L(x)}(x,\gamma^g_L(x)) - E^g_{\sigma^g_L(x+1)}(x,\gamma^g_L(x+1))}_{\geq 0 \text{ if preferences are steeper for more efficient buyers}}\end{aligned} \quad (15)$$

$$\begin{aligned}\Delta H^g_{L\setminus\{l\}}(x) - \Delta H^g_L(x+1) = &\overbrace{(E_l(x) - E_l(x-1)) - (E^g_{\sigma^g_L(x)}(x,\gamma^g_L(x)) - E^g_{\sigma^g_L(x)}(x-1,\gamma^g_L(x)))}^{\geq 0 \text{ if preferences are steeper for more efficient buyers}} \\ &+ \underbrace{(\sum_{j=1}^{x} \Delta_x E^g_{\sigma^g_L(j)}(x+1,\gamma^g_L(j)) - \sum_{j=1}^{x-1} \Delta_x E^g_{\sigma^g_L(j)}(x,\gamma^g_L(j))}_{\geq 0 \text{ under convexity property B}}.\end{aligned} \quad (16)$$

We conclude by gathering the various elements in equations (14) and (15) under convexity property A [respectively (14) and (16) under convexity property B]. **CQFD**

We first define smoothly increasing admissible preferences paths for any unit demand buyer for some set of preferences in the NUDE framework and under the non-crossing property.

Definition 13 *A smoothly increasing admissible preferences path with respect to unit demand buyer l is a mapping $\overline{V}_l(u), \overline{E}_l(.;u)$ for u on the range $[0,1]$ such that:*

1. Starting point*: $\overline{V}_l(0) = 0$, $\overline{E}_l(.;0) = E^g_{l^*}(.,y^*)$ where (l^*,y^*) corresponds to the less efficient marginal buyer with respect to usage g.*
2. Ending point*: $\overline{V}_l(1) = V_l$, $\overline{E}_l(.;1) = E_l(.)$.*
3. Smoothness*: $\overline{V}_l(u)$ and $\overline{E}_l(x;u)$ (for any $x \in [1,M]$) are continuous on $[0,1]$.*
4. Monotonicity*: $(\overline{V}_l(u) - \overline{E}_l(x;u))$ is nondecreasing in u on $[0,1]$ for any $x \in [1,M]$.*

5. Preservation of the NUDE framework and the non-crossing property*: for any $u \in [0,1]$, the preference structure if buyer l's preferences $V_l, E_l(.)$ are replaced by $\overline{V}_l(u), \overline{E}_l(.;u)$ is a NUDE framework and satisfies the non-crossing property.*

The monotonicity property guarantees that the associated coalitional value functions $w(S;u)$, $S \subset N$, are differentiable with respect to u for almost any u. The following lemma is the final step to show Proposition 3.1: we can then conclude with the integration (almost everywhere) of the inequalities derived in this proposition from the starting point to the ending point after noting that the smoothness property of the path excludes any jump at the discontinuity points of the derivatives.

Lemma A.3 *Under either the convexity property A or the convexity property B, consider a unit demand buyer l according to usage g. For any $S \subset N$, we have:*

- *If preferences are steeper for less efficient buyers, then, for any unit demand buyer $m \neq l$ according to a usage that is distinct from g, there exist a smoothly increasing admissible preferences path with respect to unit demand buyer m such that $\delta[w(S;u) - w(S \setminus \{l\};u)]/\delta u \leq 0$ for almost any u.*
- *If preferences are steeper for more efficient buyers, then, for any unit demand buyer $m \neq l$ according to the usage g, there exist a smoothly increasing admissible preferences path with respect to unit demand buyer m such that $\delta[w(S;u) - w(S \setminus \{l\};u)]/\delta u \leq 0$ for almost any u.*

Proof of lemma A.3 If $m \notin S$, then both $w(S;u)$ and $w(S \setminus \{l\};u)$ do not depend on u which guarantees the result. Now consider that $m \in S$. Let g' denote the usage according to which unit demand buyer m is interested in obtaining a unit. Let $\{k_1, \ldots, k_T\}$ the ordered list of all marginal buyers according to usage g' in coalition S (note that T depends on S) and let t_m denote the index of the marginal valuation that corresponds to unit demand buyer m. We define $k_{T+1} = (l_{T+1}, y_{T+1})$ as the marginal buyer corresponding to the starting point, i.e. with a null intrinsic valuation and an externality term that corresponds to (l^*, y^*) the less efficient marginal buyer with respect to usage g. Let $p = T - t_m + 1$ such that $p-1$ can be viewed as the number of marginal valuation in the list $\{k_1, \ldots, k_T\}$ that we should 'cross' to go from the lowest possible valuation for m to its real valuations. The path $\overline{V}_m(u), \overline{E}_m(.;u)$ is defined such that for any $k = 0, \ldots, p-1$ and $u \in [k/p, (k+1)/p]$:

$$\overline{V}_m(u) - \overline{E}_m(x;u) = (p \cdot u - k)[V^{g'}_{l_{T-k}} - E^{g'}_{l_{T-k}}(x, y_{T-k})] + (k+1-p \cdot u)[V^{g'}_{l_{T+1-k}} - E^{g'}_{l_{T+1-k}}(x, y_{T+1-k})],$$

for any $x \in [1, M]$ while $\overline{E}_l(1; u) \equiv 0$, which is a proper characterization of $(\overline{V}_m(u), \overline{E}_m(.; u))$. First we can check straightforwardly that $\overline{V}_m(u), \overline{E}_m(.; u)$ is a smoothly increasing admissible preferences path with respect to unit demand buyer m. Let $\widetilde{H}_S^g(.; u)$ and $\widetilde{E}_S^g(.; u)$ be the respective analogs of $H_S^g(.)$ and $E_S^g(.)$ given by equations (4) and (5) for the set of preferences at the point u in the path. In a second step, we show that the following properties are satisfied for any $g'' \in G$: $\widetilde{H}_S^{g''}(.; u)$ is concave for any $u \in [0, 1]$ and that $\Delta\widetilde{H}_S^{g''}(x; u) \geq \Delta\widetilde{H}_{S\setminus\{l\}}^{g''}(x; u)$ for any $x \in [1, M]$ and any $u \in [0, 1]$ if preferences are steeper for less efficient buyers [$\Delta\widetilde{H}_{S\setminus\{l\}}^{g''}(x; u) \geq \Delta\widetilde{H}_S^{g''}(x + 1; u)$ for any $x \in [1, M]$ and any $u \in [0, 1]$ if preferences are steeper for more efficient buyers]. For $g'' \neq g'$, those properties are immediately satisfied since $\widetilde{H}_S^{g''}(.; u)$ does not depend on u and since there are satisfied for $u = 1$ as an application of lemmata A.1 and A.2 to the ordered list $\{k_1, \ldots, k_T\}$. We now consider the case $g'' = g'$. On the one hand, lemmata A.1 and A.2 guarantees that those properties are satisfied for $u = k/p$ for any $k = 1, \ldots, p$ since $\widetilde{H}_S^{g'}(.; k/p)$ and $\widetilde{H}_{S\setminus\{l\}}^{g'}(.; k/p)$ correspond to $H_L^{g'}(.)$ and $H_{L\setminus\{l\}}^{g'}(.)$ where $L \in \mathcal{L}^{g'}$ is the ordered list $\{k_1, \ldots, k_{t_m-1}, k_{t_m+1}, \ldots, k_{T-k+1}, k_{T-k+1}, \ldots, k_T\}$ for $k = 1, \ldots, p-1$ ($\{k_1, \ldots, k_T\}$ for $k = p$), i.e. the list that is built from $\{k_1, \ldots, k_T\}$ by removing the marginal buyer corresponding to unit demand buyer m and by adding the marginal valuation k_{T-k+1} that may then be duplicate. Finally, the list contains at most one duplicate: $L \in \mathcal{L}^{g'}$. For $k = 0$, the same logic applies after noting that $\Delta_x\widetilde{H}_S^{g'}(.; 0)$ and $\Delta_x\widetilde{H}_{S\setminus\{l\}}^{g'}(.; 0)$ are equal respectively to $\Delta_x H_L^{g'}(.)$ and $\Delta_x H_{L\setminus\{l\}}^{g'}(.)$ where $L \in \mathcal{L}^{g'}$ is the ordered list $\{k_1, \ldots, k_{t_m-1}, k_{t_m+1}, \ldots, k_{T+1}\}$. On the other hand, for any $u \in (k/p, (k+1)/p)$ with $k = 0, \ldots, p-1$, $\widetilde{H}_S^{g'}(.; u)$ [respectively $\widetilde{H}_{S\setminus\{l\}}^{g'}(.; u)$] corresponds to a weighted sum of $\widetilde{H}_S^{g'}(.; k/p)$ and $\widetilde{H}_S^{g'}(.; (k+1)/p)$ [respectively $\widetilde{H}_{S\setminus\{l\}}^{g'}(.; k/p)$ and $\widetilde{H}_{S\setminus\{l\}}^{g'}(.; (k+1)/p)$] such that the desired properties that were shown to hold for $u = k/p$ for any $k = 0, \ldots, p$ are valid for any $u \in [0, 1]$.

The key point: on the whole, the key point we obtain is that for any set of preferences on the path, if preferences are steeper for less efficient buyers and if one unit demand buyer according to some usage g is removed then the number of units assigned to alternative usages $g'' \neq g$ do not shrink. On the contrary, if preferences are steeper for more efficient buyers and if one unit demand buyer according to some usage g is removed then the number of units assigned to the usage g does not shrink.

We are now prepared to compare $\delta w(S; u)/\delta u$ and $\delta w(S \setminus \{l\}; u)/\delta u$ to conclude the proof as sketched in Figure 1. Recall that the non-crossing property guarantees that $w(S; u)$ depends solely on the functions $\Delta_x\widetilde{H}_S^g(.; u)$. For almost every u, the partial differentials $\delta w(S; u)/\delta\Delta_x\widetilde{H}_S^g(x; u)$ and $\delta\Delta_x\widetilde{H}_S^g(x; u)/\delta u$ are well defined. Then we have:

$$\frac{\delta w(S;u)}{\delta u} = \sum_{g \in G} \sum_{x=1}^{M} \frac{\delta w(S;u)}{\delta \Delta_x \widetilde{H}_S^g(x;u)} \cdot \frac{\delta \Delta_x \widetilde{H}_S^g(x;u)}{\delta u} \tag{17}$$

We emphasize that all our arguments below are valid for those point in the path where the efficient assignment does not change.

First case: Preferences are steeper for less efficient buyers and $g \neq g'$ We have $\delta \Delta_x \widetilde{H}_S^{g''}(x;u)/\delta u = 0$ if $g'' \neq g'$. Moreover $\Delta_x \widetilde{H}_S^{g''}(x;u) = \Delta_x \widetilde{H}_{S \setminus \{l\}}^{g''}(x;u)$ for $g'' \neq g$ and so for $g'' = g'$. On the whole, after using equation (17), we obtain:

$$\frac{\delta[w(S;u) - w(S \setminus \{l\};u)]}{\delta u} = \sum_{x=1}^{M} [\frac{\delta w(S;u)}{\delta \Delta_x \widetilde{H}_S^{g'}(x;u)} - \frac{\delta w(S \setminus \{l\};u)}{\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^{g'}(x;u)}] \cdot \frac{\delta \Delta_x \widetilde{H}_S^{g'}(x;u)}{\delta u}.$$

Remark that $\delta w(S;u)/\delta \Delta_x \widetilde{H}_S^{g'}(x;u)$ is equal either to 0 or to 1 respectively if strictly less or at least x items are assigned for usage g' in the efficient assignment with respect to coalition S. Then the above key point guarantees that $\delta w(S;u)/\delta \Delta_x \widetilde{H}_S^{g'}(x;u) \leq \delta w(S \setminus \{l\};u)/\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^{g'}(x;u)$: raising an incremental opportunity for usage g' has more impact on the value of a coalition if buyer l is absent.

Second case: Preferences are steeper for more efficient buyers and $g = g'$ We have $\delta \Delta_x \widetilde{H}_S^{g''}(x;u)/\delta u = 0$ if $g'' \neq g$. Without loss of generality we can assume that buyer m is less efficient than buyer l. Let $\overline{k}$ such that buyer l is the $\overline{k}^{th}$ most efficient marginal buyer among the potential buyers in S according to usage g. We have thus $\delta \Delta_x \widetilde{H}_S^g(x;u)/\delta u = 0$ for $x \leq \overline{k}$ while $\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x;u)/\delta u = 0$ for $x \leq \overline{k} - 1$. On the whole, after using equation (17), we obtain:

$$\begin{aligned} \frac{\delta[w(S;u) - w(S \setminus \{l\};u)]}{\delta u} = & \sum_{x=\overline{k}+1}^{M} \frac{\delta w(S;u)}{\delta \Delta_x \widetilde{H}_S^g(x;u)} \cdot \frac{\delta \Delta_x \widetilde{H}_S^g(x;u)}{\delta u} \\ & - \sum_{x=\overline{k}}^{M} \frac{\delta w(S \setminus \{l\};u)}{\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x;u)} \cdot \frac{\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x;u)}{\delta u}. \end{aligned}$$

It is sufficient to show that for $x = \overline{k} + 1, \ldots, M$, we have $\delta w(S;u)/\delta \Delta_x \widetilde{H}_S^g(x;u) \cdot \delta \Delta_x \widetilde{H}_S^g(x;u)/\delta u$ is smaller than $\delta w(S \setminus \{l\};u)/\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x-1;u) \cdot \delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x-1;u)/\delta u$. First we show that $\delta \Delta_x \widetilde{H}_{S \setminus \{l\}}^g(x-1;u)/\delta u \geq \delta \Delta_x \widetilde{H}_S^g(x;u)/\delta u$. Either the left and right terms are equal to zero (if u is small enough such that the marginal valuation with respect to buyer m along the path is too small to have any impact) or those terms are respectively equal to $\frac{\delta(\widetilde{V}_m(u) - \widetilde{E}_m(x-1;u))}{\delta u}$ and $\frac{\delta(\widetilde{V}_m(u) - \widetilde{E}_m(x;u))}{\delta u}$. We then have $\frac{\delta(\widetilde{V}_m(u) - \widetilde{E}_m(x-1;u))}{\delta u} = p([V_{l_{T-k}}^{g'} -$

$E^{g'}_{l_{T-k}}(x-1,y_{T-k})] - [V^{g'}_{l_{T+1-k}} - E^{g'}_{l_{T+1-k}}(x-1,y_{T+1-k})]) \geq p([V^{g'}_{l_{T-k}} - E^{g'}_{l_{T-k}}(x,y_{T-k})] - [V^{g'}_{l_{T+1-k}} - E^{g'}_{l_{T+1-k}}(x,y_{T+1-k})]) = \frac{\delta(\widetilde{V}_m(u) - \widetilde{E}_m(x;u))}{\delta u}$ where both equalities come from how we built the path while the inequality is satisfied since preferences are steeper for more efficient buyers. On the whole we obtain the desired inequalities. In Figure 1, this insight is illustrated by the light arrow for the upgrades being longer than the dark arrow. Second, the above key point guarantees that $\delta w(S \setminus \{l\};u)/\delta\Delta_x \widetilde{H}^g_{S\setminus\{l\}}(x-1;u) \leq \delta w(S;u)/\delta\Delta_x \widetilde{H}^g_S(x;u)$. **CQFD**

B Proof of the converse parts

See Figure 2 for an illustration of the proofs.

Proof of Proposition 4.1

If an externality structure is not convex, then there exists $g^* \in G, j \in J$ and $x^* \in [1, M-2]$ such that $(x^*+2) \cdot \mathcal{E}^{g^*}_j(x^*+2) - (x^*+1) \cdot \mathcal{E}^{g^*}_j(x^*+1) < (x^*+1) \cdot \mathcal{E}^{g^*}_j(x^*+1) - x^* \cdot \mathcal{E}^{g^*}_j(x^*)$. We now generate from this externality structure a set of preferences for three potential buyers l, l', l'' such that they are single usage buyers according to the usage g^* while all the (possible) remaining potential buyers will have null preferences and such that the associated coalitional value function is not buyer submodular. Let l and l' be unit demand buyers characterized by the intrinsic valuations $V^{g^*}_l = V^{g^*}_{l'} = V$ and the corresponding externality terms $\mathcal{E}^{g^*}_j(.)$. We then characterize the third buyer l'' valuations by the intrinsic structure: $V^{g^*}_{l''}(y) = V$ if $y \in [1, x^*]$ while $V^{g^*}_{l''}(y) = 0$ if $y > x^*$ and the externality term $\mathcal{E}^{g^*}_j(.)$ for each marginal valuation. Then we chose V such that $V > \max_{u:\ u<x^*+2} (u+1) \cdot \mathcal{E}^{g^*}_j(u+1) - u \cdot \mathcal{E}^{g^*}_j(u)$. In other words, the intrinsic valuations that are positive are set high enough such that the optimal assignment of the M units among any list of marginal buyers is to assign all units to each marginal buyers with a strictly positive marginal valuation. Finally, we obtain $w(\{l,l',l''\}) - w(\{l',l''\}) = V - (x^*+2) \cdot \mathcal{E}^{g^*}_j(x^*+2) + (x^*+1) \cdot \mathcal{E}^{g^*}_j(x^*+1)$ and $w(\{l,l''\}) - w(\{l''\}) = V - (x^*+1) \cdot \mathcal{E}^{g^*}_j(x^*+1) + x^* \cdot \mathcal{E}^{g^*}_j(x^*)$. On the whole $w(\{l,l',l''\}) - w(\{l',l''\}) > w(\{l,l''\}) - w(\{l''\})$ and w is not buyer submodular.

Proof of Proposition 4.2

If a convex externality structure is not uniform, then there exists $g^* \in G, j, j' \in J, j \neq j'$ and $x^* \in [2, M]$ such that $\Delta\mathcal{E}^{g^*}_j(x^*) < \Delta\mathcal{E}^{g^*}_{j'}(x^*)$. We now generate from this externality structure a set of preferences for three potential buyers l, l', l'' such that they are single usage buyers according to the usage g^* while all the (possible) remaining potential buyers will have null preferences and such that the associated coalitional value function is not buyer submodular. Let l and l' be unit demand buyers characterized by the intrinsic valuations

$V_l^{g^*} = V_{l'}^{g^*} = V$ and the externality term $\mathcal{E}_j^{g^*}(.)$. We then characterize the third buyer l'' valuations by the intrinsic structure: $V_{l''}^{g^*}(y) = V$ if $y \in [1, x^* - 2]$, $V_{l''}^{g^*}(x^* - 1) = V^*$ while $V_{l''}^{g^*}(y) = 0$ if $y > x^* - 1$ and the externality term $\mathcal{E}_j^{g^*}(.)$ for the first $x^* - 2$ highest marginal valuations and $\mathcal{E}_{j'}^{g^*}(.)$ for the $(x^* - 1)^{th}$ highest marginal valuation. Then we chose V^* such that $V^* - \mathcal{E}_{j'}^{g^*}(x^*) = (x - 1) \cdot \Delta\mathcal{E}_j^{g^*}(x^*)$ and V is set high enough such that it is always optimal to allocate a unit to each of those marginal valuations. In other words, the intrinsic valuations are set such that: under the coalitions $\{l, l''\}$ or $\{l', l''\}$, we are indifferent between allocating $x^* - 1$ or x^* units, i.e. assigning a $(x^* - 1)^{th}$ unit to buyer l'' is neutral in term of welfare. Under the coalition $\{l''\}$, it is an efficient assignment to allocate $x^* - 1$ units since $V^* - \mathcal{E}_{j'}^{g^*}(x^* - 1) \geq (x^* - 2) \cdot \Delta\mathcal{E}_j^{g^*}(x^* - 1)$ as shown below. We consider the decomposition $V^* - \mathcal{E}_{j'}^{g^*}(x^* - 1) = V^* - \mathcal{E}_{j'}^{g^*}(x^*) + \Delta\mathcal{E}_{j'}^{g^*}(x^*)$. By replacing V^* by its definition and after using $\Delta\mathcal{E}_{j'}^{g^*}(x^*) \geq \Delta\mathcal{E}_j^{g^*}(x^*)$, we obtain $V^* - \mathcal{E}_{j'}^{g^*}(x^* - 1) \geq x^* \cdot \Delta\mathcal{E}_j^{g^*}(x^*)$. Since $x \cdot \Delta\mathcal{E}_j^{g^*}(x) - (x-2) \cdot \Delta\mathcal{E}_j^{g^*}(x-1) = \Delta[\Delta x \cdot \mathcal{E}_j^{g^*}(x)]$ and $x \to x \cdot \mathcal{E}_j^{g^*}(x)$ is convex, we obtain the desired inequality. Under the coalition $\{l, l', l''\}$, the welfare is at least as great of the one raised by the assignment of exactly x^* units. On the whole, the contribution of buyer l to the coalition $\{l, l''\}$ corresponds exactly to the value of replacing the $(x^* - 1)^{th}$ marginal valuation of l'' by buyer l conditional on the event that $x^* - 1$ items are sold: $w(\{l, l''\}) - w(\{l''\}) = [V - \mathcal{E}_j^{g^*}(x^* - 1)] - [V^* - \mathcal{E}_{j'}^{g^*}(x^* - 1)]$. The contribution of buyer l to the coalition $\{l, l', l''\}$ is at least as great as the value of replacing the $(x^* - 1)^{th}$ marginal valuation of l'' by buyer l conditional on the event that x^* items are sold: $w(\{l, l', l''\}) - w(\{l', l''\}) \geq [V - \mathcal{E}_j^{g^*}(x^*)] - [V^* - \mathcal{E}_{j'}^{g^*}(x^*)]$. Finally $\Delta\mathcal{E}_j^{g^*}(x^*) < \Delta\mathcal{E}_{j'}^{g^*}(x^*)$ implies that $w(\{l, l', l''\}) - w(\{l', l''\}) > w(\{l, l''\}) - w(\{l''\})$ and w is not buyer submodular.

Proof of the converse part of Theorem 2 If the externality structure is strongly convex while not ordered, then we can find two externality terms $\mathcal{E}_j^g(.)$ and $\mathcal{E}_{j'}^g(.)$ for a given usage $g \in G$ such that $\mathcal{E}_j^g(.) \neq \mathcal{E}_{j'}^g(.)$ while neither term is steeper than the other one. We can thus use exactly the same construction as for the proof of Proposition 4.2. The set of preferences that are generated in this way are also SFME-generated by our externality structure since SFME-generation do not impose any additional constraint compared to standard generation when neither of the underlying externality terms are not steeper than the other one.

We are then left with the case where $G > 1$ and a strongly convex ordered externality structure that is not uniform. There exists $g^* \in G, j, j' \in J, j \neq j'$ and $x^* \in [2, M]$ such that $\Delta\mathcal{E}_{j'}^{g^*}(x^*) < \Delta\mathcal{E}_j^{g^*}(x^*)$. Furthermore, for those $g^* \in G, j, j' \in J, j \neq j'$, we pick the highest possible x^*. We now SFME-generate from this externality structure a set of preferences for three potential buyers l, l', l'' such that l and l' are unit demand buyers respectively according to the usage g^* and an alternative usage $g' \neq g^*$ (which exists since $G > 1$), while all the (possible) remaining potential buyers will have null preferences and such that the associated coalitional value function is not buyer submodular. Let l and l' be characterized by the intrinsic valuations $V_l^{g^*} = V^*$ and $V_{l'}^{g} = V_+$ and the respective externality terms $\mathcal{E}_j^{g^*}(.)$ and $\mathcal{E}_j^{g'}(.)$. We then characterize the third buyer l'' valuations by the intrinsic structure (buyer l'' is not a single usage buyer, but consider both usages g^* and g' while having null preferences according to any (possible) remaining usages): $V_{l''}^{g^*}(1) = V$, $V_{l''}^{g^*}(y) = 0$ if $y > 1$ and $V_{l''}^{g^*}(y) = V_+$ if $y \leq M - x^*$, $V_{l''}^{g^*}(y) = V_+$ if $y > M - x^*$ while the externality terms are given by $\mathcal{E}_{j'}^{g^*}(.)$ and $\mathcal{E}_j^{g'}(.)$ for any marginal valuation respectively according to usage g^* and g'. Then we chose V_+, V, V^* in the following way. V is given by $V - \mathcal{E}_{j'}^{g^*}(x^*) = (x-2)\Delta\mathcal{E}_{j'}^{g^*}(x^*) + \Delta\mathcal{E}_j^{g^*}(x^*)$. In other words, within the coalition $\{l, l''\}$ we are indifferent between assigning x^* or $x^* - 1$ units to usage g^*. Since marginal valuations according to usage g^* are satisfying all convexity and steepness conditions that guarantees that the underlying maximization program is concave, then this indifference guarantees that those are optimal assignments. We then set V^* such that $V^* - \mathcal{E}_j^{g^*}(x) \geq V - \mathcal{E}_{j'}^{g^*}(x)$ for any $x \in [1, M]$ which guarantees that the preferences are SMFE-generated by the externality structure and we set V_+ high enough such that it is always optimal to assign a unit to all positive marginal valuations according to usage g'. In this way, we obtain that the optimal assignment among the coalition $\{l''\}$ is x^* units to usage g^* and the remaining $M - x^*$ units to usage g' (it comes from the point that the externality term $\mathcal{E}_j^{g^*}(.)$ is steeper than $\mathcal{E}_{j'}^{g^*}(.)$). When buyer l' is added to either coalitions $\{l''\}$ or $\{l, l''\}$, then the optimal assignment is to allocate $x^* - 1$ units to usage g^* and $M - x^* + 1$ to usage g' since V_+ is set high enough. Finally, the contribution of buyer l to the coalition $\{l, l', l''\}$ is given by $w(\{l, l', l''\}) - w(\{l', l''\}) = [V^* - \mathcal{E}_j^{g^*}(x^* - 1)] - [V - \mathcal{E}_{j'}^{g^*}(x^* - 1)]$, the contribution of buyer l to the coalition $\{l, l''\}$ is given by $w(\{l, l''\}) - w(\{l''\}) = [V^* - \mathcal{E}_j^{g^*}(x^*)] - [V - \mathcal{E}_{j'}^{g^*}(x^*)]$. Finally $\Delta\mathcal{E}_j^{g^*}(x^*) < \Delta\mathcal{E}_{j'}^{g^*}(x^*)$ implies that $w(\{l, l', l''\}) - w(\{l', l''\}) > w(\{l, l''\}) - w(\{l''\})$ and w is not buyer submodular.

C Proof of Proposition 6.4

If a convex externality structure is not uniform, then there exists $g^* \in G, j, j' \in J, j \neq j'$ and $x^* \in [2, M]$ such that $\Delta\mathcal{E}_j^{g^*}(x^*) < \Delta\mathcal{E}_{j'}^{g^*}(x^*)$. Let $A = \Delta\mathcal{E}_{j'}^{g^*}(x^*) - \Delta\mathcal{E}_j^{g^*}(x^*) > 0$. We now generate a set of preferences for three potential buyers l, l', l'' such that l' and l'' are losing bidders that can benefit from a joint deviation. Let $g' \neq g^*$ which exists since $G > 1$. We define buyer l preferences such that $V_l^{g^*}(y) = V^*$ and $E_l^{g^*}(., y) = \mathcal{E}_j^{g^*}(.)$ for any

$y \in [1, x^*]$, $V_l^{g^*}(y) = 0$ for $y > x^*$, $V_l^{g'}(y) = V$ and $E_l^{g'}(., y) = \mathcal{E}_j^{g'}(.)$ for any $y \in [1, M - x^*]$, $V_l^{g'}(y) = 0$ for $y > M - x^*$ and $V_l^g(.) \equiv 0$ for $g \neq g', g^*$. V is chosen high enough such that $V - \mathcal{E}_j^{g'}(M - x^* + 1) \geq (M - x^*) \cdot \Delta\mathcal{E}_j^{g'}(M - x^* + 1)$, i.e. such that it is always optimal to assign $M - x^* + 1$ units in presence of $M - x^* + 1$ marginal valuation $(V, \mathcal{E}_j^{g'}(.))$. V^* is chosen such that $V^* - \mathcal{E}_j^{g^*}(M - x^* + 1) = (M - x^*) \cdot \Delta\mathcal{E}_j^{g'}(M - x^* + 1) + [V - \mathcal{E}_j^{g'}(M - x^* + 1) - (M - x^*) \cdot \Delta\mathcal{E}_j^{g'}(M - x^* + 1)]$. Let l' and l'' be unit demand buyers respectively according to usages g^* and g' characterized by $V_{l'}^{g^*} = V^* - \epsilon$, $E_{l'}^{g^*}(.) = \mathcal{E}_j^{g^*}(.)$, $V_{l''}^{g'} = V - \epsilon$ and $E_{l''}^{g'}(.) = \mathcal{E}_j^{g'}(.)$ where ϵ is chosen such that $\epsilon < A/2$. The choices of V^* and V and $\epsilon > 0$ guarantees that the efficient assignment is to give all units to buyer l, more specifically x^* units to usage g^* and $M - x^*$ units to usage g'. Now consider that bidders l' and l'' jointly deviate: more specifically bidder l' reports the unit demand preferences characterized by $V_{l'}^{g^*} = V^* - \mathcal{E}_j^{g^*}(x^*) + \mathcal{E}_{j'}^{g^*}(x^*)$ and $E_{l'}^{g^*}(.) = \mathcal{E}_{j'}^{g^*}(.)$ while bidder l'' reports the unit demand preferences characterized by $V_{l''}^{g'} = V$ and $E_{l''}^{g'}(.) = \mathcal{E}_j^{g'}(.)$. Then bidders l' and l'' both obtain a unit and their joint benefit is strictly positive in the Vickrey auction as developed below. The price paid by l' is equal $V^* - \mathcal{E}_j^{g^*}(x^* - 1)$ thus that his final net payoff is equal to $-\epsilon$ (if bidder l' is removed then an efficient allocation if to assign $x^* - 1$ units to buyer l for usage g^*, $M - x^*$ units to buyer l for usage g' and one unit to buyer l'' for usage g'). The price paid by l'' is equal to the externality he imposes on the other bidders: let Z denote the externality he impose on bidder l alone, then the externality he imposes on the coalition $\{l, l'\}$ is equal to $Z - A$, i.e. by reporting a smaller externality differential when we move from $x^* - 1$ to x^* than his true externality differential bidder l' relief the price that l'' has to pay. The payoff from the final assignment for bidder l'' is given by $Z - \epsilon$ (if his true preferences were given by $V_{l''}^{g'} = V$ and $E_{l''}^{g'}(.) = \mathcal{E}_j^{g'}(.)$ then our choice for V^* is such that his payoff would correspond exactly to the externality he imposes on the other bidders). On the whole, his final net payoff is given by $A - \epsilon$. The total benefit of the losing bidders is given by $A - 2\epsilon > 0$

D Proof of Proposition 6.5

We prove indeed a much stronger robustness property to losing bidders' deviations: in any deviation, all deviators obtain a negative payoff.

Let S^{win} and S^{los} denote respectively the set of winning bidders and the set of losing bidders under truthful reporting. Under truthful reporting θ, let $x^* = \mathcal{X}_N(\theta)$ the total number of units sold. Assigning additional units is not optimal which means that for any marginal valuation (l, y) that is not a winning marginal valuation, i.e. $(l, y) \neq (\sigma_N(x), \gamma_N(x))$ for any $x \leq x^*$, we have

$$V_l(y) - E_l(x^* + 1, y) \leq \sum_{i=1}^{x^*} \Delta_x E_{\sigma_N(i)}(x^* + 1, \gamma_N(i)).$$

Since the functions $E_k(., y)$ are assumed to be convex, we thus obtain that:

$$V_l(y) - E_l(x^* + 1, y) \leq \sum_{i=1}^{x^*} \Delta_x E_{\sigma_N(i)}(u, \gamma_N(i)), \text{ for any } u > x^*. \tag{18}$$

Consider a losing bidders' deviation that leads to the joint report $\widehat{\theta}$ and let $\widehat{x}^*$ denote the number of units assigned to the bidders in S^{win}. Consider then a (previously) losing bidder $l \in S^{los}$ that now obtains $k > 0$ units. We show that the price he has to pay is greater that his private payoff from the assignment.

First event: $\widehat{x}^* + k \leq x^*$. When we remove bidder l from the coalition N, consider the allocation where all units assigned to l are now reassigned to the marginal buyers $(\sigma_N(i), \gamma_N(i))$ for $i = \widehat{x}^* + 1, \ldots, \widehat{x}^* + k$. Finally, the externality that bidder l imposes on the other bidders is greater than $\sum_{i=\widehat{x}^*+1}^{\widehat{x}^*+k} [V_{\sigma_N(i)}(\gamma_N(i)) - E_{\sigma_N(i)}(\mathcal{X}_N(\widehat{\theta}), \gamma_N(i))]$ which is greater than bidder l private payoff $\sum_{i=1}^{k} V_l(i) - E_l(\mathcal{X}_N(\widehat{\theta}), i)$ from the non-crossing assumption and since l is a losing bidder under truthful reporting.

Second event: $\widehat{x}^* + k > x^*$. When we remove bidder l from the coalition N, consider the allocation where $x^* - \widehat{x}^*$ units are reassigned to the marginal buyers $(\sigma_N(i), \gamma_N(i))$ for $i = \widehat{x}^* + 1, \ldots, x^*$ while the remaining $k - (x^* - \widehat{x}^*)$ units that were previously assigned to l are now reassigned to the seller. Finally, the externality that bidder l imposes on the other bidders is greater than $\sum_{i=\widehat{x}^*+1}^{x^*} [V_{\sigma_N(i)}(\gamma_N(i)) - E_{\sigma_N(i)}(\mathcal{X}_N(\widehat{\theta}), \gamma_N(i))] +$ $\sum_{i=1}^{x^*} [E_{\sigma_N(i)}(\mathcal{X}_N(\widehat{\theta}), \gamma_N(i)) - E_{\sigma_N(i)}(\mathcal{X}_N(\widehat{\theta}) - [k - (x^* - \widehat{x}^*)], \gamma_N(i))]$. From the inequalities (18), we have that the second term in the sum is greater than $[k - (x^* - \widehat{x}^*)] \cdot$ $\sum_{i=1}^{x^*} \Delta_x E_{\sigma_N(i)}(x^* + 1, \gamma_N(i))$. On the whole this externality is greater than $\sum_{i=1}^{k} V_l(i) - E_l(x^*, i)$ which is greater than the private payoff under the joint deviation $\sum_{i=1}^{k} V_l(i) - E_l(\mathcal{X}_N(\widehat{\theta}), i)$ (in this second event note that $\mathcal{X}_N(\widehat{\theta}) \geq x^*$).

Proof of the converse part

Take $j \in J$ such that $\mathcal{E}_j(.)$ is not convex, then there exists $x^* \in [1, M - 2]$ such that $A = [\Delta\mathcal{E}(x^* + 1) - \Delta\mathcal{E}(x^* + 2)] > 0$. We now generate a set of preferences for three potential buyers l, l', l'', while all the (possible) remaining potential buyers have null preferences, and such that l' and l'' are losing bidders that can benefit from a joint deviation. Bidder l's valuations are characterized by: the intrinsic valuation terms $V_l(y) = V$ if $y \in [1, x^*]$ and $V_l(y) = 0$ if $y > x^*$ and the externality terms $E_j(., y) = \mathcal{E}_j(.)$ for any $y \in [1, M]$. Let l' and l'' be unit demand buyers characterized by the intrinsic valuations

$V_{l'} = V_{l''} = V^*$ and the externality term $\mathcal{E}_{j'}(.) \equiv 0$. Then we chose V such that $V > \max_{u:\ u<x^*+2} (u+1) \cdot \mathcal{E}(u+1) - u \cdot \mathcal{E}_j^{g^*}(u)$ and V^* such that $x^* \cdot \mathcal{E}(x^*+2) < V^* < x^* \cdot (\Delta\mathcal{E}(x^*+2) + \Delta\mathcal{E}(x^*+1))/2$ (which is possible since $A > 0$). The efficient assignment among the coalition N is to assign x^* units to bidder l while l' and l'' are losing bidders (it comes from our choices for V^*). Now consider that bidders l' and l'' jointly deviate: more specifically bidder l' and l'' report unit demand preferences with a high intrinsic valuation, e.g. V, and no allocative externality such that the final efficient assignment is to assign x^*+2 units among the coalition N and x^*+1 among the coalitions $N \setminus \{l'\}$ and $N \setminus \{l''\}$. On the whole the externality that either l' and l'' imposes on the coalition N is equal to $x^* \cdot \Delta\mathcal{E}(x^*+2)$ such that their net final payoffs are equal to $V^* - x^* \cdot \mathcal{E}(x^*+2)$ and are thus strictly positive.

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