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Subjective Equilibrium in Repeated Games[+]

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Abstract

A player's strategy, for an n-person infinitely repeated game with discounting, is <u>subjectively rational</u> if it is a best response to his individual beliefs regarding opponents' strategies. A vector of such strategies is a <u>subjective equilibrium</u> if the play induced by it is realization equivalent to the play induced by each player's beliefs. Thus, any statistical updating can only reinforce the beliefs. It is shown that under perfect monitoring, the joint behavior at a subjective equilibrium approximates a behavior of a Nash equilibrium even when perturbations are allowed. Therefore, learning processes leading to subjective equilibrium result in approximate Nash behavior.

equilibrium as being stable with respect to learning and optimization. Players placed at such an equilibrium will not alter their beliefs and will have no incentive to alter their strategies.

Notions of subjective equilibrium are not new in economics and game thoery (see Battigalli et al. (1992) for a survey). Von Hayek (1937) already discussed the differences between subjective and objective knowledge. His test for equilibrium was "whether the individual subjective sets of data correspond to the objective data, and whether in consequence the expectations in which plans were based are born out by the facts." Hahn (1973) assumed that agents maximize their utility relative to their subjective theories about the future evolution of the economy. He defined a conjectural equilibrium as a situation where the signals generated by the economy do not alter the agents' individual theories, nor do they induce them to change their policies. Battigalli (1987) and Battigalli and Guaitoli (1988) formalized and studied the game theoretic version of Hahn's conjectural equilibrium. Rubinstein and Wolinsky (1990) defined rationalizable versions of conjectural equilibrium.

Fudenberg and Levine (1993a) introduced a notion of self-confirming equilibrium defined for finite extensive form games. A player in such a game chooses a strategy to maximize his expected payoff given his subjective beliefs about opponents' strategies. These beliefs allow the possibility that the opponents' strategies are correlated, and being defined for general extensive games this notion allows for imperfect information the players obtain and use to confirm their subjective beliefs. Fudenberg and Levine (1993) motivated this notion in a model of overlapping generations where players of different ages are randomlhy matched to play a fixed extensive subjective beliefs assign the activity used a correct payoff distribution but assign the competing unused activity a false low payoff distribution. In such a situation, his assessments are reinforced and he never finds out that he is wrong off the play path and that his play is suboptimal. In other words, he does not follow a Nash equilibrium of the complete information one person game. The current paper, however, will rule out such situations by assuming that players know their correct payoff distributions and uncertainty is restricted to be strategic, i.e., concerning opponents' strategies.

But even under strategic uncertainty alone there are serious discrepancies between behavior induced by Nash and behavior induced by subjective equilibrium. Revealing examples of extensive games exhibiting this subtle phenomenon were described by Fudenberg and Kreps (1988), and Fudenberg and Levine (1993a). In order to rule out such examples, the current paper assumes that the infinitely repeated game is played with perfect monitoring (of players' actions). This assumption, together with the earlier ones, suffices to close the gap between the behavior of Nash and of subjective equilibrium.

When we restrict ourselves to Nash equilibrium and subjective equilibrium, ignoring their approximated ϵ versions, it is easy to see, under the conditions stated above, that the two notions induce identical behavior patterns. Starting with a subjective equilibrium, one modifies the strategies of all players as follows: (1) on the support of the original play paths no modification is done; (2) in subgames following a unilateral deviation (from the support of his strategy) by player i, all players will play according to player i's subjective beliefs; (3) in subgames following showed, under assumptions closely related to ours, that coincidence of self confirming equilibrium behavior with Nash behavior is obtained.

Our objective in this paper is to describe general sufficient conditions under which subjective equilibrium behavior coincides with Nash behavior, and subjective ε-equilibrium behavior is ε-close to ε-Nash equilibrium behavior in the space of infinite play paths. Since correlations off the play paths will have to be assumed away in the statements of our main results, we prefer the simplicity gained by assuming them away in the definition of subjective equilibrium. For this reason we restrict the beliefs in a subjective equilibrium to consist entirely of (independent) behavior strategies.

It is important to note that if one player's beliefs regarding an opponent's strategy were described by a mixed strategy, i.e., believing that his opponent chose randomly one strategy from a set of behavior strategies, then by using the standard Kuhn (1953) method we could replace his beliefs with an equivalent single behavior strategy to fit the model of this paper. Disallowing correlations, as discussed above, will restrict us to the use of individually mixed strategies and thus rule out the mixing of strategies in a correlated way across players.

2. The Repeated Game

First, we briefly review the standard model of an n-player discounted repeated game with perfect monitoring. An n-person stage game is described by a set of action combinations $\Sigma = \times_{i=1}^{n} \Sigma_i$ with Σ_i denoting a finite action set of player i. Functions $u_i: \Sigma \to \mathbb{R}$ describe the stage game payoffs of the players.

We assume that each player has a discount parameter λ_i, $0 < \lambda_i < 1$, by which he evaluates the payoff received along play paths. Thus, if $z^\ell = (z_1, z_2, \ldots, z_\ell)$ is a play path, we define

$$u_i(z^\ell) = \Sigma_{t=1}^{\ell} \lambda_i^{t-1} u_i(z_t^\ell).$$

Now we complete the definition of the repeated game by defining individual payoffs for each strategy vector f^ℓ,

$$U_i(f^\ell) = Eu_i(z^\ell) = \int u_i(z^\ell) d\mu_{f^\ell}(z^\ell).$$

Equivalently, one can define the expected stage payoffs and take the discounted sum of these.

2b. <u>The Infinitely Repeated Game</u>

The set of all finite length histories is denoted by H. I.e., $H = \cup_{t=0}^{\infty} H_t$. A <u>(behavior) strategy</u> of player i in the infinitely repeated game is a function f_i from H to $\Delta(\Sigma_i)$. Notice that any f_i induces a strategy, f_i^ℓ, in the corresponding ℓ-fold repeated game. The f_i^ℓ is simply the restriction of f to the smaller domain, H^ℓ, and it is called the ℓ-truncation of f.

For every strategy vector $f = (f_1, \ldots, f_n)$ we define a probability distribution, μ_f, over the set of all infinite play paths Σ^∞. The σ-algebra for this set is defined to be the smallest one that contains all the cylinder sets, c(h). Following a standard probability formulation it suffices to assign probabilities to all the cylinder sets in order to obtain event in the game is assigned probability zero by the beliefs.

Given that a player maximizes expected utility relative to a prior probability distribution, it follows that he must be maximizing expected utility relative to his Bayes updated posterior beliefs after positive probability histories. As was shown earlier by Blackwell and Dubins (1962), and by the recent Kalai and Lehrer paper, absolute continuity guarantees that Bayes posteriors converge to the true distribution. So in the limit the players will predict the future correctly and will play a subjective equilibrium; and in finite time they will predict the future correctly only up to ϵ and will play subjective ϵ-equilibrium. We begin by considering the case of correct predictions in the future, and hence of subjective equilibrium.

As usual, we say that a strategy f_i <u>is a best response to</u> g_{-i} if $U_i(g_1,\dots,k_i,\dots,g_n) - U_i(g_1,\dots,f_i,\dots,g_n) \le 0$ for every strategy k_i. If the right side 0 is replaced by ϵ we say that f_i is an ϵ-<u>best response</u> to g_{-i}.

<u>Definition 1</u>: A <u>subjective equilibrium</u> is a strategy vector g with a beliefs matrix $(g^i_j)_{1\le i,j\le n}$ satisfying for each player i:

(0) $g^i_i = g_i$;

(1) g_i is best response to g^i_{-i}; and

(2) $\mu_g = \mu_{g^i}$.

In this case, we say that the matrix (g^i_j) sustains g.

The idea is that the i-th row, g^i, represents the subjective assessment of player i about the strategy vector that is played. Condition (0)

We omit the proof of this proposition and refer the readers to Fudenberg and Levine (1993a), and Battigalli et al. (1992) for earlier references.

Returning to the subjectively rational model, where players start with private beliefs, in any finite time the Bayesian updating process will typically become only approximately correct. This means that after finite time, even if arbitrarily long, we can only assume that the players play approximate subjective equilibria which we proceed to define.

Definition 2: Let $\varepsilon > 0$ and let μ and $\tilde{\mu}$ be two probability measures defined on the same space. We say that μ is ε-close to $\tilde{\mu}$ if there is a measurable set Q satisfying:

(i) $\mu(Q)$ and $\tilde{\mu}(Q)$ are greater than $1 - \varepsilon$; and

(ii) for every measurable set $A \subseteq Q$

$$|\mu(A) - \tilde{\mu}(A)| \leq \varepsilon\tilde{\mu}(A)$$

Remark 1: In Blackwell and Dubins' (1962) paper on merging of measures, closeness of measures was expressed by $|\mu(A) - \tilde{\mu}(A)| \leq \varepsilon$ for every event A (not just in Q). Their easy-to-state condition seems weaker, since it implies little for small probability events. For example, $\mu(A)$ could equal $2\tilde{\mu}(A)$ and still satisfy the Blackwell-Dubins closeness provided that $\mu(A)$ is sufficiently small. It turns out, however (see Kalai and Lehrer, 1990c), that the two notions are asymptotically equivalent, and the results that follow can be stated using either condition. The notion stated in

(a) $g^i_i = g_i$;

(b) g_i is a best response to g^i_{-i}; and

(c) g plays ε-like g^i.

An n-vector of strategies, f, is an <u>ε-Nash equilibrium</u>, if each f_i is an ε-best response to f_{-i}.

Our main result deals with the relation between subjective ε-equilibrium and ε-Nash equilibrium.

<u>Theorem 1</u>: In infinitely repeated games, for every $\varepsilon > 0$ there is $\bar{\eta} > 0$ s.t. for all $\eta \leq \bar{\eta}$, if g is a subjective η-equilibrium, then there exists f, s.t.

(i) g plays ε-like f, and

(ii) f is an ε-Nash equilibrium.

Theorem 1 states that a behavior induced by a subjective ε-equilibrium must be close to a behavior induced by an ε-Nash equilibrium. An ε-Nash equilibrium requires each player to choose a strategy that is ε-best response against the precise strategies used by his opponent, i.e., his payoff should be within ε of the optimally possible against theirs. On the other hand, the subjective ε-equilibrium requires precise optimization but against beliefs that are almost accurate.

The easy proof of Proposition 1 outlined in the Introduction made use of the precise coincidence of the play and conjectured play of all the players. However, in Theorem 1, with only ε-precision, this is no longer the case. Instead, our construction takes advantage of the fact that, in a number of pure strategies. Therefore, the set of behavior strategies is sequentially compact. Thus, without loss of generality, the sequences $\{g(m)\}_m$ and $\{(g(m)_j^i)\}_m$ are converging to, say, g and to (g_j^i). As the payoff functions are continuous, g is subjective equilibrium sustained by (g_j^i). Moreover, if η_m is close enough to zero g(m) ε-plays like g.

Using Proposition 1 we can find a Nash equilibrium f which plays 0-like g. Thus, if η_m is sufficiently small, g(m) ε-plays like f, which is a Nash equilibrium. This is a contradiction. //

Remark 2: In Definition 4(b) we required that g_i is a best response to g_{-i}^i. One can define δ-subjective ε-equilibrium by replacing "best response" with "δ best response." A similar proof to the one of Proposition 2 shows that, in a finitely repeated game, for every $\varepsilon > 0$ there is $\bar{\eta}$ s.t. if $\eta < \bar{\eta}$ then for every δ-subjective η-equilibrium, g, there is f s.t.

(i) g plays ε-like f, and

(ii) f is a δ-Nash equilibrium.

Notice that the relation between ε and η is independent of δ.

We are now ready to prove Theorem 1. Starting with a subjective η-equilibrium g we consider its trunctaion to the finitely repeated game of length ℓ. If ℓ is large then the truncated g is an approximate subjective η-equilibrium of the finite game, and by the above remark, it must approximately play like some Nash equilibrium f of the finite game. We extend f to the infinite game by making it coincide with g after all histories longer than ℓ. This extension makes g play close to f in the infinite game, and exploiting again the fact that ℓ is large, we conclude

$$(1 - \varepsilon)\mu_f(A) = (1 - \varepsilon) \Sigma_{C\in\mathcal{P}_\ell, C\subseteq Q} \mu_f(A \cap C) =$$
$$= (1 - \varepsilon) \Sigma_{C\in\mathcal{P}_\ell, C\subseteq Q} \mu_f(A|C)\mu_f(C) =$$
$$(1 - \varepsilon) \Sigma_{C\in\mathcal{P}_\ell, C\subseteq Q} \mu_g(A|C)\mu_f(C) \le \Sigma_{C\in\mathcal{P}_\ell, C\subseteq Q} \mu_g(A|C)\mu_g(C) = \mu_g(A).$$

For a similar argument, $\mu_g(A) \le (1 + \varepsilon)\mu_f(A)$ which concludes the proof that g plays ε-like f.

Recall that f^ℓ is $\varepsilon/2$-Nash equilibrium in the ℓ-fold repeated game. Therefore, f is $(\varepsilon/2 + \varepsilon/2)$-Nash equilibrium in the indefinite repeated game, which completes the proof of the theorem. //

Remark 3: It is easy to find examples where the behavior of an ε-Nash equilibrium is not ε-close to any Nash equilibrium under the strong notion of closeness we use.

Example: Suppose that in a two person game each player has two actions, say, a and b. Suppose furthermore that the pair (a,a) is the unique Nash equilibrium of the stage game (as in the prisoners' dilemma). Consider the following time dependent and not history dependent strategy: play always a and only at time t play b. Denote this strategy by g^t. Notice that since future payoffs are discounted, for every $\varepsilon > 0$ there is t large enough so that $g = (g^t, g^t)$ is an ε-Nash equilibrium.

As g^t is a pure strategy, μ_g is concentrated in one play path. If the discount factor is close to 0, there is no Nash equilibrium with which it plays ε-like, even for very large t's.

Notice that in the above example g^t is an ε-best response because of

Theorem 2: For any (ε, ℓ) there is $\bar{\eta}$ s.t. if $\eta < \bar{\eta}$, then every subjective η-equilibrium g plays (ε, ℓ) like some Nash equilibrium.

Proof: In view of Proposition 1, it suffices to show that g plays (ε, ℓ) like subjective equilibrium. We proceed by assuming that the theorem is incorrect. Thus, there is (ε, ℓ) and a sequence g_n of subjective η_n-equilibrium s.t. $\eta_n \to 0$ and g_n does not (ε, ℓ) play like any subjective equilibrium.

The limit of g_n, say, g, is clearly a subjective equilibrium. Moreover, on finite histories g_n and g are very close when n is sufficiently large. Therefore, g_n plays (ε, ℓ) like g when n is large enough. This is a contradiction. //

Remark 4: Two different notions of "approximate playing like" were used in the previous theorem. The weaker new one was used for the approximation as is explicitly stated, but the old one was still implicitly used in the definition of subjective η-equilibrium. One can strengthen Theroem 2 by using also the weaker notion of closeness in the definition of a subjective η-equilibrium. The added advantage of consistency, using the same definition of closeness throughout, is attractive. However, it would require the introduction of yet another version of subjective equilibrium, which we chose to avoid.

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