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Efficient and Nearly Efficient Partnerships*

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1. Introduction

We study deterministic partnerships, enterprises in which risk neutral partners jointly produce according to a nonstochastic technology, and share the resulting output among themselves. It is generally accepted that such partnerships are inefficient if the partners' actions are not verifiable. The argument is that some partner will shirk because he must share the marginal benefit of his effort, but he alone bears its cost. Holmström (1982) formalizes this argument to show that in certain differentiable, monotonic partnerships, no sharing rule can elicit an efficient set of actions.

Our first result indicates that this argument paints too dark a picture. We give a simple necessary and sufficient condition for the existence of a sharing rule for which an efficient action profile is a Nash equilibrium. An implication is that efficiency is in fact sustainable in interesting partnerships, such as generic ones with a finite number of possible actions, or those in which the partners' actions are perfect complements (Leontief partnerships). Efficiency is also sustainable if one partner is not able to affect output. This is Holmström's result (1982) that efficiency is attainable if a third party is employed to "break the budget" — such a third party is merely a partner who cannot affect output, but can share in its consumption.

Nonetheless, our characterization of efficient partnerships allows a simple proof that efficiency is unattainable in neoclassical partnerships, ones in which the production and utility functions are smooth, and efficiency requires each partner to take a productive action (this generalizes Holmström's inefficiency result). Again, however, the situation may not be too gloomy. Our second result is that quite generally, efficiency can be approximated to any desired degree through the use of simple mixed strategies, provided that the total liability of the partnership is unconstrained. This is the main result of the paper.

To prove it, we construct equilibria in which all partners but one choose their efficient actions, and that partner chooses his efficient action with high probability. Given any probability close to one, a sharing rule is found which induces such an equilibrium, with the mixing partner choosing his efficient action with that probability. Efficiency is thus approximated to any degree desired.

This result relies on the partners having unlimited liability. The sharing rule imposes large fines on some partners when the output takes on certain values. The partners who surely act efficiently do not shirk because if one of them does shirk, the sharing rule will make them all pay the mixing partner a large fine in the event that he too chooses an inefficient action. The smaller the probability that the mixing partner chooses an inefficient action, the larger must be the fine required to deter the others from shirking. An upper bound on how much a partner can be made to pay will, in general, limit the degree to which efficiency can be approximated.

We conclude that free riding causes inefficiency only to the extent that the liability of the partners is limited, or their wealth is bounded. This puts the partnership problem in a new light. It suggests, for example, that successful partnerships should be composed of wealthy individuals who can hold large liabilities.

In this paper, no partner's equilibrium payment is arbitrarily large. This differentiates our approximate efficiency result from the well-known one of Mirrlees (1974, 1975). More closely related is Gjesdal's (1976) observation that full efficiency can be achieved in a principal-agent model if the agent's action shifts the support of a random output.[1] The partners in our scheme who surely act efficiently are analogous to Gjesdal's agent: each of them views output as random, and as able to take on a value at which a punishing fine must be paid if and only if the partner chooses an inefficient action.

Other papers have shown that efficiency or near efficiency can be obtained in other kinds of partnerships, ones with a random technology,[2] or risk averse partners and random sharing rules,[3] or repeated play.[4] To our knowledge, this is the first paper to consider thoroughly the efficiency and near efficiency of one-shot, nonstochastic partnerships with risk neutral partners.

The balance of the paper starts with definitions in Section 2 and examples in Section 3. Efficient partnerships are studied in Section 4, nearly efficient partnerships in Section 5, and limited liability in Section 6. Issues and extensions are discussed in Section 7.

2. Notation and Maintained Assumptions

A *partnership* here consists of at least two partners, $N = \{1, \ldots, n\}$; a set of actions A_i for each partner; a disutility function $v_i \colon A_i \to \Re$ for each partner; and a production function $f \colon A \to \Re$, where $A = \times_{i \in N} A_i$. Action profile a results in output

$y = f(a)$. The utility of partner i is $s_i - v_i(a_i)$ if he receives share s_i of output and takes action a_i.

Each A_i is a complete metric space, and $f(\cdot)$ and each $v_i(\cdot)$ are Borel measurable. For the most part this generality does not complicate the analysis; when it does, we focus on familiar cases, such as $A_i = \Re_+$. Each $v_i(\cdot)$ is assumed bounded below by some $\underline{v} \in \Re$.

Efficient actions are those which maximize the welfare criterion,[5]

$$W(a) \equiv y(a) - \Sigma v_i(a_i). \tag{2.1}$$

An efficient action profile is assumed to exist and, mostly for expositional convenience, to be unique. We denote it as a^*, and the corresponding output as $y^* = f(a^*)$.

A *sharing rule* is a measurable map $s: f(A) \rightarrow \Re^n$ which determines each partner's share $s_i(y)$ of output, and which satisfies budget balance at all possible outputs:

$$\Sigma s_i(y) = y \quad \text{for all } y \in f(A). \tag{2.2}$$

To each sharing rule s corresponds a *partnership game,* $\Gamma(s)$. The strategy set of partner i in this game is A_i, and his payoff function is

$$u_i(a) \equiv s_i(f(a)) - v_i(a_i). \tag{2.3}$$

The central issue is whether a sharing rule exists such that a^* is a (Nash) equilibrium of the resulting partnership game. If so, *efficiency is sustainable.* *Approximate efficiency is sustainable* if for any $\varepsilon > 0$, a sharing rule s exists such that

$\Gamma(s)$ has a mixed strategy equilibrium, $P = (P_1, \ldots, P_n)$, according to which expected welfare is within ε of first best (the random variable with distribution P is denoted by $\tilde{a}$):

$$\mathcal{E}_P W(\tilde{a}) > W(a^*) - \varepsilon. \tag{2.4}$$

3. Examples

Example A (Finite Actions)

This example indicates that generic partnerships with finite action sets sustain efficiency. It illustrates two ways in which a sharing rule can elicit efficient actions.

Let $n = 3$ and each $A_i = \{0, 1\}$. Interpret $a_i = 0$ as shirking and $a_i = 1$ as working, and assume the efficient action vector is $a^* = (1, 1, 1)$. Assume the production function is generic, which means only that $f(a) \neq f(\hat{a})$ if $a \neq \hat{a}$. Because such a function is invertable, the identity of any partner who unilaterally deviates from a^* will be known. This allows the construction of a sharing rule which penalizes such deviations severely enough to deter them. To be explicit, let y^i be the output when partner i shirks but the others work. Define the sharing rule so that partner i pays a fine F to the others if y^i is realized, and all other outputs are equally shared: $s_i(y) = y/3$ if $y \notin \{y^1, y^2, y^3\}$, and otherwise, for $i \neq j \neq k$,

$$s_i(y^i) = -F \quad \text{and} \quad s_j(y^i) = s_k(y^i) = (y^i + F)/2. \tag{3.1}$$

Then no partner will shirk, given that the others work, if F is sufficiently large.

The shirker need not be identifiable. A weaker condition is that a non-shirker be identifiable after any unilateral shirking. To see this, suppose that $y^2 = y^3 \equiv \hat{y}$, but $y^1 \neq \hat{y}$. Then an observation of $\hat{y}$ does not reveal whether partner 2 or partner 3 unilaterally shirked. It does, however, reveal that partner 1 was not a unilateral shirker. Thus, shirking by 2 or 3 can be deterred, without at the same time causing 1 to shirk, by a sharing rule which requires 2 and 3 to both pay a fine to 1 if $\hat{y}$ is realized. For example, let $s(y)$ be defined as above for $y \neq \hat{y}$, and otherwise let

$$s_1(\hat{y}) = \hat{y} + 2F \quad \text{and} \quad s_2(\hat{y}) = s_3(\hat{y}) = -F. \tag{3.2}$$

The problematic case is $y^1 = y^2 = y^3$, so that the identity of a non-shirker after a unilateral deviation from a^* is not revealed. This is the case for symmetric production functions in which output depends only on the number of working partners. Then their actions are perfect substitutes, and efficiency may be unsustainable.

Example B (Leontief Production Function)

This example shows that efficiency is sustainable if the production function is Leontief, so that the partners' actions are perfect complements. Efficiency is sustained even though the condition identified in Example A is violated: here, the identity of a non-deviator is never revealed by a unilateral deviation from a^*.

Let $A_i = \Re_+$ and $f(a) = \min(a_1/\theta_1,\ldots, a_n/\theta_n)$, with each θ_i positive. Assume each disutility function is strictly convex, differentiable, and satisfies $v_i'(0) = 0$.[6] Then the efficient output is determined by $\Sigma\theta_i v_i'(\theta_i y^*) = 1$, and the efficient actions are

$a_i^* = \theta_i y^*$. Let $s_i(y) \equiv \theta_i v_i'(\theta_i y^*) y$ for each i. This defines a (linear) sharing rule: the share proportions sum to one because of the condition defining y^*. Given this rule, and given that the other partners choose their efficient actions, partner i chooses a_i to maximize

$$u_i(a_i, a_{-i}^*) = \theta_i v_i'(\theta_i y^*) \min(a_i/\theta_i, y^*) - v_i(a_i). \tag{3.3}$$

The best reply of partner i is $a_i = \theta_i y^* = a_i^*$, and efficiency is sustained.

Example C (Increasing Production Function, Compact Action Sets)

This example indicates how approximate efficiency is sustainable. Its linear-quadratic nature is unimportant; its logic is shown below (Theorem 2) to hold for any increasing production function if one action set contains extreme points.

Let $n = 2$, $A_i = [0, 2]$, $f(a) = a_1 + a_2$, and $v_i(a_i) = a_i^2/2$. Then $a^* = (1, 1)$, and it is not sustainable (Corollary 2 below). We construct a sharing rule such that the following mixed strategy profile P is an equilibrium: partner 2 chooses a_2^*, so that $P_2(1) = 1$, and partner 1 takes each extremal action with probability δ, and a_1^* with probability $1 - 2\delta$:

$$P_1(0) = P_1(2) = \delta \quad \text{and} \quad P_1(1) = 1 - 2\delta. \tag{3.4}$$

These strategies converge to the efficient a^* as $\delta \to 0$.

The interval of outputs partner 1 can generate by taking actions in $A_1 = [0, 2]$ when partner 2 chooses $a_2^* = 1$ is $[1, 3]$. So, to deter deviations by partner 1 from P, we need only define the sharing rule appropriately on [1, 3]. This is easy; for example, let

$$s_1(y) = (y-1)^2/2 \quad \text{and} \quad s_2(y) = y - s_1(y) \qquad \text{for } y \in [1, 3]. \tag{3.5}$$

Then $u_1(a_1, P_2) = 0$ for all $a_1 \in [0, 2]$, and so P_1 is a best reply to P_2.

When P is played, the smallest output with positive probability is $f(0, 1) = 1$, and the largest is $f(2, 1) = 3$. Output has positive probability of not being in [1, 3] if partner 2 deviates to any $a_2 \neq 1$. We extend the sharing rule so that then partner 2 pays partner 1 a fine:

$$s_1(y) = y + F \quad \text{and} \quad s_2(y) = -F \quad \text{for } y \notin [1, 3]. \tag{3.6}$$

Setting $F \geq (1-\delta)/2\delta$ suffices to deter partner 2 from deviating, in which case strategy P is an equilibrium.

This example suggests the importance of unlimited liability. To deter partner 2 from deviating, the fine he pays if $y \notin [1, 3]$ must be large if δ is small. To see this, fix F and $a_2 \in (0, 1)$, and observe that

$$\begin{aligned} \lim_{\delta \to 0} [u_2(P_1, a_2) - u_2(P)] &= [s_2(1+a_2) - a_2^2/2] - [s_2(2) - 1/2] \\ &= (1-a_2)a_2 > 0. \end{aligned}$$

Thus, for P_2 to remain a best reply to P_1, it is necessary for $F \to \infty$ as $\delta \to 0$.

4. Efficient Partnerships

In this section we characterize partnerships that sustain efficiency. To start, let Y_i denote the set of outputs that partner i can achieve by a unilateral deviation from a^*:

$$Y_i \equiv \{y \in \Re \mid f(a_i, a_{-i}^*) = y \text{ for some } a_i \in A_i\}. \tag{4.1}$$

The set of outputs which do not reveal the identity of a non-deviator after a unilateral deviation from a^* is then,

$$Y \equiv \bigcap Y_i. \tag{4.2}$$

Note that $y^* \in Y$. Example A indicated that efficiency is sustainable if $Y = \{y^*\}$.

The most problematic deviations are those which achieve any given output at lowest cost to the deviator. Accordingly, define a cost function for partner i on Y_i by,

$$c_i(y) \equiv \inf\{v_i(a_i) \mid f(a_i, a_{-i}^*) = y,\ a_i \in A_i\}. \tag{4.3}$$

Observe that $c_i(y^*) = v_i(a_i^*)$, since $a_i = a_i^*$ maximizes $f(a_i, a_{-i}^*) - v_i(a_i)$.

If output were shared equally, the most partner i could gain by a unilateral deviation from a^* that gave rise to output y would be,

$$g_i(y) \equiv [\, y/n - c_i(y)\,] - [\, y^*/n - v_i(a_i^*)\,]. \tag{4.4}$$

Equal sharing would sustain efficiency if $g_i(y) \leq 0$ for each $i \in N$ and $y \in Y_i$. Theorem 1 shows that a weaker sufficient condition is that the average gain from deviating be nonpositive, where the *average gain* from deviating so as to achieve $y \in Y$ is,

$$g(y) \equiv \left(\frac{1}{n}\right) \Sigma g_i(y) \;=\; \left(\frac{1}{n}\right) [y - \Sigma c_i(y) - W(a^*)]. \tag{4.5}$$

LEMMA 1: (*i*) *For each* $i \in N$, *output* y^* *maximizes* $y - c_i(y)$ *on* Y_i.

(*ii*) *For each* $y \in Y$, $g(y) \le \left(\frac{n-1}{n}\right)(y^* - y)$.

PROOF: A unilateral deviation by partner i cannot increase welfare: for $y \in Y_i$,

$$W(a^*) \;=\; y^* - \Sigma v_j(a_j^*) \;\ge\; y - c_i(y) \;-\; \sum_{j \neq i} v_j(a_j^*). \tag{4.6}$$

Hence, $y^* - v_i(a_i^*) \ge y - c_i(y)$, proving (*i*). If also $y \in Y$, these inequalities can be added to yield the following inequality, the rearrangement of which yields (*ii*):

$$(n-1)y^* + W(a^*) \;\ge\; ny - \Sigma c_i(y). \quad \blacksquare$$

THEOREM 1: *Efficiency is sustainable if and only if*

$g(y) \le 0$ *for each* $y \in Y$ *satisfying* $y < y^*$.

PROOF: (*Necessity*) Let $y \in Y$. Because partner i does not want to deviate so as to generate y,

$$s_i(y^*) - v_i(a_i^*) \;\ge\; s_i(y) - c_i(y).$$

Sum over i and use budget balance to obtain $W(a^*) \ge y - \Sigma c_i(y)$, and hence $g(y) \le 0$.

(*Sufficiency*) For $y \in Y$, define

$$s_i(y) \;\equiv\; y^*/n - v_i(a_i^*) + c_i(y) + g(y). \tag{4.7}$$

Note that $\Sigma s_i(y) = y$, by how $g(y)$ is defined in (4.5). For $y \notin Y$, the set of partners who can produce y by a unilateral deviation from a^* is $N(y) \equiv \{i \mid y \in Y_i\}$, a proper subset of N. The number of other partners is $m(y) \equiv n - |N(y)|$. For $y \notin Y$, define $s(y)$ by,

$$i \in N(y) \;\Rightarrow\; s_i(y) \equiv y^*/n - v_i(a_i^*) + c_i(y), \tag{4.8}$$

$$i \notin N(y) \;\Rightarrow\; s_i(y) \equiv \left(\frac{1}{m(y)}\right)[y - \sum_{j \in N(y)} s_j(y)]. \tag{4.9}$$

This defines a sharing rule. If a^* is played, shares are determined by (4.7) because $y^* \in Y$. So $s_i(y^*) = y^*/n$, since $g(y^*) = 0$ and $c_i(y^*) = v_i(a_i^*)$. The resulting utility of partner i is $u_i(a^*) = y^*/n - v_i(a_i^*)$. If he deviates to a_i, the output is some $y \in Y_i$, and his share is given by (4.7) or (4.8). By hypothesis and Lemma 1, $g(y) \leq 0$. The share of partner i when he so deviates is thus no greater than (4.8), and his utility satisfies,

$$\begin{aligned} u_i(a_i, a_{-i}^*) &\leq s_i(y) - c_i(y) \\ &\leq [y^*/n - v_i(a_i^*) + c_i(y)] - c_i(y) \;=\; u_i(a^*). \end{aligned}$$

This shows that a^* is an equilibrium of $\Gamma(s)$. ■

Naturally, the efficiency of Examples A and B above is consistent with Theorem 1. In Example A, the output realized after any unilateral deviation reveals the identity of a non-deviator. Hence, $Y = \{y^*\}$ and the condition of Theorem 1 holds vacuously. Holmström's (1982) observation that efficiency is sustainable if a third party is added to break budget balance is explained similarly: the third party is a partner with a single (null) action who cannot affect output, and so again $Y = \{y^*\}$. More generally, Theorem 1 implies that efficiency is sustainable if for some i, action

a_i^* minimizes $f(\cdot, a_{-i}^*)$ on A_i. In this case partner i cannot lower output by a unilateral deviation from a^*, and so y^* is minimal in Y.

In Example B production is Leontief, and so unilateral deviations from a^* achieve outputs in $Y = [0, y^*]$. To achieve $y \in Y$, partner i chooses $a_i = \theta_i y$ at cost $c_i(y) = v_i(\theta_i y)$. Because $y = y^*$ maximizes $y - \Sigma v_i(\theta_i y)$, the average gain from deviating,

$$g(y) = \left(\tfrac{1}{n}\right)[y - \Sigma v_i(\theta_i y) - W(a^*)]$$

$$= \left(\tfrac{1}{n}\right)[(y - \Sigma v_i(\theta_i y)) - (y^* - \Sigma v_i(\theta_i y^*))],$$

is nonpositive. Theorem 1 thus implies that efficiency is sustainable.

Efficiency is not sustainable in smooth partnerships, as Example C suggested. One version of this result is Holmström's (1982) Theorem 1, restated below as Corollary 2. It follows from Corollary 1, which applies to more general action spaces.

COROLLARY 1: *Efficiency is not sustainable if $y^* \in$ interior(Y) and each $c_i(\cdot)$ is differentiable at y^*.*

PROOF: The hypothesis implies that $g(\cdot)$ is differentiable at y^*. Because $g(y^*) = 0$, Lemma 1 implies that $g'(y^*) = -(\frac{n-1}{n}) < 0$. Hence, $y \in Y$ exists such that $y < y^*$ and $g(y) > 0$. Theorem 1 now implies the result. ■

COROLLARY 2: *Efficiency is not sustainable if each $A_i \subset \Re$, $a_i^* \in$ interior(A_i), $v_i(\cdot)$ and $f(\cdot)$ are C^1, and each $f_i(a^*) > 0$.*

5. Nearly Efficient Partnerships

In this section, approximate efficiency is shown to be sustainable in a broad class of partnerships, including those which Corollary 2 shows cannot sustain efficiency. The new assumptions are that actions are unidimensional and production is monotonic:

A1: $A_i \subset \Re$ for each $i \in N$,

A2: $f\colon A \to \Re$ is strictly increasing.[7]

Theorem 2 shows that approximate efficiency is sustainable if some partner has a smallest and a largest action. In the equilibrium constructed, this partner takes each extreme action with low probability, thereby insuring that with some probability, he is identifiable as a nondeviator if another partner deviates. The logic of the construction generalizes that of Example C.

THEOREM 2: *Approximate efficiency is sustainable if* A1 *and* A2 *hold, and* $\underline{a}_1 = \min(A_1)$ *and* $\bar{a}_1 = \max(A_1)$ *exist and are finite. If* $a_1^* \in (\underline{a}_1, \bar{a}_1)$ *and* $\delta \in (0, 1/2)$, *a fine* $F < \infty$ *exists such that the strategies defined by* (5.1) *are an equilibrium for the sharing rule defined by* (5.2):

$$P_1(\underline{a}_1) = P_1(\bar{a}_1) = \delta,\ \ P_1(a_1^*) = 1 - 2\delta,\ \ \textit{and}\ \ P_i(a_i^*) = 1\ \textit{for}\ i > 1. \tag{5.1}$$

$$\begin{aligned} y \in Y_1 &\ \Rightarrow\ s_1(y) = c_1(y)\ \text{ and }\ s_i(y) = (y - c_1(y))/(n-1) \text{ for } i > 1, \\ y \notin Y_1 &\ \Rightarrow\ s_1(y) = y + (n-1)F\ \text{ and }\ s_i(y) = -F \text{ for } i > 1. \end{aligned} \tag{5.2}$$

PROOF: Suppose $a_1^* = \underline{a}_1$. Then, since f is increasing in a_i, the minimal output in Y_1 and hence in Y is y^*. Full efficiency is sustainable, by Theorem 1.

Suppose $\underline{a}_1 < a_1^* < \bar{a}_1$. Let P and s be defined by (5.1) and (5.2) for some $\delta \in (0, 1/2)$ and F. If the others play P_{-1}, partner 1 can generate only outputs in Y_1, where the sharing rule exactly compensates him for his disutility. Hence, P_1 is a best reply to P_{-1}.

If P is played, output is surely in Y_1 and the fine is not paid. So, $u_i(P)$ is independent of F. Set $K = |y^* - c_1(y^*)|$, and choose F positive and so large that for each i,

$$F > [K - \underline{v} - u_i(P)]/\delta. \tag{5.3}$$

(Recall that $\underline{v}$ is a lower bound of each $v_i(\cdot)$.) Consider partner $i > 1$. For $y \in Y_1$, $s_i(y) = (y - c_1(y))/(n-1)$. Since $y - c_1(y) \leq y^* - c_1(y^*)$ by Lemma 1, $s_i(y) \leq K$. For $y \notin Y_1$, $s_i(y) = -F \leq K$. Thus, $s_i(y) \leq K$ for all y. Because f is strictly increasing, if $a_i < a_i^*$ then $f(\underline{a}_1, a_i, a_{-1,i}^*) < f(\underline{a}_1, a_{-1}^*) = \min(Y_1)$. Similarly, $f(\bar{a}_1, a_i, a_{-1,i}^*) > \max(Y_1)$ for $a_i > a_i^*$. This shows that if $a_i \neq a_i^*$, then F is paid with probability at least δ, and hence,

$$u_i(a_i, P_{-i}) \leq \delta(-F) + (1-\delta)K - v_i(a_i).$$

Use (5.3) to replace $-F$ in this inequality to obtain,

$$\begin{aligned} u_i(a_i, P_{-i}) &< -(K - \underline{v} - u_i(P)) + (1-\delta)K - v_i(a_i) \\ &\leq u_i(P) + \underline{v} - v_i(a_i) \leq u_i(P). \end{aligned}$$

This shows that P_i is a best reply to P_{-i}.

The remaining case is $\underline{a}_1 < a_1^* = \overline{a}_1$, for which the proof is the same as for the previous case after P_1 is redefined by $P_1(\underline{a}_1) = \delta$ and $P_1(a_1^*) = 1 - \delta$. ■

In Theorem 2, partner 1 takes his maximal action with small probability so as to insure that any other partner who takes too large an action will, with some probability, be detected. But if larger actions cause greater disutility, there are other ways of deterring partners from taking large actions, and no partner need have a maximal action. This is shown in Theorem 3, which is proved in the Appendix. Its assumptions strengthen A1 and A2:

A3: Each $A_i = [0,\infty)$.

A4: Each v_i is strictly increasing and C^1.

A5: The production function is strictly increasing and C^1, and the partial derivatives at a^* are positive: $f_i(a^*) > 0$ for each $i \in N$.

THEOREM 3: *Approximate efficiency is sustainable if* A3 – A5 *hold. If each* $a_i^* > 0$, *then* $\underline{\delta} > 0$ *exists such that for any* $\delta \in (0, \underline{\delta})$, $F < \infty$ *exists such that, for the s defined by (5.2) above, the following is an equilibrium*:

$$P_1(0) = \delta,\ \ P_1(a_1^*) = 1 - \delta,\ \ \textit{and}\ \ P_i(a_i^*) = 1 \textit{ for } i > 1. \tag{5.4}$$

Under the assumptions of Theorem 3, a continuous sharing rule also sustains approximate efficiency. The jump discontinuities in (5.2) can be approximated by a continuous function without destroying incentives. The details of this are left to the reader.

6. Limited Liability

If the partners cannot commit to pay large sums, they cannot adopt a sharing rule which imposes too much liability, even off the equilibrium path. If the liability each partner can bear is sufficiently large, efficiency is still sustainable in those partnerships which satisfy the condition of Theorem 1 (punishing fines are not used there). But in other partnerships limited liability does restrict the degree to which efficiency can be approximated, as we show in this section.

More notation is needed. The most that partner i could possibly pay given sharing rule s, the *liability* that s imposes on partner i, is

$$\ell_i(s) \equiv \sup \{-s_i(y) \mid y \in f(A)\}. \tag{6.1}$$

The *total liability* imposed by the sharing rule is $L(s) \equiv \Sigma \ell_i(s)$.

Only total liability is relevant for approximate efficiency (assuming again that the partners are risk neutral). This is because adding a constant to a sharing rule has no incentive effect. Thus, suppose P is an equilibrium of $\Gamma(s)$. Suppose the desired liabilities of the partners are $\hat{\ell}_1, \ldots, \hat{\ell}_n$, and the total liability of s is equal to the desired total liability: $L(s) = \Sigma \hat{\ell}_i$. Then a sharing rule $\hat{s}$ which has the desired liabilities, and for which P is also an equilibrium, is defined by $\hat{s}_i(y) \equiv s_i(y) + \ell_i(s) - \hat{\ell}_i$.

We make two simplifying assumptions. First, we strengthen slightly the assumption that the efficient action profile is unique. Assumption A6 below holds, for example, if $W(\cdot)$ is continuous and strictly concave, or if it has a finite range and a unique maximizer.

A6: For all large integers m, the sets defined by,

$$K^m \equiv \{a \in A \mid W(a) \geq W(a^*) - \frac{1}{m}\}, \tag{6.2}$$

are compact and satisfy $K^m \to \{a^*\}$ as $m \to \infty$.

The second simplifying assumption is that only a finite number of output levels can be produced. We believe this assumption can be weakened, but doing so will require delicate measure-theoretic arguments. The proof of the following result is in the Appendix.

THEOREM 4: *Assume* A6, *each* $v_i(\cdot)$ *is continuous, and the range of* f *is finite. Let* $\{(s^m, P^m)\}$ *be a sequence of sharing rules and strategies such that* P^m *is an equilibrium of* $\Gamma(s^m)$, *and* $\mathcal{E}_{P^m} W(\tilde{a}) \to W(a^*)$. *Then either the partnership sustains efficiency, or* $L(s^m) \to \infty$.

7. Issues and Extensions

We have shown that efficiency is sustainable in a significant class of deterministic partnerships, and that in an extensive class of such partnerships, the degree to which efficiency can be approximated is restricted only by liability/wealth constraints. Our concluding comments concern mixed strategies, multiple equilibria, optimal schemes, risk aversion, stochastic production, endogenous partnerships, and collusion.

Interpreting Mixed Effort Strategies

The mixed strategy used by one of the partners in an approximately efficient equilibria deserves some comment. Our preferred interpretation is that the mixing probabilities reflect the other players' uncertainty about the action of the mixing partner, not that he literally chooses stochastically. By an argument of Harsanyi (1973),[8] the equilibrium is the limit of pure strategy equilibria of incomplete information games in which the mixing partner's payoff function is unknown to the others. In our theorems, he chooses his minimal action with small probability, and perhaps also his maximal action with small probability. This suggests an intuitive game of incomplete information in which the other partners believe with small probability that the mixing partner is a "slacker" (large marginal disutility) type or a "workaholic" (small marginal disutility) type.

Multiple Equilibria

For fixed sharing rules, multiple equilibria may exist, in which case the equilibrium actually played might not be the most efficient one. For example, given the sharing rule defined by (3.5) and (3.6) in Example C, a pure strategy equilibrium is $(a_1, a_2) = (0, 1)$. This equilibrium is quite inefficient. Our feeling is that such an equilibrium is unlikely to be played, since it is Pareto dominated by the approximately efficient equilibrium, and since pre-play communication at the time the sharing rule is adopted should allow ample opportunity for the partners to coordinate on a good equilibrium to play in the production stage. That said, it would be interesting to determine whether mechanisms can be designed which induce unique approximately efficient effort strategies.

Optimal Schemes

Ex ante, the partners can be expected to agree to an optimal sharing rule and corresponding equilibrium efforts. However, if approximate efficiency but not full efficiency is sustainable, Theorems 2 – 4 indicate that an optimal sharing rule and effort equilibrium does not exist if liability is unlimited: a nonconvergent, Pareto-increasing sequence is obtained by taking the fine F to infinity and the probability δ to zero. An optimal sharing rule and effort equilibrium should exist, however, if liability is limited. We have no presumption that the optimal scheme satisfying liability constraints would resemble the kind of schemes we have discussed here. We leave the determination of optimal schemes in such circumstances to future work.

Risk Aversion

The risk imposed by a mixed strategy equilibrium is costly if partners are risk averse. Nonetheless, our approximate efficiency results hold if payoff functions are of the form $h_i(s_i) - v_i(a_i)$ or $h_i(s_i - v_i(a_i))$, with $h_i(\cdot)$ concave. The "moving support" logic still applies. To see this, note that in the equilibria of Theorems 2 and 3, the partners pay the large fine only off the equilibrium path. The fine is paid with zero probability in equilibrium, and it is the only part of the sharing rule that grows as

$\delta \rightarrow 0$. Thus, the additional welfare loss due to the risk imposed by a mixed strategy equilibrium becomes negligible as the probabilities collapse onto an efficient outcome. Details are left to the reader.[9]

Stochastic Production

If output is a stochastic function of actions, efficiency is sustainable under certain conditions.[10] When these conditions are not met, approximate efficiency may still be sustainable. We have no results on when this is the case, but it is clear that the "moving support" logic used in this paper will not apply if the support of the output distribution is the same for each action profile.

Endogenous Partnerships

Our results indicate that how well a partnership can overcome moral hazard depends on the nature of its technology and set of partners. In an environment in which these variables are endogenous, there should be a tendency for only the most efficient possible partnerships to form. The approximate efficiency results suggest a tendency for endogenous partnerships to consist of wealthy individuals who can hold large liabilities. The full efficiency results suggest a tendency for partnerships to consist of partners whose actions are strongly complementary, or to contain a partner who is productively passive, the "third-party budget breaker" of Holmström (1982).

Whether these tendencies are realized depends on the environment. Consider, for example, the tendency to bring a third party into a partnership as a budget breaker.

Efficiency can be attained in this way only if the third party is unable to decrease output (sabotage the firm). This is because the sharing rules that sustain efficiency by using a budget breaker have the property that at some outputs less than the efficient quantity, the budget breaker is paid more than he is at the efficient quantity — the budget breaker is given a preference for low output.[11] In many environments, it may be impossible to easily find a third party who will be unable to destructively interfere with working partners, or who can commit or otherwise be prevented from so interfering. Alternatively, in many environments the opportunity wage of such a third party may be too large for his or her employment to be worthwhile.[12] Casual empiricism suggests that third-party budget breakers may be difficult to find: actual third parties (banks, limited partners, etc.) rarely seem to want the profits of their partnerships to be low.[13] Consequently, the use of a budget breaker to achieve efficiency may be difficult to arrange, and approximate efficiency may be the best a partnership can achieve.

Collusion

Another issue is whether approximate efficiency is sustainable if some partners can collude. The sharing rule used to prove Theorems 2 and 3 is not robust to collusion. That rule distinguishes one partner, and has the others each pay him a fine F if he and at least one of them shirks. If F is large, as it must be if efficiency is closely approximated, and there are at least three partners, the distinguished partner and one other have an incentive to collude. The payment $(n-2)F$ from the non-colluding partners to the distinguished partner is so large that he can successfully bribe the other colluding partner to join him in shirking, outweighing the resulting loss of output. (This is also true of the scheme in which a third party is rewarded if the partnership performs poorly, as the third party has an incentive to bribe a working partner to shirk.)

Of course, as Tirole (1992) discusses, whether collusion is possible is problematic, particularly when reputation issues do not arise. Colluding partners must somehow sign (probably illegal) binding contracts that specify unverifiable (and perhaps mutually unobservable) actions and contingent side payments.

Appendix

PROOF OF THEOREM 3

By A3 – A5, the cost function of partner i for unilaterally deviating is $c_i(y) = v_i(\varphi_i(y))$, where $\varphi_i\colon Y_i \to A_i$ is the inverse function defined by $f(\varphi_i(y), a^*_{-i}) \equiv y$. The derivative $c_i'(y)$ exists in a neighborhood of y^*, and since y^* maximizes $y - c_i(y)$ on Y_i,

$$c_i'(y^*) \geq 1, \text{ with equality if } a_i^* > 0. \qquad \text{(a1)}$$

As f is increasing, Theorem 1 implies that efficiency is sustained if some $a_i^* = 0$. So assume each $a_i^* > 0$. Let P be defined by (5.4) for some $\delta \in (0, 1/2)$. Let s be defined by (5.2) for some positive F satisfying (5.3), where again $K = |\, y^* - c_1(y^*)\, |$. The arguments used to prove Theorem 2 then show that partner 1 has no incentive to unilaterally deviate from P, and each partner $i > 1$ has no incentive to deviate to some $a_i < a_i^*$. We now show that for small δ, each $i > 1$ has no incentive to deviate to some $a_i > a_i^*$.

Fix $i > 1$. The positive probability outputs if i deviates from P by choosing a_i are $x(a_i) \equiv f(0, a_i, a^*_{-1,i})$ and $z(a_i) \equiv f(a_i, a^*_{-i})$. By A5, both increase in a_i, and $x(a_i) < z(a_i)$. Observe that $Y_1 = [x(a_i^*), y^\infty)$, where $y^\infty \equiv \lim_{a_1 \to \infty} f(a_1, a^*_{-1}) \leq \infty$. If $z(a_i) \geq y^\infty$, then i pays the fine with probability at least $1 - \delta$. Since $1 - \delta > \delta$ (as $\delta < 1/2$), the argument of Theorem 2 shows that this deviation is unprofitable. Hence, the deviating interval of actions of concern is (a_i^*, a_i^∞), where $a_i^\infty \equiv \sup\{a_i \,|\, z(a_i) < y^\infty\}$.

For $a_i \in (a_i^*, a_i^\infty)$, both $x(a_i)$ and $z(a_i)$ are in Y_1, and so the shares are determined by the top line of (5.2).

The gain to i from deviating to $a_i \in (a_i^*, a_i^\infty)$ is

$$\gamma_i(a_i) \equiv u_i(a_i, P_{-i}^*) - u_i(P) = \delta\hat{A}(a_i) + (1-\delta)\hat{B}(a_i), \tag{a2}$$

where,

$$\hat{A}(a_i) \equiv (n-1)^{-1}[x(a_i) - x(a_i^*) + c_1(x(a_i^*)) - c_1(x(a_i))] + v_i(a_i^*) - v_i(a_i), \tag{a3}$$

$$\hat{B}(a_i) \equiv (n-1)^{-1}[z(a_i) - z(a_i^*) + c_1(z(a_i^*)) - c_1(z(a_i))] + v_i(a_i^*) - v_i(a_i). \tag{a4}$$

As v_1, v_i, and f are increasing, $c_1(x(a_i^*)) < c_1(x(a_i))$ and $v_i(a_i^*) < v_i(a_i)$. So, from (a3),

$$\hat{A}(a_i) \leq (n-1)^{-1}(x(a_i) - x(a_i^*)) \equiv A(a_i). \tag{a5}$$

Because $v_i(a_i^*) < v_i(a_i)$,

$$\begin{aligned}\hat{B}(a_i) &\leq (n-1)^{-1}[z(a_i) - z(a_i^*) + c_1(z(a_i^*)) - c_1(z(a_i)) + v_i(a_i^*) - v_i(a_i)] \\ &= (n-1)^{-1}[(z(a_i) - v_i(a_i)) - (z(a_i^*) - v_i(a_i^*))] \\ &\qquad - (n-1)^{-1}(c_1(z(a_i)) - c_1(z(a_i^*))).\end{aligned} \tag{a6}$$

The second square-bracketed term in (a6) is nonpositive, as a_i^* is efficient. Hence,

$$\hat{B}(a_i) \leq -(n-1)^{-1}(c_1(z(a_i)) - c_1(z(a_i^*))) \equiv B(a_i). \tag{a7}$$

So, $\gamma_i(a_i) \leq \delta A(a_i) + (1-\delta)B(a_i)$. Note that $B(a_i) < 0$, since $a_i > a_i^*$. Consequently, $\gamma_i(a_i) \leq 0$ if δ is so small that,

$$\frac{1-\delta}{\delta} \geq \frac{A(a_i)}{-B(a_i)} = \frac{x(a_i) - x(a_i^*)}{c_1(z(a_i)) - c_1(z(a_i^*))} \equiv R(a_i). \tag{a8}$$

We show that (a8) holds for small δ by showing that $R(\cdot)$ is bounded above on (a_i^*, a_i^∞). Since $R(\cdot)$ is continuous on this interval, we need only to show that it is bounded at the endpoints. Because $x(a_i) < z(a_i)$, and y^* maximizes $y - c_1(y)$ on Y_1, we know that

$$x(a_i) - c_1(z(a_i)) < z(a_i) - c_1(z(a_i)) \leq y^* - c_1(y^*).$$

Hence,

$$R(a_i) \leq \frac{c_1(z(a_i)) + y^* - c_1(y^*) - x(a_i^*)}{c_1(z(a_i)) - c_1(z(a_i^*))}. \tag{a9}$$

Now take $a_i \rightarrow a_i^\infty$. Then $z(a_i) \rightarrow y^\infty$. The right side of (a9) stays finite: it converges to 1 if $c_1(y) \rightarrow \infty$ as $y \rightarrow y^\infty$, and otherwise it converges to the finite number obtained by replacing $c_1(z(a_i))$ by its finite limit. To show that $R(\cdot)$ is bounded at the lower endpoint a_i^*, we use l'Hospital's rule, since $A(a_i^*) = B(a_i^*) = 0$. As $a_i \rightarrow a_i^*$, $z(a_i) \rightarrow y^*$. So, from A5 and (a1),

$$\lim_{a_i \rightarrow a_i^*} R(a_i) = \lim_{a_i \rightarrow a_i^*} \frac{x'(a_i)}{c_1'(z(a_i))z'(a_i)}$$

$$= \frac{f_i(0, a_{-1}^*)}{f_i(a^*)} < \infty.$$

Thus, $\delta_i \in (0, 1/2)$ exists for which (a8) holds for all $a_i \in (a_i^*, a_i^\infty)$ and $\delta \in (0, \delta_i)$. Letting $\underline{\delta}$ be the smallest δ_i finishes the proof. ■

PROOF OF THEOREM 4

Assume $\{L(s^m)\}$ is bounded above by $L < \infty$. We must show that efficiency is sustainable. We first show that $\{(s^m, P^m)\}$ converges.

Taking a subsequence if necessary, we can assume that

$$\mathcal{E}_{P^m} W(\tilde{a}) \ \geq \ W(a^*) - \frac{1}{m^2}. \tag{a10}$$

By how K^m is defined, $\mathcal{E}_{P^m} W(\tilde{a}) \ \leq \ W(a^*) - (1 - P^m(K^m))\left(\frac{1}{m}\right)$. Hence, from (a10),

$$P^m(K^m) \geq 1 - \frac{1}{m.} \tag{a11}$$

From this and A6, $P^m \to P^*$, where P^* puts all probability on a^*.

Because only total liability matters, we can assume individual liabilities are equal: $\ell_i(s^m) = L(s^m)/n$ for each m and i. Then, because of budget balance and $L(s^m) \leq L$,

$$-\left(\frac{1}{n}\right)L \ \leq \ s_i^m(y) \ \leq \ y + \left(\frac{n-1}{n}\right)L$$

for each $y \in f(A)$. Since $f(A)$ has only a finite number of points, this implies (taking a subsequence if necessary) that $\{s^m\}$ converges uniformly to a sharing rule s^*.

We now show that a^* is an equilibrium of $\Gamma(s^*)$. For some i, let $a_i \in A_i$. Because P^m is an equilibrium of $\Gamma(s^m)$,

$$\mathcal{E}_{P^m} s_i^m(f(a_i, \tilde{a}_{-i})) - v_i(a_i) \ \leq \ \mathcal{E}_{P^m}[s_i^m(\tilde{a}) - v_i(\tilde{a}_i)]. \tag{a12}$$

The rule s^* is trivially continuous on $f(A)$, since $f(A)$ is finite. This, the uniform convergence of $\{s^m\}$ to s^*, and the weak convergence of $\{P^m\}$ to the measure P^* degenerate at a^* imply that $\mathcal{E}_{P^m} s_i^m(f(a_i, \tilde{a}_{-i})) \to s_i^*(f(a_i, a_{-i}^*))$ and

$\mathcal{E}_{P^m} s_i^m(\tilde{a}) \to s_i^*(f(a^*))$. Because v_i is continuous, $\mathcal{E}_{P^m} v_i(\tilde{a}_i) \to v_i(a_i^*)$. Therefore, taking limits in (a12) yields,

$$s_i^*(f(a_i, a_{-i}^*)) - v_i(a_i) \le s_i^*(f(a^*)) - v_i(a_i^*).$$

This proves that a^* is an equilibrium of $\Gamma(s^*)$. ■

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Footnotes

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[1] This is discussed also in Holmström (1979), footnote 7.

[2] Williams and Radner (1988), Matsushima (1989), and Legros and Matsushima (1991).

[3] Rasmusen (1987).

[4] Radner (1986)

[5] Summations without indices are summations over $i \in N$.

[6] Theorem 1 below implies that efficiency is sustainable in a Leontief partnership for a wide range of disutility functions.

[7] Since f must strictly increase on the boundary of A, our approximate efficiency results do not apply to production functions like Cobb-Douglas. Still, we conjecture that they do.

[8] This is discussed on pages 230-237 in Fudenberg and Tirole (1991).

[9] If random sharing rules are feasible and partners are risk averse, full efficiency may be sustainable. This is accomplished, for example, by choosing one partner at random and having the others pay him a large fine if output is too low. See Rasmusen (1987).

[10] Williams and Radner (1988), Matsushima (1989), and Legros and Matsushima (1991).

[11] For example, in Holmström's (1982) scheme each partner pays the third party a large fine if the realized output is below its efficient level.

[12] In a general equilibrium setting, these opportunity costs are endogenous. Their equilibrium levels are determined by market interactions and initial endowments, which consequently influence the types of organization that arise. This is shown in Legros and Newman (1993).

[13] Another reason why third parties may not be used is the possibility of collusion between them and working partners, as is discussed below. Also, if the production function is stochastic with a non-moving support, adding a third party does not achieve efficiency.