Discussion Paper No. 986

CANONICAL REPRESENTATION OF SET FUNCTIONS*

by

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and

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April 1992

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*We wish to thank Gaby Debs, Dieter Denneberg, Didier Dubois, Sergiu Hart, Gerald Hanweck, Ehud Kalai, Ehud Lehrer, Bart Lipman, Abraham Neyman, and Benjamin Weiss for comments and discussions. Part of this work was done while the second author was visiting the Santa Fe Institute in Santa Fe, New Mexico, and the Autonomous University of Barcelona, Spain. The author is grateful to these institutions for their hospitality. NSF Grants Nos. SES-9113108 (Gilboa) and SES-9111873 (Schmeidler) are gratefully acknowledged.

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Abstract

The representation of a cooperative transferable utility game as a linear combination of unanimity games may be viewed as an isomorphism between not-necessarily additive set functions on the players space and additive ones on the coalitions space.

We extend the unanimity-basis representation to general (infinite) spaces of players, study spaces of games which satisfy certain properties and provide some conditions for σ-additivity of the resulting additive set function (on the space of coalitions). These results also allow us to extend some representations of the Choquet integral from finite to infinite spaces.

1. Introduction

Real valued set functions, which are not necessarily additive, are extensively used in decision theory. In one interpretation they represent a transferable utility cooperative game; in another--non-additive probabilities and belief functions. In yet other models these functions--and Choquet integration with respect to them--appear as representing decision rules for multi-criteria decision problems, and, in particular, multi-period and social choice problems.

It is well-known that the set of "unanimity" games is a linear basis for the space of such functions (i.e., games) in the case of finitely many players. In Gilboa-Schmeidler (1992) we discuss some implications and interpretations of this "canonical representation" of games, and provide several results which are all rather simple consequences of this representation.

The purpose of this paper is to extend the analysis to the general case, of possibly infinitely many players. Note that while infinitely many agents in a decision problem may be simply a matter of mathematical convenience, in the context of decisions under uncertainty infinitely many states of the world are almost a logical necessity. (See Savage (1954), who suggests that a state of the world would "resolve all uncertainty.")

The first goal is, therefore to provide a canonical representation theorem for the general case. We show that every game v can be represented as a linear combination of unanimity games according to a finitely additive signed measure μ_v (on the algebra of set coalitions). We introduce a new norm on games, and show that with respect to this ("composition") norm, the spaces of games for which μ_v is bounded and/or σ-additive are Banach spaces. Further, the space of games with bounded composition norm consists of precisely those games which are differences of totally monotone games.

We also provide sufficient conditions for μ_v to be σ-additive and show that all games v which are polynomials in measures would indeed have a σ-additive μ_v.

Finally, we provide two results reinterpreting the Choquet integral. The first shows that for every game v there are sets of finitely additive measures, C^+ and C^-, such that the integral of any function f w.r.t. (with respect to) v is simply the difference between its minimal integral w.r.t. measures in C^+ and the minimal one w.r.t. C^-. The second result states (loosely) that if v is totally monotone, the Choquet integral may be represented as minimum of means or as mean of minima.

Since all these results appear, for the finite case, in Gilboa-Schmeidler (1992), we shall not expatiate on them here. The reader is referred to the above for discussions, interpretations and many additional references.

This paper is organized as follows. Section 2 provides basic definitions and quotes some known results. Section 3 presents the main results, the proofs of which are to be found in Section 4. Finally, Section 5 concludes with a few remarks.

2. Preliminaries

Let Ω be a nonempty set of _players_ or _states of the world_ and let Σ be an algebra of _coalitions_ or _events_ defined on it. We do not assume that Σ is a σ-algebra unless specifically stated. If Σ is finite, we will assume w.l.o.g that so is Ω and that $\Sigma = 2^{\Omega}$.

The following definitions are formulated for the players space (Ω,Σ). However, they will be understood to apply to any measurable space and, in particular, to the space of coalitions to be introduced in the sequel.

A function $v: \Sigma \to \mathbf{R}$ with $v(\emptyset) = 0$ is called a <u>game</u> or a <u>capacity</u>. The space of all games will be denoted by V and will be considered as a linear space (over $\mathbf{R}$) with the natural (pointwise) operations. Similarly, the product of two or more games is to be construed as a pointwise operation.

For $v \in V$ we will use the following definitions:

(1) v is <u>monotone</u> if $A \subseteq B$ implies $v(A) \le v(B)$ for all $A,B \in \Sigma$.

(2) v is <u>normalized</u> if $v(\Sigma) = 1$.

(3) v is <u>additive</u> if $v(A \cup B) = v(A) + v(B)$ for all $A,B \in \Sigma$ with $A \cap B = \emptyset$. Such a v is also called a <u>signed finitely additive measure</u>.

(4) v is <u>σ-additive</u> if $v(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$ whenever $A_i \in \Sigma$, $\cup_{i=1}^{\infty} A_i \in \Sigma$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Such a v is also called a <u>signed measure</u>.

(5) v is <u>convex</u> if for every $A,B \in \Sigma$, $v(A \cup B) + v(A \cap B) \ge v(A) + v(B)$. It is <u>superadditive</u> if the above holds for all $A,B \in \Sigma$ with $A \cap B = \emptyset$. v is <u>concave</u> or <u>subadditive</u> if the converse inequalities hold, respectively.

(6) v is <u>nonnegative</u> if $v(A) \ge 0$ for all $A \in \Sigma$.

(7) v is <u>totally monotone</u> if it is nonnegative and, for every $n \ge 1$ and $A_1, \ldots A_n \in \Sigma$, $v(\cup_{i=1}^{n} A_i) \ge \sum_{\{I|\phi \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} v(\cap_{i \in I} A_i)$.

(8) v is a <u>finitely additive measure</u> if it is nonnegative and additive.

(9) v is a <u>measure</u> if it is nonnegative and σ-additive.

(10) v is <u>outer continuous</u> if for all $\{A_i\}_{i\geq 1} \subseteq \Sigma$ $A_{i+1} \subseteq A_i$, $\cap_{i\geq 1} A_i \in \Sigma$ $\exists \lim_{i\to\infty} v(A_i) = v(\cap_{i\geq 1} A_i)$.

Observe that additive games are totally monotone, totally monotone are convex and convex are superadditive.

Given a real valued function $f: \Omega \to \mathbf{R}$, it is said to be <u>measurable</u> if, for every $\alpha \in \mathbf{R}$, $\{\omega | f(\omega) \geq \alpha\}$ and $\{\omega | f(\omega) > \alpha\}$ are elements of Σ. The set of all bounded measurable functions will be denoted by F. In general, it does not have to be a linear space if Σ is not a σ-algebra. (This was noted by Wakker (1990).)

A function $f \in F$ is said to be <u>simple</u> if $f = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ where $\alpha_i \in \mathbf{R}$, $A_i \in \Sigma$ and 1_B is the indicator function of $B \in \Sigma$. The set of simple functions is denoted F_0.

For $v \in V$ and $f \in F$, the <u>Choquet integral</u> of f w.r.t. (with respect to) v is defined to be

$$\int f dv = \int_0^\infty v(\{\omega | f(\omega) \geq t\}) dt + \int_{-\infty}^0 [v(\{\omega | f(\omega) \geq t\}) - v(\Omega)] dt.$$

Note that it is well-defined if v is monotone and f is bounded. Also, it is always well-defined if Σ is finite. Finally, observe that this definition coincides with the standard one if v is additive.

For $v \in V$ we define the <u>core</u> to be

$$\begin{aligned} \mathrm{Core}(v) = \{p|\ &(i)\ \ p \text{ is a finitely additive measure;} \\ &(ii)\ \ p(A) \geq v(A),\ \forall\ A \in \Sigma; \\ &(iii)\ \ p(\Omega) = v(\Omega)\}. \end{aligned}$$

Note that we allow for a finitely additive measure to be identically zero. For instance, if $v = 0$, $\mathrm{Core}(v) = \{v\}$.

It will be useful to denote $\Sigma' = \Sigma \setminus \{\emptyset\}$.

For $T \in \Sigma'$, define the <u>unanimity game on T</u> to be the game $u_T \in V$ defined by

$$u_T(A) = \begin{cases} 1 & A \supseteq T \\ 0 & otherwise. \end{cases}$$

We now turn to quote some known results.

<u>Theorem 2.1</u> (Shapley (1965)): Every convex game has a nonempty core.

<u>Theorem 2.2</u> (Rosenmuller (1971, 1972), Schmeidler (1984, 1986)): A monotone game v is convex if and only if

(i) $\mathrm{Core}(v) \neq \emptyset$;

(ii) for every $f \in F_0$ ($f \in F$)

$$\int f dv = \min_{p \in \mathrm{Core}(v)} \int f dp.$$

We now turn to quote some results which, to the best of our knowledge, exist in the literature for the finite case only. The first one is the "decomposition" or "canonical representation" theorem, which is the key to many other results.

<u>Theorem 2.3</u>: Suppose Σ is finite. Then $\{u_T\}_{T \in \Sigma'}$ is a linear basis for v. The unique coefficients $\{\alpha_T^v\}_{T \in \Sigma'}$ satisfying

$$v = \sum_{T \in \Sigma'} \alpha_T^v u_T$$

are given by

$$\alpha_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) = v(T) - \sum_{\{I | \phi \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} v(\cap_{i \in I} T_i)$$

where $T_i = T \setminus \{\omega_i\}$ and $T = \{\omega_1, \ldots, \omega_n\}$.

In the sequel, $\{\alpha_T^v\}$ will refer to the above coefficients whenever Σ is finite.

<u>Theorem 2.4</u>: Suppose Σ is finite. Then v is totally monotone iff $\alpha_T^v \geq 0$ for all $T \in \Sigma'$.

Theorem 2.4 is due to Dempster (1967) and Shafer (1976). Both Theorems 2.3 and 2.4 are generalized in Gilboa-Lehrer (1991) to real-valued functions defined on arbitrary finite lattices. One of the main goals of this paper is to extend them to set functions on infinite algebras.

In the finite case the canonical representation consists of summation over elements of Σ'. In other words, we were using a measurable space $(\Sigma', 2^{\Sigma'})$ and for each $v \in V$ we implicitly defined a signed measure on it by

$$\mu_v(A) = \Sigma_{T \in A}\, \alpha_T^v.$$

Then the decomposition theorem took the form

$$v = \sum_{T \in \Sigma'} \alpha_T^v u_T = \int_{\Sigma'} u_T d\mu_v(T).$$

In the general case we therefore need an algebra on Σ' for which a similar representation holds.

This algebra will be constructed as follows: for $T \in \Sigma$, define $\tilde{T} \subseteq \Sigma$ by

$$\tilde{T} = \{S \in \Sigma' | S \subseteq T\}.$$

Denote $\Theta = \{\tilde{T} | T \in \Sigma'\} \subseteq 2^{\Sigma'}$. Let Ψ be the algebra generated by Θ, and let $\hat{\Psi}$ be the σ-algebra generated by it. Thus $\Psi \subseteq \hat{\Psi} \subseteq 2^{\Sigma'}$. It is easy to see that these inclusions would be strict for large spaces (say, if $\Omega = [0,1]$ and $\Sigma = B([0,1])$.)

In case Σ is finite, we define the <u>composition norm</u> of v to be

$$\|v\| = \sum_{T \in \Sigma'} |\alpha_T^v|$$

<u>Theorem 2.5</u> (Gilboa-Schmeidler (1992)): Suppose Σ is finite. Then for every $v \in V$ there are unique totally monotone $v^+, v^- \in V$ such that

$$v = v^+ - v^-$$

and

$$\|v\| = \|v^+\| + \|v^-\|.$$

Furthermore, $\|v\| = v(\Omega)$ iff v is totally monotone.

We now extend the definition of the composition norm to the general case. Given a sub-algebra $\Sigma_0 \subseteq \Sigma$, and $v \in V$, let $v|_{\Sigma_0}$ denote the restriction of v to Σ_0. Then define, for $v \in V$,

$$\|v\| = \sup\{\|v|_{\Sigma_0}\| \mid \Sigma_0 \text{ is a finite sub-algebra of } \Sigma\}.$$

It is simple to check that $\|\bullet\|$ is indeed a norm. When no confusion is likely to arise, we will refer to it as "the norm."

We also note without proof that if v is additive, the composition norm of v coincides with the variation norm as defined, say, in Dunford-Schwartz (1957).

3. Statement of the Main Results

In this section we state the main results, which extend the canonical representation theorem and some of its implications.

Theorem A: For every $v \in V$ there exists a unique signed finitely additive measure μ_v on (Σ', Ψ) such that

$$(*) \qquad v = \int_{\Sigma'} u_T \, d\mu_v(T).$$

Furthermore $\|v\| = \|\mu_v\|$ and the mapping $v \mapsto \mu_v$ is linear and continuous. Conversely, every additive μ_v on Ψ defines $v \in V$ by $(*)$. Finally, v is totally monotone iff μ_v is nonnegative.

In the sequel, μ_v will always refer to the (signed finitely additive) measure on Ψ defined by v.

Let us now introduce the following subspaces of V:

$$V^b = \{v \in V \mid \|v\| < \infty\}$$

$$V^\sigma = \{v \in V \mid \mu_v \text{ is a } \sigma\text{-additive signed measure}\}.$$

(Equivalently, $V^\sigma = \{v \in V \mid \mu_v$ has a (unique) σ-additive extension to $\hat{\Psi}\}$.)

$$V^{b\sigma} = V^b \cap V^\sigma.$$

<u>Theorem B</u>: V^b, V^σ and $V^{b\sigma}$ are Banach spaces with respect to $\|\bullet\|$. Furthermore, $V^b = \{v^+ - v^- \mid v^+, v^-$ are totally monotone$\}$.

<u>Corollary</u> ("Min Minus Min"): Let $v \in V^b$. then there exist two sets of finitely additive measures, C^+ and C^-, which are convex and closed in the w*-topology, such that for all $f \in F_0$

$$\int_\Omega f dv = \min_{p \in C^+} \int_\Omega f dp - \min_{p \in C^-} \int_\Omega f dp.$$

(The proof uses Theorems B and 2.2.)

The following two theorems provide sufficient conditions for v to belong to V^σ.

First let us introduce the subspace of polynomials in (σ-additive) measures: define

$$p\sigma A = \{\textstyle\sum_{i=1}^{N} \alpha_i \prod_{j=1}^{K_i} \lambda_{ij} \mid N \geq 1, K_i \geq 1$$
$$\text{and } \alpha_i \in \mathbf{R} \text{ for } i \leq N \text{ and } \lambda_{ij} \text{ is a measure on } \Omega\}.$$

With this definition we may state:

<u>Theorem C</u>: $p\sigma A \subsetneq V^{b\sigma}$.

Next we consider the special case in which Ω is countable.

<u>Theorem D</u>: If Ω is countable, the mapping $v \mapsto \mu_v$ is a bijection from

$$\{v \in V \mid v \text{ is totally monotone and outer continuous}\}$$

onto

$$\{\mu \mid \mu \text{ is a measure on } \Psi\}.$$

We now proceed to discuss two additional results relating to the Choquet integral. These were also presented in Gilboa-Schmeidler (1992) for the case of a finite Σ.

<u>Theorem E</u>: (Related results appear in Choquet (1953-54), Murofushi-Sugeno (1989), and Wasserman (1990).) Let $v \in V^{b\sigma}$ and $f \in F$. Then

$$\int_\Omega f dv = \int_{\Sigma'} [\inf_{\omega \in T} f(\omega)] d\mu_v(T).$$

<u>Corollary</u> ("Mean of Mins and Min of Means"): Assume that $v \in V^\sigma$ is totally monotone and that $f \in F_0$. Then

$$\int_\Omega f dv = \int_{\Sigma'} [\inf_{\omega \in T} f(\omega)] d\mu_v(T) = \min_{p \in \mathrm{Core}(v)} \int_\Omega f dp.$$

4. <u>Proofs and Related Analysis</u>

4.1 <u>Proof of Theorem A</u>:

Let us define a <u>basic element</u> to be a subset of Σ' of the form

$$\tilde{A} \setminus \cup_{i=1}^{n} \tilde{B}_i$$

for some $A, B_i \in \Sigma'$ and $n \geq 0$. Note that $\tilde{\Omega} = \Sigma'$ is a basic element. We will agree that a representation of a basic element as above presupposes that $\emptyset \neq B_i \subsetneq A \ \forall\, i \leq n$, $B_i \not\subset B_j$ for $i \neq j$. Under these assumptions, the representation is unique.

Lemma 4.1.1: Every member of Ψ can be represented as the union of finitely many disjoint basic elements.

Proof: Recall that Ψ is the algebra generated by

$$\Theta = \{\tilde{A} \mid A \in \Sigma'\}.$$

Hence, every element of Ψ can be written in its disjunctive normal form as

$$\dot{\bigcup_{1 \leq i \leq n}} [(\bigcap_{j=1}^{k_i} \tilde{A}_{ij}) \cap (\bigcap_{j=1}^{\ell_i} (\tilde{B}_{ij})^c)]$$

(where $\dot{\cup}$ denotes disjoint union.) W.l.o.g. assume $k_i \geq 1$. (If $k_i = 0$, introduce $A_{i1} = \Omega$.)

It only remains to note that if $A = \cap_\alpha A_\alpha$, then $\tilde{A} = \cap_\alpha \tilde{A}_\alpha$. ∎

It will be useful to denote, for $\{B_i\}_{i=1}^n \subseteq \Sigma$,

$$\Delta_v(\{B_i\}_{i=1}^n) = \sum_{\emptyset \neq I \subseteq \{1\ldots n\}} (-1)^{|I|+1} v(\cap_{i \in I} B_i).$$

Let us now define μ_v on basic elements by

$$\mu_v(\tilde{A} \setminus \cup_{i=1}^{n} \tilde{B}_i) = v(A) - \Delta_v(\{B_i\}).$$

Next, extend μ_v to Ψ by additivity. Notice that this definition implies linearity of μ_v in v.

<u>Lemma 4.1.2</u>: μ_v is well-defined and additive on Ψ.

<u>Proof</u>: Let Σ_0 be a finite sub-algebra of Σ and let Ψ_0 be the corresponding sub-algebra of Ψ. By Theorem 2.3 one can see that for $A, B_1, \ldots, B_n \in \Sigma_0$,

$$\mu_v(\tilde{A} \setminus \cup_{i=1}^{n} \tilde{B}_i) = \sum_{\{T \in \Sigma_0 | T \subseteq A, T \not\subseteq B_i, i \leq n\}} \alpha_T^v$$

Since every two members of Ψ belong to Ψ_0 for some finite Σ_0, the desired conclusion follows. ■

<u>Lemma 4.1.3</u>:

$$(*) \qquad v = \int_{\Sigma'} u_T d\mu_v(T)$$

<u>Proof</u>: For every $s \in \Sigma$,

$$\int_{\Sigma'} u_T(S) d\mu_v(T) = \mu_v(\{T | u_T(S) = 1\}) = \mu_v(\{T | T \subseteq S\}) = \mu_v(\tilde{S}) = v(S). \quad ■$$

Next we have

<u>Lemma 4.1.4</u>: μ_v is the unique measure on Ψ satisfying (*).

Proof: Let μ be a measure satisfying (*). Obviously,

$$\mu(\tilde{S}) = \mu_v(\tilde{S}) = v(S) \text{ for all } S \in \Sigma.$$

Next consider a basic element $(\tilde{A} \setminus \cup_{i=1}^n \tilde{B}_i)$. Since μ is additive, it has to satisfy

$$\begin{aligned}\mu(\tilde{A} \setminus \cup_{i=1}^n \tilde{B}_i) \quad &= \mu(\tilde{A}) - \mu(\cup_{i=1}^n \tilde{B}_i) = \\ &= \mu(\tilde{A}) - \textstyle\sum_{\emptyset \neq I \subseteq \{1 \ldots n\}} (-1)^{|I|+1} \mu(\cap_{i \in I} \tilde{B}_i) \\ &= v(A) - \Delta_v(\{B_i\}_{i=1}^n) = \mu_v(\tilde{A} \setminus \cup_{i=1}^n \tilde{B}_i).\end{aligned}$$

which also implies $\mu_v = \mu$ throughout Ψ. ■

Next, observe that for any finite sub-algebra $\Sigma_0 \subseteq \Sigma$ and its corresponding $\Psi_0 \subseteq \Psi$, the norms of v and μ_v restricted to Σ_0 and Ψ_0, respectively, are equal. This implies that $\|v\| = \|\mu_v\|$.

Since the map $v \mapsto \mu_v$ is linear, it is also continuous.

The fact that every μ on Ψ induces a $v \in V$ is immediate. We are therefore left with

Lemma 4.1.5: v is totally monotone iff μ_v is nonnegative.

Proof: Assume v is totally monotone. Consider a basic element $(\tilde{A} \setminus \cup_{i=1}^n \tilde{B}_i)$ and a finite algebra Σ_0 containing $\{A\},\{B\}_{i=1}^n$. Then $\mu_v(\tilde{A} \setminus \cup_{i=1}^n \tilde{B}_i)$ is the sum of several coefficients $\{\alpha_T^v\}$ (as in 4.1.2 above) all of which are nonnegative.

Conversely, suppose μ_v is nonnegative, and consider $B_1,\ldots,B_n \in \Sigma$. We need to show that

$$v(\cup_{i=1}^{n} B_i) \geq \Delta_v(\{B_i\}_{i=1}^{n}).$$

However, denoting $B = \cup_{i=1}^{n} B_i$,

$$v(B) - \mu_v(\{B_i\}_{i=1}^{n}) = \mu_v(\tilde{B} \setminus \cup_{i=1}^{n} \tilde{B}_i) \geq 0. \qquad \blacksquare$$

This completes the proof of Theorem A.

4.2 Proof of Theorem B

Let us start with the claim that V^b, V^σ are Banach spaces. It is obvious that V^b and V^σ are linear subspaces of V, and we have noted that $\|\bullet\|$ is a norm. We therefore need to check only completeness.

Lemma 4.2.1: V^b is complete.

Proof: Standard.

Lemma 4.2.2: V^σ is complete.

Proof: Let $\{v_n\}_{n\geq 1}$ be a Cauchy sequence in V^σ. Let v be the pointwise limit of $\{v_n\}$. By standard arguments, v is well-defined and $\|v - v_n\| \to 0$ as $n \to \infty$. We only need to show that μ_v is σ-additive.

However,

$$\|\mu_v - \mu_{v_n}\| = \|\mu_{(v-v_n)}\| = \|v - v_n\|.$$

Hence, μ_v is the limit (in the variation norm) of $\{\mu_{v_n}\}_n$. The latter being σ-additive, so is the former. ∎

Note also that since both V^σ and V^b are Banach spaces, so is $V^{b\sigma} \equiv V^b \cap V^\sigma$.

Next we wish to show that V^b consists of all differences between totally monotone functions. It is obvious that if v is totally monotone,

$$\|v\| = v(\Omega) < \infty$$

and therefore $\{v^+ - v^- \mid v^+, v^- \text{ totally monotone}\} \subseteq V^b$.

The converse is given by

Lemma 4.2.3: Assume $v \in V^b$. Then there are totally monotone v^+, v^- such that $v = v^+ - v^-$. Furthermore, there are unique such v^+, v^- satisfying $\|v\| = \|v^+\| + \|v^-\|$.

Proof: Given $v \in V^b$, notice that $\|\mu_v\| < \infty$, i.e., μ_v is bounded. Then, by Jordan's decomposition theorem, there are finitely additive measures μ^+, μ^- such that $\mu_v = \mu^+ - \mu^-$ and

$$\|\mu_v\| = \|\mu^+\| + \|\mu^-\|.$$

Defining v^+, v^- by μ^+, μ^- respectively yields the representation of v. Furthermore, the uniqueness of μ^+, μ^- (satisfying the norm equation) implies that of v^+, v^-. ∎

The characterization of V^b as differences of totally monotone set functions reminds one of the space BV, defined and discussed in Aumann-Shapley (1974) for $\Omega = [0,1]$ and $\Sigma = B([0,1])$. They define the variation norm to be

$$\|v\|_{var} = \sup\{\textstyle\sum_{i=0}^{n} |v(S_{i+1}) - v(S_i)| \mid \emptyset = S_0 \subseteq \ldots \subseteq S_{n+1} = \Omega\},$$

and BV to be the Banach space of all games with bounded variation norm.

Another clone is the summation norm, defined in Gilboa (1989) as

$$\|v\|_{sum} = \sup\{\textstyle\sum_{i=1}^{n} |v(S_i)| \mid \{S_1,\ldots,S_n\} \text{ is a partition of } \Omega\}$$

BS denotes the Banach space of all bounded summation games.

It is easy to see that our (composition) norm dominates both the variation and the summation norms. Indeed, an equivalent definition of the composition norm is

$$\|v\| = \sup\{\textstyle\sum_{i=1}^{n} |v(A_i) - \Delta_v(\{B_i^j\}_{j=1}^{k_i})| \mid \{(\tilde{A}_i \setminus \cup_{j=1}^{k_i} \tilde{B}_i^j)\}_{i=1}^{n} \text{ is a partition of } \tilde{\Omega}\}.$$

Considering all the partitions of $\tilde{\Omega} = \Sigma'$ into finitely many basic elements $\{(\tilde{A}_i \setminus \cup_{j=1}^{k_i} \tilde{B}_i^j)\}_i$ one may focus on those for which $k_i = 1$, i.e., partitions of the form $\{(\tilde{A}_i \setminus \tilde{A}_{i-1})\}$. For these

$$|v(A_i) - \Delta_v(\{B_i^j\}_{j=1}^{k_i})| = |v(A_i) - v(A_{i-1})|$$

and the supremum over sums of such expressions reduces to the variation norm.

On the other hand, one may consider only partitions of the form

$$\{(\tilde{\Omega} \setminus \cup_{j=1}^{k} \tilde{B}^j), \tilde{B}^1,\ldots,\tilde{B}^k\}$$

where $(B^j)_j$ is a partition of Ω. In this case

$$|v(\Omega) - \Delta_v((B^j)_{j=1}^k)| = |v(\Omega) - \textstyle\sum_{j=1}^k v(B^j)|$$

and the supremum over sums of such expressions is bounded between $\|v\|_{sum}$ and $3\|v\|_{sum}$.

We therefore conclude that

$$V^b \subseteq BV \cap BS.$$

To see that the converse does not hold, consider the following

Example 4.2.4: Let $\Omega = \mathbf{N}$ and $\Sigma = 2^\Omega$. Define

$$v(A) = \begin{cases} 1 & if\ |A^c| \le 1 \\ 0 & otherwise. \end{cases}$$

It is easy to check that

$$\|v\|_{var} = \|v\|_{sum} = 1.$$

However, $\|v\|$ is unbounded: for each k, consider $A_i = \Omega \setminus \{i\}$ for $1 \le i \le k$ and a partition of $\tilde{\Omega}$ containing the basic element $(\tilde{\Omega} \setminus \cup_{i=1}^k \tilde{A}_i)$. Obviously, $\|v\| \ge |v(\Omega) - \Delta_v((A_i)_{i=1}^k)| = (k - 1)$. ∎

This example may suggest that a bounded Δ_v is the crucial property of games in V^b. Yet we note that

<u>Remark 4.2.5</u>: There are games v for which Δ_v is bounded yet $\|v\|$ is not.

<u>Proof</u>: Consider the following example: $\Omega = \mathbf{N}$, $\Sigma = 2^{\Omega}$, and define, for $A \subseteq \Omega$,

$$m(A) = \begin{cases} \infty & \text{if } A = \mathbf{N} \\ 0 & \text{if } A = \emptyset \\ \max\{n \mid \{1, \ldots, n\} \subseteq A\} & \text{otherwise.} \end{cases}$$

Next, define

$$v(A) = \begin{cases} 0 & \text{if } m(A) = 0 \\ f(0) & \text{if } m(A) = \infty \\ f\left(\frac{1}{m(A)}\right) & \text{otherwise.} \end{cases}$$

where f: $[0,1] \to [0,1]$ is some function with unbounded variation satisfying $f(0) = f(1) = 0$.

For $A \subseteq \mathbf{N}$, let

$$A' = \{n \mid \{1, \ldots, n\} \subseteq A\}.$$

It is easy to check that, for all $\{B_i\}_{i=1}^k \subseteq \Sigma$,

$$\Delta_v(\{B_i\}_{i=1}^k) = \Delta_v(\{B_i'\}_{i=1}^k).$$

Assuming, w.l.o.g., $B_1' \supseteq B_2' \supseteq \ldots \supseteq B_k'$, and using Lemma 4.2.7 below,

$$|\Delta_v(\{B_i'\}_{i=1}^k)| = |v(B_1')| \leq 1.$$

Hence, Δ_v is bounded. Yet $v \notin BV$ and, perforce, $v \notin V^b$. ∎

We conclude this sub-section with two facts about the function Δ_v which, in particular, will complete the proof of Remark 4.2.5.

<u>Fact 4.2.6</u>: For any $T \in \Sigma'$, $\{B_i\}_{i=1}^k \subseteq \Sigma$

$$\Delta_{u_T}(\{B_i\}_{i=1}^k) = \begin{cases} 1 & \text{if } \exists\, i \le k \text{ s.t. } T \subseteq B_i \\ 0 & \text{otherwise.} \end{cases}$$

<u>Proof</u>: Given T, $\{B_i\}_i$, assume w.l.o.g. that $T \subseteq B_i$ for $i \le j$ and $T \not\subseteq B_i$ for $i > j$. Obviously, if $j = 0$, $\Delta_{u_T}(\{B_i\}_i) = 0$. Assume, then, $j > 0$. In this case

$$\begin{aligned}\Delta_{u_T}(\{B_i\}_{i=1}^k) &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\dots,k\}} (-1)^{|I|+1} u_T(\cap_{i\in I} B_i) = \\ &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\dots,j\}} (-1)^{|I|+1} u_T(\cap_{i\in I} B_i) = \\ &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\dots,j\}} (-1)^{|I|+1} = 1.\end{aligned}$$ ∎

<u>Fact 4.2.7</u>: For all $v \in V$ and $\{B_i\}_{i=1}^k \subseteq \Sigma$, if $B_k \subseteq B_{k-1}$, then

$$\Delta_v(\{B_i\}_{i=1}^k) = \Delta_v(\{B_i\})_{i=1}^{k-1})$$

<u>Proof</u>: By fact 4.2.6, it is easy to check that the conclusion holds for every unanimity game $v = u_T$.

For an arbitrary v, consider the finite sub-algebra of Σ generated by $\{B_i\}_{i=1}^k$. On this sub-algebra v is a linear combination of finitely many unanimity games. Δ_v being linear in v, the claim is proved. ∎

4.3 Proof of Theorem C

In order to show that $p\sigma A \subsetneq V^{b\sigma}$, it suffices to show that for any measures $\lambda_1,\ldots,\lambda_k$ on Ω, $v \equiv \Pi_{i=1}^{k} \lambda_i$ is in $V^{b\sigma}$. This proof is done by construction.

Given $\lambda_1,\ldots,\lambda_k$, let $\lambda_{1\ldots k}$ be the product measure on Ω^k, i.e., $\lambda_{1\ldots k} = \lambda_1 \times \lambda_2 \times \ldots \times \lambda_k$ on Σ^k.

Let us define μ on Ψ as follows: for every $\mathscr{E} \in \Psi$, let

$$\mu(\mathscr{E}) = \lambda_{1\ldots k}(\{(\omega_1,\ldots,\omega_k) \in \Omega^k | \{\omega_1,\ldots,\omega_k\} \in \mathscr{E}\}).$$

That is to say, for each Ψ-measurable subset of coalitions $\mathscr{E}$, we consider all coalitions of size k or less in $\mathscr{E}$, and the measure of $\mathscr{E}$ is defined to be the $\lambda_{1\ldots k}$-measure of all k-tuples in Ω^k consisting of members of one of those coalitions.

We need to show that μ is well-defined, that it is a measure and that $\mu = \mu_v$.

We first note that

Lemma 4.3.1: For all $A \in \Sigma'$, $\mu(\tilde{A})$ is well-defined and equals $v(A)$.

Proof: First notice that

$$\{(\omega_1,\ldots,\omega_k) \in \Omega^k | \{\omega_1,\ldots,\omega_k\} \in \tilde{A}\} =$$
$$= \{(\omega_1,\ldots,\omega_k) \in \Omega^k | \omega_i \in A \text{ for } 1 \le i \le k\} = A^k.$$

Hence, for all $A \in \Sigma'$, this set is $\lambda_{1\ldots k}$-measurable and

$$\mu(\tilde{A}) = \lambda_{1\ldots k}(A^k) = \Pi_{i=1}^k \lambda_i(A) = v(A). \qquad \blacksquare$$

Next we have

Lemma 4.3.2: μ is well-defined and σ-additive.

Proof: Consider a basic element $\tilde{A} \setminus \cup_{j=1}^n \tilde{B}_j$.

$$\{(\omega_1,\ldots,\omega_k) \in \Omega^k \mid \{\omega_1,\ldots,\omega_k\} \in \tilde{A} \setminus \cup_{j=1}^n \tilde{B}_j\} =$$
$$= \{(\omega_1,\ldots,\omega_k) \in \Omega^k \mid \omega_i \in A \text{ for all } 1 \le i \le k\} \setminus$$
$$\cup_{j=1}^n \{(\omega_1,\ldots,\omega_k) \in \Omega^k \mid \omega_i \in B_j \text{ for all } 1 \le i \le k\} =$$
$$= A^k \setminus \cup_{j=1}^n (B_j)^k.$$

Notice that these sets are $\lambda_{1\ldots k}$-measurable in Ω^k.

Furthermore, disjoint unions of basic elements are mapped to disjoint unions of sets of the form above. Hence μ is well-defined on Ψ. Its σ-additivity follows from that of $\lambda_{1\ldots k}$ on Ω^k. $\blacksquare$

Noticing that $\lambda_{1\ldots k}$ is nonnegative, we conclude that μ is a measure on Ψ. Together with the conclusion of Lemma 4.3.1 (and using the uniqueness result in Theorem A), $\mu = \mu_v$ follows. This completes the proof of Theorem C.

A few remarks may be in order. First, notice that the definition of μ can be described as follows: first, consider only nonempty subsets with no more than k elements:

$$B_k = \{T \subseteq \Omega | 1 \le |T| \le k\} = \{\{\omega_1, \ldots, \omega_k\} | \omega_i \in \Omega \text{ for } i \le k\}.$$

Next, map every such $T \in B_k$ on all the $|T|!$ $|T|$-tuples in $\Omega^{|T|}$. Finally, for each $\mathscr{E} \in \Psi$ consider the $\lambda_{1\ldots k}$-measure of the image of $(\mathscr{E} \cap B_k)$.

Note, however, that in general B_k need not be a subset of Σ. Furthermore, even if $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$, the set B_k need not be Ψ-measurable. For instance, if $\Omega = [0,1]$ and $\Sigma = B([0,1])$, B_k is not Ψ-, nor even $\hat{\Psi}$-measurable.

Yet, in the sense described above, we may say that μ_v is "concentrated" on B_k. We therefore conclude that for $v \in p\sigma A$, μ_v is "concentrated" only on finite coalitions in Ω. It is obvious, therefore, that polynomials in measures are only a "small" subset of the spaces we are interested in.

4.4 Proof of Theorem D

W.l.o.g. assume $\Omega = \mathbf{N}$, $\Sigma = 2^{\Omega}$. Let us first show that a measure μ on Ψ induces a totally monotone and outer continuous game v. Total monotonicity obviously follows from nonnegativity of μ. To prove outer continuity, let $A_n \supseteq A_{n+1}$ and $A = \cap_{n=1} A_n$. Consider the (countable) partition of $\tilde{\Omega}$ given by

$$\{(\tilde{\Omega} \setminus \tilde{A}_1)\} \cup \{(\tilde{A}_n \setminus \tilde{A}_{n+1})\}_n \cup \{\tilde{A}\}.$$

By σ-additivity,

$$\begin{aligned} v(\Omega) &= [v(\Omega) - v(A_1)] + \textstyle\sum_{n\ge1} [v(A_n) - v(A_{n+1})] + v(A) \\ &= v(\Omega) - \lim_{n\to\infty} v(A_n) + v(A) \end{aligned}$$

and the result follows. Next, we show that if v is totally monotone and outer continuous, μ_v is σ-additive.

Let then be given such a game v. Define $C_i = \mathbb{N} \setminus \{i\}$ and let Ψ_0 be the algebra generated by $\{\tilde{C}_i\}_{i\geq 1}$. We first wish to show

<u>Lemma 4.4.1</u>: μ_v is σ-additive on Ψ_0.

<u>Proof</u>: Let $\{(\tilde{A}_i \setminus \cup_{j=1}^{k_i} \tilde{B}_i^j)\}_{i\geq 1}$ be a Ψ_0-measurable partition of $\tilde{\Omega}$. We need to prove that

$$\sum_{i\geq 1} v(A_i) - \Delta(\{B_i^j\}_{j=1}^{k_i}) = v(\Omega).$$

Notice that each of $\{A_i, B_i^j\}_{i,j}$ is the intersection of finitely many C_i's, hence it is co-finite.

We will now enumerate the A_i's according to "layers," such that each A_i in layer ℓ is a subset of some A_j in layer $(\ell - 1)$, and $\tilde{A}_i$ is subtracted from the basic element corresponding to A_j.

First notice that there is exactly one i for which $A_i = \Omega$. Assume w.l.o.g. this is A_1 and call $\{A_1\}$ "layer 1." Next consider the sets $\{B_1^j\}_j$. Since $B_1^j \in \tilde{\Omega}$, each of them has to appear as A_j for some j. Assume w.l.o.g. these are $A_2, A_3, \ldots, A_{k_1+1}$, and let us refer to these as "layer 2." Continue in this fashion, and notice that for every i,j, there is a k such that $A_k = B_i^j$. (Note, however, that many pairs (i,j) may correspond to the same k.)

Next we claim that by this enumeration all the sets $\{A_i\}_{i\geq 1}$ are exhausted. Indeed, if this were not the case, there is a set A_i which is contained in an infinite decreasing sequence of other A_j's. Yet, since they are all different, such A_i cannot be co-finite.

Let us therefore assume that our enumeration is $\{A_{\ell i}\}_{\ell, i}$ where $\ell \geq 1$, $1 \leq i \leq M_\ell$ for each ℓ, and $\{A_{\ell i}\}_i$ is layer i.

Let us further assume w.l.o.g. that for every ℓ, $i \neq j$, $A_{\ell ij} \equiv A_{\ell i} \cap A_{\ell j}$ is contained in $A_{(\ell+1)r}$ for some r. That is, that the intersection of every two members of a certain layer, or a superset thereof, appears in the next one. (This also means that the basic element corresponding to $A_{\ell i}(A_{\ell j})$ has an empty intersection with $\tilde{A}_{\ell ij}$.) Note that, given the layer structure, one may always introduce these intersections and redefine the layers accordingly, so as to satisfy this condition, to which we refer as the "intersection condition."

We now introduce

<u>Claim</u>: For every $L \geq 1$, $\cup_{\ell \leq L} \cup_{1 \leq i \leq M_L} (\tilde{A}_{\ell i} \setminus \cup_{j=1}^{k_{\ell i}} \tilde{B}_{\ell i}^j) = \boldsymbol{\Omega} \setminus \cup_{r=1}^{M_{L+1}} \tilde{A}_{(L+1)r}$.

Loosely, what this equality means is that one may "get rid" of the layers successively, and, instead of subtracting $\tilde{A}_{\ell i}$ and adding their basic elements, we may ignore $\tilde{A}_{\ell i}$ and subtract the next layer sets directly.

<u>Proof of Claim</u>: The proof is, obviously, by induction on L. For the induction step it suffices to show that, under our conditions,

$$(\boldsymbol{\Omega} \setminus \cup_{i=1}^{k} \tilde{A}_i) \cup \cup_{i=1}^{k} (\tilde{A}_i \setminus \cup_{j=1}^{k_i} \tilde{B}_i^j) = (\boldsymbol{\Omega} \setminus \cup_{i=1}^{k} \cup_{j=1}^{k_i} \tilde{B}_i^j).$$

To show the inclusion $\supseteq$, assume that $T \in \tilde{\Omega}$ but $T \notin \tilde{B}_i^j$ for all i,j. Then either $T \in (\tilde{\Omega} \setminus \cup_{i=1}^{k} \tilde{A}_i)$, or else $T \subseteq A_i$ for some $i \leq k$. But then $T \in \tilde{A}_i \setminus \cup_{j=1}^{k_i} \tilde{B}_i^j$.

Conversely, suppose that a coalition T belongs to the LHS. If $T \in (\tilde{\Omega} \setminus \cup_{i=1}^{k} \tilde{A}_i)$, $T \not\subseteq A_i$ for all $i \leq k$ and, since $B_i^j \subseteq A_i$, $T \not\subseteq B_i^j$ for all i,j. Next consider T such that $T \in (\tilde{A}_i \setminus \cup_{j=1}^{k_i} B_i^j)$ for some i. We contend that this may be true for at most one such index i. Indeed, if $T \in \tilde{A}_i \cap \tilde{A}_j$, $T \subseteq A_i \cap A_j \equiv A_{ij}$. But then the basic element corresponding to A_i will have a nonempty intersection with $\tilde{A}_{ij}$, in contradiction to our intersection condition.

Hence, $T \not\subseteq A_j$ for $j \neq i$, and, perforce, $T \not\subseteq B_j^s$ for $j \neq i$ and $1 \leq s \leq k_j$. Since we also know that $T \not\subseteq B_i^s$ for $1 \leq s \leq k_i$, $T \in (\Omega \setminus \cup_{i=1}^{k} \cup_{j=1}^{k_i} B_i^j)$.

This concludes the proof of the claim.

In order to complete the proof of Lemma 4.4.1, let us consider the expression

$$\sum_{\ell=1}^{L} \sum_{i=1}^{M_L} \left(v(A_{\ell i}) - \Delta_v(\{B\ell_i^j\}_{j=1}^{k_{\ell i}})\right)$$

By the claim and the additivity of μ_v, it equals

$$v(\Omega) - \Delta_v(\{A_{(L+1)r}\}_{r=1}^{M_{L+1}})$$

Since v is totally monotone,

$$\Delta_v(\{A_{(L+1)r}\}_{r=1}^{M_{L+1}}) \leq v(\cup_{r=1}^{M_{L+1}} A_{(L+1)r}).$$

Denote $A_L = \cup_{r=1}^{M_L} A_{Lr}$. It suffices to show that $v(A_L) \to 0$ as $L \to \infty$. However, we know that

$$\{(\Omega \setminus \cup_{r=1}^{M_L} \tilde{A}_{Lr})\}_{L \geq 1}$$

is an increasing sequence of sets whose union is $\tilde{\Omega}$. Focusing on singletons, we conclude that

$$\{(\Omega \setminus A_L)\}_{L \geq 1}$$

is also an increasing sequence whose union is Ω. In other words, $A_{L+1} \subseteq A_L$ and $\cap_{L\geq 1} A_L = \emptyset$. By outer continuity of v, however, $v(A_L) \to 0$ and the lemma is proved. ∎

We continue with the proof of Theorem D. We know that for a totally monotone and outer continuous v, μ_v is σ-additive on Ψ_0. This implies that μ_v has a unique σ-additive extension $\hat{\mu}_v$ to the σ-algebra generated by $\{\tilde{C}_i\}_{i\geq 1}$. We note that this σ-algebra contains Ψ (hence, also $\hat{\psi}$), since for every $A \in \Sigma$

$$\tilde{A} = \cap_{i \in A^c} \tilde{C}^i.$$

So we only need to show that $\hat{\mu}_v = \mu_v$ on Ψ.

Let $\hat{v}$ be the game (on Ω) induced by $\hat{\mu}_v$. Since $\hat{\mu}_v$ is σ-additive, $\hat{v}$ is outer-continuous. But for every co-finite A,

$$\hat{v}(A) = \hat{\mu}_v(\tilde{A}) = \mu_v(\tilde{A}) = v(A).$$

Since both v and $\hat{v}$ are outer continuous, $v = \hat{v}$, which also implies that $\hat{\mu}_v = \mu_v$ on all Ψ.

Thus we have proved that v is totally monotone and outer continuous iff μ_v is a measure. The fact that the map $v \mapsto \mu_v$ is a bijection was already proven in Theorem A. This concludes the proof of Theorem D.

4.5 Proof of Theorem E

Given $v \in V^{b\sigma}$ and $f \in F$, assume w.l.o.g. that $f \geq 0$. Then

$$\begin{aligned} \int_\Omega f dv &= \int_0^\infty v(\{\omega | f(\omega) \geq \alpha\}) d\alpha \\ &= \int_0^\infty [\int_{\Sigma'} u_T(\{\omega | f(\omega) \geq \alpha\}) d\mu_v(T)] d\alpha. \end{aligned}$$

In order to use Fubini's theorem, we need to show that the function $g\colon \mathbf{R}_+ \times \Sigma' \to \mathbf{R}$ defined by

$$g(\alpha,T) = u_T(\{\omega | f(\omega) \geq \alpha\})$$

is $B(\mathbf{R}_+) \times \hat{\Psi}$-measurable. In other words, the set

$$A = \{(\alpha,T) | f(\omega) \geq \alpha \text{ for all } \omega \in T\} =$$
$$= \{(\alpha,T) | \alpha \leq \inf_{\omega\in T} f(\omega)\}$$

has to be $B(\mathbf{R}_+) \times \hat{\Psi}$-measurable.

Notice that for every $\alpha \in \mathbf{R}_+$

$$\{T | f(\omega) \geq \alpha \text{ for all } \omega \in T\} = \tilde{S}$$

where

$$S = \{\omega | f(\omega) \geq \alpha\} \in \Sigma,$$

and measurability of A follows by a standard construction. Thus we have

$$\int_\Omega f dv = \int_{\Sigma'} \int_0^\infty u_T(\{\omega | f(\omega) \geq \alpha\}) d\alpha \; d\mu_v(T) =$$
$$= \int_{\Sigma'} [\inf_{\omega\in T} f(\omega)] d\mu_v(T). \qquad \blacksquare$$

5. Remarks

5.1 Updating Non-Additive Probabilities

The map $v \mapsto \mu_v$ suggests a procedure for updating a non-additive probability v: map v onto an additive μ_v (on Ψ), update the latter and project the updated μ_v into V. It is simple to check that one obtains

$$v(B|A) = v(B \cap A)/v(A).$$

(While using the dual games of $\{u_T\}$ as a basis would give rise to Dempster-Shafer's rule

$$v(B|A) = \frac{v((B \cap A) \cup A^c) - v(A^c)}{v(\Omega) - v(A^c)}.)$$

5.2 Radon-Nikodym Theorem

The isomorphism between non-additive set functions on Ω and additive ones on Σ' also suggests a "Radon-Nikodym" theorem for non-additive set functions (interpreted as non-additive probabilities or as games). This was also discussed (for the finite case) in Gilboa-Schmeidler (1992). Such a theorem would take the following form: if $v, w \in V^{b\sigma}$ and w is "absolutely continuous" w.r.t. v, then there exists a function $g: \Sigma' \to \mathbf{R}$ such that for all $f \in F$,

$$\int_\Omega f dw = \int_{\Sigma'} [\inf_{\omega \in T} f(\omega)] g(T) d\mu_v(T).$$

For such a theorem to hold, one needs to have an appropriate definition of "absolute continuity" of w w.r.t. v, which would imply the absolute continuity of μ_w w.r.t. μ_v.

At present we are unaware of any reasonably-elegant conditions on v and w that would guarantee this property.

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