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OPTIMAL PROCUREMENT MECHANISMS

by

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Abstract

The procurement of supplies is often conducted through the buyer analogue of an auction. Sealed bids are submitted and the contract is awarded to the lowest bidder. Although this method may be an optimal way of *selling* an object, an additional complication arises in the case of purchasing a good. When sellers are privately informed about the quality of the good to be sold, these mechanisms typically result in the provision of the lowest quality object. This paper characterizes optimal mechanisms in environments where sellers are privately informed about quality. It shows that the commonly used auction mechanism is privately or socially optimal in only a small class of environments. In another plausible set of environments the optimal mechanism is simply to order potential suppliers and to tender take-it-or-leave-it offers to each sequentially. We use the duality theorem of linear programming to provide a methodology by which necessary and sufficient conditions can be derived to determine when any incentive compatible trading environment maximizes social or private surplus.

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1 Introduction

The efficiency and simplicity of auctions is a major reason for their popularity both in theoretical research and practical implementation. First price or uniform price bidding systems are common ways both to purchase goods and services and to sell them. The U.S. Department of Defence, for example, procures as much as thirty percent of its supply needs via what amounts to a first price sealed bid auction. Unfortunately, while auctions are efficient and (if properly conducted) profit-maximizing for certain environments with incomplete information, in others, they can be strikingly inefficient.

In this paper, we illustrate that if quality uncertainty is a part of the environment, auctions are only rarely the best mechanism. In fact, in a significant class of environments, the best mechanism is to ignore the possibility of competition among potential suppliers and simply to choose one supplier to provide the good in all events. More generally, we provide a methodology by which necessary and sufficient conditions can be derived to determine when any incentive compatible trading environment maximizes either social surplus or private surplus. We focus on the cases of reserve price auctions and sequential take-it-or-leave-it mechanisms but our techniques can be applied to a much wider class of institutions. The approach uses the duality theorem of linear programming. Although it has long been recognized that such optimal mechanism design problems are linear programs, to our knowledge this is the first direct application of the duality theorem to this problem.

Our model generalizes an independent private values auction environment. Typically, the private information of participants on one side of the market is not directly related to the valuation of the single uninformed seller (or buyer) on the opposing side. In the case in which a single seller attempts to sell his product to one of many buyers, this formulation may be convincing – a seller may well be expected to understand her opportunity costs of yielding a good in a trade. However, in the alternative formulation, where a single buyer attempts to obtain an object, it is no longer so obvious that the private information of the provider, which may include the marginal costs of provision or her use value of the good, are independent of the valuation of the buyer. We allow the valuations of the (informed) seller and the (uninformed) buyer to depend on each other and ask what will be the optimal trading mechanism in this case. The problem can be examined with differing objective functions – maximizing ex ante social surplus, ex ante aggregate seller surplus, and ex ante buyer surplus.

Much of the literature on procurement we are aware of concentrates on moral hazard issues, typically issues of incentives for investment in research and development or some other costly effort which may affect quality. (See Rob (1986), Laffont and Tirole (1991) or Lang and Rosenthal (1991)). Our analysis is concerned only with the adverse selection aspect of procurement and, as such, is most closely related to papers by Myerson (1981) and Samuelson (1983). In an environment where an uninformed seller faces many privately informed buyers, Myerson (1981) characterizes the now well-known results on incentive compatibility and individual rationality in direct revelation mechanisms which we exploit. He also provides a sufficient condition for when a reserve price auction maximizes the expected surplus of an uninformed agent. Samuelson examines the case of a single uninformed buyer and an informed seller and describes the class of mechanisms which maximize ex ante social surplus. We develop a technique which provides necessary and sufficient conditions which can be used to determine when *any* given trading institution is optimal according to either criterion, ex ante social surplus or private surplus of the uninformed agent. In doing so, we illustrate how small is the class of environments in which auctions are optimal and suggest that very often relatively non-competitive institutions perform better.

2 An Example

Consider an environment where a potential buyer attempts to purchase a good of uncertain quality from many potential suppliers via an auction where the lowest bidder wins the contract. If the bids submitted are increasing in the quality of the good, then while an auction may ensure trade, it also ensures trade always with the low-quality provider. When quality is important either socially, or privately to the buyer, such an outcome can be highly undesirable as the following example illustrates.

A potential buyer wishes to purchase a single object from one of S potential suppliers. Each supplier has privately known opportunity cost, q_s, which is, ex ante, distributed uniformly and independently over $I = [0,1]$. An object from a seller with cost, q_s, generates a use value for the buyer of $v(q_s) = 1/2 + 3/2q_s$ and, therefore, increases social surplus by $w(q_s) = v(q_s) - q_s = (1 + q_s)/2$. If the allocation mechanism is conducted by one of many possible auction-like institutions which allocate the trade to the lowest bidder, then trade will always occur with the lowest cost and lowest quality seller. A simple computation reveals that any such procedure generates expected social surplus, $(1 + 1/(S+1))/2$. As S becomes large, then the procedure almost certainly ensures trade with the lowest possible quality seller and expected social surplus converges to $1/2$.

The poor performance of the auction suggests that we might look for other simple mechanisms which dominate the auction. For example, if a single seller is chosen at random and offered a (credible) take-it-or-leave-it price of one, then it is subgame perfect for the seller to accept and the expected social surplus is $3/4$ independent of S. In fact, as Theorem 4.2 illustrates, this mechanism achieves the highest possible social surplus in this example.

Why are auctions so popular? Consider a variation on the above where $v(q_s) = 2$ for all q_s. This is the corresponding many-seller-single-buyer analogue of the typical many-buyer- single-seller auction model. In this case, an auction mechanism is socially optimal. It ensures trade with the lowest quality seller as before but now such trade generates the highest possible surplus. In this paper, we extend the intuition in these examples to specify exactly when auctions are optimal procurement procedures and when other simple mechanisms are optimal.

We generalize the above examples to a broad range of environments. The economic interpretation covers many situations of adverse selection: a government attempting to purchase goods of undeterminable quality from many potential suppliers; a firm hiring from a pool of workers with differing skills; an insurance company purchasing risk from clients with privately known probabilities of accident. In all of these situations, the private information of a supplier affects her reservation value and may be directly relevant for the ex post valuation of the buyer. It is not necessary that the potential sellers enjoy precise information about the true value of q_s. We will require, though, that their private information be independent of that of other sellers.

The examples in this section motivate an investigation into the best way of procuring supplies. We conduct the analysis using the methodology provided by mechanism design. In this paper, we concentrate on two simple mechanisms. A low-price or auction-like trading mechanism with maximum reserve prices, k_s, generates trade via a second price bidding system with the lowest bidder selling her object at either the second lowest bid or at her reserve price as long as her own bid lies below her reserve price, k_s. Alternatively, a sequential offer institution is one in which the buyer offers a take-it-or-leave-it price of k_1 to seller 1. If it is accepted, trade takes place and the game ends. If it is rejected, the buyer makes an offer, k_2 to seller 2 and so on until all sellers have rejected, in which cases no trade takes place. Both mechanisms are incentive compatible and are easily represented as direct revelation mechanisms. [1] Observe that both of these mechanisms reflect our assumption that the mechanism designer can commit to a mechanism which may result in no trade even though trade might be ex post desirable. We characterize necessary and sufficient conditions that the economic environment must satisfy for the institutions to be optimal. In doing so, an algorithm emerges which indicates how this methodology can be applied to other mechanisms, as well.

The paper is organized as follows. Section 3 describes the environment. Section 4 presents the general linear program which constitutes the basis of our results. In Sections 5 and 6 we provide interpretations of the conditions for when auctions and sequential offer mechanisms are optimal and in Section 7 we provide examples of other types of mechanisms which may be optimal in other environments.

[1]There are many institutions which implement the same equilibrium outcomes as those described here. Some researchers have objected to the mechanism design approach on the grounds that many games which implement these mechanisms also possess Nash equilibria other than those described by the revelation game. (Postlewaite and Schmeidler (1986)). In the case of the mechanisms we analyse though there are well-known institutions for which either a unique Nash or subgame perfect equilibrium exist and which implement the desired direct revelation mechanism. A second price mechanism possesses a unique equilibirium in dominant strategies . The sequential offer game possesses a unique subgame perfect equilibrium.

3 Notation and the Model

We index sellers by s , $s = 1, \ldots, S$ and abuse notation by letting S represent the set of sellers as well; thus $s \in S$. Every seller s observes some private information $q_s \in I = [0,1]$. We will often refer to q_s as quality. Each individual parameter q_s is independently distributed according to a *continuous and strictly positive* density function $f_s(q_s)$; $F_s(q_s)$ represents the cumulative distribution.

For any measurable set $A \subset I$, $\mathbf{1}_A$ represents the indicator function of the set A, that is the function that takes value one if q_s is in A and zero otherwise.

Agents' preferences are defined over money and the use of the good in the standard way: If a seller with quality q_s engages in trade and receives a money transfer of m, her net payoff is given by $m - u(q_s)$. If no trade occurs, then the net payoff is zero. Ordering indices so that $u(\cdot)$ is strictly decreasing and selecting units appropriately, we may assume, without loss of generality, that $u(q_s) = q_s$. Thus, q_s represents the opportunity costs to seller s of parting with her good.

The buyer has a potential use value for a good of quality q_s given by $v_s(q_s)$. If he gives a money transfer of m and receives an object of quality q_s, his net payoff is given by $v_s(q_s) - m$. No trade yields a payoff of zero for the buyer as well. Procuring the object from seller s with quality q_s, generates a *total surplus* of $w_s(q_s) = v_s(q_s) - q_s$.

For technical reasons, we assume that $v_s : I \mapsto \Re$ is essentially bounded ($v_s \in L_\infty(I)$). [2] We identify by $L_{\infty+}$ the non-negative orthant of L_∞.

The vector $q = (q_1, \ldots, q_s) \in I^S$ is a profile of types. Given $q \in I^S$, q_{-s} denotes the projection of q to $I^{S \setminus \{s\}}$. We define $f(q) = f_1(q_1) \times f_2(q_2) \times \ldots \times f_S(q_S)$ and for any $i \in S$, $f_{-i} = \prod_{s \neq i} f_s(q_s)$.

Note that while consumption of the good allows the buyer to perceive the quality of the object, it is assumed that the features of quality we focus on are not verifiable by a court so that it is not possible to contract contingent on an object's true quality.

Definition of Mechanism: A mechanism is a pair of integrable functions (p_s,t_s) for each possible seller $s \in S$, where:

$$(i) \quad p_s : I^S \mapsto I,\ t_s : I^S \mapsto \Re,$$

and

$$(ii) \quad \textstyle\sum_{s=1}^{S} p_s(q) \leq 1, \forall q \in I^S.$$

This functions should be interpreted as follows: if sellers report $q = (q_1, \ldots, q_s)$, the probability that seller s will trade his or her good is $p_s(q)$ and the transfer payment to/from seller s is $t_s(q)$.

[2] In order to use the duality approach we need a space with a non-empty positive orthant, hence the requirement that the relevant functions lie in $L_\infty(I)$

Given a mechanism, the expected utility of seller s from reporting type z_s when her type is q_s is given by

$$\pi_s(z_s \mid q_s) = \int_{I^{S\setminus s}} [t_s(z_s, q_{-s}) - q_s\, p_s(z_s, q_{-s})]\, f_{-s}(q_{-s})\, dq_{-s}$$

and the expected utility for the buyer (when seller s, with type q_s reports $z_s(q_s)$) is

$$\pi_b = \sum_{s=1}^{S} \int_I [p_s(z(q))\, v_s(q_s) - t_s(z(q))]\, f(q)\, dq.$$

where $z(q) = (z_1(q_1),\, z_2(q_2), \ldots,\, z_S(q_S))$.

It follows from the revelation principle, that for any Nash equilibrium of any trading game with outcomes in terms of payments and probability of trade, there exists a direct revelation mechanism with truthtelling as a Nash equilibrium and which generates the same outcome. We, therefore, consider only mechanisms in which agents report truthfully; the mechanism must be incentive compatible (IC). For any $s \in S$,

$$\textbf{(IC)} \quad \pi_s(q_s \mid q_s) \geq \pi_s(x_s \mid q_s), \quad \forall q_s \in I, \forall x_s \in I.$$

In addition, one may require that it be optimal for agents to participate in this mechanism.[3] Without loss of generality, let zero be the value for all agents who do not participate in the mechanism. Thus voluntary participation requires

$$\textbf{(IR)} \quad \pi_s(q_s \mid q_s) \geq 0, \quad \forall q_s \in I\ \forall s \in S,\ \textit{and}\ \pi_b \geq 0.$$

For any $z(q) \in L_\infty(I^S)$, define $T_s : L_\infty(I^S) \mapsto L_\infty(I)$, by

$$T_s z\,(q_s) = \int_{I^{S\setminus s}} z(q_s, q_{-s})\, f_{-s}(q_{-s})\, dq_{-s}$$

Thus, $(T_s p_s)(q_s)$ represents the expected probability of trade of seller s, conditional on her report q_s.

The proof of the following theorem is essentially the proof in Myerson and Satterthwaite (1983).[4]

Theorem 3.1 *Let* $\{p_s\}_{s\,\in\,S}$, $p_s : I^S \mapsto I$ *and* $\sum_{s=1}^{S} p_s \leq 1$. *Then,*

(a) *There exists a collection of transfer functions* $\{t_s(q)\}_{s\in S}$, *such that* $\{p_s, t_s\}_{s\in S}$ *is a mechanism satisfying IC if and only if* $T_s p_s(q_s)$ *is non-increasing in* q_s $\forall s \in S$.

[3] In cases where the mechanism designer has great authority, such a constraint may not be present; the designers may be able to force the buyer to participate in a scheme which yields an expected surplus less than zero. We present the more general framework in which it may pose a restriction. Consideration of problems without the constraint are a straightforward adaptation of the analysis.

[4] Samuelson (1984) provides a clear explanation of the following results for the case where there is only one seller.

(b) *Suppose that* $\forall s \in S$, $T_s p_s$ *is non-increasing in* q_s. *Then there exists a collection of transfer functions* $\{t_s(q)\}_{s \in S}$, *such that* $\{p_s, t_s\}_{s \in S}$ *is a mechanism satisfying IC and IR if and only if*

$$\sum_{s=1}^{S} \int_{I^S} p_s(q)[(v_s(q_s) - q_s)f(q) - F(q_s)f_{-s}(q_{-s})]\, dq \geq 0.$$

(c) *Suppose that* $\forall s \in S$, $T_s p_s$ *is non-increasing in* q_s. *Then the expected buyer surplus is given by*

$$\pi_b = \sum_{s=1}^{S} \int_{I^S} p_s(q)[(v_s(q_s) - q_s)f(q) - F_s(q_s)f_{-s}(q_{-s})]\, dq - \sum_{s=1}^{S} \pi_s(1 \mid 1)$$

Remark: The continuity of f_s is only used in this theorem. Myerson and Satterthwaite assume continuity but their theorem holds as well with piece-wise continuity of f_s, as do the remainder of our results.

As a consequence of Theorem 3.1, finding an optimal mechanism can be reduced to solving a linear optimization problem in infinite dimensional spaces. The decision variables are the probability-of-trade functions $\{p_s\}_{s \in S}$. The objective function will be of the form

$$\sum_{s=1}^{S} \int_{I^S} h_s(q)\, p_s(q)\, dq$$

where

$$h_s(q) = (v_s(q_s) - q_s)f(q) = w_s(q_s)\, f(q),$$

when the objective is to maximize the expected social surplus, and

$$h_s(q) = (w_s(q_s) - \frac{F_s(q_s)}{f_s(q_s)})\, f(q)$$

when the objective is to maximize the expected buyer's surplus. The last case follows from Theorem 3.1 (c), by setting $\pi(1 \mid 1)$ equal to zero and using the fact (Myerson and Satterthwaite 1983) that IC also implies that $\frac{d\pi(q_s|q_s)}{dq_s} \leq 0$. Note that with this formulation we will require that $\frac{F_s}{f_s} \in L_\infty(I)$.

The $\{p_s\}_{s \in S}$ must satisfy some constraints. They must not add up to more than one because only one good is to be traded; the expected probabilities of trade $T_s p_s$ must be non-increasing in q_s because IC must hold. In addition, individual rationality may also be required. The IR constraint will take the form

$$\sum_{s=1}^{S} \int_{I^S} g_s(q)\, p_s(q)\, dq \geq 0$$

where

$$g_s(q) = [w_s(q_s) - \frac{F_s(q_s)}{f_s(q_s)}]\, f(q) \tag{1}$$

when maximizing the social surplus, and

$$g_s(q) = 0$$

in the buyer's surplus case, because the IR constraint is already incorporated in the objective function.

We examine mechanisms that maximize the buyer's expected surplus, the aggregate expected surplus of the S sellers, and total expected surplus. It is straightforward to show that any mechanism which maximizes total expected surplus also maximizes ex ante aggregate seller surplus so we focus only on expected buyer surplus and expected total surplus in what follows.

4 The Linear Program

A few more definitions will help us state the optimization problem. Given a Banach space V, we denote its dual, the space of bounded linear operators on V, by V^*, and we will write $\langle \cdot, \cdot \rangle$ to identify the bilinear mapping from $V \times V^*$ to $\Re$.

We use D to represent the differential operator that assigns its derivative to any function $g : I \mapsto \Re$ differentiable almost everywhere. More precisely,

$$Dg(x) = \frac{dg}{dx}(x)$$

whenever it is defined.

We define

$$H_s = \{p \in L_\infty(I^S) : DT_s p \in L_\infty(I)\}.$$

With the norm $\|p\| = \|p\|_\infty + \|DT_s p\|_\infty$, H_s is a Banach space, similar to a Sobolev space. (See, for instance, Ziemer 1989.) Thus, the operator D is well defined on any element $T_s p$ $\forall p \in H_s$. We denote by $T_s{}^*$ and D^* the adjoint operators of T_s and D respectively.[5] We also define the following convex subset of H_s,

$$G_s = \{p_s \in H_s \mid p_s \geq 0,\ T_s p_s \text{ is piecewise-}C^1\text{, and lower semi-continuous}\}$$

The lower-semicontinuity rules out mechanisms of the form $p_s = \mathbf{1}_{q_s > k_s}$ which do not satisfy incentive compatibility despite having a non-positive derivative almost everywhere.

Given $\{(h_s, g_s)\}_{s \in S}$, consider the following linear program $(\mathcal{P})$:

$$\begin{aligned}
\underset{\{p_s\}_{s \in S}}{Max} \quad & \textstyle\sum_{s=1}^{S} \langle p_s, h_s \rangle \\
s.t. \quad & \\
\textstyle\sum_{s=1}^{S} p_s &\leq 1 \\
DT_s p_s &\leq 0, \forall s \in S \\
\textstyle\sum_{s=1}^{S} \langle p_s, -g_s \rangle &\leq 0. \\
p_s \in G_s, & \quad \forall s \in S
\end{aligned}$$

[5] Given an operator $O : V \mapsto W$, the adjoint O^* of O is the operator $O^* : W^* \mapsto V^*$ defined by $\langle Ox, y \rangle = \langle x, O^* y \rangle$, $\forall x \in V$, and $\forall y \in V^*$.

The Lagrangian corresponding to $\mathcal{P}$ is

$$\mathcal{L} = \langle 1, \gamma \rangle + \sum_{s=1}^{S} \left(\langle p_s, h_s + \alpha g_s - \gamma \rangle - \langle DT_s p_s, \lambda_s \rangle \right)$$

which, using the adjoint operators, can be rewritten as

$$\mathcal{L} = \langle 1, \gamma \rangle + \sum_{s=1}^{S} \langle p_s, h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s \rangle$$

where $\gamma \in L_\infty{}^*(I^S), \lambda_s \in L_\infty{}^*(I) \quad \forall s \in S, \alpha \in \Re$. This representation makes the formulation of the dual program to $\mathcal{P}$ transparent.

The constraint space, L_∞, has a positive cone with non-empty interior. Also, since h_s is in L_∞ and p_s is bounded by 1 $\forall s \in S$, the optimal value is finite. As is common in optimization problems, in what follows, we require that a regularity condition be satisfied.

Regularity Condition: For all s, there exists $z_s \in G_s$, sucht that the constraints in are satisfied with strict inequality.

Remark: The regularity condition is only necessary for the problem of maximizing total surplus since it is straightforward to show that in the case of maximizing buyer surplus, the condition is automatically satisfied. It is easy to find a feasible $z_s \in G_s$, $\forall s \in S$ such that $\sum_{s=1}^{S} z_s < 1$ and $DT_s z_s < 0$. Therefore, whether the regularity condition holds, depends fundamentally on the IR constraint. When maximizing social surplus, this will depend on the characteristics of g_s, defined in (1). If there exists an $\epsilon > 0$ such that $w_s(q_s) \geq \epsilon$, $\forall q_s < \epsilon$, the regularity condition will be satisfied.

It follows from the duality theory for linear programs (Theorem 1 and Corollary 1, page 219, and Theorem 1 page 220, Luenberger 1969) that [6]

Theorem 4.1 *Suppose the regularity condition is satisfied.*

A feasible $\{p_s\}_{s \in S}$ attains the Max in $\mathcal{P}$ if and only if there exists a $\gamma \in L_\infty{}^*(I^S)$, $\lambda_s \in L_\infty{}^*(I) \quad \forall s \in S$, $\alpha \in \Re$ such that

$$\begin{aligned}
\langle 1, \gamma \rangle &= \sum_{s=1}^{S} \langle p_s, h_s \rangle && (2)\\
\langle x, -h_s - \alpha g_s + \gamma + T_s^* D^* \lambda_s \rangle &\geq 0, \quad \forall s \in S,\ \forall x \in F_s, && (3)\\
\langle x, \gamma \rangle &\geq 0, \quad \forall x \in L_\infty(I^S)_+ && (4)\\
\langle y, \lambda_s \rangle &\geq 0, \quad \forall y \in L_\infty(I)_+, \forall s \in S && (5)
\end{aligned}$$

[6] Anderson and Nash 1987 discuss in detail the duality theory for linear optimization problems in infinite dimensional spaces.

Complementary slackness implies, in addition, that if $\{p_s\}_{S \in S}$ is a solution to $\mathcal{P}$ then

$$\begin{aligned}
\textstyle\sum_{s=1}^{S} \langle p_s, \alpha g_s \rangle &= 0, & (6)\\
\langle DT_s p_s, \lambda_s \rangle &= 0 \quad \forall s \in S, & (7)\\
\langle p_s, h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s \rangle &= 0 \quad \forall s \in S, \ \textit{and} & (8)\\
\langle 1 - \textstyle\sum_{s=1}^{S} p_s, \gamma \rangle &= 0. & (9)
\end{aligned}$$

Remark: G_s is pointwise dense in the space of non-increasing, non-negative functions in H_s. Thus, whenever $\{p_s\}_{S \in S}$ attains the maximum in $\mathcal{P}$ over G_s, $\{p_s\}_{S \in S}$ also attains the maximum over the space of non-increasing and non-negative functions in H_s.

Our algorithm to characterize optimal mechanisms relies on the previous theorem. First, we postulate a given mechanism or candidate solution for $\mathcal{P}$. Then, we obtain necessary and sufficient conditions for the existence of the dual operators of Theorem 4.1. This procedure identifies the environments, if there are any, in which the candidate mechanism is optimal.

We now apply this anlysis to the two simple trading mechanisms referred to in the introduction, the sequential offer mechanism and the auction. In the former, a take-it-or-leave-it offer of a given amount k_1 is tendered to the first potential seller $s = 1$, if accepted, the transaction takes place and the game ends. If it is rejected, then another offer, k_2 is made to the next seller $s = 2$. This process continues until one seller accepts the offer. If no seller accepts then no trade takes place.

Several mechanisms will produce the same equilibrium outcomes. For instance, a mechanism may include two types of trivial offers. First, an offer of $k_s = 0$ implies that seller s trades with probability zero and, therefore, seller s could be made her offer at any time during the procedure. To avoid this indeterminacy in the position of the traders, we relabel sellers so that those that never trade come last. Second, when a seller s is assigned a potential offer of one, all sellers that follow will not trade. Hence, any potential offer to $s' > s$ will produce the same outcome. In this case we assume that $k_{s'} = 0$ for all $s' > s$. Finally, if no seller accepts an offer, no trade takes place. To represent this possibility conveniently, we introduce an artificial seller with zero value who, if called to trade receives an offer of one and sells the good for certain.

Formally, we define the canonical representation of the trading mechanism,

Sequential Offer Mechanism: Define the artificial player $\tilde{s}$ with $\tilde{h}_{\tilde{S}} = \tilde{g}_{\tilde{S}} = 0$, $f_{\tilde{S}}(q_{\tilde{S}}) = 1$, and let $k_{\tilde{S}} = 1$. Let $S' = S \cup \{\tilde{s}\}$. A sequential offer mechanism with offered prices of $\{k_s\}_{s \in S'}$, is represented by the probability of trade functions $\{\tilde{p}_s\}_{s \in S'}$ where

$$\tilde{p}_s(q) = \begin{cases} 1 & \text{if } q_s \le k_s, \ q_i > k_i, \ \forall i < s \\ 0 & \text{otherwise} \end{cases}$$

and

(i) $k_s = 0 \Rightarrow s > s'$, $\forall s' \in S'$ with $k_{s'} \neq 0$.

(ii) Define $\hat{s} = min\{s \mid k_s = 1\}$. Then if $i > \hat{s}, k_i = 0$.

Theorem 4.2 *: Suppose h_s and g_s satisfy $h_s(q) = f(q)\tilde{h}_s(q_s)$ and $g_s(q) = f(q)\tilde{g}_s(q_s)$ and $\tilde{h}_s$,$\tilde{g}_s$ in $L_\infty(I)$ for all s. Let $\{\tilde{p}_s\}_{s\in S'}$ be a sequential offer mechanism. Then $\{\tilde{p}_s\}_{s\in S}$ is a solution to $\mathcal{P}$ if and only if there exists $\alpha \geq 0$ such that for any s and $s' \in S'$*

(a) $\langle \mathbf{1}_{q_s \leq r}\, \tilde{p}_s, h_s + \alpha g_s\rangle \leq \langle \tilde{p}_s, h_s + \alpha g_s\rangle \quad \forall r \in [0, k_s]$,

(b) $\langle \mathbf{1}_{q_s \leq r}\, \mathbf{1}_{r' < q_{s'} \leq k_{s'}}\, \tilde{p}_{s'}, h_s + \alpha g_s - h_{s'} + \alpha g_{s'}\rangle \leq 0$, $\forall r \in [0,1]$ *and* $r' \in [0, k_{s'}]$.

(c) $\sum_{s=1}^{S} \langle \tilde{p}_s, \alpha g_s\rangle = 0$, *and* $\sum_{s=1}^{S} \langle \tilde{p}_s, g_s\rangle \geq 0$.

Proof: ($\Rightarrow$) We will use repeatedly the fact that $\langle x, T_s^* D^* \lambda_s\rangle = \langle DT_s x, \lambda_s\rangle$. Let $z \in G_s$ be such that $z \leq 1$ and $T_s\tilde{p}_s z$ is constant over $[0, k_s]$. Let $r' \in [0, k_s]$. By linearity and equation (8),

$$\langle \tilde{p}_s \mathbf{1}_{q_s \leq r'}\, z, h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s\rangle + \\ \langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s\rangle + \langle \tilde{p}_s(1-z), h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s\rangle = 0.$$

Since $(T_s\tilde{p}_s - T_s\tilde{p}_s z)$ is constant over $[0, k_s]$ and zero elsewhere, $\tilde{p}_s(1-z) \in G_s$ and the first and third terms are non-positive, by (3). Therefore, the second term must be non-negative,

$$\langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, \gamma\rangle \leq \langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, h_s + \alpha g_s - T_s^* D^* \lambda_s\rangle.$$

By complementary slackness (7), $\langle \tilde{p}_s \mathbf{1}_{q_s \leq r'}\, z, T_s^* D^* \lambda_s\rangle + \langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, T_s^* D^* \lambda_s\rangle = 0$. The first term is non-positive by (5); the second term must be non-negative. Thus,

$$\langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, \gamma\rangle \leq \langle \tilde{p}_s \mathbf{1}_{r' < q_s \leq k_s}\, z, h_s + \alpha g_s\rangle. \tag{10}$$

Choose any $x \in G_s$ such that $DT_s x \leq 0$. The first order condition (3) and the non-negativity of λ_s imply

$$\langle x, h_s + \alpha g_s\rangle \leq \langle x, \gamma\rangle \tag{11}$$

First, let $x = \mathbf{1}_{q_s \leq r}\, \tilde{p}_s$ for any $r \in [0, k_s]$. Then (11) becomes

$$\langle \mathbf{1}_{q_s \leq r}\, \tilde{p}_s, h_s + \alpha g_s\rangle \leq \langle \mathbf{1}_{q_s \leq r}\, \tilde{p}_s, \gamma\rangle \leq \langle \tilde{p}_s, \gamma\rangle,$$

where the second inequality follows by the non-negativity of γ. By (10) with $r' = 0$ and $z = 1$ the previous inequality implies (a).

Second, let $x = \mathbf{1}_{q_s \leq r} \, \mathbf{1}_{r' < q_{s'} \leq k_{s'}} \, \tilde{p}_{s'}$ for any $r \in [0,1]$. Using (10) for s' with $z = \mathbf{1}_{q_s \leq r}$, (10) and (11) yield (b).

Finally, (c) is formed by a constraint of $\mathcal{P}$ and its complementary slackness condition and it is therefore necessary.

($\Leftarrow$) Using (c), it is clear that $\{\tilde{p}_s\}_{s \in S'}$ is feasible. We will show that if (a)-(c) hold, then (2), (3), (4) and (5) in Theorem 4.1 are satisfied.

We begin with a Lemma establishing the existence of certain functions that will be used to define the variables of the dual problem.

Lemma 1 *Assume (a)-(c) hold. Then, for any $s \in S$ there exists a function $\bar{h}_s \in L_\infty(I)$ such that*

$$\begin{aligned} \langle \mathbf{1}_{q_s \leq r} \, , \bar{h}_s f \rangle &\geq \langle \mathbf{1}_{q_s \leq r} \, , h_s + \alpha g_s \rangle, \; \forall r \in [0, k_s] & (12) \\ \langle \mathbf{1}_{q_s \leq k_s} \, , \bar{h}_s f \rangle &= \langle \mathbf{1}_{q_s \leq k_s} \, , h_s + \alpha g_s \rangle & (13) \\ \bar{h}_{s'}(q_{s'}) \geq \bar{h}_s(q_s) &\geq 0, \; \forall s' \leq s, \; q_s \in [0, k_s], \; q_{s'} \in [0, k_{s'}]. & (14) \end{aligned}$$

Proof of Lemma: Choose any s and let

$$b = Sup_{r \leq k_s} \frac{\langle \mathbf{1}_{q_s \leq r} \, , h_s + \alpha g_s \rangle}{F_s(r)} \geq Inf_{r' \leq k_s} \frac{\langle \mathbf{1}_{r' < q_s \leq k_s} \, , h_s + \alpha g_s \rangle}{F_s(k_s) - F_s(r)} = a \geq 0, \quad (15)$$

where the first inequality follows by letting $r = k_s$ and $r' = 0$ and the second one by (b).

Let $\tilde{r}$ satisfy

$$\langle \mathbf{1}_{q_s \leq k_s} \, , h_s + \alpha g_s \rangle = F_s(\tilde{r}) \, b + (F_s(k_s) - F_s(\tilde{r})) \, a = F_s(\tilde{r}) \, (b - a) + F_s(k_s) \, a. \quad (16)$$

Dividing both sides by $F_s(k_s)$, it is immediate that such an $0 \leq \tilde{r} \leq k_s$ exists.

Define,

$$\bar{h}_s(r) = \begin{cases} b & \text{if } r \leq \tilde{r} \\ a & \text{if } r > \tilde{r} \end{cases}$$

By definition, $\bar{h}_s$ is non-negative and, by definition of $\tilde{r}$, (13) holds. We will now show that $\bar{h}_s$ satisfies (12).

For any $r \in [0, \tilde{r}]$, by definition of b,

$$\langle \mathbf{1}_{q_s \leq r} \, , \bar{h}_s f \rangle = F_s(r) \, b \geq F_s(r) \, \frac{\langle \mathbf{1}_{q_s \leq r} \, , h_s + \alpha g_s \rangle}{F_s(r)} = \langle \mathbf{1}_{q_s \leq r} \, , h_s + \alpha g_s \rangle.$$

For any $r \in [\tilde{r}, k_s]$,

$$\begin{aligned} \langle \mathbf{1}_{q_s \leq r} \, , \bar{h}_s f \rangle &= F_s(\tilde{r}) \, b + (F_s(r) - F_s(\tilde{r})) \, a = F_s(\tilde{r}) \, (b - a) + F_s(r) \, a \\ &= \langle \mathbf{1}_{q_s \leq k_s} \, , h_s + \alpha g_s \rangle - (F_s(k_s) - F_s(r)) \, a \end{aligned}$$

$$
\begin{aligned}
&= \langle \mathbf{1}_{q_s \leq r}, h_s + \alpha g_s \rangle + \langle \mathbf{1}_{r < q_s \leq k_s}, h_s + \alpha g_s \rangle - (F_s(k_s) - F_s(r))\, a \\
&= \langle \mathbf{1}_{q_s \leq r}, h_s + \alpha g_s \rangle + (F_s(k_s) - F_s(r))\, [\frac{\langle \mathbf{1}_{r < q_s \leq k_s}, h_s + \alpha g_s \rangle}{F_s(k_s) - F_s(r)} - a] \\
&\geq \langle \mathbf{1}_{q_s \leq r}, h_s + \alpha g_s \rangle
\end{aligned}
$$

where the first three lines follow by definition of $\bar{h}_s$ and $\tilde{r}$ equation (16); the inequality by definition of a.

Therefore, for $s' < s$, $x_{s'} \in [0, k_{s'}]$ and $x_s \in [0, k_s]$,

$$
\bar{h}_{s'}(x_{s'}) \geq Inf_{r' \leq k_{s'}} \frac{\langle \mathbf{1}_{r' < q_{s'} \leq k_{s'}}, h_{s'} + \alpha g_{s'} \rangle}{F_{s'}(k_{s'}) - F_{s'}(r')} \geq Sup_{r \leq k_s} \frac{\langle \mathbf{1}_{q_s \leq r}, h_s + \alpha g_s \rangle}{F_s(r)} \geq \bar{h}_s(x_s)
$$

where the first and last inequalities follow from the definition of $\bar{h}_s$ and $\bar{h}_{s'}$ and the intermediate one from (b).

QED

We continue with the proof of the Theorem. Let α be as in the hypothesis.

Let γ be

$$
\gamma(q) = \sum_{s=1}^{S} \bar{h}_s(q_s) f(q) \tilde{p}_s(q)
$$

By construction, $\gamma \in L_\infty(I^S)$ and therefore it may be regarded as an element of $L_\infty{}^*(I^S)$. Since $\bar{h}_s$ is non-negative,γ is non-negative, that is (4) is satisfied.

Note that $\langle 1, \tilde{p}_s \bar{h}_s f \rangle = \langle \tilde{p}_s, h_s + \alpha g_s \rangle$. Then, using the definition of γ and condition (c), we obtain

$$
\langle 1, \gamma \rangle = \textstyle\sum_{s=1}^{S} \langle \tilde{p}_s, h_s + \alpha g_s \rangle = \sum_{s=1}^{S} \langle \tilde{p}_s, h_s \rangle.
$$

Thus, the value of the dual equals the value of the primal, equation (2).

Define, $\hat{h} = \bar{h}_{\hat{s}}(1)$. Observe that $\hat{h} = 0$ if $\hat{s} = \tilde{s}$ and, in general, takes on the lowest value that γ ever acquires.

Define,

$$
\lambda_s(q_s) = \begin{cases} \langle \mathbf{1}_{x \leq q_s}, \bar{h}_s f - h_s - \alpha g_s \rangle & \text{if } q_s \leq k_s \\ \langle \mathbf{1}_{k_s \leq x \leq q_s}, \hat{h} f - h_s - \alpha g_s \rangle & \text{if } q_s > k_s \end{cases}
$$

Condition (b) and the definition of $\bar{h}_s$ imply the non-negativity of λ_s for $q_s \in [0, k_s]$. To check the non-negativity of λ_s for $q_s > k_s$, replace s' with $\hat{s}$ in (b) and consider first any $s < \hat{s}$. Since for $r \leq k_s$ we have $\mathbf{1}_{q_s \leq r}\, \tilde{p}_{\hat{S}} = 0$, (b) may be rewritten for $r > k_s$ as,

$$
\begin{aligned}
&[F_{\hat{S}}(k_{\hat{S}}) - F_{\hat{S}}(r')]\, \langle \mathbf{1}_{k_s < q_s \leq r}\, \tilde{p}_{\hat{S}}, h_s + \alpha g_s \rangle \\
&\quad \leq \; [F_s(r) - F_s(k_s)]\, \langle \mathbf{1}_{r' < q_{\hat{S}} \leq k_{\hat{S}}}\, \tilde{p}_{\hat{S}}, h_{\hat{S}} + \alpha g_{\hat{S}} \rangle
\end{aligned}
$$

Dividing both sides by $[F_{\hat{s}}(k_{\hat{s}}) - F_{\hat{s}}(r')]$, and since the expression above holds for all $r' \leq k_{\hat{s}}$, it holds for its infimum a. (a is the infimum for $\hat{s}$ defined in (15).) Thus, since $\hat{h} = \bar{h}_{\hat{s}}(1) = a$,

$$\langle \mathbf{1}_{k_s < q_s \leq r}, h_s + \alpha g_s - \hat{h} f \rangle \leq 0.$$

For $s > \hat{s}$, $k_s = 0$, and (b) directly implies that that the second part of the definition of λ_s is non-negative. Therefore (5) is satisfied.

For any $z \in G_s$, let $\{c_l\}_{l=1}^{\infty}, \{b_l\}_{l=1}^{\infty}$ denote the points of discontinuity of $T_s z$ (if any) in $[0, k_s]$ and $(k_s, 1]$ respectively and define for $x \in [0,1]$,

$$\Delta(x) = \mathrm{Lim}_{\epsilon \mapsto 0} T_s z(x - \epsilon) - \mathrm{Lim}_{\epsilon \mapsto 0} T_s z(x + \epsilon). \tag{17}$$

Integrating by parts and using separability we have

$$\begin{aligned} \langle DT_s z, \lambda_s \rangle \ &= \ \sum_{l=0}^{\infty} \Delta(c_l) \langle \mathbf{1}_{q_s \leq c_l}, \bar{h}_s f - (h_s + \alpha g_s) \rangle + \langle z, h_s + \alpha g_s - \bar{h}_s f \rangle \\ &\quad + \sum_{l=0}^{\infty} \Delta(b_l) \langle \mathbf{1}_{k_s \leq q_s \leq b_l}, \hat{h} f - h_s - \alpha g_s \rangle + \langle z, h_s + \alpha g_s - \hat{h} f \rangle \end{aligned}$$

By the lower semicontinuity of z, the two summation terms are non-negative, call them $K_s z$.

Finally, to prove that λ_s, α and γ satisfy (3), for $z \in G_s$,

$$\begin{aligned} &\langle z, h_s + \alpha g_s - T_s^* D^* \lambda_s - \gamma \rangle \\ &= \ \langle \mathbf{1}_{q_s \leq k_s} z, h_s + \alpha g_s - T_s^* D^* \lambda_s - \gamma \rangle + \langle \mathbf{1}_{q_s > k_s} z, h_s + \alpha g_s - T_s^* D^* \lambda_s - \gamma \rangle \\ &= \ \langle \mathbf{1}_{q_s \leq k_s} z, \bar{h}_s f - \gamma \rangle + \langle \mathbf{1}_{q_s > k_s} z, \hat{h} f - \gamma \rangle - K_s z \\ &\leq \ \langle \mathbf{1}_{q_s \leq k_s} z, \bar{h}_s f - \textstyle\sum_{j=1}^{S} \tilde{p}_j \bar{h}_j f \rangle + \langle \mathbf{1}_{q_s > k_s} z, \hat{h} f - \textstyle\sum_{j=1}^{\hat{s}} \tilde{p}_j \bar{h}_j f \rangle \\ &= \ \sum_{i < s} \langle \mathbf{1}_{q_s \leq k_s} \tilde{p}_i z, (\bar{h}_s - \bar{h}_i) f \rangle + \sum_{i=1}^{\hat{s}} \langle \mathbf{1}_{q_s > k_s} \tilde{p}_i z, (\hat{h} - \bar{h}_i) f \rangle \\ &\leq \ 0, \end{aligned}$$

where the terms disappear by substituting the definition of $T_s^* D^* \lambda_s$, using the fact that $\tilde{p}_{s'} \mathbf{1}_{q_s \leq k_s} = 0$, $\forall s' > s$, and $\mathbf{1}_{q_s \leq k_s} \sum_{i=1}^{S} \tilde{p}_i = \mathbf{1}_{q_s \leq k_s}$. The final inequality follows from the definition of $\bar{h}_s$ and $\hat{h}$, because $\tilde{p}_s = 0$ for $s > \hat{s}$ and by (14) applied also to $\hat{h}$.

QED

Auction: Let $k_s \in [0,1]$, $\forall s \in S$ and order s so that $k_1 \geq k_2 \geq ... k_S$. Low price bidding institutions with reserve prices, $\{k_s\}$ are represented as a direct revelation mechanism by the probability of trade functions $\{\hat{p}_s\}_{s \in S}$ as follows,

$$\hat{p}_s(q) = \begin{cases} 1 & \text{if } q_s < Min\{q_{s'}, k_s\},\ \forall s' \\ 0 & \text{otherwise} \end{cases}$$

This mechanism can be implemented by a second price bidding scheme in which the lowest bidder sells the good at either the price bid by the second lowest bidder if that bid is below the second lowest bidder's reserve price, otherwise, if all other bidders submit bids above their reserve prices, the lowest bidder receives in payment, her reserve price. Auction mechanisms with $k_1 > k_2$ have additional necessary and sufficient conditions which resemble those in Theorem 4.2. For conciseness, we concentrate on the case where $k_1 = k_2$. In addition, the proof is simpler if we assume that $k_1 < 1$. The extension to the more general class requires some additional cases to examine which we leave to the reader. [7]

Theorem 4.3 : *Suppose h_s and g_s satisfy $h_s(q) = f(q)\ \tilde{h}_s(q_s)$, $g_s(q) = f(q)\ \tilde{g}_s(q_s)$ and $\tilde{h}_s$, $\tilde{g}_s \in L_\infty(I)$ for all s. An auction, $\{\hat{p}_s\}_{s\ \in\ S}$, with $k_1 = k_2 < 1$ is a solution to $\mathcal{P}$ if and only if there exists $\alpha \geq 0$ such that for all s and i in S,*

(a) $\langle \mathbf{1}_{q_i \leq k_i}\ \mathbf{1}_{q_s \leq k_s}\ \hat{p}_i z, h_s + \alpha g_s - h_i - \alpha g_i \rangle \leq 0$, *and*
$\langle \mathbf{1}_{q_i \leq k_i}\ \hat{p}_i z, -h_i - \alpha g_i \rangle \leq 0$, $\forall z \in L_\infty(I^S)$.

(b) $\langle \mathbf{1}_{k_s < q_s \leq r}\ , h_s + \alpha g_s \rangle \leq 0$, $\forall r > k_s$

(c) $\sum_{s=1}^{S} \langle \hat{p}_s, \alpha g_s \rangle = 0$, $\langle \hat{p}_s, -g_s \rangle \leq 0, \forall s$.

Proof: $\Rightarrow$ First, we show that for any $x \in G_s$,

$$\langle \mathbf{1}_{q_s \leq k_s}\ x, T_s^* D^* \lambda_s \rangle = 0. \tag{18}$$

Observe that $\hat{p}_s(q)$ can be rewritten as (If $s = 1$, replace $\prod_{i<s}(.)$ with 1.)

$$\hat{p}_s = \mathbf{1}_{q_s \leq k_s}\ [\prod_{i>s} \mathbf{1}_{q_i \geq k_i}\ \prod_{i<s} \mathbf{1}_{q_i \geq q_s}\ + \prod_{i>s} \mathbf{1}_{k_i \geq q_i \geq q_s}\ \prod_{i<s} \mathbf{1}_{q_i \geq q_s}\ ]$$

Thus

$$T_s\hat{p}_s = \mathbf{1}_{q_s \leq k_s}\ [\prod_{i>s}(1 - F_i(k_i)) \prod_{i<s}(1 - F_i(q_s)) + \prod_{i>s}(F_i(k_i) - F_i(q_s)) \prod_{i<s}(1 - F_i(q_s))]$$

Note that $f_s(q_s) > 0\ \forall q_s$ implies $DT_s\hat{p}_s(q_s) < 0$ provided $q_s \leq k_s$. Then, complementary slackness, (7), implies (18).

[7] It will be evident in the proof that the following theorem actually holds for a much broader class of auction-like mechanisms. Theorem 4.3 applies to any similar mechanism so long as $T_s\hat{p}_s$ is strictly decreasing over $[0, k_s]$.

Let z be such that $T_s \hat{p}_s z \in C^1(I)$ and $1 \geq z \geq 0$. Then (8) implies

$$\langle \hat{p}_s z, h_s + \alpha g_s - T_s^* D^* \lambda_s - \gamma \rangle + \langle \hat{p}_s(1-z), h_s + \alpha g_s - \gamma - T_s^* D^* \lambda_s \rangle = 0$$

We may sign both terms to be non-positive using (3) as we did in Theorem 4.2. Therefore,

$$\langle \hat{p}_s z, T_s^* D^* \lambda_s \rangle = \langle \hat{p}_s z, h_s + \alpha g_s - \gamma \rangle,$$

and using (18) with $x = \hat{p}_s z$ we have

$$\langle \hat{p}_s \mathbf{1}_{q_s \leq k_s} z, h_s + \alpha g_s - \gamma \rangle = 0, \forall z, \; with \, T_s(\hat{p}_s \mathbf{1}_{q_s \leq k_s} z) \in C^1(I). \tag{19}$$

The non-negativity of γ and the Lebesgue Convergence Theorem imply the second part of (a).

Let z be such that $T_s(\hat{p}_s \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} z)$, $T_{s'}(\hat{p}_s \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} z) \in C^1(I)$. By (18), $\langle \hat{p}_s \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} z, T_{s'}^* D^* \lambda_{s'} \rangle = 0$. Then, the first order condition (3) applied to s' implies

$$\langle \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} \hat{p}_s z, h_{s'} + \alpha g_{s'} - \gamma \rangle \leq 0 \tag{20}$$

Combining (19) and (20),

$$\langle \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} \hat{p}_s z, h_{s'} + \alpha g_{s'} \rangle \leq \langle \mathbf{1}_{q_s \leq k_s} \mathbf{1}_{q_{s'} \leq k_{s'}} \hat{p}_s z, h_s + \alpha g_s \rangle,$$

The Lebesgue Convergence Theorem implies that the expression above holds for all $z \in L_\infty(I^S)$, yielding condition (a).

(b) Since $\hat{p}_s + \mathbf{1}_{k_s < q_s \leq r} \; (1 - \sum_{i=1}^S \hat{p}_i) \in G_s$, by (3),

$$\begin{aligned}
0 &\leq \langle \hat{p}_s + \mathbf{1}_{k < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), -h_s - \alpha g_s + T_s^* D^* \lambda_s + \gamma \rangle \\
&= \langle \mathbf{1}_{k_s < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), -h_s - \alpha g_s + T_s^* D^* \lambda_s + \gamma \rangle \\
&= \langle \mathbf{1}_{k_s < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), -h_s - \alpha g_s + T_s^* D^* \lambda_s \rangle \\
&\leq \langle \mathbf{1}_{k_s < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), -h_s - \alpha g_s \rangle
\end{aligned}$$

Complementary slackness, (8), yields the first equality, and (9) the second one. The final inequality is established by observing that from (7) and (5),

$$\langle \mathbf{1}_{k_s < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), T_s^* D^* \lambda_s \rangle = \langle \hat{p}_s + \mathbf{1}_{k_s < q_s \leq r} \, (1 - \textstyle\sum_{i=1}^S \hat{p}_i), T_s^* D^* \lambda_s \rangle \leq 0$$

since $T_s(\hat{p}_s + \mathbf{1}_{k_s < q_s \leq r} \, (1 - \sum_{i=1}^S \hat{p}_i))$ is decreasing.

Condition (c) is immediate.

($\Leftarrow$) We show that there exists a solution, α , λ_s and γ , to the dual of $\mathcal{P}$, with the same value as $\mathcal{P}$. Let α be such that (a) through (c) hold.

We first define γ,

$$\gamma = \sum_{s=1}^{S} \hat{p}_s(h_s + \alpha g_s).$$

By (a), γ is non-negative, and using (c), the value of the dual equals the value of the proposed primal.

Second,

$$\lambda_s(q_s) = \begin{cases} 0 & \text{if } q_s \leq k_s \\ \langle \mathbf{1}_{k_s \leq x_s \leq q_s}, -h_s - \alpha g_s \rangle & \text{otherwise} \end{cases}$$

The non-negativity (5) of λ_s follows from (b).

For any $z \in G_s$, let $\{b_l\}_{l=0}^{\infty}$, $b_0 = k_s$ denote the points of discontinuity of $T_s z$ in $(k_s, 1]$ if any. Integrating by parts,

$$\langle DT_s z, \lambda_s \rangle = \sum_{l=0}^{\infty} \Delta(b_l)\langle \mathbf{1}_{k_s \leq q_s \leq b_l}, -h_s - \alpha g_s \rangle + \langle T_s \mathbf{1}_{q_s > k_s}\, z, h_s + \alpha g_s \rangle$$

where $\Delta(b_l)$ is defined as in (17) in Theorem 4.2.

The separability of h_s and g_s implies the second term can be written as $\langle z, h_s + \alpha g_s \rangle$. From (b) and the lower semi-continuity of $T_s z$ it follows that the summation term is non-negative, call it $K_s z$.

We now show that the first order condition (3) holds. Let $z \in G_s$. Using the definition of λ_s,

$$\begin{aligned} &\langle z, h_s + \alpha g_s - T_s^* D^* \lambda_s - \gamma \rangle \\ &= \langle \mathbf{1}_{q_s \leq k_s}\, z, h_s + \alpha g_s - \gamma \rangle + \langle \mathbf{1}_{q_s > k_s}\, z, -\gamma \rangle - K_s z \\ &\leq \langle \hat{p}_s \mathbf{1}_{q_s \leq k_s}\, z, h_s + \alpha g_s - \gamma \rangle + \langle (1 - \hat{p}_s) \mathbf{1}_{q_s \leq k_s}\, z, h_s + \alpha g_s - \gamma \rangle + \langle \mathbf{1}_{q_s > k_s}\, z, -\gamma \rangle \end{aligned}$$

The first term is zero by definition of γ and the last term is negative because γ is non-negative. Consider the second term. Observe that $\mathbf{1}_{q_s \leq k_s}\, (1 - \hat{p}_s) = \mathbf{1}_{q_s \leq k_s} \sum_{i \neq s,\, i \in S} \hat{p}_i$ and $(\sum_{i \neq s,\, i \in S} \hat{p}_i) \times (\sum_{j=1}^{S} \hat{p}_j) r_i = \sum_{i \neq s,\, i \in S} \hat{p}_i r_i$. Therefore, we can rewrite the middle term in the equation above as

$$\begin{aligned} &\langle \mathbf{1}_{q_s \leq k_s} \textstyle\sum_{i \neq s,\, i \in S} \hat{p}_i z, h_s + \alpha g_s - \gamma \rangle \\ &= \langle \mathbf{1}_{q_s \leq k_s}\, z \textstyle\sum_{i \neq s,\, i \in S} \hat{p}_i, h_s + \alpha g_s - \sum_{j=1}^{S} \hat{p}_j (h_j + \alpha g_j) \rangle \\ &= \textstyle\sum_{i \neq s,\, i \in S} \langle \hat{p}_i \mathbf{1}_{q_s \leq k_s}\, z, h_s + \alpha g_s - h_i - \alpha g_i \rangle. \end{aligned}$$

By (a), this term is always non-positive.

QED

5 Optimal Auctions

In this section, we discuss Theorem 4.3. Theorem 4.2 is analyzed in the next section. Necessary and sufficient conditions for an auction with reserve prices, $\{k_s\}$, to maximize buyer surplus and social surplus are given in the following corollaries. We assume through out the section that the regularity condition is satisfied. Given a function z, let $\mathbf{supp}(z)$ represent the support of z.

Corollary 5.1 *A low-price bidding institution with reserve prices,* $\{k_s\}_{s \in S}$, $1 > k_1 = k_2$, *maximizes expected* buyer surplus *if and only if for all s and i in S,*

(a) $[w_s(q_s) - F_s(q_s)/f_s(q_s)]\ f(q) \leq [w_i(q_i) - F_i(q_i)/f_i(q_i)]\ f(q)$ *(a.e.) in* $\mathbf{supp}(\hat{p}_i\ \mathbf{1}_{q_s \leq k_s})$

(b) $E[w_s(q_s) - F_s(q_s)/f_s(q_s) \mid k_s < q_s < r] \leq 0,\ \forall r \in (k_s, 1]$

Corollary 5.2 : *A low-price bidding institution with reserve price,* $\{k_s\}_{s \in S}$, $1 > k_1 = k_2$, *maximizes expected* social surplus *if and only if there is an* $\alpha > 0$ *such that for all s in S,*

(a) $w_s(q_s) + \alpha(w_s(q_s) - F_s(q_s)/f_s(q_s)) \leq w_i(q_i) + \alpha(w_i(q_i) - F_i(q_i)/f_i(q_i))$ *(a.e.) in* $\mathbf{supp}(\hat{p}_i\ \mathbf{1}_{q_s \leq k_s})$

(b) $E[w_s(q_s) + \alpha(w_s(q_s) - F_s(q_s)/f_s(q_s)) \mid k_s < q_s < r] \leq 0,\ \forall r \in (k_s, 1]$

(c) $\sum_{s=1}^{S} \langle \hat{p}_s, \alpha(w_s - F_s/f_s)\rangle = 0, \sum_{s=1}^{S} \langle \hat{p}_s, w_s - F_s/f_s\rangle \geq 0$

Condition (a) illustrates how limited are the environments in which auctions are optimal. When considering buyer's surplus, (a) implies that $w_s(q_s) - F_s(q_s)/f_s(q_s)$ must be non-increasing almost everywhere in $[0, k_s]$; When maximizing social surplus, (and provided the IR constraint does not bind,) an auction will not be optimal unless $w_s(q_s)$ is decreasing over $[0, k_s]$. If the IR constraint binds, for an auction to be optimal there must exists a non-negative α such that $w_s(q_s) + \alpha(w_s(q_s) - F_s(q_s)/f_s(q_s))$ is decreasing in quality over $[0, k_s]$. Condition (b) determines the seller types, $q_s > k_s$, who never trade. After rejecting a seller, s, for submitting a bid above k_s, a further take-it-or-leave-it price of $r > k_s$ must not generate positive surplus. The extra condition in Corollary 5.2 ensures that the buyer's expected surplus, given that he is trading with the lowest seller type in the subset $[0, k_1] \times [0, k_2] \times \ldots \times [0, k_S]$, is non-negative. In the symmetric case, where Q_1 denotes the random variable which is the lowest order statistic of realized seller types, condition (c) requires

$$E[w(Q_1) - F(Q_1)/f(Q_1) \mid Q_1 \leq k] \geq 0.$$

Myerson (1981) shows that $w_s(q_s) - F_s(q_s)/f_s(q_s)$ decreasing everywhere is sufficient for a reserve price auction to maximize buyer surplus. Corollary 5.1 proves that this condition is necessary only over the range $[0, k_s]$ and that the weaker condition (b) need be satisfied over the set of non-trading seller types.

Observe that in the analogue to the many buyer-single seller symmetric auction environment, $w(q) = \bar{v} - q$. Conditions (a) to (c) are automatically satisfied for 5.2 as long as $\bar{v} \geq 1$. The standard result that auction-like mechanisms are, at least, Pareto efficient stems from these special features of the environment.

With such strong conditions for the optimality of an auction, it is natural to ask what other simple mechanisms might solve the optimization problem? In the next section, we discuss the conditions for the optimality of sequential offer mechanisms.

6 Optimal Sequential Offer Mechanisms

The power of the optimal auction theorem lies in the very simple tests that are required to determine when low-price mechanisms are optimal. A similar simplicity also underlies the sequential offer theorem. Corollary 6.1 illustrates that to determine if and when a given sequential offer mechanism is optimal, all that is required is a comparison of the proposed mechanism against a relatively small class of other mechanisms. We state the corollary for the case of maximizing expected social surplus without the IR constraint: the case of maximizing expected buyer surplus is covered by replacing $w_s(q_s)$ with $(w_s(q_s) - F_s(q_s)/f_s(q_s))$; the case of binding IR constraints is discussed later in the section.

Corollary 6.1 *If IR does not bind, a sequential take-it-or-leave-it mechanism with offered prices of* $\{k_s\}_{s \in S'}$*, maximizes the expected* social surplus *if and only if for all* s' *with* $k_{s'} > 0$ *and for all* $s > s'$

(a) $\int_0^{k_{s'}} w_{s'}(q_{s'}) f_{s'}(q_{s'}) dq_{s'} \geq \int_0^{r'} w_{s'}(q_{s'}) f_{s'}(q_{s'}) dq_{s'}, \forall r' \leq k_{s'}$

(b') $E[w_{s'}(q_{s'}) \mid r' \leq q_{s'} \leq k_{s'}] \geq E[w_s(q_s) \mid q_s \leq r], \forall r \leq k_s, \ r' \leq k_{s'}$.

(b") $E[w_{s'}(q_{s'}) \mid k_{s'} \leq q_{s'} \leq \ r'] \geq E[w_s(q_s) \mid q_s \leq r], \forall r \leq k_s, \ r' \leq k_{s'}$.

Proof: Condition (a) corresponds directly to Theorem 4.2(a). Condition (b') comes from 4.2(b) with $s' < s$ since

$$\begin{aligned}
\langle \mathbf{1}_{q_s \leq r} \mathbf{1}_{r' < q_{s'} \leq k_{s'}} \tilde{p}_{s'}, h_{s'} + \alpha g_{s'} \rangle &= \prod_{i<s'} (1 - F_i(k_i)) \int_0^r \int_{r'}^{k_{s'}} w_{s'}(q_{s'}) f_{s'} f_s dq_{s'} dq_s \\
&= F_s(r) \prod_{i<s'} (1 - F_i(k_i)) \int_{r'}^{k_{s'}} w_{s'}(q_{s'}) f_{s'} dq_{s'} \\
&\geq \langle \mathbf{1}_{q_s \leq r} \mathbf{1}_{r' < q_{s'} \leq k_{s'}} \tilde{p}_{s'}, h_s + \alpha g_s \rangle \\
&= \prod_{i<s'} (1 - F_i(k_i)) \int_0^r \int_{r'}^{k_s} w_s(q_s) f_{s'} f_s dq_{s'} dq_s \\
&= (F_{s'}(k_{s'}) - F_{s'}(r')) \prod_{i<s'} (1 - F_i(k_i)) \int_0^r w_s(q_s) f_s dq_s
\end{aligned}$$

Rearranging terms yields (b'). For (b"), since $\tilde{p}_s \mathbf{1}_{q_{s'} \leq k_{s'}} = 0$ for $s' < s$, replace s with s' in 4.2(b) to get

$$\begin{aligned}
\langle \mathbf{1}_{r<q_s\leq k_s}\, \mathbf{1}_{q_{s'}\leq r'}\, \tilde{p}_s, h_{s'} + \alpha g_{s'}\rangle &= \prod_{i<s, i\neq s'} (1 - F_i(k_i)) \int_r^{k_s} \int_{k_{s'}}^{r'} w_{s'}(q_{s'}) f_{s'} f_s dq_{s'} dq_s \\
&= (F_s(k_s) - F_s(r)) \prod_{i<s, i\neq s'} (1 - F_i(k_i)) \int_{k_{s'}}^{r'} w_{s'}(q_{s'}) f_{s'} dq_{s'} \\
&\leq \langle \mathbf{1}_{r<q_s\leq k_s}\, \mathbf{1}_{k_{s'}<q_{s'}\leq r'}\, \tilde{p}_s, h_s + \alpha g_s\rangle \\
&= (F_{s'}(r') - F_{s'}(k_{s'})) \prod_{i<s, i\neq s'} (1 - F_i(k_i)) \int_r^{k_s} w_s(q_s) f_s dq_s
\end{aligned}$$

QED

Since the mechanism includes the 'no-trade' seller, $S+1$, condition (b') implies that every seller type who trades with positive probability yields positive surplus. Condition (a) ensures that any other offer, r' yields lower surplus than the offer, k_s. Notice that if $k_s < 1$ for every seller, condition (a) becomes the same as condition 4.3(b) in the determination of the reserve price in the optimal auction.

Condition (b) contains the substance of the Theorem. It shows that the expected surplus loss from lowering the offer to seller s' (to say, r') must be greater than the expected surplus to be made by making any offer (say r) to any seller, s, farther on the sequence. Sellers who yield the higher conditional expected surplus are made offers earlier. Thus, (b) both helps to determine the order in which offers are to be made and establishes when the simple sequential offer mechanism is the best way to procure the object. Since the alternative mechanisms are feasible, (a) and (b) are clearly necessary conditions. Corollary 6.1 indicates they are also sufficient.

The conditions of Theorem 4.2 become even simpler if sellers are assumed to be symmetric, as in the example in Section 2. Sequential take-it-or-leave- it offers of k to each seller in turn is an optimal mechanism, if and only if

$$E[w(q_1) \mid q_1 \leq k] \geq E[w(q_1) \mid q_1 \leq r],\ \forall r \leq k,$$

$$E[w(q_1) \mid k \leq q_1 \leq r] \leq 0\ \forall r \geq k \textit{ and } E[w(q_1)] \geq 0.$$

That is, the term, expected social surplus conditional on $q_s \leq r$, must be maximized at k and any higher offer must yield non- positive surplus. Since in the example in Section Two, the conditional expected surplus is strictly increasing in r for all $r \in [0,1]$, and since IR is satisfied at $k_1 = 1$, the mechanism which maximizes social surplus is that which consists of a take-it-or-leave-it offer of one to the first (arbitrarily selected) seller.

It is worth emphasizing that the actual number of potential sellers does not play a role in the conditions of Theorem 4.2. In the Section Two example, even if there is a large number of sellers, so the ex ante probability of a high quality seller is very high, and even though high quality is very valuable, the best that can be achieved is the unconditional average quality.

When the IR constraint binds, the interpretations of the conditions are similar to Corollary 6.1, though the general problem of maximizing social surplus (or equivalently, ex ante aggregate seller surplus) is complicated by this constraint. When comparing alternative offers r' and r, the potential costs of violating the individual rationality constraint $E[w_s(q_s) - F_s(q_s)/f_s(q_s) \mid q_s \leq r]$ must be included. To allow for this constraint, a 'pseudo-objective function', $w_s(q_s) + \alpha(w_s(q_s) - F_s(q_s)/f_s(q_s))$, must be constructed where α is the multiplier on this constraint. This 'pseudo-objective function' replaces $w_s(q_s)$ in Corollary 6.1 and the IR constraint itself enters as an additional condition. [8]

In the general case, with $\alpha > 0$ and asymmetric sellers, it is not necessarily true that each seller who trades yields the same expected social surplus. Nor is it necessary that trade with each seller yield the buyer a non-negative expected surplus. An optimal mechanism may involve cross-subsidization —strictly positive buyer surplus from trade with one seller might be used to offset socially optimal but privately suboptimal trade with another seller in a mechanism which maximizes social surplus. Thus, some sellers may be chosen to trade more in order to exploit their usefulness in relaxing this constraint rather than for their direct social contribution. Note that it is possible, then, that offers to some sellers yield the buyer a negative expected surplus while offers to others yield strictly positive surplus. The question arises as to whether a buyer would continue to participate in such a mechanism if the sellers who yield him positive surplus are made offers first. If rejections occur early and the continuation of the game promises a negative expected surplus, the buyer might refuse to continue to participate. This issue can be addressed by adding a further sequential rationality constraint to the optimization problem of the form that the seller offers be ordered so that for every s

$$\sum_{i>s} \langle p_i, w_i(q_i) - F_i/f_i \rangle \geq 0$$

Since this is also a linear constraint it is easily incorporated in the linear program.

Theorem 4.2 sheds some light on a question left open by Samuelson (1984). Samuelson's model can be considered a special case of this model where there is one true seller, $S = 1$ along with the artificial seller. Samuelson's result that take-it-or- leave-it offers maximize the buyer's expected surplus is directly implied by Theorem 4.2. Samuelson also shows that maximizing *social surplus* requires probability of trade functions with at most two steps. Theorem 4.2 indicates precisely when single step mechanisms are optimal and can thus be used to determine when single or double stepped mechanisms are required.

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[8] An example with a sequential offer game in which the IR constraint binds is

(a) $f_s(q_s) = 1, \ \forall s \in S, \ \forall q_s \in I,$

(b) $w(q_s) = \begin{cases} 1/2q_s + 1/8, & \text{if } q_s < 1/2, \\ 1/4, & \text{otherwise} \end{cases}$

Note that this yields sequential offers of $1/2$, $k = 1/2$ and $\alpha = 1$.

To illlustrate the relationship between Theorems 4.2 and 4.3, consider the problem of maximizing social surplus when $w(q) = c$, a constant for all q. In this case, trade with any seller type yields the same social surplus. Furthermore, if c is high enough the IR constraints do not bind. Either a no reserve price auction or a sequential offer mechanism, trading with the first seller with probability one, is optimal. However, for low values of c, the IR constraint is more likely to bind. Since allocation of trade is not an issue in this example, an auction is typically better because such mechanisms are more likely to satisfy the participation constraint — the constraint affects the buyer only and the competition from the sellers in the auction generally yields higher surplus to the buyer.

Theorem 4.2 indicates that to determine when and if a take-it-or-leave-it mechanism offering $\{k_s\}$ is optimal, two tests are required. First, we must evaluate each seller offer against other potential offers to the same seller. These returns are easily computed. Second, the conditional expected surplus lost from any slightly lower offer to any early seller must exceed the conditional expected surplus from any offer to any later seller. The conditions of Theorem 4.2 do not depend directly on the number of potential sellers. Only characteristics of the utility functions and distribution of uncertainty determine whether a sequential offer mechanism is optimal.

7 Other Mechanisms

The usefulness of the approach in Section 4 is limited only by our ability to devise trading institutions which we can represent as direct revelation mechanisms.The proofs of Theorems 4.2 and 4.3 provide insight about how to characterize necessary and sufficient conditions for when any incentive compatible mechanism is optimal. Let $\{p_s\}_{S \in S}$ characterize a direct revelation mechanism which is incentive compatible and construct the expected probability of trade function, $T_s p_s(q_s)$. The proofs suggest that dual operators, λ_s, must be zero where $T_s p_s(q_s)$ is strictly increasing and otherwise be given by

$$\langle x, T_s^* D^* \lambda_s \rangle = \langle x, h_s + \alpha g_s - \bar{h}_s \rangle$$

for an appropriate $\bar{h}_s$. The rest of the algorithm consists of finding conditions on the h and g functions so that the feasibility conditions of Theorem 4.1 are satisfied. In this section, we provide two examples where different mechanisms from those analyzed above are optimal.

Example 1 Consider a hybrid take-it-or-leave-it mechanism in which a buyer makes a take-it-or-leave-it offer to all sellers in turn and, if all reject, then conducts an auction. In this example with two sellers ($s = 1,\ 2$), we use the sufficiency part of the linear programming theorem to show one case in which this constitutes an optimal mechanism:

$$w(q_s) = \begin{cases} 1/2 + 1/2q_s, & \text{if } q_s \in [0, 1/2] \\ 5/4 - q_s & \text{otherwise} \end{cases}$$

and

$$f(q_s) = 1$$

In this example, the necessary and sufficient conditions for a take-it-or-leave-it mechanism to be optimal hold for low q's and those for an auction-like mechanism hold for q's greater than $1/2$.

Claim 7.1 *The mechanism which consists of offering a take-it-or-leave-it price of* $5/8$ *to selller one and* $3/4$ *to seller two and conducting a second price auction in the event of two rejections, maximizes social surplus in the above example.*

Proof: Define $\alpha = 0$, $\langle x, T_s^* D^* \lambda_s \rangle = \langle \mathbf{1}_{q_s \leq k}\, x, w(q_s) - \bar{w} \rangle$ where $k = 1/2$ and $\bar{w} = 2 \int_I \mathbf{1}_{q_s \leq k}\, w(q_s)\, dq_s$, and

$$\gamma(q_1, q_2) = \begin{cases} \bar{w}, & \text{if } q_1 \leq 1/2 \\ \bar{w}, & \text{if } q_1 > 1/2,\ q_2 \leq 1/2 \\ w(q_1), & \text{if } 1/2 < q_1 \leq q_2 \\ w(q_2), & \text{if } 1/2 < q_2 < q_1 \end{cases}$$

It is straightforward to show that the value of the dual to $\mathcal{P}$ equals the value of the primal with this definition. Let x be any function in G_s

$$\langle x, w(q_s) - T_s^* D^* \lambda_s - \gamma \rangle = \langle \mathbf{1}_{q_s \leq k}\, x, 0 \rangle + \langle \mathbf{1}_{q_s > k}\, x, w(q_s) - \gamma \rangle$$

The first term on the right side follows by definition of λ_s and γ. The second term is negative by definition of γ and because $w(q_s)$ is decreasing in q for $q \geq 1/2$. Since $w(q_s)$ is increasing in q for $q \in [0, 1/2]$, for any x such that $T_s x$ is decreasing,

$$\langle D T_s x, \lambda_s \rangle = \langle x, T_s^* D^* \lambda_s \rangle \leq 0.$$

Thus, (p_s, λ_s, γ, α) form a saddle-point of the Lagrangian of $\mathcal{P}$.

The prices are determined by noting that sellers of type $1/2$ must be indifferent between accepting the offered price and waiting for the auction. The expected price from the auction is $3/4$. Seller one can expect her rejection to result in an auction with probability $1/2$ while seller two knows that her rejection results in an auction for sure.

QED

Example 2 Sequential offer mechanisms could also be of the following form: the first seller is made one offer, upon rejection, the second seller is made an offer and if that is rejected, the first seller is made a second, higher offer. The next example shows that this mechanism may also be an optimal mechanism. Let the environment be

$$w_2(q_2) = \begin{cases} 3q_2, & \text{if } q_2 \in [0, 1/2] \\ 0 & \text{otherwise} \end{cases} \quad w_1(q_1) = \begin{cases} 4q_1, & \text{if } q_1 \in [0, 1/2] \\ 1/4 & \text{otherwise} \end{cases}$$

and

$$f_s(q_s) = 1, \forall s$$

Claim 7.2 *The mechanism which consists of offering a take-it-or-leave-it price of* 3/4 *to seller one, if that is rejected an offer of* 1/2 *to seller two and in the event of another rejection an offer of* 1 *to seller one, maximizes social surplus in the above example.*

Proof: The proof follows the same lines as in Claim 7.1. We simply provide definitions of the dual variables and leave the rest to the reader.

$$\alpha = 0$$

$$\gamma(q_1, q_2) = \begin{cases} 1, & \text{if } q_1 \leq 1/2 \\ 3/4, & \text{if } q_1 > 1/2,\ q_2 \leq 1/2 \\ 1/4, & \text{else} \end{cases}$$

$$\langle x, T_s^* D^* \lambda_s \rangle = \langle x, w(q_s) - \gamma \rangle$$

QED

Notice that the mechanism described in Claim (7.2) is from a larger class of mechanisms in which the class of seller types are partitioned into intervals, low seller types are made (appropriately chosen) offers in turn, then higher seller types are made offers in turn and so on until potentially all seller types are made offers. Some reflection will reveal that as the partition is made finer, such mechanisms actually approximate low-price mechanisms.

8 Extensions

It is worth emphasizing that we analyze a pure adverse selection model and that this yields substantial simplicity. The moral hazard problem where quality choice is endogenous would also be of interest. However, our environment may not be the best model for such a study. A natural way to introduce moral hazard would be to allow for an initial, costly investment, e which affects the distribution of quality. Since there is no screening mechanism to enable an uninformed buyer to distinguish among sellers who have invested in different levels of e, a trading mechanism only increases the likelihood that the seller will not actually be the end user of the product. The possibility of a trade exacerbates the incentive to shirk. A more general optimization problem can be constructed which takes this behavior into account but, of course, the linearity of the objective function disappears.

A different issue arises from the question of whether or not the constraint that the buyer can purchase at most one object is reasonable. Although the buyer has use value for a single object, the possibility exists of purchasing one object, learning its true value and, conditional on the discovery that it is in fact of poor quality, proceeding to attempt to purchase another one. Such a mechanism is incentive compatible since, subject to a completed sale, a seller has no incentive to misrepresent the true quality. Since the constraint $\sum_{s=1}^{S} p_s \leq 1$ binds, it is reasonable to conjecture that allowing the buyer to engage in more purchases will increase welfare. Such mechanisms are of interest since a type of optimal search behaviour could result in which a buyer continues to purchase until an acceptable quality object is found. On the other hand, it is fair to note the objection that information revelation in this circumstance relies rather strongly on the fact that the seller who has just sold his good has no incentive not to lie. It may also be unrealistic to expect that government agencies will have the political latitude that would allow them to purchase objects and then dispose of them in the event they turn out to be of low quality.

The techniques developed here can be extended to analyze this possibility and indeed can be applied to any trading institution that can be described as a direct revelation mechanism and to any additional constraints on outcomes that can be incorporated as linear constraints on the probability of trade functions. The marriage of two powerful theorems, the linear programming theorem and the theorem of implementable mechanisms, yields a methodology which we feel can be fruitfully applied to a wide range of adverse selection problems.

9 References

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