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We'd Rather Fight Than Switch:

Trying to Understand "Let's Make a Deal"

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Author's Note:

This paper examines the traditional Let's Make a Deal problem as well as a family of similar problems. It introduces a possible bias in decision making that may lead to the common mistake in Let's Make a Deal and summarizes the results of experiments testing this bias. As will be elaborated in the paper, the Let's Make a Deal problem strikes a mathematical nerve in a predominantly innumerate world. Debates over its solution enliven mathematicians, probabilists, decision scientists and laypersons alike.

1 Introduction

Thirty-two years ago Martin Gardner featured a mathematical puzzle about three prisoners on death row in his popular *Scientific American* column (Gardner 1959). He noted that it was "making the rounds" in the academic community. Recently the problem has resurfaced as the Let's Make a Deal problem, the name being derived from a television game show. Its return has merited among other mentions a front page New York Times article (Tierney 1991), a series of letters in *Parade* (Vos Savant 1991), and discussion in mainstream economic journals (Nalebuff 1987, 1990). An enormous amount of informal evidence from elementary school classrooms, graduate decision theory seminars, dinner conversations, and personal experience supports a single conclusion: most people get the wrong answer.

In its present form the Let's Make a Deal problem (hereafter LMAD) consists of three stages. In the first stage the contestant randomly selects one door from among three with the goal of locating the one door which hides a prize. In the second stage, the host, Monty Hall, opens an incorrect door from among the two unchosen doors. Finally, in the third stage, Monty asks the contestant whether she would like to switch from her initial choice to the remaining unopened door. At the completion of the three stages, Monty opens the remaining two doors awarding the prize only if the contestant's door conceals the prize. While the problem's formulation is simple enough, explaining its answer - if the contestant knows all three stages of the game she should switch - is not. Many people, at least initially, believe that switching confers no advantage. Two doors remain. The correct door was chosen randomly; therefore, both doors have equal probability of being correct. The error in this logic will be discussed later in the paper.

Perhaps more confusing than LMAD's answer is its appeal. Mathematical story problems are almost universally abhorred. A favorite FAR SIDE cartoon by Gary Larson depicts Hell's Library as consisting of two books, *Story Problems* and *More Story Problems*. LMAD's simple formulation well known by a generation raised on television and its incorrect salient solution make it exceptional and interesting. Moreover, the correct solution's subtle proof never convinces some of its veracity which adds to LMAD's allure. Finally, for those who grow exasperated as their logical arguments fail to convince others of the solution, the problem provides a hint as to how mathematicians can get so excited with just chalk and symbols.

In light of LMAD's rebirth, this paper investigates the problem with the hope of gaining a better understanding of the bias in decision making that causes people to get it wrong. The remainder of the paper is organized as follows. It begins with a formal mathematical proof and an exploration of the incorrect "logic" that most people employ to arrive at the wrong answer. The next section summarizes experiments on people previously unexposed to LMAD. These experiments support two conclusions; first, increasing the number of doors makes the problem easier; and second, people have difficulty learning LMAD. The paper also considers problems similar to LMAD which elucidate the bias leading to incorrect decisions. The paper concludes with a brief recapitulation and an intriguing offer. Three appendices follow which include proofs for some of the more complicated claims made in the paper and a detailed explanation of the experiments.

2. And Behind Door #1...

LMAD confuses almost everyone. The fact that mathematicians have difficulty shows that the error cannot be due to an inability to update probabilities. Anyone who knows Bayes' rule can recognize the advantage of switching if provided with the correct initial and conditional probabilities. Errors arise because people misinterpret the information resulting from the revelation of an incorrect door. Accordingly, mathematicians' debates over LMAD typically center on how to update and not on whether to update. This improper updating represents a "systematic violation" of rationality and leads Nalebuff (1987) to ask whether we "should look for alternatives to Bayes Rule."

In defense of those people who arrived at the incorrect answer, imprecise wording of LMAD may make switching no better than standing pat. As will be formalized later, slight changes in the wording affect the updating and the optimal strategy. The proper formulation of LMAD and its solution are given below:

LMAD: There are three doors, one of which was chosen randomly to conceal a prize. A contestant is asked to select which door she thinks hides the prize. After the contestant selects, Monty Hall randomly reveals one door from among those doors she did not choose which do not conceal the prize. After revealing an incorrect door, Monty Hall offers the contestant the opportunity to switch to the remaining unchosen door. If the prize is behind her selected door, she receives it. If the contestant knows all of the stages of the game should she switch?

To best analyze the problem, each stage will be analyzed individually.

There are three doors, one of which was chosen randomly to conceal a prize.

The correct door was assigned randomly; therefore, the probability that any given door is correct is one third:

DOOR #1	DOOR #2	DOOR #3
p(1) = 1/3	p(2) = 1/3	p(3) = 1/3

where p(i) = probability that door i is correct.

A contestant is asked to select which door she thinks hides the prize.

Let door i be the door that she chooses. can write the probabilities that she is correct and incorrect as:

DOOR i	DOOR j and DOOR k
p(i)= 1/3	p(j) + p(k) = 2/3
(Correct)	(Incorrect)

After the contestant selects, Monty Hall randomly reveals one door from among those doors she did not choose which do not conceal the prize.

We provide two explanations of how to correctly update the probabilities. The first uses Bayes' rule and the second a reversal of ordering argument.

Bayes' Rule: There are two possible cases, either Monty reveals door j or he reveals door k. Without loss of generality assume that he reveals door j. There are two events that could have led to door j being revealed.

Event 1: Door i conceals the prize (probability 1/3) and Monty randomly chose door j from among from door j and door k (probability 1/2). The probability of this event is $(1/3)\cdot(1/2) = 1/6$.

Event 2: Door k conceals the prize (probability 1/3), forcing Monty to reveal door j (probability 1). The probability of this event is $(1/3)\cdot(1) = 1/3$.

The conditional probability that door i, the initially chosen door, is correct given that Monty reveals door j is: $(1/6)/[(1/3) + 1/6)] = 1/3$. The probability that door k, the unchosen unopened door is correct is 2/3.

Reversal of ordering: In LMAD, Monty Hall shows the contestant the incorrectness of one of the two unchosen doors before offering the switch. Suppose instead that he offers the contestant both unchosen doors before revealing an incorrect door. Clearly, the contestant should switch. She doubles her odds. After switching she knows that both of her doors cannot conceal the prize. If Monty (knowingly) reveals an incorrect door from among her two doors, he does not alter her probability of winning. The timing of the revelation, specifically whether it occurs before or after the offer to switch, does not effect the optimal choice. In either case she should choose both doors, even if one has been shown incorrect.

After revealing an incorrect door, Monty Hall offers the contestant the opportunity to switch to the remaining unchosen door. If the prize is behind her selected door, she receives it. If the contestant knows all of the stages of the game should she switch?

The probability that the door she chose, door i, is correct is only 1/3, and the probability that the other unopened door is correct is 2/3. Switching doubles her odds of winning, so yes she should switch.

As mentioned earlier, even students of probability theory can believe that switching affords no advantage. The error in computing probabilities occurs not because of an inability to update but because of improper updating. The probability that the initially selected door is correct is updated when it should not be. The opening of an incorrect door should only be seen as a signal as to which, if either, of the unchosen doors is correct. Instead, the revelation causes an (unwarranted) increased belief that the initial choice is correct. An examination of the canonical incorrect argument clarifies the point.

The Incorrect Argument: *Originally there were three doors, each equally likely. One door was shown not to conceal the prize, therefore, the two remaining doors each have probability 1/2 of being correct. Switching does not increase the probability of winning.*

This argument misinterprets how the information affects the probabilities of the two remaining doors. At the point in the game when Monty offers the contestant the switch, one of the unchosen doors has not been revealed. Remember that according to the rules, the initially chosen door could not have been opened, while the remaining unopened door could have been, but only *if* it was incorrect. Therefore, the unchosen door has survived a test of revelation which the initial door has not. The LMAD bias can be characterized as an *inability to recognize that the revelation only tests the likelihood of the unchosen door*. The revelation does not test the likelihood of the initial door which could not have been opened in stage two.

The LMAD bias should be distinguished from what Dawes (1988) refers to as "distributing ignorance equally across verbally defined categories." For example, if two coins are flipped then either 0, 1, or 2 heads could occur. Someone who distributes ignorance equally would deduce that each event has probability 1/3. The LMAD bias, not recognizing that an event has undergone an additional test of likelihood, differs entirely. In Dawes' treatment subjects count the events and assign equal probability to each. In LMAD contestants begin with correct initial probabilities (all of the doors are equally likely to conceal the prize), but upon receipt of more information, mistakenly treat the revelation of a door as though its effect was symmetric and maintain equal probability of the events.

If in fact the LMAD bias causes people to stand pat rather than switch, fewer people should commit errors as the existence of the test becomes clearer. As has been suggested by Littlechild and others (Nalebuff 1990) increasing the number of doors makes the test more obvious. If the contestant chooses from among one hundred doors and in stage two sees ninety-eight incorrect doors from among the ninety-nine unchosen doors, she should be more likely to recognize the asymmetry of the test. The unchosen door survived a test while her initial choice was guaranteed safe harbor. Even if she cannot compute the probabilities at all, much less precisely, the logic of switching should be more apparent. As will be detailed in the next section, increasing the number of doors to one hundred made the bias less commonplace.

Generally, biases can be overcome when the numbers are taken to extremes. Stretching the Dawes example above, few people would believe that one hundred tosses of a fair coin are as likely to yield no heads as fifty. In the case of another well known bias, only Sherlock Holmes recognized that one dog not barking was an important clue, but anyone would have deduced that only a friendly intruder could slip by one hundred and one dalmatians.[1]

Before proceeding to the results of the experiments, no treatment of LMAD would be complete without mentioning that if the three stages of the game are not known to the player then switching need not be optimal. Suppose that Monty offers the switch only when the contested has initially selected the correct door. In other words, Monty Hall only gives the option of taking "the other door" when "the other door" is wrong . In this scenario, whenever Monty offers a door the contestant should refuse.

3. Experiments

We ran two sets of experiments for real money to test our hypothesis. The first experiment asked whether LMAD becomes easier as the number of doors increases. We considered three cases, n = 3, 10 and 100, where n equals the number of doors. These games will be referred to as LMAD(3), LMAD(10) and LMAD(100) respectively. To minimize framing effects (Kahneman and Tversky 1979), the second stage of the game was cast as "n-2 incorrect doors have been randomly selected from the n-1 unchosen doors" as opposed to "the correct door must be either the initial door or door k." Furthermore, our wording of the crucial part of the experiment agrees with the game show which emphasizes revealing bad outcomes. For a precise characterization of the experiments see Appendix 3.

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[1] I would like to thank Max Bazerman for suggesting this line of argument.

Our expectation that the bias would be decreasing in the number of doors was strongly supported. The subjects were fifty MBA students in a decision making class. Winners, those who had the guessed the correct door at the end of the game, received five dollars. Losers received nothing. Only two of seventeen switched in LMAD(3). In LMAD(10), eight of seventeen switched, while in LMAD(100) fourteen of sixteen switched. Table 1 gives the proportion of individuals who switched and 95% confidence intervals for these results.

Table 1

# Doors	#Switch	p(Switch)	.95 CI
3	2 out of 17	.116	$.02 \leq p \leq .33$
10	8 out of 17	.471	$.26 \leq p \leq .69$
100	14 out of 16	.875	$.66 \leq p \leq .98$

The subjects were also asked to estimate the probability that their initial and final selections were correct: the former to guarantee they understood the game's formulation the latter its solution. In LMAD(3), six of seventeen (35%) correctly estimated the odds of that the initial door was correct after the revelation at 1/3. However, four of these people did not switch. Discussion following as well as written comments made during the experiment hinted that many of these subjects thought that switching did not matter. In short, subjects neglected to update the probabilities so that they added to one. In LMAD(10) five of seventeen (29%) correctly estimated the probability of the other door at 9/10 and all of the rest except one believed the odds were even. In LMAD(100) exactly half of the 16 subjects switched and correctly estimated their probability of winning to be 99/100. Of the rest, all of whom claimed that the probability of winning after switching was one half, three quarters switched. This switching in the face of possible regret implies that even though the subjects could not correctly compute probabilities, they acted "as if" they were Bayesians. Moreover, these results support our characterization of the bias. As the number of doors increased, subjects were more likely to recognize the additional test survived by the alternative door.

The second set of experiments used forty-six MBA students who recently spent three weeks studying probability. The subjects were divided into two groups. Group 1 played the traditional LMAD(3) with winners paid $5 and losers nothing, and group 2 simultaneously played both LMAD(3) and LMAD(100) for $3 and $2 respectively. Group 1 was used as a control. Briefly, only three of twenty-four subjects in Group 1 switched and a different set of three correctly computed the probability of winning. As was the case in the previous experiment, the written comments of the subjects with correct probabilities show that they believed switching neither improved nor decreased their probability of winning. Of the three who switched, two volunteered during discussion that they did so capriciously.

The subjects in Group 2, who played the LMAD(3) and LMAD(100) simultaneously, had difficulty learning LMAD(3). Even though the wording of the two games was identical (except for the number of doors) the results for LMAD(3) differ only slightly from the control group's. Only four of twenty-two in Group 2 switched in LMAD(3). While those four individuals also switched in LMAD(100) all estimated the probability incorrectly in not only LMAD(3) but also in LMAD(100). None of the subjects who correctly computed the probabilities in LMAD(100) recognized the similarities in the games and "learned" to switch in LMAD(3). In other words, no one learned through mathematical reasoning. Eighteen of the twenty-two subjects playing both games switched in LMAD(100). Twelve of the eighteen (66%) who switched estimated their probability of winning at 1/2 and only five (28%) computed the correct probability. Table 2 below summarizes the results.

Table 2

Group#	# Doors	#Switch	p(Switch)	.95 CI
1	3	3 of 24	.125	$.03 \le p \le .29$
2	3	4 of 22	.182	$.06 \le p \le .37$
2	100	18 of 22	.875	$.63 \le p \le .94$

Discussion following the experiments indicated those subjects who felt that the odds were even in LMAD(3) may have remained with their initial choice out of either inertia or regret avoidance. In LMAD(100), many "knew" switching was better despite their inability to prove so mathematically. Again, subjects intuitively recognized the optimal strategy without being able to formally justify it. Arguments in favor of switching emphasized the unlikelihood that the initial door was correct and the obvious increase in the likelihood of the alternative. No coherent explanations were offered as to why the same logic did not hold in LMAD(3). Comments typically took the form "they seemed different somehow." This difference drives the bias. In LMAD(100), the opening of ninety-eight incorrect unchosen doors makes transparent the asymmetric affects on the likelihood of the initial door and the remaining unchosen door. In LMAD(3), revealing one incorrect unchosen door does not appear to impact the likelihood of the initial door and the remaining door asymmetrically.

4. Variations on a Theme

LMAD can take many equivalent forms. Gardner's (1959) original story tells of three prisoners, one of whom was randomly selected to be sent to the gallows. The Warden has not yet announced the outcome of the selection process. The first prisoner asks the warden for the name of one of the others who will not be killed. After being told that the third prisoner will not be killed, the first prisoner's probability of being selected is..? Additional verbal representations of mathematically equivalent problems, while fun, add less to our understanding than other mathematical forms which create the same difficulties for decision makers. Alan Truscott (1991) recently dealt (pun intended) with just such a problem in his bridge column. His example will be summarized below (for a full treatment see Appendix 2). Another problem first mentioned by Gardner (1959) which may be even more difficult than LMAD(3) will also be discussed. Gardner's problem will be extended to three "Myrtle problems", the name Myrtle being liberated from Paulos (1988) who elaborates on a variant of the puzzle in his fun little book *Innumeracy*. Finally, conditions on random variables used by Kirschenhieter (1991) will be shown to generate many LMAD type problems.

The essence of Truscott's bridge problem can be understood without the formal argument. Briefly, a card player must guess whether Player 2 or Player 4 holds the jack of hearts. Ignoring other information, Player 4 can be shown to be more likely to hold the card using simple counting arguments. Suppose though that on the previous trick Player 4 laid the queen of hearts. Experienced card players know that if Player 4 held both the jack and the queen, he would have been indifferent between which card he played first. Therefore, the likelihood that Player 4 holds the jack decreases, given that were he to have it, he might have played it. Player 2, having given no such signal, is more likely to hold the jack. (See Appendix 2)

A distinction can be drawn between Truscott's problem and LMAD. In the latter, two alternatives are not equally likely because one has undergone an additional test which increases its probability relative to the other. In the former, the additional test (that Player 4 could have tossed the jack previously) *decreases* the probability of one event (Player 4 holding the jack), making the other event (Player 2 holding it) more likely.

The next set of problems may be even more difficult than LMAD(3). Evidence from informal pre-tests supports the conjecture that the Myrtle problems trip up even the most formal decision makers.

The Myrtle Problems: *A census worker and amateur probabilist arrives at a household which claimed two children but neglected to specify their gender. Assume that boys and girls are equally likely. Consider the following three separate scenarios:*

Scenario 1: The Census worker meets the oldest child, named Myrtle, what is the probability that she has a younger sister?

Scenario 2: The census worker asks the mother, "do you have at least one daughter?" The mother replies that she does. What is the probability that she also has a son?

Scenario 3: Randomly a child, Myrtle, enters the living room where the interview is being conducted. What is the probability that the other child is a boy?

The answer to Scenario 1 is 1/2. Boys and girls are equally likely. The younger child has equal probability of being either. The answer to Scenario 3 is also 1/2. The child that the census worker sees was randomly selected. Therefore, the other child, also randomly selected, is equally likely to be either a boy or a girl.

The answer to Scenario 2 is 2/3. How can this be? The answer can be arrived at in two ways. First, using formal probability: Initially there were four equally likely cases, bb, bg, gb, and gg, where gb means that the older child is a girl and the younger a boy. The information that the mother has a daughter rules out the case bb. However, it does not make any of the other three more or less likely than the other. The three remaining cases, bg, gb, and gg are all equally likely. In two of the three cases the other child is a boy, thus the probability of 2/3.

The second explanation uses the idea of an additional test. In Scenario 1 the gender of the older child does not test in any way the gender of the younger. Similarly, in Scenario 3, the gender of random child does not test in any way the gender of the other child. However, in Scenario 2, verifying that there exists a girl implies that both children's genders may have been tested. Suppose that the census worker was asked by the Census Bureau to verify the mother's answer. He could select the children in any order: older-younger, taller-shorter, randomly, etc. There are three cases of interest:

Case 1: The first child in the ordering is a girl

Case 2: The first child is a boy, and the second a girl.

Case 3: Both children are boys.

Only in Cases 1 and 2 could the census worker report that the mother's answer had been truthful. Given an affirmative response, with positive probability both children have been viewed (Case 2). More important, if they were, then the first child was a boy. An affirmative answer implies that with positive probability two children were viewed and one was a boy, but with probability zero two children were viewed and both were girls. Therefore, the probability that the mother has a son exceeds one-half.

If confusing LMAD and Myrtle type problems only occurred in game shows, bridge tournaments, and math examinations, their importance would be primarily diversionary. However, similar problems can arise in the analysis of more important phenomena. Kirschenheiter (1991) compares incentive aspects of disclosing market and historical costs of assets to shareholders. He asks when a firm might prefer to withhold one of the two valuations from the potential shareholders. He shows, given technical assumptions, that withholding occurs if $p_{xy} \cdot p_{yz} > p_{xz}$ where p_{xy} equals the correlation coefficient between x and y, and the variables x, y, z represent the market cost, historical cost, and shareholder value of an asset respectively. He further requires that all of the p_{ij}'s are greater than or equal to zero.

How does this relate to LMAD? Consider a simple example where $p_{xy} \cdot p_{yz} > p_{xz}$ and all of the p_{ij}'s ≥ 0. Let x and z be independent and equal 1 with probability 1/2 and 0 with probability 1/2. There are four possible states {x,z} = {0,0}, {0,1}, {1,0}, and {1,1} each equally likely with probability 1/4. Let y be a random variable whose value depends on x and z. If x=z then y=x=z with probability 1. If, on the other hand, x = (1-z), then y equals 1. A simple calculation shows that $p_{xy} = p_{yz} = 1/2$, and $p_{xz} = 0$, satisfying the assumptions. More to the point, we can now show the equivalence of the example and Scenario 2 of the Myrtle problems.

Let x be the older child's gender and z the younger's, with 1's denoting girls and 0's boys. Let the variable y represent the question "do you have at least one daughter?", with 1 representing a yes and 0 representing a no. Clearly, x and z are independent each equalling 1 and 0 with probability 1/2. Also, the two conditions for y are met: If x=z, then y=x=z, (if both children are girls the mother answers yes, if both boys no) and if x=(1-z) then y=1 (if either is a girl, the mother answers yes). Scenario 2 can be restated as finding p(x=0 or z=0|y=1). (See Appendix 1)

Many LMAD type problems can be written by choosing random variables x, y, z with $p_{xy} \cdot p_{yz} > p_{xz}$ and all of the p_{ij}'s ≥ 0. The easiest have x, y, and z taking only the values 0 and 1. We will present one at the end of this section. A generic LMAD type problem does not necessarily fit into this form. Necessary conditions for a LMAD type problem are:

(1) $\quad \text{prob}(B) \geq \text{prob}(A) > 0$

(2) $\quad 0 < \text{prob}(B|y) < \text{prob}(A|y)$

The first inequality says that event B is at least as probable as event A before the signal y occurs. The second says that while y is consistent with either A or B, A is more likely given that y occurred. In other words, y offers a stronger test of A's likelihood than of B's. All of the problems considered can be put in this form. In LMAD, let B be the event that the door chosen at the Beginning is correct, and A be the event that the Alternative door is correct. In Truscott's problem, let A be the event where Player 2 holds the jack, and B where Player 4 does.

To use random variables x,y,z which satisfy $p_{xy} \cdot p_{yz} > p_{xz}$ and p_{ij}'s ≥ 0 to form LMAD type problems, let events A and B represent specific values of x and z respectively or combinations of values of x and z. Let the random variable y be positively correlated with both x and z but be asymmetrically consistent with values of x and z. Consider the following example:

Bowling Romans: *Romulus and his identical twin Remus have both gone bowling. Romulus being the happiest of men always smiles. Remus on the other hand smiles every other day. His mood today is not known. There are two alleys in town, One Alley Lane and Big Zero's Gutter House. Assume that each brother independently flips a coin to determine which alley to visit. Richard Nixon, yet another avid bowler, appears with a One Alley Lane scorecard and upon being asked if he saw a smiling person fitting Romulus/Remus' physical description, Nixon responds (honestly!) that he did. What is the probability that both Romulus and Remus are at One Alley Lane?*

Appendix 2 provides the answer to Bowling Romans, we leave the formal proof to the motivated reader. Bowling Romans was created from random variables satisfying $p_{xy} \cdot p_{yz} > p_{xz}$ and p_{ij}'s ≥ 0. Let x be the alley chosen by Remus (either One or Zero), z be the alley chosen by Romulus, and y be the event that a smiling person was seen at One Alley Lane. Finally, let A be the probability that one brother went to Big Zero's given that the other went to One Alley Lane, and B be the probability that both brothers went to One Alley Lane given that one brother went there. Initially A and B have equal probability. Following the revelation of y, they do not. Additional LMAD type problems are easily made from this basic mathematical form.

Conclusion:

As expected, careful experimentation with monetary incentives supported earlier anecdotal evidence that people cannot correctly solve LMAD; most of the graduate students tested opted not to switch. More interesting than those results was that as the number of doors increased LMAD became less difficult. This result supports our conjecture that people do not recognize that the unopened door has undergone an additional test of likelihood. Written comments during the experiments and discussions following further supported this conjecture.

Also of interest was the result that playing LMAD(100) and LMAD(3) simultaneously did not lead to greater understanding of either problem. LMAD(3) appears not to be easily learned. The hope that subjects might be able to arrive at the solution deductively proved to be in vain. Clearly, repeated playing of LMAD, where the improved odds resulting from switching will be manifested in the distribution of wins and losses, would induce learning. Learning through repeated playing only suggests that people can recognize an unfair coin after enough flips, not that they can learn LMAD. Being convinced that the probabilities are unequal through experiments implies acceptance but not understanding of LMAD's proof.

Another intriguing result was that most subjects made the correct decision in LMAD(10) and LMAD(100) even though they could not compute correct probabilities. The subjects in LMAD(10) and LMAD(100) estimated the same probability of being correct as those who played LMAD(3), but only the former groups switched. Probabilistic arguments that led to equal likelihood were overridden by logical arguments that the unchosen door had survived an additional test in LMAD(10) and LMAD(100). Ironically, decision making instructors stress the opposite; using formal probabilistic reasoning to correct flawed human logic. On a postive note, advocates of rational choice (a.k.a. economists) can find evidence of "as if" rationality in our subjects' decisions.

In sum, we were able to make LMAD easier by increasing the number of doors, but we were not able to induce deductive learning through our experiments. Whether this rather lengthy paper can teach LMAD depends partially on the transparency of the following..

Monty Hall arranges a deck of cards face down before you on a table. You are told that if you select the Ace of spades you win $100. If you select any other card you win $0. After pointing at your selection, you notice that there are 54 cards on the table. You exclaim to Monty that he must have left the jokers in the deck. Monty offers to check, peeking at the cards that you did not select. Finally, he flips one. It is a joker. He says, "you are right. Would you like to choose a different card?"

Well, would you?

Appendix 1

This appendix proves three claims presented in the paper. The first two claims prove that the host cannot by signalling make staying with the initial choice the preferred decision. The third claim formally proves Myrtle Scenario 2.

Claim 1: In LMAD(3) switching is always at least as good as standing pat regardless of the probabilities with which the host randomizes between the doors.

pf: Without loss of generality let door 1 be the door initially chosen. Let p be the probability that the host shows door 2 if door 1 was correct. If door 3 was correct he must show door 2, and equivalently, if 2 was correct than 3 must be revealed. The following probabilities are easily computed:

Correct door	prob(correct)	revealed door	prob(revealed)
1	1/3	2	p
		3	1-p
2	1/3	3	1
3	1/3	2	1

There are two cases to be considered. For notational purposes let $p(1c|2R)$ equal the probability that door 1 was correct given that door 2 was revealed and let $p(2R|1c)$ equal the probability that door 2 was revealed given that door 1 was correct.

Case 1: *Door 2 revealed:* Using Bayes's rule:

$$p(1c|2R) = p(2R|1c)/[p(2R|1c) + p(2R|3c)] = p/[p+1]$$

Since $p \le 1$, it follows that $p(1c|3R) \le 1/2$

Case 2: *Door 3 revealed:* Using Bayes's rule:

$$p(1c|3R) = p(3R|1c)/[p(3R|1c) + p(3R|3c)] = (1-p)/[2-p]$$

Since $p \le 1$, it follows that $p(1c|3R) \le 1/2$

Therefore, in both cases switching guarantees at least as high of a probability of winning as standing pat.

Claim 2: In LMAD(n) switching is always at least as good as standing pat regardless of the probabilities with which the host randomizes between the doors.

pf: Without loss of generality assume that door 1 was initially chosen and that all doors except door 2 have been revealed to not contain the prize. Let $p(2r|1c)$ equal the probability that 2 was the *remaining* door and door 1 was correct, and let $p(1c|2r)$ equal the probability that door 1 was correct and door 2 remaining. Using Bayes' rule:

$$p(1c|2r) = p(2r|1c)/[p(2r|1c) + p(2r|2c)]$$
$$p(2c|2r) = p(2r|2c)/[p(2r|1c) + p(2r|2c)]$$

The denominators are the same so only the numerators need be compared. Switching is preferred iff $p(2r|2c) \geq p(2r|1c)$, but $p(2r|2c) = 1$ and $p(2r|1c) \leq 1$, which proves the claim.

Claim 3: If x and z are independent random variables each taking the values 0 and 1 with probability 1/2 and if y is distributed as follows:

$$y = x \qquad \text{if } x = z$$
$$y = 1 \qquad \text{if } x = 1-z$$

then $p(x=0 \text{ or } z=0|y=1) = 2/3$.

pf: There are eight possible outcomes. The states and their probabilities are listed below:

Outcome {x,y,z}	Prob {x,y,z}
000	1/4
001	0
010	0
011	1/4
100	0
101	0
110	1/4
111	1/4

Using Bayes' rule:

$$p(x=0 \text{ or } z=0|y=1) = \frac{p(010) + p(011) + p(110)}{p(010) + p(011) + p(110) + p(111)}$$

$$= [0 + 1/4 + 1/4]/[0 + 1/4 + 1/4 + 1/4]$$
$$= 2/3$$

Appendix 2

Truscott's Problem: Bridge is played with a standard deck of playing cards and four players. Players 1 and 3 comprise one team and Players 2 and 4 the other. Player 1 leads the first card followed by Player 2, and so on. After all four have played cards, whomever wins the "trick" leads the first card of the next one. Player 3's cards are placed face up on the table after Player 1's initial lead, and Player 1 chooses the order in which Player 3 plays his cards. There are two simple rules which determine who wins a trick and how to play. Assume no trump.

Rule 1. The highest numbered card in the suit led wins.
Rule 2. If possible a player must lay a card of the same suit as the card led.

Player 1 knows both his cards and the cards of Player 3.

Player 1	Player 3
♥A 8 7 5 2	♥K 10 6 3
♦9 7 2	♦A 3
♣K J 3	♣A Q 10 6 4
♠A 3	♠K 8

Player 1 leads ♥A followed by ♥4 from Player 2, ♥3 from Player 3 and ♥Q from Player 4. Player 1 then leads ♥8 followed by ♥9 from Player 2. Should Player 3 lay ♥K or ♥10? In other words, who has the greater probability of holding ♥J, Player 2 or Player 4? If Player 2 (Player 4) is more likely to hold ♥J, then Player 3 should lay ♥10 (♥K). There are two cases:

Case A		Case B	
Player 2	Player 4	Player 2	Player 4
♥J 9 4	♥Q	♥9 4	♥Q J

In each case, twenty two cards remain to be allocated to the two players. Simple counting arguments show that in Case A there are 646,646 possible hands and in Case B, 705,432. In Case B can assume that Player 4 randomly chooses which card to play on the first trick. (If the same player holds consecutive cards they have equal value!) Therefore, if Player 4 held ♥J then with probability 1/2 he would have thrown it on the first trick. ♥J Case B can be subdivided into Cases B1 and B2. In Case B1, Player 4 holds both ♥Q and ♥J and plays ♥Q first. In Case B2, Player 4 again holds both but ♥J is played first. There are only 352,716 possible hands in Case B1. Since the correct comparison is between Case A and Case B1, the optimal strategy for Player 3 is to play ♥10, because Player 2 is more likely to hold ♥J.

Bowling Romans Answer: 2/5

Appendix 3

The Experiments: Subjects were told that they were going to play a game for real money. The proctor presented a sealed enveloped marked "Scott". In total the subjects were given two sheets of paper. The first sheet explained the problem and asked a subject to both make an initial selection and estimate the probability his selection was correct. The second sheet revealed incorrect numbers, offered the opportunity to switch, and again asked subjects to estimate the probability they were correct. After presenting the envelope, the proctor handed out Sheet # 1.

SHEET # 1

The envelope marked "Scott" on the chalkboard contains one of the following numbers:

1 2 3 4 5 6 7 8 9 10

Your goal is to guess the number that is in the envelope. If you are correct, you will win $5. The game that we are going to play proceeds in two stages. In the first stage, you will be asked to guess the number. Once you have made your guess, wait until no one else is at Scott's desk and go hand your sheet to him. In the second stage of the game, he will reveal eight incorrect numbers from among those nine numbers that you did not choose. You will then be given the opportunity to switch your initial guess to the other possible number.

You will be asked to estimate the probability that your guess is correct at each stage. There is no payoff for estimating the correct probability but it is important for our purposes.

(1) What is your initial guess? ___

(2) What is the probability that your initial guess is correct? ___

TURN IN YOUR PAPER TO SCOTT WHEN HE IS AVAILABLE

After completing Sheet # 1, a subject brought it forward and received Sheet #2 from the proctor, who would cross out all but two numbers on Sheet #2. If the subject had guessed the correct number the proctor randomly chose another number to leave as a possibility. If, on the other hand, the subject was incorrect, the initial choice and the correct choice were left as the remaining possibilities. Subject then took Sheet #2 back to their desks to complete the experiment.

SHEET # 2

The numbers which are crossed out below are definitely not the correct number

1 2 3 4 5 6 7 8 9 10

(3) What is your final guess? (this is the one that counts for the money) ____

(4) What is the probability that your final guess is correct? ____

After the correct number was revealed from the envelope, subjects were encouraged to both defend their decisions in both written and oral arguments.

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