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THE MISSING EQUILIBRIA IN HOTELLING'S LOCATION GAME

by

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Abstract

We study the location equilibrium in Hotelling's model of spatial competition. As d'Aspremont et al. (1979) have shown, with quadratic consumer transportation cost the two sellers will seek to move as far away from each other as possible. This generates a coordination problem which the literature typically ignores by restricting firm 1 to locate in the first half and firm 2 in the second half of the market. We study the non-cooperative outcome in the absence of such a coordination device and find that the location game possesses an infinity of mixed strategy Nash equilibria. In these equilibria coordination failure invalidates the principle of 'maximum differentiation' and firms may even locate at the same point.

Keywords: Spatial competition, Hotelling's location model, coordination games.
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1 Introduction

The location problem of firms selling homogeneous goods is attributed to Hotelling (1929). In his seminal paper, Hotelling presents a model of two firms competing over locations and then prices in a two-stage subgame perfect equilibrium. Since then, Hotelling's spatial model has triggered an increasing flow of research in industrial organization (imperfect competition) and marketing (choice of new product). Indeed, either firms compete over physical locations in the geographical space, or over product design in the characteristic space. In the latter case transportation cost measures the disutility of not purchasing the ideal product.

Assuming that transportation costs are linear in distance, Hotelling (1929) argues that each firm gets higher profits by moving closer to its competitor so that in equilibrium both locate at the center of the market. Yet, d'Aspremont, Gabszewicz and Thisse (1979) point out that this argument contains a flaw because the price subgame in Hotelling's model fails to have a pure strategy equilibrium if firms are located too close to each other (but not at the same location). Indeed, in general one should not expect 'minimum differentiation' as advocated by Hotelling. When the firms are not spatially differentiated, Bertrand competition in the pricing subgame will reduce their profits to zero. By selecting different locations, however, they can ensure themselves positive profits.

This intuition is confirmed in a number of articles that study various formulations of Hotelling's location problem. Osborne and Pitchick (1987) study Hotelling's original model using a result of Dasgupta and Maskin (1986) that guarantees the existence of a mixed strategy price equilibrium. They show that the overall game has a subgame perfect equilibrium with pure strategies in the location stage. D'Aspremont et al. (1979) introduce a quadratic transportation cost function to sidestep nonexistence of a pure price equilibrium. Lederer and Hurter (1986) study the location game when firms use discriminatory pricing to differentiate between consumers at different locations. In Bester (1989) buyers and sellers bargain over prices after the sellers have chosen their locations. In all these versions of Hotelling's spatial competition model the firms wish to avoid identical locations. An exception is the model by de Palma et al. (1985) who confirm the principle of 'minimum differentiation' when consumer choices are probabilistic enough, or equivalently, preferences are sufficiently dispersed.

When the firms do not want to locate at the same point, the literature generally imposes a coordination device concerning the ranking of the firms' locations along the market segment. Typically, firm 1 is assumed to be to the left of firm 2. This device can be interpreted as a collusive rule which restricts the firms' strategy spaces. In the absence of this restriction the duopolists find themselves in a coordination game. This results in a number of possible equilibrium configurations that have been overlooked in the literature.

To make our point, we focus on Hotelling's model in the version of d'Aspremont et al. (1979) with quadratic consumer transportation cost. This has the advantage that the pricing subgame has a unique pure strategy equilibrium for all locations and that the firms' payoffs in the location game can easily be explicitly computed. Also, this version is of special interest since the firms will seek to move away from each other as far as possible. Under the above mentioned coordination device this leads to 'maximum differentiation' as the firms will locate at the endpoints of the market. We study the non-cooperative outcome without coordination and find that there is an infinity of mixed strategy equilibria. In these equilibria 'maximum differentiation' does not occur because of coordination failure. Indeed when the firms adopt identical location strategies, they may end up being located at the same point with positive probability. Our subgame perfect equilibria involve mixed strategies over location and pure strategies over prices. Here location can be interpreted as product design, about which the opponent has no information in the design phase. Once products have been developed and presented to the public, their characteristics are revealed and then firms compete in prices.

We describe the model in Section 2. Section 3 characterizes various types of asymmetric equilibria, in which the two players adopt different location strategies. Section 4 demonstrates that there is a unique player-symmetric equilibrium in mixed strategies and provides a characterization of the equilibrium distribution function. Concluding remarks are gathered in Section 5.

2 The Game

Hotelling's (1929) model can be viewed as a three-stage game: In the first stage there are two firms that simultaneously select a location at which to operate. Then, having observed location decisions, the duopolists simultaneously post prices. In the final stage, the consumers take their purchasing decisions conditioned on the firm's locations and prices.

The market region $A \equiv [0,1]$ is represented by a line segment of length normalized to one. The two firms offer products that are identical in all respects except for the location of availability. Both firms employ the same constant returns to scale technology and production costs are normalized to zero. Initially, each of the duopolists chooses a location in A; let x denote the location of firm 1, and let y denote the location of firm 2. Consumers are uniformly distributed on A; we identify consumer $a \in A$ with his initial location. Each consumer seeks to buy a single unit of the good. To make a purchase he has to visit the store of one of the sellers'. He faces a transportation cost $t(\cdot)$ that is a function of Euclidean distance d. Accordingly, he buys the good from the firm for which price plus travel cost is the lowest. Let p_i be the price charged by firm i. Then the set of all consumers who buy from firm 1 is given by

$$D_1(p_1, p_2, x, y) \equiv \{a \in A \mid p_1 + t(d(x,a)) \leq p_2 + t(d(y,a))\}. \qquad (1)$$

Each consumer $a \in D_2(p_1, p_2, x, y) \equiv A - D_1(p_1, p_2, x, y)$ purchases the good from firm 2. Thus the payoff of firm i is

$$R_i(p_1, p_2, x, y) \equiv \int_{D_i(p_1, p_2, x, y)} p_i \, da. \qquad (2)$$

Following d'Aspremont et al. (1979) we assume that transportation costs are quadratic, i.e. $t(d) = d^2$. This guarantees that the price setting subgame between the duopolists has a unique equilibrium for any given location pair (x,y). Indeed, d'Aspremont et al. (1979) computed the price equilibrium (p_1^*, p_2^*) and obtained the solution

$$p_1^*(x,y) = (y-x)(2+x+y)/3, \; p_2^*(x,y) = (y-x)(4-x-y)/3 \quad \text{if } x \leq y. \qquad (3)$$

By symmetry we get the equilibrium prices

$$p_1^*(x,y) = (x-y)(4-x-y)/3,\ p_2^*(x,y) = (x-y)(2+x+y)/3 \quad \text{if } x \geq y. \tag{4}$$

With quadratic transportation costs one obtains $D_1(p_1^*, p_2^*, x, y) = \{a \in A | a(y-x) \leq 0.5(p_2^* - p_1^* + y^2 - x^2)\}$. This allows us to compute each firms i's payoff in the location stage, $\Pi_i(x,y) \equiv R_i(p_1^*, p_2^*, x, y)$, as a function of location decisions. These payoffs are

$$\Pi_1(x,y) = (y-x)(2+y+x)^2/18 \text{ if } x \leq y\ ,$$
$$\Pi_1(x,y) = (x-y)(4-x-y)^2/18 \text{ if } x \geq y \tag{5}$$

for firm 1, and

$$\Pi_2(x,y) = (y-x)(4-x-y)^2/18 \text{ if } x \leq y\ ,$$
$$\Pi_2(x,y) = (x-y)(2+x+y)^2/18 \text{ if } x \geq y \tag{6}$$

for firm 2. Notice that payoffs are symmetric in the sense that

$$\Pi_1(x,y) = \Pi_2(1-y, 1-x),\ \Pi_1(x, 1-x) = \Pi_2(x, 1-x). \tag{7}$$

The remainder of our analysis is devoted to studying the Nash equilibria of the game where firm 1 and 2 choose $x \in A$ and $y \in A$, respectively, with payoffs given by (5) and (6).

3 Asymmetric Equilibrium

This section studies asymmetric equilibria where the two firms adopt different location strategies that may involve randomization. D'Aspremont et al. (1979) observed that each firm can increase its profit by moving further away from the location of its competitor. This immediately implies the following result.

Proposition 1: *There are exactly two pure strategy equilibria. These are* $(x^*, y^*) = (0, 1)$ *and* $(x^*, y^*) = (1, 0)$.

Proof: Consider all (x, y) such that $x \leq y$. Then one has

$$\partial\Pi_1(x,y)/\partial x = -(2+y+x)(2+3x-y)/18 < 0$$
$$\partial\Pi_2(x,y)/\partial y = (4-y-x)(4+x-3y)/18 > 0. \tag{8}$$

Therefore there is exactly one equilibrium such that $x^* \leq y^*$, namely $(x^*, y^*) = (0, 1)$. By symmetry of payoffs there is exactly one equilibrium such that $x^* \geq y^*$, namely $(x^*, y^*) = (1, 0)$. Q.E.D.

The literature typically imposes the restriction that firm 1 locates in the first half and firm 2 in the second half of A. With this restriction the equilibrium is obviously unique. Removing this restriction generates a second pure strategy equilibrium by symmetry of the game. Yet, this is not the only consequence. The duopolists' game can be viewed as a coordination game; both gain an advantage from moving as far away as possible. In this situation, the restriction $x \leq 0.5 \leq y$ works as a coordination device. Without such coordination the firms may end up at locations in the same half of the market. In what follows we adopt a purely non-cooperative view to analyze equilibrium configurations when there is a possibility of coordination failure.

Proposition 2: *There is a mixed strategy equilibrium in which firm 2 chooses* $y^* = 0$ *with probability* $1/2$ *and* $y^* = 1$ *with probability* $1/2$ *and firm 1 chooses* $x^* = 1/2$. *Symmetrically, there is an equilibrium in which firm 1 chooses* $x^* = 0$ *with probability* $1/2$ *and* $x^* = 1$ *with probability* $1/2$ *and firm 2 chooses* $y^* = 1/2$.

Proof: To prove the first part, we first show that, given the behavior of firm 2, firm 1 cannot gain by deviating from $x^* = 1/2$. Indeed, firm 1's payoff from choosing $x \in [0, 1/2]$ is

$$\varphi(x) \equiv 0.5\Pi_1(x,1) + 0.5\Pi_1(x,0) = (1-x)(3+x)^2/36 + x(4-x)^2/36. \tag{9}$$

Accordingly for $x \in (0, 1/2)$ one has

$$\varphi'(x) = (4 - x)(4 - 3x)/36 - (3 + x)(1 + 3x)/36 = (13 - 26x)/36 > 0. \quad (10)$$

This proves that $x^* = 1/2$ maximizes $\varphi(x)$ subject to $x \in [0, 1/2]$. A symmetric argument establishes that $x^* = 1/2$ also maximizes $[\Pi_1(x, 1) + \Pi_1(x, 0)]/2$ subject to $x \in [1/2, 1]$. As a result $x^* = 1/2$ is an optimal response of firm 1 to firm 2's strategy.

Using the computations in the proof of Proposition 1, one has

$$\partial\Pi_2(x^*, y)/\partial y > 0 \text{ for } y > x^*, \ \partial\Pi_2(x^*, y)/\partial y < 0 \text{ for } y < x^*. \quad (11)$$

As $\Pi_2(x^*, 0) = \Pi_2(x^*, 1)$, this implies that both $y^* = 0$ and $y^* = 1$ maximize firm 2's payoff. This proves that randomizing over $y^* = 1$ and $y^* = 0$ is a best reply of firm 2 to firm 1's strategy.

The second part of the Proposition follows by symmetry. Q.E.D.

The distance between the firms' locations in the equilibrium of Proposition 2 is only one half of that in the equilibrium with 'Maximum Differentiation' in Proposition 1. As a result, their payoffs are decreased from 1/2 to 49/148. This welfare loss is due to coordination failure. Proposition 2 shows that when one firm chooses both endpoints of some interval with positive probability, its opponent may wish to locate strictly in the interior of this interval. This suggests equilibrium configurations in which firm 1 will have an incentive to locate at a point strictly between any adjacent set of firm 2's locations. As a consequence, locations of firm 1 and 2 alternate. This intuition is rigorously proved in Proposition 3 and 4, in which the nature of these equilibria is also described. The following result shows that for any arbitrary number n there is an equilibrium such that one of the firms randomizes over n locations and the other over $n - 1$ locations.

Proposition 3: *For any number $n \geq 2$ there is a pair of location vectors $\xi = (x_1, ..., x_n)$ and $v = (y_1, ..., y_{n-1})$ such that firm 1 chooses x_i with probability $q_i > 0$ and firm 2 chooses y_i with probability $r_i > 0$. Moreover, $x_i < y_i < x_{i+1}$ for all $i = 1, ..., n - 1$.*

Proof: Define $Z \subset \mathbb{R}^{2n-1}$ by $Z \equiv \{\xi, v | 0 \le x_i \le y_i \le x_{i+1} \le 1$ for all $i = 1, ..., n-1\}$. Clearly, Z is convex and compact. Let

$$\varphi_1(x, v, r) \equiv \Sigma_j \Pi_1(x, y_j) r_j,$$
$$\varphi_2(y, \xi, q) \equiv \Sigma_j \Pi_2(x_j, y) q_j. \tag{12}$$

Note that $\partial^2 \Pi_1(x,y)/\partial x^2 < 0$ for $x > y$ and $x < y$. Therefore $\varphi_1(., v, r)$ is a strictly concave function of x for all $x \in (y_{i-1}, y_i)$, where $y_0 \equiv 0$ and $y_n \equiv 1$. This together with the Maximum Theorem implies that

$$f_{1i}(v, r) \equiv \text{argmax}_{\, x \in [y_{i-1}, y_i]} \, \varphi_1(x, v, r) \tag{13}$$

is a continuous function of (v, r). Similarly

$$f_{2i}(\xi, q) \equiv \text{argmax}_{\, y \in [x_i, x_{i+1}]} \, \varphi_2(y, \xi, q) \tag{14}$$

is a continuous function of (ξ, q). Define $f_1(.) \equiv [f_{11}(.), ..., f_{1n}(.)]$ and $f_2(.) \equiv [f_{21}(.), ..., f_{2,n-1}(.)]$.

Define $S_1 \equiv \{q \in \mathbb{R}^n | \Sigma_i q_i = 1\}$ and $S_2 \equiv \{r \in \mathbb{R}^{n-1} | \Sigma_i r_i = 1\}$. Then

$$g_1(\xi, v, r) \equiv \text{argmin}_{q \in S_1} \Sigma_i q_i \Sigma_j \Pi_1(x_i, y_j) r_j,$$
$$g_2(\xi, v, q) \equiv \text{argmin}_{r \in S_2} \Sigma_i r_i \Sigma_j \Pi_2(x_j, y_i) q_j, \tag{15}$$

are convex valued, upperhemicontinuous correspondences. As a result, the correspondence $h(\xi, v, r, q) \equiv f_1(v, r) \times f_2(\xi, q) \times g_1(\xi, v, r) \times g_2(\xi, v, q)$ maps $Z \times S_1 \times S_2$ into itself. Also, it is convex-valued and upperhemicontinuous so that by Kakutani's Theorem it has a fixed point (ξ^*, v^*, q^*, r^*). We will show that (ξ^*, v^*, q^*, r^*) satisfies the conditions of Proposition 3.

First we show that $\varphi_1(x_i^*, v^*, r^*) = \varphi_1(x_{i+1}^*, v^*, r^*)$ for all $i = 1, ..., n$. Suppose the contrary. Note that by definition of $g_1(.)$ one has $q_i^* = 0$ for all i such that $\varphi_1(x_i^*, v^*, r^*) > \min_j \varphi_1(x_j^*, v^*, r^*)$. Suppose there is a $k > 1$ such that $\varphi_1(x_i^*, v^*, r^*) > \min_j \varphi_1(x_j^*, v^*, r^*)$ for all $i < k$ and $\varphi_1(x_k^*, v^*, r^*) = \min_j \varphi_1(x_j^*, v^*, r^*)$. Then $\varphi_2(y, \xi^*, q^*)$ is strictly decreasing over $[x_1^*, x_k^*]$ because $q_i^* = 0$ for $i < k$. Accordingly, by definition of $f_{2i}(.)$ one has

$y_i^* = x_i^*$ for all $i < k$. Therefore, by definition of $f_{1k}(.), x_k^*$ must maximize $\varphi_1(x, v^*, r^*)$ subject to $x_{k-1}^* \leq x \leq y_k^*$. As $\varphi_1(x, v^*, r^*)$ is strictly concave over $[x_{k-1}^*, y_k^*]$ this yields a contradiction to $\varphi_1(x_{k-1}^*, v^*, r^*) > \varphi_1(x_k^*, v^*, r^*)$. The same argument shows that there cannot be a $k < n$ such that $\varphi_1(x_i^*, v^*, r^*) > \min_j \varphi_1(x_j^*, v^*, r^*)$ for all $i > k$.

Suppose there is a k and an l such that $k < l-1$ and $\varphi_1(x_i^*, v^*, r^*) > \min_j \varphi_1(x_j^*, v^*, r^*)$ for all $k < i < l$, and $\varphi_1(x_k^*, v^*, r^*) = \varphi_1(x_l^*, v^*, r^*) = \min_j \varphi_1(x_j^*, v^*, r^*)$. Then $\varphi_2(y, \xi^*, q^*)$ is strictly concave over $[x_k^*, x_l^*]$ and so one has $y_k^* = x_{k+1}^*$ and/or $y_{l-1}^* = x_{l-1}^*$. In the first case, x_k^* must maximize $\varphi_1(x, v^*, r^*)$ subject to $y_{k-1}^* \leq x \leq x_{k+1}^*$. But then $\varphi_1(x_{k+1}^*, v^*, r^*) > \varphi_1(x_k^*, v^*, r^*)$ leads to a contradiction because $\varphi_1(., v^*, r^*)$ is strictly concave over $[y_{k-1}^*, x_{k+1}^*]$. In the second case, a similar argument yields a contradiction. This proves $\varphi_1(x_i^*, v^*, r^*) = \varphi_1(x_{i+i}^*, v^*, r^*)$ for all $i = 1, ..., n$. The same arguments as above can be used to show that $\varphi_2(y_i^*, \xi^*, q^*) = \varphi_2(y_{i+1}^*, \xi^*, q^*)$ for all $i = 1, .., n-1$.

Next we show that $x_i^* < y_i^* < x_{i+1}^*$ for all $i = 1, ..., n-1$. Clearly, one cannot have $x_1^* = x_n^*$ because otherwise lowering x_1 or increasing x_n would increase firm 1's profit $\varphi_1(x, v^*, r^*)$. Suppose there is a k such that $x_k^* = y_k^* < x_{k+1}^*$. Then x_{k+1}^* must maximize $\varphi_1(x, v^*, r^*)$ subject to $x_k^* \leq x \leq y_{k+1}^*$. As $\varphi_1(x, v^*, r^*)$ is strictly concave over $[x_k^*, y_{k+1}^*]$ this leads to a contradiction to $\varphi_1(x_k^*, v^*, r^*) = \varphi_1(x_{k+1}^*, v^*, r^*)$. By the same argument one can rule out that $x_k^* < y_k^* = x_{k+1}^*$ for some k. Finally, $y_{k-1}^* = x_k^* < y_k^*$ or $y_k^* < x_{k+1}^* = y_{k+1}^*$ would contradict that y_k^* maximizes $\varphi_2(y, \xi^*, q^*)$ subject to $x_k^* \leq y_k \leq x_{k+1}^*$ because $\varphi_2(y, \xi^*, q^*)$ is strictly concave over $[x_k^*, x_{k+1}^*]$ and $\varphi_2(y_i^*, \xi^*, q^*) = \varphi_2(y_{i+1}^*, \xi^*, q^*)$ for all i.

It remains to show that $q_i^* > 0$ and $r_i^* > 0$ for all i. Suppose $q_1^* = 0$. Then $\varphi_2(y, \xi^*, q^*)$ is strictly decreasing over $[x_1^*, x_2^*]$ and so $y_1^* = x_1^*$, a contradiction to our above result that $x_i^* < y_i^*$ for all $i = 1, ..., n-1$. Similarly, we can rule out $q_n^* = 0$. Suppose there is a k and an l such that $k < l-1$ and $q_k^* > 0, q_l^* > 0$, and $q_i^* = 0$ for all $k < i < l$. Then $\varphi_2(y, \xi^*, q^*)$ is strictly concave over $[x_k^*, x_l^*]$ and so $y_k^* = x_{k+1}^*$ and/or $y_{l-1}^* = x_{l-1}^*$. This again contradicts our above result. The same argument proves that $r_i^* > 0$ for all

$i = 1, .., n-1$. Q.E.D.

Again, by symmetry of payoffs we can simply reassign the firms' indices to show that there also is an equilibrium in which firm 1 randomizes over $n-1$ and firm 2 over n locations. Moreover, the same arguments as in the proof of Proposition 3 can be used to prove existence of an equilibrium in which both firms randomly select one of n locations.

Proposition 4: *For any number $n \geq 2$ there is a pair of location vectors $\xi = (x_1, ..., x_n)$ and $v = (y_1, ..., y_n)$ such that firm 1 chooses x_i with probability $q_i > 0$ and firm 2 chooses y_i with probability $r_i > 0$. Moreover, $x_i < y_i < x_{i+1} < y_{i+1}$ for all $i = 1, ..., n-1$.*

For the case $n = 2$, the equilibrium described by Proposition 3 is the one identified in Proposition 2. One can also easily compute the equilibrium of Proposition 4 for $n = 2$. Here, firm 1 chooses $x_1^* = 0$ with probability q_1^* and some $0.5 < x_2^* < 1$ with probability $q_2^* = 1 - q_1^*$. The equilibrium strategy of firm 2 is symmetric; it chooses $y_1^* = 1 - x_2^*$ with probability $r_1^* = q_2^*$ and $y_2^* = 1$ with probability $r_2^* = q_1^*$. Accordingly, x_2^* must maximize $(1 - q_1^*)\Pi_1(x, 1 - x_2^*) + q_1^*\Pi_1(x, 1)$. This yields the first-order condition

$$q_1^*(3 + x_2^*)(1 + 3x_2^*) = (1 - q_1^*)3(5 - 4x_2^*). \tag{16}$$

The second equilibrium condition requires that firm 1 is indifferent between locating at x_1^* and x_2^*, i.e. one must have $q_1^*\Pi_1(0, 1) + (1 - q_1^*)\Pi_1(0, 1 - x_2^*) = q_1^*\Pi_1(x_2^*, 1) + (1 - q_1^*)\Pi_1(x_2^*, 1 - x_2^*)$ which is equivalent to

$$9q_1^* + (1 - q_1^*)(1 - x_2^*)(3 - x_2^*)^2 = q_1^*(1 - x_2^*)(3 + x_2^*)^2 + (1 - q_1^*)9(2x_2^* - 1). \tag{17}$$

Solving equations (16) and (17) for q_1^* and x_2^* gives the solution

$$q_1^* \simeq 0.344, \quad x_2^* \simeq 0.730. \tag{18}$$

Equation (18) together with $x_1^* = 0$ and $q_2^* = 1 - q_1^*$ defines firm 1's equilibrium strategy. Clearly, firm 2's behavior $(y_1^*, y_2^*) = (1 - x_2^*, 1), (r_1^*, r_2^*) = (1 - q_1^*, q_1^*)$ is optimal simply by symmetry. As an interesting property of the equilibrium each firm is more likely to select a location in the interior of the market than at the endpoint. Indeed, their expected distance in location in this equilibrium is approximately 0.437, whereas it is 1 in the equilibria of Proposition 1 and 0.5 in the equilibria of Proposition 2. This points to a reduction in the expected distance of location as the number of points in the support of the equilibrium strategies increases. It is a likely conjecture that letting n go to infinity in both Proposition 3 and 4 leads to convergence of the respective equilibria to an equilibrium with full support on $[0, 1]$. Unfortunately, we were not able to confirm this conjecture. However, the following Section shows that an equilibrium with this property exists and is symmetric in the sense that both firms adopt identical (mixed) strategies.

4 Symmetric Equilibrium

Proposition 1 reveals that there is no pure strategy equilibrium in which both firms adopt identical location strategies. However since the players' payoffs are symmetric, one should expect that there is such an equilibrium in mixed strategies. This is confirmed by the following result which also establishes uniqueness of the player-symmetric equilibrium. The proof employs four lemmas that are proven in the Appendix. To state the result, we define the coefficients $(a_{1n}, a_{2n}), n = 0, 1, 2, ...$, recursively by

$$a_{10} = 0, \quad a_{11} = 1, \quad a_{20} = 1, \quad a_{21} = 0, \tag{19}$$

$$a_{in} = \frac{2(n+1)}{5n} a_{i(n-1)} - \frac{2n^2 + 2n - 1}{5n(n-1)} a_{i(n-2)}, \quad n \geq 2, i = 1, 2.$$

Furthermore, define c_1 and c_2 by

$$c_1 = \frac{104 + 39 \sum_{n=0}^{\infty} a_{2n}/(n+1)}{200 + \sum_{n=0}^{\infty}(40a_{2n} - 28a_{1n})/(n+1)}, \quad c_2 = \frac{39 - 40c_1}{28}. \tag{20}$$

Proposition 5: *There exists a unique (player-) symmetric equilibrium of the location game. It has each player playing a mixed strategy with c.d.f. F over $[0, 1]$. On $(0, 1)$, F*

is strictly increasing, continuously differentiable, and has a density $f = F'$ *given by*

$$f(x) = \sum_{n=0}^{\infty}(c_1 a_{1n} + c_2 a_{2n})x^n.$$

F *has mass points at* $x = 0$ *and* $x = 1$ *given by*

$$F(0) = F(1) - F(1^-) = \frac{5}{2}c_2 - \frac{9}{8}.$$

Approximately, we have $c_1 = 0.61$, $c_2 = 0.52$, *and* $F(0) = 0.18$.

Proof: The location game is symmetric, and the payoffs Π_1 and Π_2 of firm 1 and 2 from choosing $(x, y) \in [0, 1]$ are continuous and defined on the product of non-empty, compact subsets of $\mathbf{R}$. Hence, the location game has a symmetric mixed strategy equilibrium (Dasgupta and Maskin (1986), Lemma 7). Let $F : [0, 1] \to [0, 1]$ denote the cumulative distribution function of the (common) equilibrium strategy. By definition, F is non-decreasing and continuous from the right. Recall that

$$\frac{\partial}{\partial x}\Pi_1(x, y) = \frac{1}{18}\begin{cases} -(2 + x + y)(2 + 3x - y) & < 0 \text{ if } x < y; \\ (4 - x - y)(4 - 3x + y) & > 0 \text{ if } x > y; \end{cases} \tag{21}$$

$$\frac{\partial^2}{\partial x^2}\Pi_1(x, y) = -\frac{1}{9}\begin{cases} 4 + 3x + y & < 0 \text{ if } x < y; \\ 8 - 3x - y & < 0 \text{ if } x > y. \end{cases} \tag{22}$$

Let

$$P(x) \equiv \int_0^1 \Pi_1(x, y)dF(y) \tag{23}$$

denote firm 1's payoff from choosing x, given that firm 2 plays its equilibrium strategy. Note that P is continuous on $[0, 1]$. In the sequel, in order to characterise F, we shall first prove some regularity properties of P and F.

Lemma 1: For all $x \in (0, 1)$, $\partial P(x)/\partial x_-$ and $\partial P(x)/\partial x_+$ exist, and

$$\partial P(x)/\partial x_\sigma = \int_0^1 \partial\Pi_1(x, y)/\partial x_\sigma dF(y), \quad \sigma = +, -.$$

Lemma 2: F is continuous on $(0,1)$ and P is differentiable on $(0,1)$.

As P is differentiable on $(0,1)$, $P'(x) = \int_0^1 \partial\Pi_1(x,y)/\partial x_\sigma dF(y)$, for $\sigma = +,-$. By Lemma 2, P' is continuous on $(0,1)$. Furthermore, we have for $0 < x+h < 1$

$$18\frac{P'(x+h)-P'(x)}{h} = \int_0^x (6x+2y-16+3h)dF(y) \tag{24}$$

$$-\int_{x+h}^1 (6x+2y+8+3h)dF(y) + \frac{1}{h}\int_x^{x+h} (x+y-4+h)(3x-y-4+3h)dF(y)$$

$$+\frac{1}{h}\int_x^{x+h} (x+y+2)(3x-y+2)dF(y).$$

Hence, for any $x \in (0,1)$ the existence of $\lim_{h\to 0}\frac{1}{h}(P'(x+h)-P'(x))$ implies the existence of $\lim_{h\to 0}\frac{1}{h}(F(x+h)-F(x))$ and vice versa, and the two are related by

$$\lim_{h\to 0}\frac{P'(x+h)-P'(x)}{h} = \int_0^1 \frac{\partial^2}{\partial x^2}\Pi_1(x,y)dF(y) \tag{25}$$

$$+\frac{2}{9}(2x^2-2x+5)\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}.$$

Lemma 3: F is strictly increasing on $[0,1]$.

Lemma 3 implies that P is constant on $[0,1]$. In particular, $P'' = 0$ on $(0,1)$, whence, by (25), F is differentiable (everywhere) on $(0,1)$. Since F is non-decreasing, its derivative is integrable, and Theorem 8.21 in Rudin (1974, p. 179) implies

$$F(b) - F(a) = \int_a^b F'(t)dt \quad \forall 0 < a < b < 1. \tag{26}$$

Hence, F has an integrable density $f \equiv F'$ on $(0,1)$. Denote F's possible mass in $x=0$ and $x=1$ by $p \equiv F(0)$ and $q = F(1) - F(1^-)$, respectively. (The proof of Lemma 3 shows that p and q must, in fact, be strictly positive.) Then Lemma 3 implies

$$px(4-x)^2 + \int_0^x (x-y)(x+y-4)^2 f(y)dy + \int_x^1 (y-x)(x+y+2)^2 f(y)dy \tag{27}$$

$$+q(1-x)(3+x)^2 = \text{ const}$$

on $[0,1]$. Differentiating (27) twice yields, for $x \in [0,1]$

$$\int_0^x (3x-y-4)(x+y-4)f(y)dy - \int_x^1 (3x-y+2)(x+y+2)f(y)dy \tag{28}$$

$$+p(x-4)(3x-4) - q(x+3)(3x+1) = 0,$$

$$\int_0^x (8-3x-y)f(y)dy \quad + \int_x^1 (4+3x+y)f(y)dy - (4x^2-4x+10)f(x)$$
$$+(8-3x)p + (3x+5)q = 0. \tag{29}$$

(29) shows that f is even differentiable. Differentiating (29) twice yields

$$3\int_0^x f(y)dy - 3\int_x^1 f(y)dy + (4x^2-4x+10)f'(x) + 8(2x-1)f(x) + 3p - 3q = 0, \tag{30}$$

$$(2x^2-2x+5)f''(x) + (12x-6)f'(x) + 11f(x) = 0. \tag{31}$$

In addition to the information contained in the differential equation (31), the fact that (28) - (30) hold in the endpoints of $[0,1]$ gives additional information. Letting $x=0$ and $x=1$ in (30) and using

$$\int_0^1 f(y)dy + p + q = 1 \tag{32}$$

yields

$$8f(0) - 10f'(0) - 6p + 3 = 0, \tag{33}$$

$$8f(1) + 10f'(1) - 6q + 3 = 0. \tag{34}$$

Letting $x=0$ and $x=1$ in (29) gives

$$\int_0^1 yf(y)dy = 10f(0) - 4p - q - 4, \tag{35}$$

$$10f(0) + 10f(1) - 4p - 4q - 9 = 0. \tag{36}$$

From (28) we then obtain

$$\int_0^1 y^2 f(y)dy = -20p - q + 4, \tag{37}$$

$$20f(0) + 12p - 20q - 9 = 0. \tag{38}$$

Each symmetric mixed strategy equilibrium hence has to satisfy (31) - (38). From the theory of second order ordinary differential equations we know that each solution of (31) has the form

$$f = c_1 f_1 + c_2 f_2, \tag{39}$$

where $c_1, c_2 \in \mathbb{R}$ are constants, and f_1 and f_2 independent solutions of (31), i.e. solutions satisfying

$$f_1 f_2' - f_1' f_2 \neq 0 \text{ on } [0,1]. \tag{40}$$

Considering equation (31) in the complex plane shows that its solutions are analytic on the open disk around 0 with radius $\frac{1}{2}\sqrt{10}$. (The radius is determined by the zeros of the leading coefficient, $2x^2 - 2x + 5$). Hence their restrictions to the real interval $[0,1]$ can be expressed as convergent power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \;\; x \in [0,1]. \tag{41}$$

By substituting (41) in (31) and rearranging terms, we find that for any solution f the coefficients $\{a_n\}_{n=0}^{\infty}$ have to obey the following recursion formula:

$$a_n = \frac{2(n+1)}{5n} a_{n-1} - \frac{2n^2 + 2n - 1}{5(n-1)n} a_{n-2}, \;\; n \geq 2. \tag{42}$$

For the remainder of the proof fix two independent solutions $\{a_{1n}\}_{n=0}^{\infty}$ and $\{a_{2n}\}_{n=0}^{\infty}$ by setting $a_{10} = 0, \;\; a_{11} = 1, \;\; a_{20} = 1, \;\; a_{21} = 0$. Let

$$A_i \equiv \sum_{n=0}^{\infty} a_{in}, \;\; B_i \equiv \sum_{n=0}^{\infty} n a_{in}, \;\; C_i \equiv \sum_{n=0}^{\infty} \frac{a_{in}}{n+1}, \;\; i = 1, 2, \tag{43}$$

(These sums exist because the solutions to (31) are analytic on $[0,1]$.) Any f obtained from (39) then satisfies

$$f(0) = c_2, \quad f(1) = A_1 c_1 + A_2 c_2, \quad f'(0) = c_1, \quad f'(1) = B_1 c_1 + B_2 c_2. \tag{44}$$

By means of (44) we can restate (32) - (38) as a system of seven equations in the unknowns c_1, c_2, p, and q. The equilibrium described by F will be unique if this system has a unique solution. In order to prove uniqueness, let us single out the four equations (32), (33), (36), and (38) and show that they already determine a unique solution. (Note that it is not possible that the seven equations contradict each other, since we know a forteriori that there is a symmetric equilibrium and hence a solution to (31) - (38).)

$$C_1c_1 + C_2c_2 + p + q - 1 = 0 \tag{32'}$$

$$8c_2 - 10c_1 - 6p + 3 = 0 \tag{33'}$$

$$10A_1c_1 + 10(A_2+1)c_2 - 4p - 4q - 9 = 0 \tag{36'}$$

$$20c_2 + 12p - 20q - 9 = 0 \tag{38'}$$

By eliminating p and q this system reduces to

$$5(15A_1 + 16)c_1 + (75A_2 - 19)c_2 = 78, \quad 5(3C_1 - 8)c_1 + (15C_2 + 47)c_2 = 39/4. \tag{45}$$

The determinant of system (45) is

$$D \equiv 75(15(A_1C_2 - A_2C_1) + 47A_1 + 40A_2 + \frac{19}{5}C_1 + 16C_2 + 40). \tag{46}$$

The following lemma, which is an exercise in numerical mathematics, implies the desired uniqueness result.

Lemma 4: $D > 0$.

Having established the uniqueness of F we immediately conclude that the distribution given by F must be symmetric around $x = 0.5$. For, if this were not so, the distribution function

$$G(x) = \begin{cases} 1 - F(1^-) & \text{if } x = 0, \\ 1 - F(1-x) & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1 \end{cases} \tag{47}$$

would, by the symmetry of the location game, define a different (player-)symmetric equilibrium, which would contradict the uniqueness of F.

It follows that $p = q$, that the equations (36′) and (38′) are identical, and, from solving (32′)-(36′), that F is given by

$$c_1 = \frac{39C_2 + 104}{200 - 28C_1 + 40C_2}, \quad c_2 = \frac{1}{28}(39 - 40c_1), \quad p = \frac{5}{2}c_2 - \frac{9}{8}. \tag{48}$$

A numerical calculation using ($A8$) in the Appendix yields $C_1 \simeq .5091, C_2 \simeq .6293$ so that $c_1 \simeq .61,\ c_2 \simeq .52,\ p \simeq .18.$ Q.E.D.

At the symmetric equilibrium the two firms end up at the same location with a probability of approximately 0.065. Thus, with some chance the outcome yields 'minimum differentiation' instead of 'maximum differentiation', which occurs in the pure strategy equilibrium described by Proposition 1.

5 Conclusion

We have demonstrated that Hotelling's (1929) model with quadratic consumer transportation costs possesses an infinity of equilibria in which the duopolists randomize over locations. These equilibria have been overlooked in the literature because the coordination problem underlying the location game has not been recognized. Interestingly, most of our results in Section 3 do not rely on the specific form of the firms' payoff functions. The proofs in this Section essentially require that payoffs satisfy certain symmetry properties, that each player's payoff is increasing in the distance from the other player, and that payoffs are strictly concave over each interval in which the other player is not located. Unfortunately, we have not been able either to rule out or to verify equilibria where the players' strategies are represented by a distribution function with an infinite number of masspoints. With this exception, our results provide a full characterization of all possible equilibrium configurations.

6 Appendix

Lemma 1: For all $x \in (0,1)$, $\partial P(x)/\partial x_-$ and $\partial P(x)/\partial x_+$ exist, and

$$\partial P(x)/\partial x_\sigma = \int_0^1 \partial \Pi_1(x,y)/\partial x_\sigma dF(y), \quad \sigma = +,-.$$

Proof: Let $x \in (0,1)$. Consider any sequence $\{h_n\}_0^\infty$ such that $0 \le h_n \le 1-x$ $\forall n$ and $h_n \to 0$. Let

$$f_n(x,y) \equiv \frac{1}{h_n}(\Pi_1(x+h_n,y) - \Pi_1(x,y)).$$

By (21) and since Π_1 is continuous, we have

$$f_n(x,y) \to \partial \Pi_1(x,y)/\partial x_+ \quad \forall y \in [0,1], \qquad |\, f_n(x,y) \,| \le \text{const} \quad \forall n, \forall y \in [0,1].$$

By Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} \int_0^1 f_n(x,y)dF(y)$ exists, and $\partial P(x)/\partial x_+ = \int_0^1 \partial \Pi_1(x,y)/\partial x_+ dF(y)$. The argument for $\partial P(x)/\partial x_-$ is analogous. Q.E.D.

Lemma 2: F is continuous on $(0,1)$ and P is differentiable on $(0,1)$.

Proof: By Lemma 1, for $x \in (0,1)$,

$$\frac{\partial}{\partial x_+}P(x) - \frac{\partial}{\partial x_-}P(x) = (\frac{\partial}{\partial x_+}\Pi_1(x,x) - \frac{\partial}{\partial x_-}\Pi_1(x,x))(F(x) - F(x^-)). \qquad (A1)$$

By (21), the first factor on the right hand side of $(A1)$ is strictly positive. Since P is continuous, in order for firm 1 to put a strictly positive mass $F(x) - F(x^-) > 0$ on x, it would be necessary that $\partial P(x)/\partial x_+ \le \partial P(x)/\partial x_-$. Hence $F(x) = F(x^-)$. This together with $(A1)$ implies that P is differentiable on $(0, 1.)$ Q.E.D.

Lemma 3: F is strictly increasing on $[0,1]$.

Proof: Suppose there is an $\alpha > 0$ such that $F(x) = F(0)$ for $x \in [0,\alpha]$. By continuity, there exists a maximal interval $[0,a] = F^{-1}(F([0,\alpha]))$ on which F is constant. Again by continuity, there exists an $\epsilon > 0$ such that F is strictly increasing on $[a, a+\epsilon]$. Since F

defines an optimal mixed strategy, firm 1 must be indifferent between all $x \in [a, a+\epsilon]$, hence $P(x) = K \quad \forall x \in [a, a+\epsilon]$. Also, $P(x) \leq K \quad \forall x \in (0, a)$. By (21) and Lemma 1, for $x \in (0,1)$

$$P'(x) = F(0)\frac{\partial}{\partial x}\Pi_1(x,0) + \int_0^1 \frac{\partial}{\partial x}\Pi_1(x,y)dF(y) \tag{A2}$$

$$= \frac{1}{18}\Big(F(0)(4-x)(4-3x) - \int_a^1 (x+y+2)(3x-y+2)dF(y)\Big).$$

If $F(0) = 0$, $(A2)$ implies that $P'(x) < 0$, hence $P(x) > K$ on $(0, a)$, a contradiction. If $F(0) > 0$, the distribution given by F puts positive weight on $x = 0$, hence $P(0) = K$. Since $F'(x) = 0$ everywhere on $(0, a)$, (25) implies that P'' exists everywhere on $(0, a)$ and

$$P''(x) = \int_0^1 \frac{\partial^2}{\partial x^2}\Pi_1(x,y)dF(y). \tag{A3}$$

By (22) and $(A3)$, $P''(x) < 0$ on $(0, a)$. This, together with $P(0) = P(a) = K$ implies $P(x) > K$ on $(0, a)$, again a contradiction.

An analogous argument proves that there is no $\beta < 1$ such that F is constant on $[\beta, 1]$, and finally (only using $(A3)$) that F is strictly increasing on every interval $[\alpha, \beta]$, $0 < \alpha < \beta < 1$. Q.E.D.

Lemma 4: $D > 0$.

Proof: For $N \geq 9$, $N \in \mathbb{N}$, choose δ such that

$$\frac{N^2+N-0.5}{5(N-1)N}\Big(1 + \sqrt{1 + \frac{10(N-1)N}{N^2+N-0.5}}\,\Big) < \delta < 1. \tag{A4}$$

(Note that the left hand side of $(A4)$ is smaller than 1 for $N \geq 9$). Suppose that for $n \geq N$

$$|\, a_{n-2} \,| < \delta^{n-2} \text{ and } |\, a_{n-1} \,| < \delta^{n-1}. \tag{A5}$$

Then, by (42), the choice of δ implies

$$\begin{aligned} |\, a_n \,| \quad &< \frac{2}{5}\,\Big(\frac{n+1}{n}\delta^{n-1} + \frac{n^2+n-\frac{1}{2}}{(n-1)n}\delta^{n-2}\Big) \\ &< \frac{2}{5}\,\frac{N^2+N-\frac{1}{2}}{(N-1)N}\,\delta^{n-2}(1+\delta) < \delta^n. \end{aligned} \tag{A6}$$

Using the identity

$$\sum_{n=0}^{\infty} a_{in} = \sum_{n=0}^{N-3} a_{in} + \sum_{n=N-2}^{\infty} a_{in}$$

we get from $(A6)$ for all N and δ satisfying $(A4)$ and $(A5)$

$$\sum_{n=0}^{N-3} a_{in} - \frac{\delta^{N-2}}{1-\delta} < \sum_{n=0}^{\infty} a_{in} < \sum_{n=0}^{N-3} a_{in} + \frac{\delta^{N-2}}{1-\delta} \qquad (A7)$$

and

$$\sum_{n=0}^{N-3} \frac{a_{in}}{n+1} - \frac{\delta^{N-2}}{1-\delta} < \sum_{n=0}^{\infty} \frac{a_{in}}{n+1} < \sum_{n=0}^{N-3} \frac{a_{in}}{n+1} + \frac{\delta^{N-2}}{1-\delta} \qquad (A8)$$

for $i = 1, 2$. A numerical calculation shows that $(A4)$ and $(A5)$ are satisfied for $N = 30$ and $\delta = .9015$, and that the estimates for A_i and C_i given in $(A7)$ and $(A8)$ yield $D > 0$. Q.E.D.

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