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THRESHOLD EXTERNALITIES AND ECONOMIC DEVELOPMENT: A NOTE*

by

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1. Introduction.

In an interesting paper recently published on this journal, (Azariadis-Drazen [1990]), Costas Azariadis and Allan Drazen have put forward an OLG model of physical and human capital accumulation which seems appropriate for studying the role that education and the creation of human capital have in the process of economic development. In particular the two authors aim at explaining the "stylized fact" according to which certain underdeveloped countries appear stuck in a "poverty trap" where neither type of capital is accumulated and personal income stagnates in spite of the fact that they are in possession of the same technology that allows other countries to grow persistently.

The basic idea of Azariadis and Drazen (A-D from now on) is that if the aggregate amount of human capital has a positive external effect in the education process there would exist two types of stationary equilibria. In the first one (the "no-training equilibrium" in their terminology) the initial amount of human capital is so low that the marginal return from the resources invested in education is below the opportunity cost of withdrawing them from the production of physical goods. The stock of human capital therefore never increases and the economy settles to a stationary state without growth in income per capita. At the second equilibrium the stock of human capital is quite larger and the return from additional education is therefore high enough to sustain a constant investment in this activity. The economy will then grow along a balanced path where the ratio between the two stocks is kept constant and per-capita income increases exponentially.

If one could prove that both types of equilibria are locally asymptotically stable then it would be reasonable to claim that a threshold level exists for the stock of human capital, such that countries starting below it sink into the poverty trap whereas countries above it move toward boundless growth.

Unfortunately the formal set-up adopted by A-D is not enough to deliver the desired results and the two propositions given at page 515, vol. CV of this Journal, need several qualifications to be correct. This is what the present note will try to do, taking the A-D model as given and without questioning their economic assumptions. It will turn out, nevertheless, that the formal equivalent of the heuristic description given above cannot be proved in the A-D framework and that some radical modification of the basic economic assumptions is needed. I will not attempt this here.

2. Model, Assumptions and Propositions.

The reader interested in a detailed description of the model should consult the original article. I will concentrate here on the study of the asymptotic behavior of the equilibria of an A-D economy. The latter are given by all the non-negative sequences $\{x_t, k_t, \tau_t\}_{t=0}^{\infty}$, with $0 \leq \tau_t \leq 1$, which are generated by iterating the dynamical system (1) from some initial condition (x_0, k_0).

$$(1.a) \qquad [x_{t+1}(2-\tau_{t+1})]k_{t+1} = s[f'(k_{t+1}), w(k_t)x_t(1-\tau_t), w(k_{t+1})x_{t+1}]$$

$$(1.b) \qquad x_{t+1} = x_t[1 + \gamma(x_t)\tau_t]$$

$$(1.c) \qquad f'(k_{t+1})w(k_t) \geq w(k_{t+1})\gamma(x_t), \qquad \text{(with = when } \tau_t > 0).$$

The notation is the usual: x is the aggregate stock of human capital, k is the physical to human capital ratio, τ is the percentage of available time that the young generation invest in education, $f(k)$ is a strictly concave neoclassical production function satisfying Inada-type conditions on the boundaries, $s(R,y,y')$ denotes saving as a function of the expected rate of return $R = f'(k)$, current income y and future income y', it is increasing in (R,y) and decreasing in y', $w(k)$ is the wage rate equal to $[f(k) - kf'(k)]$ and, finally, the gross rate of human capital accumulation is given by $1+\gamma(x)\tau$. It is assumed that γ is non-negative and monotone non-decreasing with $\hat{\gamma} = \max\{\gamma(x);\ x\geq 0\} = \gamma(x)$ for all $x \geq \hat{x}$ and, further, that $s(R,y,y')$ is increasing in all arguments and linearly homogeneous in (y,y') for any given R.[1]

We consider first the no-training equilibrium. This is the object of proposition 1 in the A-D paper. My modified version is:

Proposition 1. Under the maintained assumptions there exists a level $\underline{x} > 0$ for the stock of human capital and a level $\underline{k} > 0$ for the physical stock such that all the triples $(x,\underline{k},0)$ with $0 \leq x \leq \underline{x}$ are fixed points of (1). Denote the set composed of all such triples as $\underline{K}$. Points in $\underline{K}$ are "asymptotically stable" in the following sense: if $\{x,k_t,0\}_{t=0}^{\infty}$ is an equilibrium then $(x,k_t,0) \rightarrow (x,\underline{k},0)$.

Proof. Set $\tau_t = 0$, $x_t = x$ and $k_t = k$ for all t in (1). Then (1.b) can be dropped and (1.a) reduces to $2k/w(k) = s[f'(k),1,1]$. The latter has a unique solution $\underline{k} > 0$ because f strictly concave implies that $2k/w(k)$ is strictly increasing and $s[f'(k),1,1]$ is strictly decreasing in k. Now let $f'(\underline{k}) = \underline{\gamma}$, and set $\underline{x} = \text{argmax}\ \{x\colon \gamma(x) \leq \underline{\gamma}\}$. Such an $\underline{x}$ exists and is non-negative; clearly all the triples $(x,\underline{k},0)$ with $0 \leq x \leq \underline{x}$ are no-training stationary states because they satisfy inequality (1.3). A local analysis of (1.a) and

(1.b) shows that all the equilibrium paths $\{x_t, k_t, \tau_t\}_{t=0}^{\infty}$ with $\tau_t = 0$ and $x_t = x \in [0,\underline{x}]$ for all t's larger than some finite t' will converge to $(x,\underline{k},0)$. Q.E.D.

Notice what this result <u>does not say</u>: it does not say that there exists an open neighborhood of $\underline{K}$ in $\Re^3$ such that all equilibria beginning within such a neighborhood will converge to a point in $\underline{K}$. In other words the proposition does not imply that the no-training equilibria are dynamically stable, at least not in the usual sense. The reason is simple: potentially the model described by (1.a)-(1.c) has multiple equilibrium paths departing from a common initial conditions. It is a third order system for which only two initial conditions, (x_0,k_0), are available: the initial value τ_0 is chosen in equilibrium by looking forward to the future value of k_1. This is the typical situations in which, depending on individual expectations, different values of k_1 may be selected, each one of them consistent with a distinct equilibrium sequence. While the complexity of the model makes it quite hard to work out explicit examples, it should be clear why one is prevented from proving a strong stability result.

A better understanding of the dynamic behavior of the model can be obtained by the following geometrical construct. Denote as NT the subset of $\Re^2$ composed of all the pairs (x,k) satisfying $f'(k) \geq \gamma(x)$, clearly $\underline{K} \subset NT$ as in figure 1. Trajectories starting within NT and which also remain within NT can only move along vertical lines to keep the initial value of x_0 constant. As the arrows indicate this implies that orbits beginning in the shaded subsets of NT cannot remain in NT and, a-fortiori, cannot converge to $\underline{K}$ while those starting in the area I indicated as $B(\underline{K})$ may converge to $\underline{K}$ if they remain in $B(\underline{K})$. Nevertheless there may exist equilibria beginning in $B(\underline{K})$ and leaving it after a finite number of periods: these are "optimistic" equilibria along which at time t a value k_{t+1} is selected which is feasible from k_t and such that (x_t, k_{t+1}) is not in NT anymore. Notice that, on the other hand, there may exist equilibria that start outside both $B(\underline{K})$ and NT and that jumps into $B(\underline{K})$ at a finite t and remain there forever: these are "pessimistic" equilibria along which physical capital is de-accumulated and the incentives for investing in education disappear. From a qualitative point of view the economic phenomena I am describing are not much different from those already encountered in other models with externalities, see e.g. Boldrin (1990), Boldrin-Kiyotaki-Wright (1991), Matsuyama (1991), Mortensen (1991), etc. .

I shall now move on to the second type of stationary equilibria studied by A-D: these are equilibria departing from initial conditions (x_0, k_0) where $x_0 \geq \hat{x}$, (so that $\gamma(x_0) = \hat{\gamma}$), and k_0 is large enough to make it feasible to select accumulation sequences satisfying (1.c) with equality. A stationary equilibrium will then be a balanced growth path along which $x_{t+1} = (1+\hat{\gamma}\tau^*)x_t$, $k_t = k^*$ and $\tau_t = \tau^* > 0$ for all t. With regard to their existence one can prove the following:

<u>Proposition 2.</u> Let $\underline{k}$ be the stationary value defined in proposition 1 and denote with k^* the solution to $f'(k) = \hat{\gamma}$. Then:

(i) if $\underline{k} > k^*$ a value $\tau^* \in (0,1)$ exists such that when $x_0 \geq \hat{x}$ $\{x_0(1+\hat{\gamma}\tau^*)^t, k^*, \tau^*\}_{t=0}^{\infty}$ is an equilibrium with initial condition (x_0, k^*) .

(ii) if $\underline{k} < k^*$ then no such value of τ exists in $[0,1]$.

<u>Proof.</u> It is rather simple. In case (ii) concavity of f implies that $k^*/w(k^*) > \underline{k}/w(\underline{k})$ and $f'(\underline{k}) > \hat{\gamma}$ are both true. Then one has the following

sequence of inequalities:

$$2k^*/w(k^*) > 2\underline{k}/w(\underline{k}) = s[f'(\underline{k}),1,1] > s[\hat{\gamma},1,1]$$

On the other hand a stationary equilibrium exists if there is a value of τ between zero and one which solves

$$(2-\tau)k^*/w(k^*) = s[\hat{\gamma},(1-\tau)/(1+\hat{\gamma}\tau),1] \qquad (*)$$

The previous inequalities imply, though, that the left hand side of (*) is strictly above the right hand side of (*) at both $\tau = 0$ and $\tau = 1$. By comparing the slopes of the two functions one can also see that the same is true for all the other admissible values of τ and therefore an equilibrium does not exist.

In case (i) one needs again a solution to (*). It is easy to see that the right hand side is now larger than the left hand side for $\tau = 0$ and smaller than it at $\tau = 1$. At least one stationary value of τ will therefore exists.

Q.E.D.

To study the dynamic stability of the stationary point (k^*,τ^*) one has to overcome the same difficulties illustrated earlier for the no-training equilibrium: multiple solutions to system (1) are possible for a given initial condition in a neighborhood of (k^*,τ^*) and therefore a general statement is not possible. Moreover a close inspection of the young agents maximization problem shows that interior solutions with $0 < \tau < 1$ and $s(R,y,y') > 0$ are very "tenuous" and it is hard to characterize the subset of υσιπμετ $(\omega_\upsilon,\lambda_\upsilon,\lambda_{\upsilon+1})$ ιξ $\mathbf{R}^3$ at which they are realized, let alone showing that the latter is an attracting set for the dynamical system (1).

It is true, though, that a linearization of (1.a) and (1.c) with respect to k and τ in a neighborhood of (k^*, τ^*), (and with $x_{t+1} = x_t(1+\hat{\gamma}\tau^*)$), shows that the latter has the local structure of a saddle, as A-D had already pointed out in their article. This leaves the question of stability open: a saddle point is dynamically "stable" only if there are compelling economic reasons for which, at any given initial condition, the equilibrium path lies on the stable manifold. This is the case in most models of intertemporal optimization because one can show that all the paths different from the stable manifold violate either a feasibility condition or the transversality condition. There is no transversality condition to be imposed in the present contest. Therefore one would have to prove that, for a given pair (x,k_0) with $x \geq \hat{x}$ and k_0 near k^*, the only value of τ_0 that induces a feasible orbit under repeated iterations of (1.a)-(1.c) is the one that puts (k_0,τ_0) on the stable manifold of (k^*,τ^*). This seems quite hard to prove given the full generality of the model, even if it might be possible for some particular cases.

I should also note that examples for which only the no-training steady state exists are very easy to construct as the reader will easily figure out with some algebraic manipulation of the following economy: $u(c_1,c_2) = \ln(c_1) + \ln(c_2)$; $f(k) = k^\alpha$; $\gamma(x) = (\beta+1)x/(1+x)$ for $0 \leq x \leq \beta$ and $\gamma(x) = \beta$ otherwise.

3. Conclusions.

I have argued that, with some additional assumptions, one can show that two types of stationary equilibria may exist in the Azariadis-Drazen model of human capital accumulation. These stationary positions would correspond to the "no-training" and "training" equilibria described in the original paper. The asymptotic stability of either one of them remains, nevertheless, a very open issue.

As I have suggested the model displays multiplicity of equilibria which makes it impossible to determine a unique equilibrium sequence for any given initial condition. Furthermore system (1) is a third order, non-linear dynamical system with very general functional forms. In order to understand its asymptotic behavior one needs to be able to carry on a global analysis of the system. This goes beyond my present abilities.

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1. A-D assume only that lim $\gamma(x) = \hat{\gamma}$ as x goes to infinity, but this would not be enough to guarantee that one of the critical assumptions of proposition 2 is always satisfied.

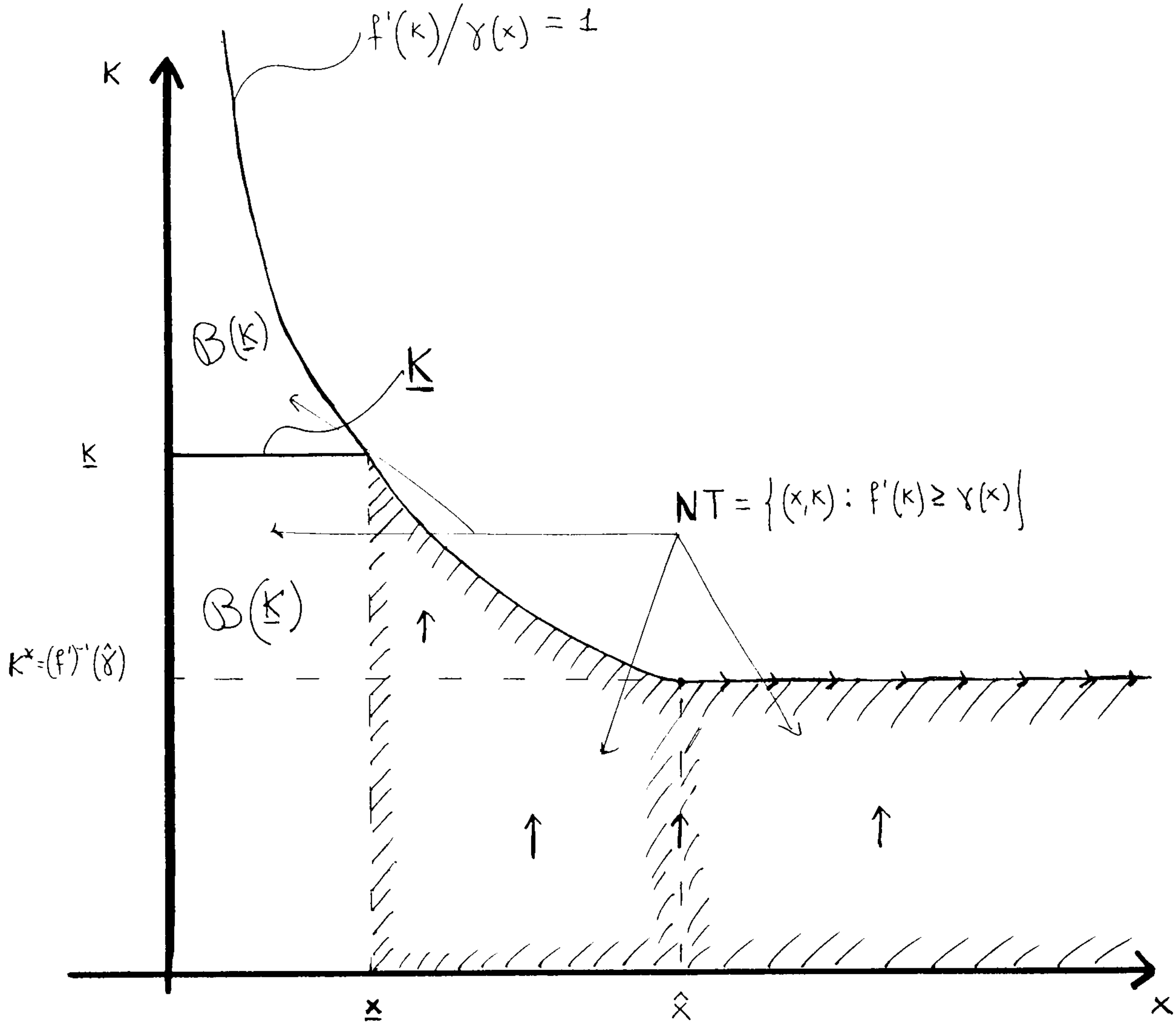

Figure 1.