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CONVERGENCE TO EFFICIENCY IN A SIMPLE MARKET WITH INCOMPLETE INFORMATION*

by

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Abstract

An independent private values model of trade with m buyers and m sellers is considered in which a double auction sets price to equate revealed demand and supply. In a symmetric Bayesian Nash equilibrium, each trader acts not as a price-taker, but instead strategically misrepresents his true demand/supply to influence price in his favor. This causes inefficiency. We show that the amount by which a trader misreports is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$. By comparison, inefficiency is $O(1/m)$ for a dual price mechanism and $O(1/m^{1/2})$ for a fixed price mechanism. Price-taking behavior and its associated efficiency thus quickly emerge in the double auction despite the asymmetric information and the noncooperative behavior of traders.

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1. INTRODUCTION

A trader who privately knows his own preferences may demand more favorable terms than he is in truth willing to accept. Such behavior, which is the essence of bargaining, may lead to an impasse that delays or lessens the gains from trade. A trader who acts as a price-taker, by contrast, honestly responds to prices with his true demand. Price-taking behavior together with market-clearing prices guarantee efficient allocations. Two assumptions of price theory are many traders and complete information. The former justifies price-taking behavior by diminishing any single trader's impact on prices, and the latter makes plausible the discovery of market-clearing prices. Price theory and its assertion that trade at market-clearing prices is efficient, however, provide insight into a far greater variety of situations than the strength of these assumptions would suggest. The problem is to explain how this is possible.

Our contribution is to consider a finite market in which the rules for trading are explicit, the number of traders can be small, and each trader privately knows his own preferences. We show that strategic noncooperative behavior in this market converges rapidly to price-taking behavior as the number of traders increases. Bilateral trade can be very inefficient because of strategic behavior; we show that this inefficiency quickly becomes inconsequential. Numerical evidence even suggests that markets with as few as twelve traders can be almost fully efficient.

The mechanism we consider is a simple model of a <u>call market</u>. A call market collects bids and offers from traders, constructs supply and demand curves, fixes a market-clearing price, and executes the indicated trades. The daily opening price of each stock listed on the New York Stock Exchange is set by a call market that aggregates the bids and offers that have arrived overnight. Call markets twice a day fix copper and gold prices in London. The Wunsch system of computerized trading conducts periodic call markets in each of the listed stocks of the New York Stock Exchange. Of all the institutions that mediate trade, call markets come closest to operationalizing Marshall's supply-demand diagram, which is so central in microeconomic thought.[1]

An independent private values model. There are m buyers and n sellers who meet to trade units of an indivisible good. Each buyer wants to buy one unit and each seller wants to sell one unit. A trader's preferences are determined by his redemption value.[2] We use v (for value) to denote a buyer's redemption value and c (for cost) to denote a seller's redemption value. A seller's payoff when he trades is the difference p-c between the price p and his cost c. A buyer's payoff when he trades is the difference v-p between his value v and the price p. A trader who fails to trade has a payoff of zero.[3]

Each seller's cost is independently drawn from a distribution F and each buyer's value is independently drawn from a distribution G. While a trader privately observes the draw of his own redemption value, he remains ignorant of those of others beyond the distributions from which they are drawn. The distributions F and G are common knowledge. We assume that F

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[1] See Schwartz (1988) for further discussion of the use of call markets.

[2] This term is suggested by the experimental literature. We use it instead of "reservation value", which connotes a rule of behavior.

[3] The traders are thus risk neutral. Allowing risk aversion complicates the proofs but does not change the essential results (see Rustichini, Satterthwaite and Williams (1990)).

and G are C^1 functions on [0,1] whose respective densities f and g are positive on [0,1].

The mechanism and its incentives. Each of the n sellers submits an offer while each of the m buyers submits a bid. The bids and offers are aggregated into demand and supply functions. The crossing of their graphs determines an interval [a,b] from which a market-clearing price may be selected. The price p is $(1-k)a + kb$, where each choice of $k \in [0,1]$ defines a different mechanism. Trade occurs among buyers who bid at least p and sellers whose offers were no more than p. The market then disperses. This mechanism is called the <u>k-double auction</u> (or <u>k-DA</u>) because it is a two-sided auction of bidding and offering.

An example shows the mechanics of this institution, the incentives it provides, and the inefficiency that may result. Suppose $k = 0.5$, $m = n = 3$, the three buyers bid 0.95, 0.50, and 0.42, and the three sellers offer 0.10, 0.35, and 0.53. As Figure 1.1 shows, any price in the interval [0.42, 0.50] is market-clearing. Given that $k = 0.5$, price is set at 0.46 and two units trade among the sellers who made the two lowest offers and the buyers who made the two highest bids.

Two points deserve emphasis. First, traders can affect the price and this affects their behavior. For example, the buyer who bid 0.50 regrets ex post that he did not bid 0.43, for if he had, then the price would have been 0.425. This possibility causes each trader to strategically shade his offer/bid away from his redemption value. Second, this strategic misrepresentation may lead to inefficiently few units being traded. In this example, the seller who offered 0.53 and the buyer who bid 0.42 did not trade. If, as is plausible, this seller's cost is 0.47 and this buyer's value is 0.51, then efficiency mandates that they should trade. The possibility that a trader's attempt to affect price may backfire and prevent a profitable trade is what limits each trader's misrepresentation.

Equilibrium. Harsanyi's (1967-68) Bayesian model is used to analyze the effect of incentives and incomplete information upon trade. Because a trader only knows the distributions from which the redemption values of others are drawn, he bids against each other trader's rule for selecting his offer/bid as a function of his redemption value. In turn, the others bid against his rule. A trader's strategy is a Lebesgue measurable function that specifies an offer/bid for each of his possible redemption values. A set of strategies, one for each trader, defines a Bayesian-Nash equilibrium if, at each redemption value of each trader, the offer/bid specified by his strategy maximizes his conditional expected utility given that the other traders are using their specified strategies. For simplicity, we only consider equilibria that are symmetric in the sense that all traders on the same side of the market use the same strategy. Let S be the common strategy of sellers, B the common strategy of buyers, and let $\langle S,B \rangle$ denote the use of S by each seller and B by each buyer. Two more restrictions upon $\langle S,B \rangle$ are added in section 2; one implies that neither S nor B is a dominated strategy and the second that trade occurs with positive probability in $\langle S,B \rangle$. "Equilibrium" in the remainder of the paper means a pair $\langle S,B \rangle$ that satisfies these restrictions.

Example. In the 1-DA price is set at the top of the interval of possible market clearing prices. It is the exceptional case in which a seller cannot favorably influence the price at which he trades. Consequently price-taking is his unique dominant strategy, i.e., $\tilde{S}(c) = c$.

For the case of uniform F and G, $B(v) = mv/(m+1)$ is the unique smooth function such that $<\tilde{S},B>$ is an equilibrium (Williams (1990)). While the 1-DA and the 0-DA are exceptional in that traders on one side of the market have no incentive to misreport, the amount by which traders on the other side of the market misrepresent is typical of all k-DAs.

Results. The meaning of price-taking behavior in this model is subtle; passive response to an existing price is meaningless here because price is determined simultaneously by all offers/bids. Nevertheless, if a trader were to ignore the possibility that he might affect price, then his best offer/bid would be his redemption value, for that guarantees he would trade whenever the realized price yields him gains from trade. This is analogous to a trader in a competitive market who takes the market price as given and chooses his purchase to maximize his utility without taking into account the small effect his purchase has on price. Within our model, price-taking behavior is thus honest reporting of one's redemption value.

Our first convergence result describes how quickly price-taking behavior emerges as the market increases in size. Stated here for the simplest case in which $m = n$, the maximal amount by which any trader distorts his redemption value is $O(1/m)$: there exists a $\kappa(F,G)$, independent of m and n, such that

$$v - B(v) < \kappa/m \quad \text{and} \quad S(c) - c < \kappa/m \tag{1.1}$$

for any equilibrium $<S,B>$ in the market of size $m = n$.

The emergence of price-taking behavior as the number of traders increases makes the market increasingly efficient. Our second result describes the rate at which this happens. To make this precise, for any sample of redemption values the <u>potential gain from trade</u> is the total gain that can be achieved by reallocating the n units to those n traders who most highly value them. This is the amount that would be achieved in a k-DA if each trader acted as a price-taker. The *expected potential gain from trade* is the expected value of this random variable. Given an equilibrium <S,B>, the *expected gain from trade* is the total expected gain received by the traders when each follows the prescribed strategy. Finally, the *expected efficiency* of an equilibrium <S,B> is the expected gain divided by the expected potential gain and the *expected inefficiency* is one minus the expected efficiency. Stated here for the simple case of $m = n$, our second convergence result states that the expected inefficiency of any equilibrium is $O(1/m^2)$.

Comparison with rates at which other mechanisms converge to efficiency shows that this is fast. In the final section we compare the k-DA to three other mechanisms including the optimal mechanism, which by construction has the fastest possible rate of convergence. The conclusion is that the k-DA converges as fast as the optimal mechanism for some, but not all, distributions F and G and it strictly dominates the other two mechanisms by converging at a faster rate.

Antecedent work on the k-DA. The k-DA has served as a simple model for investigating trade in finite markets. A competitive market can be modelled as a continuum of sellers whose costs are distributed according to F together with a continuum of buyers whose values are distributed according to G. The competitive price is the solution p* to the supply/demand equation $F(p) = 1-G(p)$. Telser (1978, p.300) used the environment and institution we consider here to investigate the meaning of p* for a finite market consisting of a sample of m buyers' values and m sellers' costs. He assumed price-taking behavior and showed that the sample price converges in distribution to normality with mean p^* and variance $O(1/m)$. Telser related this result to Marshall's (1949, V.I.4, p.273) "great law" that "the larger the market for a commodity the smaller generally are the fluctuations in its price". Our work complements Telser's test of Marshall's law by showing that an increase in market size is significant not only because the samples of redemption values will better approximate the market fundamentals F and G, but also because it increases competition and thus more tightly constrains strategic behavior.

Wilson (1985) initiated study of strategic behavior in the multilateral k-DA. Assuming the existence of equilibria with certain regularity properties, he proved that such equilibria are interim incentive efficient in the Holmstrom-Myerson (1983) sense when the market is sufficiently large. This means that after traders have learned their own redemption values but before they submit offers/bids, it cannot be common knowledge that a change in equilibrium or the institution would be Pareto-improving. Wilson's result thus helps to explain the endurance of simple institutions such as the k-DA.

Satterthwaite and Williams (1989b) and Williams (1990) studied convergence to price-taking behavior in the buyer's bid double auction, which is essentially the same as the 1-DA. This paper subsumes both these papers as special cases and completes them by establishing the rate of convergence to efficiency. In addition, the 1-DA is contrived so that price-taking is a dominant strategy for each seller. For the bilateral case this implies that a unique equilibrium exists because the seller's dominant strategy induces a unique best response from the buyer. This contrasts with the continuum of equilibria that exist in the bilateral k-DA for $k \in (0,1)$ (Satterthwaite and Williams (1989a)). The k-DA is thus sufficiently rich to model the classic intuition that increasing the number of traders resolves the indeterminacy of bargaining when traders are few.

Future directions. A simple model such as ours could be generalized in many ways.[4] Its most distinctive feature is its modeling of the strategic use of private information in the context of a plausible mechanism for organizing trade in a finite market. This suggests that it would be most interesting to vary both the mechanism and the information structure. With regards to the mechanism, we envision a comparative theory of market institutions, analogous to auction theory, that would explain the suitability of a mechanism for a given environment. The comparisons of Section 4 are a modest step towards this goal. As to the information structure, we have worked an example with correlated values in which the k-DA's rates of convergence continue to hold. This suggests that the independent private values model is not a knife-edge case. Akerlof's (1970) famous "market for lemons", however, epitomizes a variety of examples that show how adverse selection can cause market failure when the traded commodity has a common value component. Such examples emphasize the importance of the information structure to market efficiency and suggests the scope of the problem that remains.

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[4] Obvious candidates include the possibility of asymmetric equilibria and the various ways in which market power can persist even as the market increases in size.

2. THE MODEL

The rules of the k-DA. We now define the k-DA in a form that facilitates analysis. List the n+m offers/bids as $s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(n+m)}$.[5] For fixed $k \in [0,1]$, set $p = (1-k)s_{(m)} + ks_{(m+1)}$. Table 2.1 is used to explain exactly who trades at this price. Because there are m buyers, $t + u = m$. If $s_{(m)} \neq s_{(m+1)}$, then $s + u = m$. The supply s at price p therefore equals the demand t (i.e., p is a market-clearing price) whenever $s_{(m)} \neq s_{(m+1)}$. When $s_{(m)} = s_{(m+1)}$, shortages or surpluses may exist at p. In this case the allocation is carried out as far as possible by assigning priority to sellers whose offers were smallest and buyers whose bids were largest. If this does not complete the allocation, then a fair lottery determines which of the remaining traders on the long side of the market trade.

Restrictions on equilibria. In addition to symmetry, we only consider equilibria $\langle S,B \rangle$ such that:

$\{c \mid S(c) < 1\}$ has positive F-measure and $\{v \mid B(v) > 0\}$ has positive G-measure; (2.1)

at every $c, v \in [0,1]$, $S(c) \geq c$ and $B(v) \leq v$. (2.2)

Assumption (2.1) states that it is a positive probability event that traders on one side of the market make offers/bids at which traders on the other side can profitably trade. This rules out "no-trade" equilibria, e.g., $B(v) = 0$ and $S(c) = 1$ for all $c, v \in [0,1]$. Assumption (2.2) rules out equilibria in which traders use dominated strategies. Neither (2.1) nor (2.2) restricts the strategies that are available to any trader as he

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[5] Throughout the paper, $s_{(q)}$ denotes the q<u>th</u> order statistic (i.e., the q<u>th</u> smallest value) in a specified sample of offers/bids.

attempts to maximize his conditional expected utility; rather, they restrict the equilibria for which we prove results.

Basic properties of equilibria. Figure 2.1 depicts an equilibrium $\langle S,B \rangle$. As shown below in Theorem 2.1, this figure typifies equilibria in three respects:

for each $\langle S,B \rangle$, there exists values $\underline{v}$, $\bar{c}$ such that a seller with cost c trades with positive probability if and only if $c < \bar{c}$ and a buyer with value v trades with positive probability if and only if $\underline{v} < v$; (2.3)

S and B are increasing over $[0,\bar{c})$ and $(\underline{v},1]$, respectively; (2.4)

$$\lim_{v \downarrow \underline{v}} B(v) = \underline{v} = \lim_{c \downarrow 0} S(c) \text{ and } \lim_{c \uparrow \bar{c}} S(c) = \bar{c} = \lim_{v \uparrow 1} B(v). \quad (2.5)$$

The intervals $(\underline{v},1]$ and $[0,\bar{c})$ are called the <u>intervals from which serious offers/bids are made</u>. Because a seller whose cost is above $\bar{c}$ almost never trades, he can costlessly submit a large number as his offer; similarly, a buyer whose value is below $\underline{v}$ may bid a negative number. Misrepresentation thus cannot be bounded for values in $[0,\underline{v}]$ and costs in $[\bar{c},1]$. It is over the intervals from which serious offers/bids are made that equilibrium tightly constrains the strategies S and B.

Point (2.4) implies that S and B are differentiable almost everywhere[6] in $[0,\bar{c})$ and $(\underline{v},1]$. This permits the first order approach we use to prove our convergence results. Finally, by tying $\underline{v}$ to the smallest serious offers and $\bar{c}$ to the largest serious bids, (2.5) allows the inefficiency non-serious offers/bids cause to be bounded.

Some notation is needed to make (2.3-2.5) precise. Let λ denote an offer/bid of a trader. Given $\langle S,B \rangle$, define:

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[6] See, for instance, Royden (1968, p.96).

$P_b(\lambda)$ = probability that a buyer trades when he bids λ, all sellers use S, and the other m-1 buyers use B;

$P_s(\lambda)$ = probability that a seller trades when λ is his offer, all buyers use B, and the other n-1 sellers use S;

$\pi_b(v,\lambda)$ = a buyer's expected payoff when v is his value, λ is his bid, all sellers use S, and the other m-1 buyers use B;

$\underline{v}$ = $\inf \{v \mid P_b(B(v)) > 0\}$;

$\bar{c}$ = $\sup \{c \mid P_s(S(c)) > 0\}$;

$\bar{b}$ = $\sup \{B(v) \mid v < 1\}$;

$\underline{s}$ = $\inf \{S(c) \mid c > 0\}$.

Observe that $P_b(\lambda)$ is nondecreasing and $P_s(\lambda)$ is nonincreasing in λ. Assumption (2.1) implies that $P_b(B(v)) > 0$ near $v = 1$ and $P_s(S(c)) > 0$ near $c = 0$. The values $\underline{v}$, $\bar{c}$ are thus well-defined and satisfy $\underline{v} < 1$, $\bar{c} > 0$.

Theorem 2.1. For $c' < c''$, $v' < v''$ in [0,1], the following statements are true for an equilibrium <S,B> that satisfies (2.1) and (2.2):

$P_b \cdot B$ is nondecreasing and $P_s \cdot S$ is nonincreasing on [0,1]; (2.6)

if $c' < \bar{c}$, then $S(c') < S(c'')$; (2.7)

if $\underline{v} < v''$, then $B(v') < B(v'')$; (2.8)

$\lim_{v \downarrow \underline{v}} B(v) = \underline{v} = \underline{s}$ and $\lim_{c \uparrow \bar{c}} S(c) = \bar{c} = \bar{b}$. (2.9)

The proof is in the Appendix.

Theorem 2.1 implies that the outcome of the k-DA is almost always a market-clearing price. Random allocation is necessary only if (i) $s_{(m)} = s_{(m+1)} = p$, (ii) some offers are no more than p, and (iii) some bids are as large as p. Statements (2.7-2.8) imply that (i) occurs with positive probability only for p below $\underline{v}$ or above $\bar{c}$. In either of these events, (2.9) implies that (ii) occurs with probability zero. The possibility of random allocation is therefore ignored in the remainder of the paper.

The dual market. Table 2.2 defines the dual market to the market defined thus far in the paper. Its middle column summarizes our notation. Theorem 2.2 establishes a symmetry between equilibria of a market and its dual. This symmetry means that a bound on misrepresentation by buyers implies a bound for sellers.

Theorem 2.2. If <S,B> is an equilibrium satisfying (2.1) and (2.2) in the given market, then <S*,B*> is an equilibrium in the dual market that also satisfies (2.1) and (2.2).

This follows directly from a change of variable in the integral representation of a buyer's expected payoff. Details can be found in Rustichini, Satterthwaite and Williams (1990).

3. MAIN RESULTS

A buyer's first order condition. The bound on misrepresentation that drives our two convergence results follows from a first order condition for equilibrium. Consider an equilibrium <S,B>. Pick $v \in (\underline{v},1)$, and $c \in (0,\bar{c})$ such that $B'(v)$ and $S'(c)$ both exist and $B(v) = S(c) \equiv \lambda$. A buyer's first order condition at such a (c,λ,v) triple is[7]

$$0 = \frac{\partial \pi_b(v,\lambda)}{\partial \lambda} = (v-\lambda)\left[nK_{n,m}(\lambda)\frac{f(c)}{S'(c)} + (m-1)L_{n,m}(\lambda)\frac{g(v)}{B'(v)}\right] - kM_{n,m}(\lambda), \tag{3.1}$$

[7] A seller's first order condition is similar and can be found in Rustichini, Satterthwaite and Williams (1990).

where:

$K_{n,m}(\lambda)$ = the probability that offer/bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-1 buyers using strategy B and n-1 sellers using S;

$L_{n,m}(\lambda)$ = the probability that bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-2 buyers using strategy B and n sellers using S;

$M_{n,m}(\lambda)$ = the probability that bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-1 buyers using strategy B and n sellers using S.

Formulas (A.6-A.8) in the Appendix define these probabilities.

This equation can be derived by following Satterthwaite and Williams (1989b, Thm. 3.1). It is helpful to develop some intuition here by explaining how (3.1) equates a buyer's marginal expected gain from changing his bid with his marginal expected cost. If the bid λ is too small to include him among those who trade, then by increasing it an incremental amount $\Delta\lambda$ he may surpass other bids and offers and move into the set of buyers who trade. The sum in brackets times $\Delta\lambda$ is the probability that this occurs: the first term in the sum is the marginal probability of acquiring a unit by passing a seller's offer and the second term is the marginal probability of acquiring a unit by passing another buyer's bid. The profit from such a trade is between $(v-\lambda)$ and $(v-\lambda-\Delta\lambda)$. Therefore the marginal expected profit for a buyer who raises his bid is $(v-\lambda)$ times the term in brackets. On the other side of the ledger, if the bid λ is large enough to include him among those who trade, then increasing it by $\Delta\lambda$ may simply increase the price he pays by $k\Delta\lambda$ through the price-setting rule $(1-k)s_{(m)} + ks_{(m+1)}$. $M_{n,m}$ is the probability that the buyer increases the price he pays by $k\Delta\lambda$. Therefore $kM_{n,m}$ is the buyer's marginal expected loss from increasing his bid above λ.

For $v \in [\underline{v},1]$, the first order condition (3.1) of a buyer may fail if either (i) B(v) is outside of the range of S or (ii) S'(c) does not exist for the value of c that solves S(c) = B(v). Nevertheless as long as B'(v) exists the inequality

$$(v-\lambda)(m-1)L_{n,m}(\lambda)\frac{g(v)}{B'(v)} - kM_{n,m}(\lambda) \le 0 \qquad (3.2)$$

holds because in equilibrium the marginal expected gain from passing a buyer (disregarding the possibility of passing a seller) as a result of raising one's bid surely can not exceed the marginal expected cost from raising the price. Theorem 2.2's result that B' exists almost everywhere in $[\underline{v},1]$ therefore implies that (3.2) holds almost everywhere in that interval. This inequality is the basis for our first convergence result.

Convergence to price-taking behavior. The bounds on misrepresentation that Theorem 3.1 establishes are stated in terms of the function

$$q(n,m) \equiv \max\left\{ \frac{1}{n}\left[1+\frac{m}{n}\right], \frac{1}{m}\left[1+\frac{n}{m}\right] \right\}.$$

Informally the theorem states that (i) a trader's equilibrium misrepresentation is O(q(n,m)) on the interval from which he makes serious offers/bids and (ii) the complement of this interval, in which misrepresentation can not be bounded, has length O(q(n,m)). To develop some intuition concerning these rates, consider a sequence of markets in which n/m is bounded both above and away from zero. When n/m is restricted in this way, the equality O(q(n,m)) = O(1/n) = O(1/m) holds and describes the rate at which (i) price-taking behavior emerges on the intervals over which serious offers/bids are made and (ii) these intervals grow to include the entire range [0,1] of possible redemption values.

Theorem 3.1. Suppose F and G are C^1 distributions on $[0,1]$ with positive densities over this interval and $k \in [0,1]$. Consider any equilibrium $\langle S,B \rangle$ of the k-DA in a market with m buyers and n sellers that satisfies (2.1) and (2.2). There exists a constant $\kappa(F,G) > 0$, which is independent of $\langle S,B \rangle$, m, and n, such that

$$v - B(v) \le \kappa q(n,m) \tag{3.3}$$

for all $v \in (\underline{v},1]$,

$$S(c) - c \le \kappa q(n,m) \tag{3.4}$$

for all $c \in [0,\bar{c})$, and

$$\underline{v},\ 1 - \bar{c} \le \kappa q(n,m). \tag{3.5}$$

Proof. We prove below that

$$v-B(v) \le \frac{\tau \cdot k}{m-1}\left[1 + \frac{n}{m}\right] \tag{3.6}$$

for $v \in (\underline{v},1]$, where

$$\tau \equiv 2 \sup_{x \in [0,1]} \max\left\{ \frac{G(x)}{g(x)}, \frac{(1-G(x))F(x)}{g(x)(1-F(x))} \right\}. \tag{3.7}$$

where x is a dummy variable. The rest of the theorem then follows easily. Applying (3.6) to the equilibrium $\langle S^*,B^* \rangle$ of the dual market described in Theorem 2.2 implies

$$S(c)-c = (1-c) - B^*(1-c) \le \frac{\tau^*(1-k)}{n-1}\left[1 + \frac{m}{n}\right] \tag{3.8}$$

for $c \in [0,\bar{c})$, where τ^* is defined by (3.7) for the dual market as

$$\tau^* \equiv 2 \sup_{x \in [0,1]} \max\left\{ \frac{1-F(x)}{f(x)}, \frac{F(x)(1-G(x))}{f(x)G(x)} \right\}. \tag{3.9}$$

To obtain the bounds (3.3-3.4) in the theorem from (3.6) and (3.8), we note that $n/(n-1)$ and $m/(m-1)$ are both less than 2 and set

$$\kappa \equiv 2 \max(\tau k, \tau^*(1-k)). \tag{3.10}$$

To bound $\underline{v}$, apply Theorem 2.1 and the bound just established to deduce that $\underline{v} - \underline{s} = \lim_{c\downarrow 0} S(c) = \lim_{c\downarrow 0}(S(c)-c) \le \kappa q(n,m)$. A similar argument bounds $1-\bar{c}$.

For $v \in (\underline{v},1]$, let $\lambda = B(v)$. Solving (3.2) for v-B(v) gives

$$v-B(v) \le \frac{kM_{n,m}(\lambda)}{(m-1)L_{n,m}(\lambda)g(v)} B'(v), \tag{3.11}$$

which holds for almost all $v \in (\underline{v},1]$. In the Appendix it is shown that

$$\frac{M_{n,m}(\lambda)}{L_{n,m}(\lambda)} \le 2 \left[G(v) + \frac{n}{m} \frac{(1-G(v))F(v)}{1-F(v)}\right], \tag{3.12}$$

from which it follows that

$$v-B(v) \le \frac{\tau \cdot k}{m-1} \left[1 + \frac{n}{m}\right] B'(v) \tag{3.13}$$

for all $v \in (\underline{v},1]$ at which $B'(v)$ exists. This implies (3.6) for v at which $B'(v) \le 1$. The remainder of the proof shows that (3.6) also holds for $v \in (\underline{v},1]$ at which either (i) $B'(v) > 1$ or (ii) $B'(v)$ fails to exist.

We first note the following. Consider an increasing sequence $\{v_i\}$ that has as its limit $v \in (\underline{v},1]$. Suppose $v_i - B(v_i) \le \kappa q(n,m)$ for each element of the sequence. Then $v-B(v) \le \kappa q(n,m)$ because $\lim_{i\to\infty} v_i = v$ and, since B is increasing, $B(v) \ge \lim_{i\to\infty} B(v_i)$.

Consider now a $v'' \in (\underline{v},1]$ at which $B'(v'') > 1$. At some value v in the interval $[\underline{v},v'')$ the derivative $B'(v)$ exists and is not more than one. To see this, recall from Theorem 2.1 that $\lim_{v\downarrow\underline{v}} B(v) = \underline{v}$. If at almost all $v \in (\underline{v},v'')$ the derivative $B'(v)$ were more than one, then at all v in this interval B(v) would exceed v, which violates (2.2). The value $v' = \sup\{v \in (\underline{v},v'') \mid B'(v) \le 1\}$ therefore exists. Our result from immediately above implies that $v'-B(v') \le \kappa q(n,m)$. Almost everywhere in (v',v'') the derivative $B'(v)$ exists and exceeds one. Since B is increasing, it follows that

$$v''-v' \le \int_{v'}^{v''} B'(v)dv \le B(v'') - B(v'),$$

which upon rearrangement gives the desired result: $v''-B(v'') \le v'-B(v') \le \kappa q(n,m)$.

Next consider values of $v \in (\underline{v},1]$ at which $B'(v)$ does not exist. Because B is differentiable almost everywhere, any such value is a limit point of an increasing sequence of values at which B' exists. The desired bound holds wherever B' exists and hence at every value in the sequence. As argued above it therefore holds at v. Q.E.D.

Two observations should be made about the bound (3.6) on buyer's misrepresentation. First, it is increasing in k, which is intuitive because k measures a buyer's potential influence upon the price at which he trades. For $k = 0$, in fact, (3.6) implies that $B(v) = v$, which is the buyer's dominant strategy for the only k-DA in which he cannot influence price to his advantage. Second, formula (3.7) for τ suggests that convergence may lag at values v where $g(v)$ is small. This is intuitive, for a small density means that it is unlikely that the values of other buyers are near v, i.e., competition is less intense. On the other hand, (3.6) does not fully reflect global incentive constraints because it is mainly derived from a first order condition. Perturb G on some small interval $(v-\delta,v+\delta)$ so that g is made very small near v. While this causes (3.6) at v to explode, it may not radically change the equilibria, for the strict monotonicity of B above $\underline{v}$ means that $B(v)$ is constrained by how B is defined in $[\underline{v},v-\delta)$ where (3.6) hasn't changed. One should thus be cautious in using (3.6) for comparative statics.

Convergence to efficiency. The function q(n,m) becomes infinite and Theorem 3.1's bounds become ineffective as n/m approaches either zero or infinity. Consequently our convergence to efficiency result is restricted to sequences in which n/m is bounded both above and away from zero.

Theorem 3.2. Suppose F and G are C^1 distributions on [0,1] with positive densities over this interval and let K > 1. For m and n such that

$$1/K < n/m < K, \tag{3.14}$$

consider equilibria <S,B> of the k-DA with m buyers and n sellers that satisfy (2.1-2.2). There exists a number $\xi(K,F,G)$, independent of m and n, such that the expected inefficiency of any such equilibrium is no more than ξ/m^2.

Proof. A lower bound on the denominator of expected inefficiency is computed by pairing off each of $n \wedge m$ ($\equiv \min(n,m)$) buyers with a seller and noting: (i) the expected potential gain from trade within each pair is some positive number η; (ii) the expected potential gain from trade among all n+m traders is at least as large as the amount that can be achieved through pairwise trading. The expected potential gain from trade is therefore at least $\eta(n \wedge m)$.

The proof is thus reduced to showing that the numerator of expected inefficiency is O(1/m). The idea is as follows. A seller and a buyer inefficiently fail to trade at a given price only if the offer/bid of each is on the wrong side of the price. Because misrepresentation is O(1/m), their redemption values must be within O(1/m) of the price. The value of a missed trade is thus O(1/m), and the expected number of missed trades is bounded by the expected number of the n+m redemption values that lie within

O(1/m) of the price, which can be bounded by a finite number that depends upon F and G but not n and m. For simplicity, the calculation in the formal proof below is conditioned on the mth smallest redemption value (hereafter denoted as $t_{(m)}$) instead of the price. Also, the bounds in this argument are ineffective when $t_{(m)}$ is near zero or one, so the losses in this event are bounded using a separate argument that rests upon the exponentially fast decline of the probability that $t_{(m)}$ is so extreme.

The following notation is needed for the formal proof:

t ≡ a sample of n+m redemption values;

μ ≡ the distribution of $t_{(m)}$;

L(**t**) ≡ the total value of trades that inefficiently fail to occur given **t** and <S,B>.

The goal is to show that E[L(t)] is O(1/m). We write

$$E[L(t)] = \int E[L(t)|t_{(m)}]\, d\mu(t_{(m)}) \tag{3.15}$$

and then bound the value of this integral over the intervals $[0,\varepsilon)$, $[\varepsilon,1-\varepsilon]$, and $(1-\varepsilon,1]$, where $\varepsilon > 0$ is chosen so that

$$F(\varepsilon),\ 1-F(1-\varepsilon),\ G(\varepsilon),\ 1-G(1-\varepsilon) \le (1/2)^{2K+1}. \tag{3.16}$$

The desired bound is then an immediate consequence of the following lemmas:

Lemma 3.1. The probability that $t_{(m)}$ is below ε is $O(2^{-m})$ and the probability that $t_{(m)}$ is above $1-\varepsilon$ is $O(2^{-n})$.

Lemma 3.2. There exists a number $\nu(\varepsilon,F,G)$, independent of $t_{(m)}$, such that $E[L(\mathbf{t})|t_{(m)}] \le \nu(\varepsilon,F,G)/m$ for $t_{(m)} \in [\varepsilon,1-\varepsilon]$.

Because the total value of missed trades is no more than $m \wedge n$, Lemma 3.1 implies that the integral in (3.15) over $[0,\varepsilon)$ and $(1-\varepsilon,1]$ is O(1/m). Lemma 3.2 then provides the needed bound over $[\varepsilon,1-\varepsilon]$. Q.E.D.

Proof of Lemma 3.1. Two inequalities are needed:

$$\max(m,n) \le \min(Km,Kn), \tag{3.17}$$

and

$$\binom{m+n}{m} \le 4^{\max(m,n)}. \tag{3.18}$$

Inequality (3.17) follows immediately from (3.14); (3.18) is proven in the Appendix.

The probability that the redemption values of a specific set of m traders all lie below ε is no more than $[\max(F(\varepsilon),G(\varepsilon))]^m$, and there are $\binom{m+n}{m}$ such sets of traders. The probability that $t_{(m)}$ is below ε is thus no more than

$$\binom{m+n}{m} [\max(G(\varepsilon),F(\varepsilon))]^m \le 4^{\max(m,n)} \frac{1}{2^{(2K+1)m}} = \frac{4^{\max(m,n)}}{4^{Km}} 2^{-m} \le 2^{-m}$$

where the first inequality follows from (3.18) and the choice of ε and the second from (3.17). A similar argument shows that the probability that $t_{(m)}$ lies above $1-\varepsilon$ is $O(2^{-n})$. Q.E.D.

Proof of Lemma 3.2. For the value of κ given by Theorem 3.1, define

$$\gamma \equiv \kappa \cdot K(1 + K). \tag{3.19}$$

Given (3.14), it follows that

$$v - B(v) < \gamma/m \text{ for } v \in (\underline{v},1], \tag{3.20}$$

$$S(c) - c < \gamma/m \text{ for } c \in [0,\bar{c}), \text{ and} \tag{3.21}$$

$$\underline{v},\ 1-\bar{c} \le \gamma/m. \tag{3.22}$$

We restrict our attention to values of m sufficiently large that $\gamma/m < \varepsilon$, so that $\underline{v} \in [0,\varepsilon)$ and $\bar{c} \in (1-\varepsilon,1]$.

We first bound the value of a missed trade given a sample **t** for which $t_{(m)} \in [\varepsilon,1-\varepsilon]$. Inequalities (3.20-3.21) imply

$$t_{(m)} - \gamma/m \le s_{(m)} \le t_{(m)} + \gamma/m. \tag{3.23}$$

A buyer with value v and a seller with cost c inefficiently fail to trade only if:

$B(v) \le s_{(m)}$, so the buyer does not trade; (3.24)

$S(c) \ge s_{(m+1)} \ge s_{(m)}$, so the seller doesn't either; (3.25)

$v > c$, so a profitable trade exists between the (3.26)

the buyer and the seller.

Statement (3.24) implies $v \le s_{(m)} + \gamma/m$, (3.25) implies $c \ge s_{(m)} - \gamma/m$, and these together with (3.26) imply

$$s_{(m)} - \gamma/m < c < v < s_{(m)} + \gamma/m. \tag{3.27}$$

The value v-c of a missed trade is thus no more than $2\gamma/m$.

The proof is now completed by bounding the expected number of missed trades given $t_{(m)}$. Statements (3.27) and (3.19) imply that the redemption values v, c of a missed trade satisfy

$$t_{(m)} - 2\gamma/m \le c < v \le t_{(m)} + 2\gamma/m.$$

The expected number of missed trades conditional upon $t_{(m)}$ is thus bounded by the expected number of redemption values that lie within $2\gamma/m$ of $t_{(m)}$. This is bounded by fixing $t_{(m)}$ and summing over the number i of buyers' values above $t_{(m)}$. Given $t_{(m)}$ and i, these i values are independently distributed according to $[G(\cdot)-G(t_{(m)})]/[1-G(t_{(m)})]$, whose density is $g(\cdot)/[1-G(t_{(m)})]$. Similarly, the remaining n-i costs are independently distributed according to $[F(\cdot)-F(t_{(m)})]/[1-F(t_{(m)})]$, whose density is $f(\cdot)/[1-F(t_{(m)})]$. Because $t_{(m)} \le 1-\varepsilon$ and f and g are continuous, these densities are bounded above by some number $\alpha(F,G,\varepsilon)$ that is independent of m. Conditional upon $t_{(m)}$, the expected number of redemption values above and within $2\gamma/m$ of $t_{(m)}$ is thus no more than $n(2\gamma/m)(\alpha(F,G,\varepsilon))$. A similar argument shows that for some $\beta(F,G,\varepsilon)$ the expected number of redemption values below and within $2\gamma/m$ of $t_{(m)}$ is no more than $m(2\gamma/m)(\beta(F,G,\varepsilon))$. Using (3.14) we thus obtain a constant bound on the expected number of redemption values within $2\gamma/m$ of $t_{(m)}$ that holds for all $t_{(m)} \in [\varepsilon, 1-\varepsilon]$, which completes the proof. Q.E.D.

Existence of equilibria. This paper does not contain an existence proof. Nevertheless the ease of computing examples suggests that equilibria do exist.[8] For $k \in (0,1)$, the buyer and seller first order conditions are linear differential equations in $1/S'$ and $1/B'$ that can be solved using standard numerical techniques. In brief, an initial point $(c", \lambda", v")$ such that $0 < c" < \lambda" < v" < 1$ determines a smooth solution $<S,B>$ such that $S(c") = \lambda" = B(v")$. In the bilateral case (given some assumptions on F and G to insure the sufficiency of the first order approach[9]) each such solution is an equilibrium (Satterthwaite and Williams (1989a)). Because price-taking behavior emerges as m and n increase, every initial point cannot determine an equilibrium as the market increases in size. What happens numerically in the multilateral case is that if an initial point does not determine an equilibrium, then B′ or S′ turns negative somewhere along the solution the point generates.

The set of smooth equilibria can be approximated by starting with a grid of initial points and discarding those solutions that do not represent

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[8] Existence has been proven for the multilateral 1-DA (Williams (1991)). Computation of equilibria is also important because it provides a target for experimentation. Numerical solution of the first order conditions and the possibility of experimentally testing our results are discussed in Satterthwaite and Williams (1992).

[9] It is sufficient that $c + F(c)/f(c)$ and $v + (G(v) - 1)/g(v)$ are increasing functions, which is true in the uniform case. Sufficiency of the first order conditions is discussed in Satterthwaite and Williams ((1989a, Thm. 3.1) and (1989b, p.495)).

equilibria. Figures 3.1 and 3.2 show bundles of equilibrium strategies for $m = n = 2$ and $m = n = 6$, respectively, in the case of uniform F and G. Our experience is that, for $k \in (0,1)$, the choice of an initial point is robust, which suggests that an infinite set of equilibria exists for each choice of m and n.

4. COMPARISON OF MECHANISMS

Speed is relative, and mechanisms may be ranked according to how quickly expected inefficiency converges to zero. For the case of $m = n$ we rank three mechanisms relative to the k-DA and find that optimal mechanisms dominate the k-DA, the k-DA strictly dominates a dual price double auction when $F = G$, and each of these mechanisms strictly dominates a fixed price mechanism.

Optimal mechanisms. The fastest possible convergence is given by the optimal mechanisms, where an optimal mechanism for a given F, G, m and n is one that maximizes the expected gain from trade subject to the constraints of incentive compatibility and individual rationality. Gresik and Satterthwaite (1989) used the revelation principle to characterize optimal mechanisms. Investigation of their proofs shows that the optimal mechanism in the case of uniform F and G and $m = n$ has an expected inefficiency of at least σ/m^2 for some $\sigma > 0$.[10] The k-DA thus achieves the fastest possible rate of convergence in this case. For other choices of F and G, however, the optimal mechanism can have an expected inefficiency of zero for finite m

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[10] The key step is in their (7.23), which in the case of uniform F and G establishes (in their notation) that $\alpha(\tau) \geq \delta/\tau$ for some positive constant δ.

and n, which is not true for the k-DA.[11] The optimal mechanism thus dominates the k-DA.

Optimal mechanisms, however, are implausible as rules for market organization because they are tailored to the environment m, n, F and G. As Wilson (1987, p.36) emphasized, the rules of real markets "are not changed as the environment changes; rather they persist as stable, viable institutions." Notice that the k-DA's rules are independent of the environment. In our view optimal mechanisms are benchmarks for measuring performance. An open question is whether or not some robust institution exists that surpasses the k-DA by converging as fast to efficiency in every environment and faster in some.

A dual price double auction. McAfee (1992) defined a mechanism that is similar to a Groves mechanism in that it elicits truthful revelation of redemption values by giving each side of the market its own price. It has been cleverly designed so that (i) a single price is sometimes determined, and (ii) a surplus is generated instead of a deficit, which makes it more plausible than a Groves mechanism.

The mechanism works as follows in the case of $m = n$. Let $C_{(i)}$ denote the ith smallest offer and let $V_{(j)}$ denote the jth largest bid, i.e.,

$$V_{(0)} \geq V_{(1)} \geq V_{(2)} \geq \ldots \geq V_{(m)} \geq V_{(m+1)},$$

and

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[11] According to eq. (3.1) of Gresik and Satterthwaite (1989), the optimal mechanism is efficient if, in their notation, $G(\alpha,\tau) > 0$ for $\alpha = 0$. For two buyers, one seller, and redemption values drawn from the distribution $\ell n(1+x)/(\ell n\ 2)$, we have checked that $G(0,\tau) > 0$. This example contrasts with the impossibility result of Myerson and Satterthwaite (1983) which established for the bilateral case that expected inefficiency is positive in an independent private values model. Finally, McAfee and Reny (1992) showed that the expected inefficiency can be zero in the bilateral case if redemption values are correlated in a particular way.

$$C_{(0)} \leq C_{(1)} \leq C_{(2)} \leq \ldots \leq C_{(m)} \leq C_{(m+1)},$$

where $C_{(m+1)} \equiv 1 \equiv V_{(0)}$ and $V_{(m+1)} \equiv 0 \equiv C_{(0)}$. The quantity q is defined by $V_{(q)} \geq C_{(q)}$ and $V_{(q+1)} < C_{(q+1)}$ and the price p as $[V_{(q+1)}+C_{(q+1)}]/2$. Trade then follows one of two rules, depending on the value of p:[12]

if $p \in [C_{(q)},V_{(q)}]$, then trade is carried out at a price of p between the q buyers who bid at least p and the q sellers whose offers were no more than p; (4.1)

if $p \notin [C_{(q)},V_{(q)}]$, then each buyer who submitted one of the q-1 largest bids buys and pays $V_{(q)}$, while each seller who submitted one of the q-1 smallest offers sells and receives $C_{(q)}$. (4.2)

Three points should be noted. First, a trader cannot affect the price at which he trades. McAfee used this to prove that honest revelation is a dominant strategy. We thus assume that $C_{(i)} = c_{(i)}$ (the ith smallest seller's cost) and $V_{(j)} = v_{(j)}$ (the jth largest buyer's value). Second, the mechanism is not efficient. In event (4.2) a trade of value $v_{(q)}-c_{(q)}$ is not made. Third, the mechanism generates a monetary surplus of size $(q-1)(v_{(q)}-c_{(q)})$ in event (4.2). McAfee postulates a nonstrategic "specialist" who absorbs this as his profit.[13]

Notice that at most one profitable trade is not made in this mechanism, and it is the least profitable trade. McAfee proves that the expected value of this lost trade is O(1/m), which is intuitive. This implies an expected

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12 Random allocation may be needed in case of ties at $V_{(q)}$ or $C_{(q)}$. See McAfee (1992) for details.

13 This surplus could be returned to the traders, i.e., payments that are independent of the offers/bids can be devised so that the surplus on average is zero. Such payments must depend on F and G, however, which runs afoul of Wilson's critique by making the mechanism dependent upon the environment

inefficiency of $O(1/m^2)$ if one counts the profit of the specialist as part of the gains from trade. A specialist, however, is normally justified by his role in facilitating trade and not in terms of his own profit. We show in the Appendix for the case of F = G that his expected profit is bounded below for all m by some positive constant. Counting only the gain received by the traders (which is appropriate for comparison to the k-DA), the expected inefficiency of the dual price double auction is therefore $O(1/m)$.

The fixed-price mechanism. As in the introduction, let p* denote the competitive price of the limiting continuum market (i.e., the solution to $1-G(p) = F(p)$ in the case of m = n). In the fixed-price mechanism, which Hagerty and Rogerson (1985) discussed, trade occurs between buyers and sellers who indicate their willingness to trade at p*, with traders on the long side of the market randomly given the right to trade. Gresik and Satterthwaite (1989) showed that the expected inefficiency of this mechanism is $O(1/m^{1/2})$. Its poor performance is due to its failure to rank traders according to their redemption values, which means that it often fails to execute those trades of greatest value.

A numerical comparison. For the case of uniform F and G and m ranging from 1 to 8, Table 4.1 lists the expected inefficiencies of the mechanisms discussed above together with those of both the least and the most efficient equilibria of the 0.5-DA. Even for such small m, the rates we discussed above are evident for the optimal mechanism and the 0.5-DA, with the expected inefficiency decreasing by a factor of 4 as m doubles. The rates for the other two mechanisms are slower for small m than the rates they eventually achieve.

APPENDIX

Proof of Theorem 2.1. Arguments of Satterthwaite and Williams (1989b, Thm. 2.2) and Chatterjee and Samuelson (1983, Thm. 1) generalize to prove that $P_b \cdot B$ is nondecreasing and $P_s \cdot S$ is nonincreasing on [0,1]. Because the proofs of (2.7) and (2.8) are so similar, the proof of (2.7) is omitted.

The following notation is needed to prove (2.8):

$C_b(\lambda) \equiv$ a buyer's expected payment when he bids λ, all sellers use S, and the other m-1 buyers use B.

Note that

$$\pi_b(v,\lambda) = P_b(\lambda)v - C_b(\lambda),$$

and that $C_b(\lambda)$ is nondecreasing in λ.

We now assume that $B(v') > B(v'')$ and derive a contradiction. Because P_b and $P_b \cdot B$ are both nondecreasing, it must be true that $P_b(B(v'')) = P_b(B(v'))$, i.e., a lower value buyer (v') bids more than the higher value buyer (v") even though it doesn't increase the probability that he trades. We show that this contradicts the assumption that B(v') is an optimal bid for a buyer with value v'. The argument rests upon the following facts:

the set $\{c \mid S(c) \le B(v'')\}$ has positive F-measure; (A.1)

for $\lambda \in (B(v''),B(v'))$, the set $\{c \mid S(c) > \lambda\}$ has positive F-measure; (A.2)

the set $\{v \mid B(v) < B(v'')\}$ has positive G-measure. (A.3)

Statement (A.1) is true because $P_b(B(v'')) > 0$, (A.2) is true because $S(c) \ge c$ (by (2.2)) and $\lambda < B(v') \le 1$, and (A.3) is true because $B(v'') > 0$ and $B(v) \le v$ (by (2.2)). Statements (A.1-A.3) imply that given a bid of $\lambda \in (B(v''),B(v'))$ by the selected buyer, it is a positive probability event that $s_{(m)} < \lambda < s_{(m+1)}$, where the sample is now the n+m-1 offers/bids of the other traders. In this event, the selected buyer affects the price at which he trades, from which it follows that $C_b(\lambda)$ is increasing over the interval $(B(v''),B(v'))$. This and (2.6) imply that

$$\pi_b(v',B(v')) = P_b(B(v'))v' - C_b(B(v'))$$
$$< P_b(B(v''))v' - C_b(B(v'')) = \pi_b(v',B(v'')),$$

which contradicts the optimality of the bid $B(v')$ for the buyer with value v'. We conclude that $B(v') \leq B(v'')$.

We next show that $B(v') < B(v'')$ by supposing instead that $B(v') = B(v'')$ and deriving a contradiction. Because B is nondecreasing over $(\underline{v},1]$, to show that it is increasing over this interval it is sufficient to show that it is increasing over $(\underline{v},1)$. We can therefore assume that $v'' < 1$. From above, we know $B(v) = B(v'')$ for $v \in [v',v'']$, and, because $B(v) \leq v$ and $P_b(B(v'')) > 0$, it is clear that $0 < B(v'') < v'' < 1$. We now argue that $P_b(\lambda)$ has a jump discontinuity at $\lambda = B(v'')$, from which a contradiction easily follows.

Consider a buyer with value v''. Because $P_b(B(v'')) > 0$, the set $\{c \mid S(c) \leq B(v'')\}$ has positive F-measure. Because $S(c) \geq c$ and $B(v'') < 1$, the set $\{c \mid S(c) > B(v'')\}$ also has positive F-measure. It therefore is a positive probability event that (i) all other buyers bid $B(v'')$, (ii) some offers are no more than $B(v'')$ and some are strictly more.[14] Given this event, if the selected buyer bids $B(v'')$ then the market price is $B(v'')$ and the available supply must be rationed among the m buyers. By raising his bid above $B(v'')$ the selected buyer obtains a unit with probability one in

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[14] The assumption that $m \geq 2$ is needed here.

this event rather than with probability less than one due to rationing. For this reason, there exists an $\varepsilon > 0$ such that

$$P_b(\lambda) > P_b(B(v")) + \varepsilon$$

for all $\lambda > B(v")$. An argument from Satterthwaite and Williams (1989b, p.483) establishes the following bound on the change in the buyer's expected payment as he raises his bid from $B(v")$ to $\lambda > B(v")$:

$$C_b(\lambda) - C_b(B(v")) \leq [P_b(\lambda) - P_b(B(v"))]\lambda + [\lambda - B(v")].$$

Combining these inequalities, for $\lambda > B(v")$ we have

$$\begin{aligned}\pi_b(v",\lambda) - \pi_b(v",B(v")) &= [P_b(\lambda) - P_b(B(v"))]v" + C_b(B(v")) - C_b(b) \\ &> \varepsilon\ (v" - \lambda) + [B(v") - \lambda].\end{aligned}$$

This expression is positive for λ near $B(v")$, which contradicts the assumption that $B(v")$ is an optimal bid for a buyer with value $v"$.

We conclude by proving that $\lim_{v\to\underline{v}+} B(v) = \underline{v} = \underline{s}$; the proof that $\lim_{c\to\bar{c}-} S(c) = \bar{c} = \bar{b}$ is omitted because it is so similar to this argument. The equality $\underline{v} = \underline{s}$ is established by proving that both of the inequalities $\underline{v} < \underline{s}$, $\underline{s} < \underline{v}$ lead to contradictions. If $\underline{v} < \underline{s}$, then $B(v) \leq v < \underline{s}$ for values of v that are greater than but sufficiently near $\underline{v}$. As a consequence, $P_b(B(v)) = 0$ at such values of v, which contradicts the definition of $\underline{v}$. If $\underline{s} < \underline{v}$, then $c \leq S(c) < v < \underline{v}$ for c near zero and v less than but sufficiently near $\underline{v}$. This implies that $P_b(B(v)) > 0$ for such values of v, which also contradicts the definition of $\underline{v}$.

The equality $\lim_{v\to\underline{v}+} B(v) = \underline{v}$ is established by noting first that $B(v) > \underline{s}$ for $v > \underline{v}$ because a buyer's bid must exceed $\underline{s}$ if he is to have a positive probability of trading. Therefore, as v approaches $\underline{v}$ from above, $\lim_{v\to\underline{v}+} B(v) \geq \underline{s} = \underline{v}$. But $v \geq B(v)$ for all v. Therefore $\lim_{v\to\underline{v}+} B(v) = \underline{s} = \underline{v}$. Q.E.D.

Probabilities in the first order condition (3.1). For $\lambda \epsilon (\underline{v},1]$, define the function $S^{-1}(\lambda)$ by the formula

$$S^{-1}(\lambda) \equiv \inf \{c \mid S(c) \geq \lambda\}, \tag{A.4}$$

and for $\lambda \epsilon [0,\bar{c})$ define the function $B^{-1}(\lambda)$ by

$$B^{-1}(\lambda) \equiv \sup \{v \mid B(v) \leq \lambda\}. \tag{A.5}$$

Using $S^{-1}(\lambda) = c$ and $B^{-1}(\lambda) = v$, the probabilities $K_{n,m}(\lambda)$, $L_{n,m}(\lambda)$, and $M_{n,m}(\lambda)$ can be written as functions of c and v:

$$K_{n,m}(\lambda) = \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} \binom{m-1}{i}\binom{n-1}{j} G(v)^{i}F(c)^{j}(1-G(v))^{m-1-i}(1-F(c))^{n-1-j}, \tag{A.6}$$

$$L_{n,m}(\lambda) = \sum_{\substack{i+j=m-1\\0\leq i\leq m-2\\0\leq j\leq n}} \binom{m-2}{i}\binom{n}{j} G(v)^{i}F(c)^{j}(1-G(v))^{m-2-i}(1-F(c))^{n-j}, \tag{A.7}$$

$$M_{n,m}(\lambda) = \sum_{\substack{i+j=m\\0\leq i\leq m-1\\0\leq j\leq n}} \binom{m-1}{i}\binom{n}{j} G(v)^{i}F(c)^{j}(1-G(v))^{m-1-i}(1-F(c))^{n-j}. \tag{A.8}$$

Proof of (3.12). We show that the ratio $M_{n,m}(\lambda)/L_{n,m}(\lambda)$ satisfies

$$\frac{M_{n,m}(\lambda)}{L_{n,m}(\lambda)} \leq 2G(v) + \frac{2\ n}{m}\ \frac{(1-G(v))F(c)}{1-F(c)} \tag{A.9}$$

$$\leq 2G(v) + \frac{2\ n}{m}\ \frac{(1-G(v))F(v)}{1-F(v)}\ , \tag{A.10}$$

where (A.10) implies (3.12). Inequality (A.10) follows from (A.9) because $c \leq \lambda \leq v$ and $F(c)/(1-F(c))$ is increasing in c. We therefore focus upon (A.10).

Define

$Y_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-2 buyers using the strategy B and n sellers using S.

We first show that

$$M_{n,m} = (1 - G(v))Y_{n,m} + G(v)L_{n,m}, \tag{A.11}$$

or equivalently

$$\frac{M_{n,m}}{L_{n,m}} = (1 - G(v)) \frac{Y_{n,m}}{L_{n,m}} + G(v). \tag{A.12}$$

The bound (A.9) will be obtained by bounding $Y_{n,m}/L_{n,m}$ and then substituting into (A.12). The probability $M_{n,m}$ is defined for a sample of offers/bids from m-1 buyers using the strategy B and n sellers using the strategy S. Select a buyer. The event that defines $M_{n,m}$ is the disjoint union of the following two events:

the selected buyer bids at least λ and λ lies between $s_{(m)}$ and $s_{(m+1)}$ in the sample of offers/bids from the remaining m-2 buyers and n sellers;

the selected buyer bids less than λ and λ lies between $s_{(m-1)}$ and $s_{(m)}$ in the sample of offers/bids from the remaining m-2 buyers and n sellers.

The selected buyer bids at least λ with probability 1-G(v) and less than λ with probability G(v). Equation (A.11) then follows from the definitions of $Y_{n,m}$ and $L_{n,m}$.

To bound $Y_{n,m}/L_{n,m}$, we partition the events that define these probabilities according to the number i of buyers' bids that are no more than λ. For $0 \le i \le m-2$, define

$Y^i_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-2 buyers using the strategy B and n sellers using S, and exactly i of the offers/bids at or below λ are buyers' bids;

$L^i_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-2 buyers using strategy B and n sellers using S, and exactly i of the offers/bids at or below λ are buyers' bids.

It is clear that

$$Y_{n,m} = \Sigma_{i=0}^{m-2} Y^i_{n,m}, \tag{A.13}$$

$$Y^i_{n,m} = \binom{m-2}{i} \binom{n}{m-i} G(v)^i F(c)^{m-i} (1-G(v))^{m-2-i} (1-F(c))^{n-m+i}, \tag{A.14}$$

$$L_{n,m} = \Sigma_{i=0}^{m-2} L^i_{n,m}, \text{ and} \tag{A.15}$$

$$L^i_{n,m} = \binom{m-2}{i} \binom{n}{m-1-i} G(v)^i F(c)^{m-1-i} (1-G(v))^{m-2-i} (1-F(c))^{n-m+i+1}. \tag{A.16}$$

The identity

$$i \binom{m-2}{i} \equiv (m-1-i) \binom{m-2}{i-1}$$

and formulas (A.14) and (A.16) imply that

$$\frac{Y^i_{n,m}}{L^{i-1}_{n,m}} \le \frac{G(v)}{1-G(v)} \quad \text{for } (m-1)/2 \le i \le m-2. \tag{A.17}$$

The identity

$$(m-i) \binom{n}{m-i} \equiv [(n+1) - (m-i)] \binom{n}{m-1-i}$$

and the bound

$$\frac{(n+1) - (m-i)}{(m-i)} \le \frac{2\ n}{m} \quad \text{for } 0 \le i \le (m-2)/2$$

together with formulas (A.14) and (A.16) imply that

$$\frac{Y^i_{n,m}}{L^i_{n,m}} \le \frac{2\ n}{m} \frac{F(c)}{1-F(c)} \quad \text{for } 0 \le i \le (m-2)/2. \tag{A.18}$$

It follows that

$$\frac{Y_{n,m}}{L_{n,m}} = \frac{\Sigma^{m-2}_{i=0} \ Y^i_{n,m}}{\Sigma^{m-2}_{i=0} \ L^i_{n,m}} \tag{A.19}$$

$$= \frac{\Sigma_{0\le i\le(m-2)/2} \ Y^i_{n,m}}{\Sigma^{m-2}_{i=0} \ L^i_{n,m}} + \frac{\Sigma_{i>(m-1)/2} \ Y^i_{n,m}}{\Sigma^{m-2}_{i=0} \ L^i_{n,m}} \tag{A.20}$$

$$\le \frac{\Sigma_{0\le i\le(m-2)/2} \ Y^i_{n,m}}{\Sigma_{0\le i\le(m-2)/2} \ L^i_{n,m}} + \frac{\Sigma_{i>(m-1)/2} \ Y^i_{n,m}}{\Sigma_{i>(m-1)/2} \ L^{i-1}_{n,m}} \tag{A.21}$$

$$\le \frac{2\ n}{m} \frac{F(c)}{1 - F(c)} + \frac{G(v)}{1 - G(v)}, \tag{A.22}$$

where the left and right terms in (A.21) are bounded by first rewriting (A.18) and (A.17) as upper bounds on $Y^i_{n,m}$ and then substituting into the numerators. Q.E.D.

Proof of (3.18). This is established in the case of m = n by

$$\binom{2\ m}{m} = \frac{\Pi^m_{i=1}\ 2i \ \cdot \ \Pi^m_{i=1}(2i-1)}{\Pi^m_{i=1} i \ \cdot \ \Pi^m_{i=1} i} \le 4^m. \tag{A.23}$$

Because of the symmetry of (3.18) in m and n, we need only prove it for $n \ge m$. This is done by induction on n: assuming (3.18) is true for n and m with $n \ge m$, then

$$\binom{n+1+m}{m} = \frac{n+1+m}{n+1} \binom{n+m}{m} \le 2 \cdot 4^{\max(m,n)} < 4^{\max(m,n+1)}$$

since $(n+1+m)/(n+1) < 2$ whenever $n \ge m$. Q.E.D.

The specialist's profit in the dual price double auction. Because a lower bound is sought, we needn't consider the entire event (4.2) in which profit is made. Consider the subevent defined by $0 < v_{(q)} < p < c_{(q+1)} < 1$ and $c_{(q)} < v_{(q+1)}$, with $0 < q < m$. The profit for a sample in this subevent is $(q-1)(v_{(q)}-c_{(q)})$, which is at least $(q-1)(v_{(q)}-v_{(q+1)})$. Letting $x \equiv c_{(q+1)}$, $y \equiv v_{(q+1)}$ and $z \equiv v_{(q)}$, the expected profit in this subevent is at least

$$\int_{y<z<(x+y)/2} \sum_q (q-1)(z-y)\binom{m-2}{q-1}\binom{m-1}{q} \tag{A.24}$$

$$\cdot \quad F(y)^{m-1}(1-F(z))^{q-1}(1-F(x))^{m-q-1}\, m^2(m-1)\, f(x)f(y)f(z)\, dxdydz.$$

Integrating with respect to x produces

$$\int_{y<z<(1+y)/2} \sum_q \frac{(q-1)}{(m-q)}\, (z-y)\, \binom{m-2}{q-1}\binom{m-1}{q} \tag{A.25}$$

$$\cdot \quad F(y)^{m-1}(1-F(z))^{q-1}(1-F(2z-y))^{m-q}\, m^2(m-1)\, f(y)f(z)\, dydz.$$

Replace $1-F(z)$ with the smaller value $1-F(2z-y)$ and then perform the change of variable $w \equiv 2z-y$, $y \equiv y$ to obtain[15]

$$\int_{y<w<1} \sum_q \frac{(q-1)}{(m-q)}\, \frac{(w-y)}{4}\, \binom{m-2}{q-1}\binom{m-1}{q} \tag{A.26}$$

$$\cdot \; F(y)^{m-1}(1-F(w))^{m-1} m^2(m-1)\, f(y)f((w+y)/2)\, dydw.$$

---

[15] The equality in (A.28) can be shown as follows. Divide a set A consisting of 2m-3 objects in a set B with m-1 objects and a set C with m-2 objects. Selecting m objects from A requires selecting q from B and m-q from C for some q. The number of ways of choosing m objects from A can thus be computed by summing over q the number of ways of choosing q from B and m-q from C, which gives the desired formula.

Replace $f((w+y)/2)$ with $\chi f(w)$, where $\chi \equiv \inf_{0\le a,b\le 1} f(a)/f(b)$, and substitute using

$$\frac{(q-1)}{(m-q)} \binom{m-2}{q-1} = \binom{m-2}{m-q}, \tag{A.27}$$

$$\sum_q \binom{m-2}{m-q}\binom{m-1}{q} = \binom{2m-3}{m} \ge \frac{1}{4}\binom{2m-2}{m-1}, \tag{A.28}$$

and

$$m^2 \ge \frac{2m(2m-1)}{4} \tag{A.29}$$

to obtain

$$\frac{\chi(m-1)}{64} \int_{y<w<1} \binom{2m-2}{m-1} (w-y)F(y)^{m-1}(1-F(w))^{m-1}(2m)(2m-1)f(y)f(w)dydw. \tag{A.30}$$

This integral is the expected difference between the (m+1)st and the mth redemption values in a sample of 2m independent draws from F. Because f is positive on [0,1], this difference is bounded below by a constant times 1/(2m+1) (David (1981), 34-35). It then follows that the seller's expected profit is bounded below by a positive constant that is independent of m. Q.E.D.

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Table 2.1

The array of offers and bids in a k-double auction.

	offers	bids
No. $\geq s_{(m+1)}$	r	t
No. $\leq s_{(m)}$	s	u

Table 2.2

The dual market to a given market.

	given market	dual market
number of sellers	n	$n^* \equiv m$
number of buyers	m	$m^* \equiv n$
seller's cost	c	$c^* \equiv 1-v$
buyer's value	v	$v^* \equiv 1-c$
seller's distribution	F	$F^*(x) \equiv 1-G(1-x)$
buyer's distribution	G	$G^*(x) \equiv 1-F(1-x)$
seller's strategy	S	$S^*(x) \equiv 1-B(1-x)$
buyer's strategy	B	$B^*(x) \equiv 1-S(1-x)$
double auction	k	$k^* \equiv 1-k$

Table 4.1.

Expected inefficiencies of the optimal mechanism, the least and most inefficient equilibria of the 0.5-double auction, the dual price mechanism, and the fixed price mechanism for different market sizes in the case of uniform F and G.

m=n	Optimal Mechanism	0.5-DA Least	0.5-DA Most	Dual Price Mechanism	Fixed Price Mechanism
1	0.16	0.16	1.00	0.25	0.25
2	0.056	0.056	0.063	0.21	0.22
4	0.015	0.015	0.016	0.16	0.18
6	0.0069	0.0069	0.0070	0.12	0.16
8	0.0039	0.0039	0.0039	0.099	0.15

Notes. The values of the optimal and fixed price mechanisms are taken from Tables I and II respectively of Gresik and Satterthwaite (1989). We calculated the values for the dual price mechanism by a direct probability calculation; our values for m = 2 and 4 agree with the values from a simulation that McAfee (1992) reported in his Table I. Finally, the values for the 0.5-DA were obtained by numerically integration using the equilibria that we computed employing the procedure described in the last paragraph of Section 3. Calculation of the values for the 0.5-DA posed numerical difficulties; consequently values are reported to only two significant digits.

Figure 1.1. The three offers and three bids on the left imply the supply and demand curves on the right. A market-clearing price can be chosen between 0.42 and 0.50. The 0.5-DA selects 0.46.

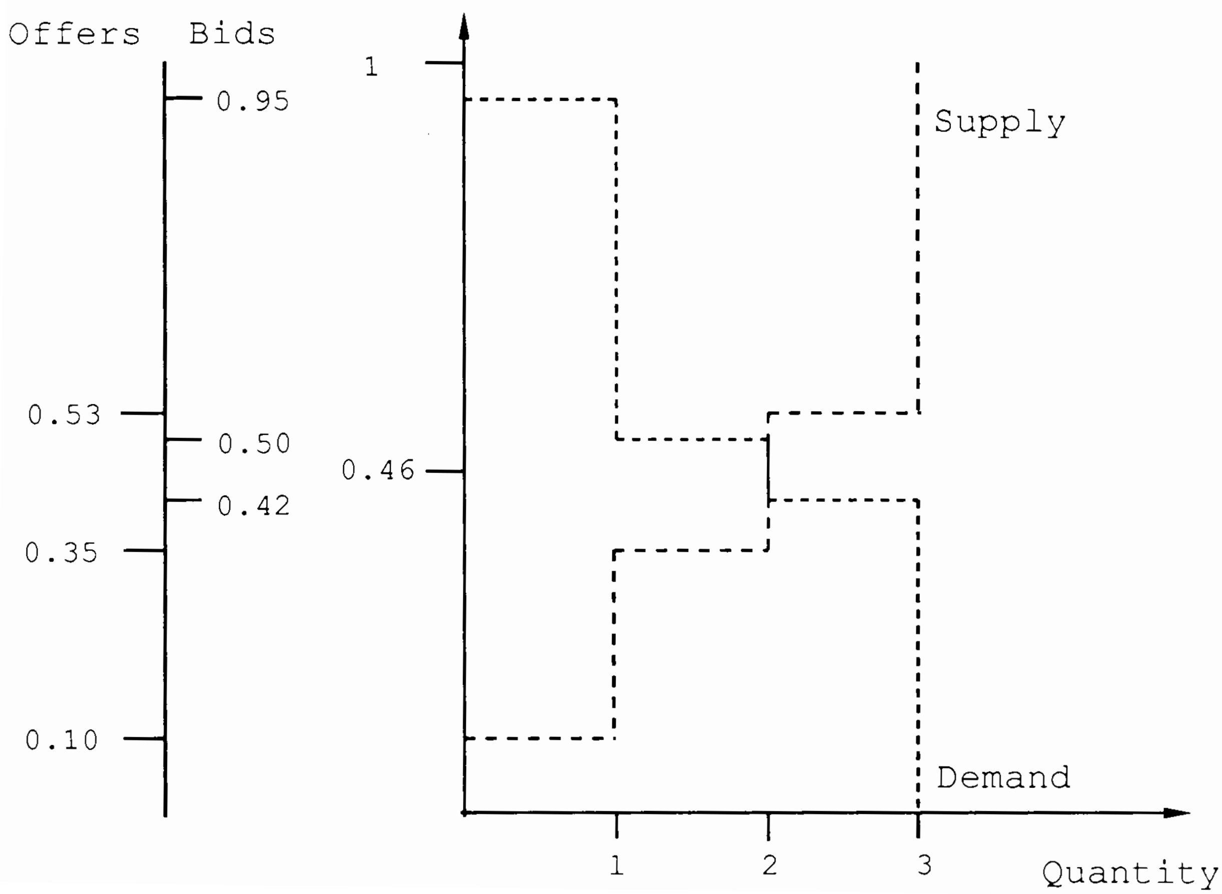

Figure 2.1. S(c) and B(v) are an equilibrium pair of strategies for the 0.5-DA in the case of uniform F and G and m = n = 2. For this equilibrium $\underline{v}$ = 0.172 and $\bar{c}$ = 0.828.

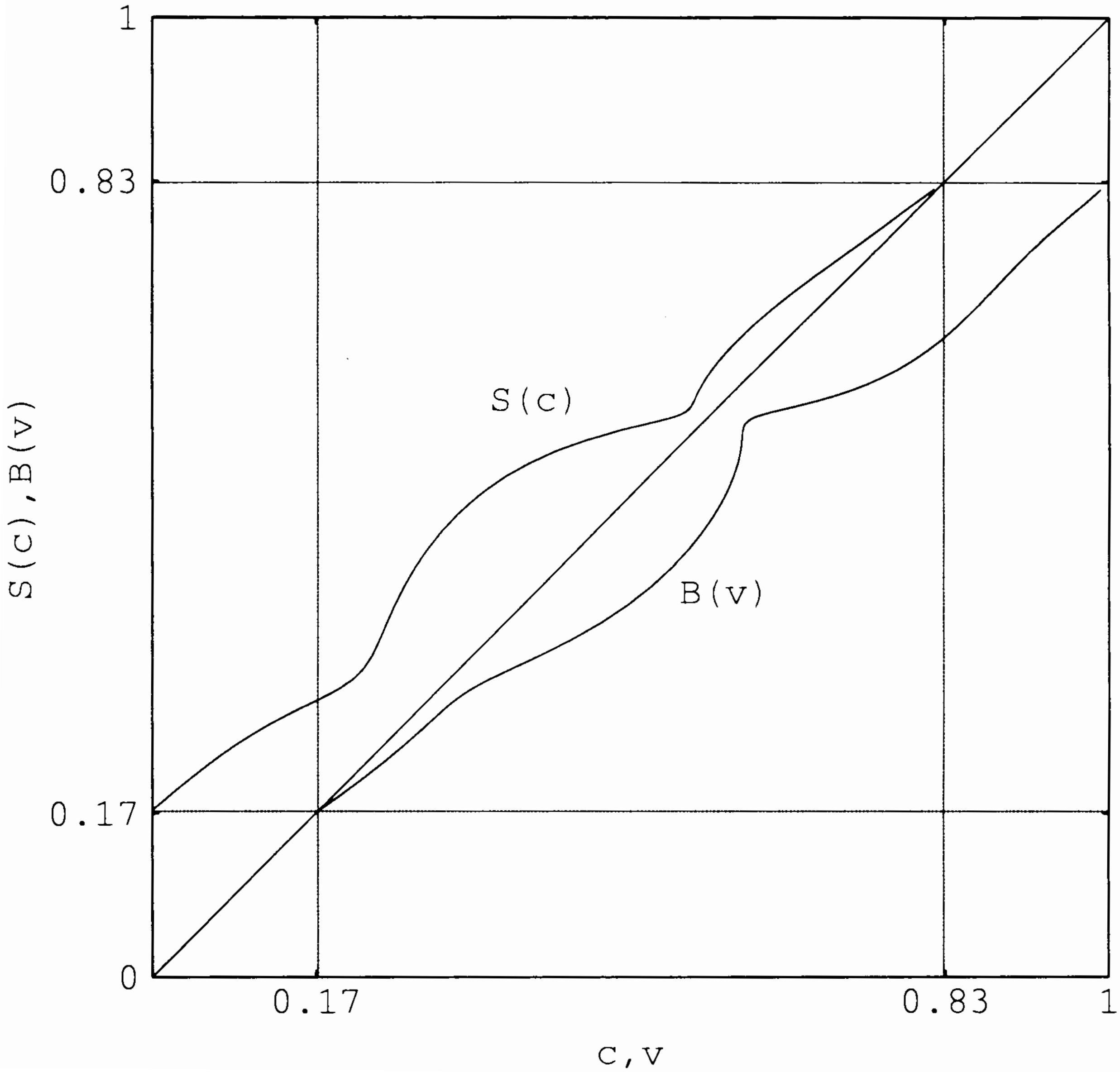

Figure 3.1. A bundle of equilibrium strategies in the 0.5-DA for uniform F and G and m = n = 2.

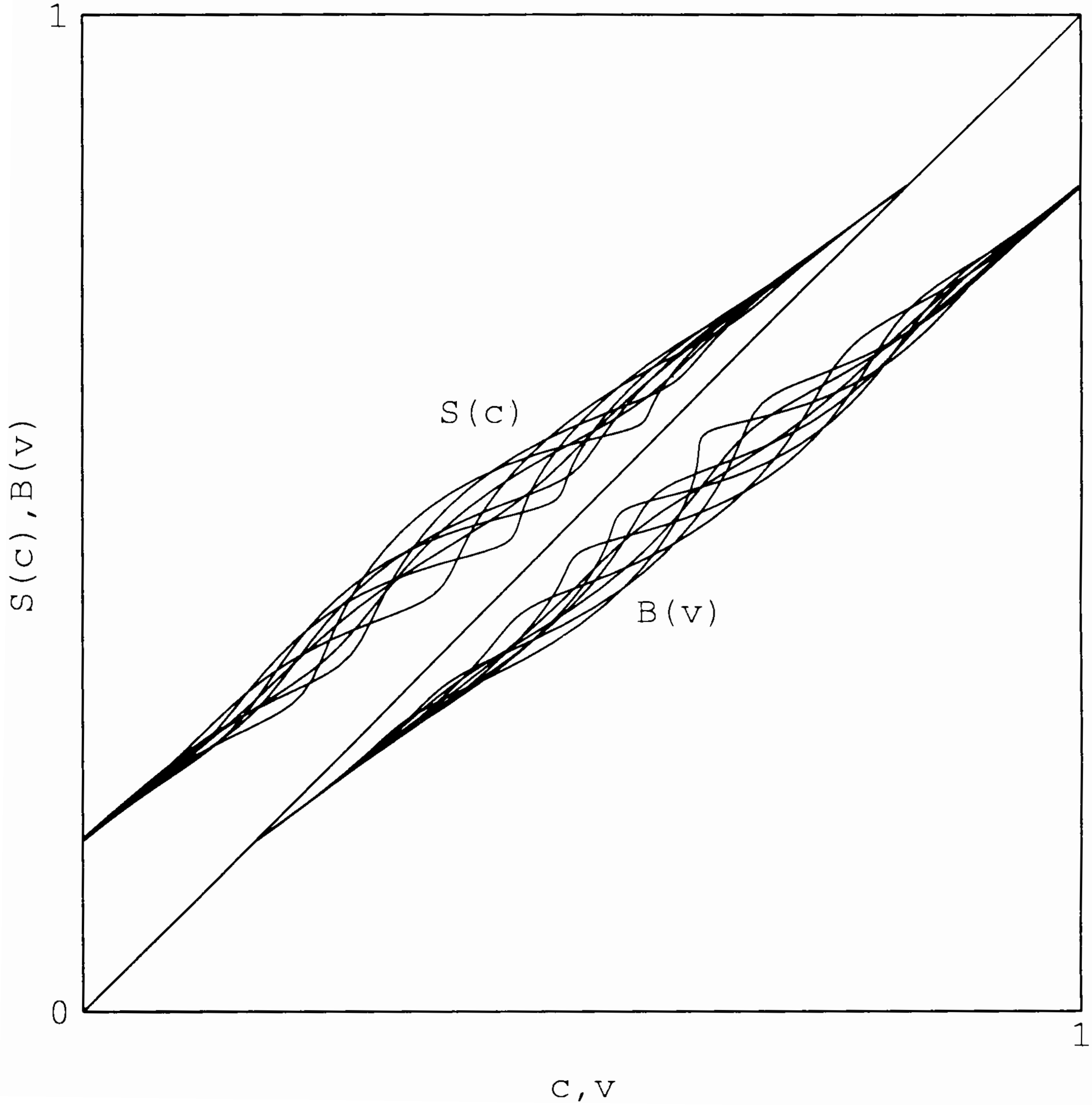

Figure 3.2. A bundle of equilibrium strategies in the 0.5-DA for uniform F and G and m = n = 6.