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A GAME THEORETICAL APPROACH TO THE INTERNATIONAL DEBT OVERHANG*

by

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Abstract:

This paper considers an international financial problem of a sovereign country called debt overhang. The term "debt overhang" expresses the situation where a sovereign country has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments for some period. We formulate this problem as a noncooperative game with n lender banks as players where each decides either to sell its loan exposure to the debtor country at the present price of debt on the secondary market, or to wait and keep its exposure. There are many pure and mixed strategy Nash equilibria in this game. However we show that in any Nash equilibrium, the resulting secondary market price remains almost the same as the present price when the number of banks is large. We also obtain the comparative statics result that in a mixed strategy equilibrium, banks with smaller loan exposures have a greater tendency to sell than banks with larger loan exposure. In addition, we discuss the structure of the set of Nash equilibria.

1. Introduction

The "debt overhang" has recently been observed in international financial relations between sovereign countries and foreign commercial banks. The term "debt overhang" expresses the situation where a sovereign country has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments for some period. The existence of the debt overhang is a serious problem for the debtor country, which keeps the country in a bad economic situation and prevents from growing. The debtor country should reduce its debt to have an access to the international financial market which is necessary for its economic growth. Many proposals for resolution of the debt overhang have been discussed (cf. Versluysen (1989)).[1] For any kind of the resolution, however, first it is necessary to understand the nature of the debt overhang. The present paper investigates this problem from the game theoretical viewpoint.

Before describing the details of our game theoretical investigation, we explain the economic background of the problem of debt overhang. The debt overhang is closely related to the presence of the secondary market for debts. The secondary market has emerged as a result of the countries' economic inability to make full repayments. On the market, loan exposures are traded by lender banks and other financial institutions, and each dollar of debt is priced much below one. For instance, in the case of Bolivia, the price of one dollar of debt was 5¢ in 1985 and 6¢ in 1986, and in the case of Peru, it was 19¢ in 1986 and 6¢ in 1988. Trade on the secondary market means that if a lender sells its loan exposure to some other financial institution at the price, say 5¢, then the lender obtains 5% of the lent money and gives up remaining 95%, but the other institution will take over the right to the loan.

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There are several attempts to investigate some problems related to the phenomenon of debt overhang. They focus mainly on the bargaining over debt reduction between the country and the banks. For example, Fernandez and Kaaret (1988) considered a situation where one country borrowed money from two banks, one big and one small. Adopting the Nash bargaining solution, they investigate possible agreements on debt reduction. Fernandez and Rosenthal (1989) and Bulow and Rogoff (1986) provided different bargaining models with one country and one bank. Thus those authors considered bargaining over repayments between a borrowing country and a lending bank or banks, given the existing situation of debt overhang.

It is especially important in this paper to remark that currently the debtor country could be a possible buyer of debts on the secondary market in addition to banks and other financial institutions, while at the beginning of the secondary market, the debtor was excluded from trade.[2] The trade between a debtor country and a lender bank (debt buyback) at the price, say 5¢ again, on the secondary market is regarded as 95% forgiveness, since the lender recoups 5% of its loan and the country is not indebted any more. Lender banks have given up the possibility of recouping their total amounts of loan exposures since it has been practically impossible to expect the total repayments. Although countries have had possibilities to buy back their debts at discounted prices, many of them have not succeeded in reducing their indebtedness significantly.

In addition to the above economic background, we mention a few important empirical facts on the debt overhang. In many cases, a debtor country has borrowed money from many foreign banks at the same time. For example, in 1985 Mexico held loans from about 700 banks, Argentina from 370, and Venezuela from 460 banks (Bouchet, 1987, p.75). However, there has been some tendency for the number of lender banks to decline. Nevertheless, many debtor countries have not reduced their indebtedness, because of the compounded interests.

Keeping in mind the background of the problem and the above empirical facts, we explain our endurance competition game and the main result. We consider a situation with one country and its creditor banks in a short period. We formulate the problem as a one-shot game with creditor banks as players. The debtor country is treated as a part of the environment. We also assume the existence of a price function which gives the secondary market price of country's debt.

In our game, each bank has to decide either to sell its loan exposure to the country at the present secondary market price or to wait and keep its exposure. If a bank sells its exposure, it obtains a payoff equal to the value of the exposure determined by the present secondary market price of debt. If a bank waits, its payoff is assumed to be the present value of the loan exposure determined by the resulting secondary market price.

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2 For historical and institutional description of the secondary markets for debts, see, for example, Sachs and Huizinga (1987), Huizinga (1989, pp.5-8), and Hajivassiliou (1989).

On one hand, if many banks sell their loan exposures and the country's debt becomes smaller by paying the discounted amount of the loan exposures, then the secondary market price of the next period may become higher. If this is the case, the banks who wait may receive a higher payoff by the increased price, which implies that there are some incentives for the banks to wait. On the other hand, if only few banks sell, the total outstanding debt may increase with the accrued interest. In this case, the price of debt may fall, which implies that there are some incentives for the banks to sell. These two opposite tendences balance in equilibrium.

We consider the Nash equilibrium concept to represent the strategically stable behavior of banks. In fact, there are many pure and mixed strategy Nash equilibria. However, independently of a choice of a Nash equilibrium, we can draw a definite conclusion on the behavior of the secondary market price. It says that the resulting secondary market price of debt remains almost the same as the present price for a large number of banks. Equivalently the total outstanding debt of the country remains almost unchanged. When the distribution of loan exposures is relatively equal, this theorem gives the prediction that the proportion of banks selling is approximated by the interest rate. When the loan exposures are unequal, one of our results states that in a mixed strategy equilibrium, banks with smaller loan exposures have higher probabilities of selling than banks with larger loan exposures. Therefore the previous prediction is adjusted so that the proportion of banks selling may exceed the interest rate. Section 3 discusses these results.

Although the nature of our problem is dynamic, we focus on the behavior of banks in one period. Prokop (1991) formulates this problem as a dynamic game with an infinite horizon in the case of two banks, and shows that the set of subgame perfect equilibria for the game is quite limited. In each period, some subgame perfect equilibria give the same outcome as those of our one-period approach and the others also give very similar outcomes. Thus we have the decomposition property of the whole dynamics into the one-period problems, which means that we do not lose much of the dynamic nature of the debt overhang problem by our one-period approach. The one-period and the dynamic approaches are regarded as mutually complementary in that the one-period approach is simpler and enables us to discuss complex problems, and that the dynamic approach captures a long-run behavior but it is too complicated to consider some problems such as effects of a large number of banks.

The paper is organized as follows. Section 2 gives a description of the endurance competition game and the structure of the set of pure strategy Nash equlibria. In Section 3, the main limit theorem is given, which states that in any (pure or mixed strategy) Nash equilibrium, the secondary market price of debt is almost constant for a large number of banks. We also discuss comparative statics on mixed strategy equilibria. In Section 4, we consider the structure of the set of mixed strategy equilibria.

2. Endurance Competition Game

In an endurance competition game G, we consider decision making of banks in one particular period. The economic situation of the game G is described as a triple (N, $\{D_i\}$, P). The symbol N denotes the set of banks 1, 2, ..., n, who have lent some amount of money to a foreign country, and D_i denotes the present loan exposure of bank i. The symbol P denotes a real-valued continuous function on $[0, +\infty)$, which gives the secondary market price $\mathrm{P}[d]$ when the total outstanding debt is d. The present secondary market price is given as $\mathrm{P}[D]$, where $D = \sum_{i \in N} D_i$. One additional element is the market interest factor, denoted β (= 1 + the interest rate) > 1. We assume that

$$(2.1) \qquad \mathrm{P}[D] > \mathrm{P}[\beta D] \quad \text{and} \quad \mathrm{P}[D] \leq \mathrm{P}[0]$$

The first inequality of (2.1) states that if the country does not buy back any debt in the present period, then the total outstanding debt increases to βD by the accrued interest and its market price declines. The second inequality states that if all banks sell their loan exposures at the present price $\mathrm{P}[D]$, then in the next period the price of (an arbitrarily small) debt is higher than or equal to the present price $\mathrm{P}[D]$.

Throughout the paper, we assume that the price function P is fixed, but the set N of banks and their loan exposures $\{D_i\}$ may vary. Therefore we denote our game G by (N, $\{D_i\}$). In this game, the banks are players and the country is treated as a part of the environment.

Each bank $i \in N$ has two pure strategies 0 - to sell its exposure at the present price $P[D]$ and 1 - to wait and postpone the decision to the next period. We denote the strategy space {0, 1} of player i by S_i. Then $S_1 \times ... \times S_n$ is the outcome space. When each bank i chooses its strategy s_i ($i \in N$) and the market transactions are completed, the price of the next period becomes

$$(2.2) \qquad \mathrm{P}[\beta d], \quad \text{where } d = \sum_{j \in N} s_j D_j.$$

If bank i keeps its loan exposure D_i by the next period, then the loan exposure D_i increases to βD_i, and the new price is $P[\beta d]$. Thus the present value of the exposure which will be sold in the next period becomes

$$\frac{1}{\beta}(\beta D_i P[\beta d]) = D_i P[\beta d]. \tag{2.3}$$

We assume that this present value $D_i P[\beta d]$ is the payoff to bank i if it does not sell its loan exposure in the present period. We also assume that if bank i sells D_i in the present period, then its payoff is simply $D_i P[D]$. We do not take into account the possibilities of the banks' revenues by postponing selling its exposure after the next period.[3] Thus the payoff function of bank i is given as

$$h_i(s) = h_i(s_1, \ldots, s_n) = \begin{cases} D_i P[\beta \sum_{j \in N} s_j D_j] & \text{if } s_i = 1 \\ D_i P[D] & \text{if } s_i = 0. \end{cases} \tag{2.4}$$

If all lender banks sell their exposures to the country, then the country must pay the total of $DP[D]$. We assume that the country is able to afford these repayments, which implies that the current price $P[D]$ is small enough for the repayments or the total amount of debts is not so large to prevent the total repayment.[4] In fact, the price function $P[\cdot]$ may also depend upon the country's disposable income (i.e. income left after repayments). Since it is assumed that repayments are made according to the current market price $P[D]$, the disposable income is automatically determined by the market transactions. The price function $P[\cdot]$ is interpreted as determined by taking this consideration into account.

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[3] This possibility is taken into account in the dynamic description of this problem with two banks in Prokop (1991). However, he shows that this does not substantially change the structure of the equilibria.

[4] Formally we need this assumption. However, practically, we need the assumption that the country is able to afford to pay a little more than the payment given by Theorem 1.

In addition to pure strategies, we allow the banks to play mixed strategies. Denote the set of mixed strategies of bank i by T_i , i.e. $T_i = \{p_i : 0 \le p_i \le 1\}$ for $i \in N$, where p_i is the probability of waiting by bank i. Define the expected payoff to bank i by

$$(2.5)\quad H_i(p) = p_i \left[\sum_{S \subseteq N - \{i\}} \prod_{\substack{j \notin S \\ j \in N - \{i\}}} p_j \prod_{j \in S} (1 - p_j) D_i P[\beta(D - \sum_{j \in S} D_j)] \right] + (1 - p_i)\, D_i\, \mathrm{P}[D]$$

$$\text{for } \mathrm{p} = (p_1, \ldots, p_n) \text{ in } T_1 \times \ldots \times T_n.$$

The first term of (2.5) is the expected payoff to bank i from waiting and the second term $(1 - p_i) D_i \mathrm{P}[D]$ is the expected payoff from selling.

We apply the Nash equilibrium concept as a solution of our endurance competition game. A strategy n-tuple $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ is called a Nash equilibrium if for all $i \in N$,

$$H_i(\hat{p}) \ge H_i(\hat{p}_{-i}, p_i) \quad \text{for all} \quad p_i \in T_i\,,$$

where $\hat{p}_{-i} = (\hat{p}_1, \ldots, \hat{p}_{i-1}, \hat{p}_{i+1}, \ldots, \hat{p}_n)$.

In the case of two banks, under the condition

$$\mathrm{P}[\beta D] < \mathrm{P}[D] < \mathrm{P}[\beta D_i] \text{ for } i = 1, 2,$$

our endurance competition game G has three equilibria:

$$(\hat{p}_1, \hat{p}_2) = (1, 0); \quad (\hat{p}_1, \hat{p}_2) = (0, 1)\ ; \ \text{and} \quad \hat{p}_i = \frac{P[\beta D_j] - P[D]}{P[\beta D_j] - P[\beta D]} \text{ for } i, j = 1, 2 \ \ (i \ne j).$$

These three equilibria are given by subgame perfect equilibria in Prokop's (1991) dynamic formulation with infinite horizon, i.e., the above equilibria occur in each period in the subgame perfect equilibria. In Prokop (1991), the pure strategy equilibria are eliminated for the reason of a continuation of the situation. He finds some additional subgame perfect equilibria whose outcomes are almost the same as our third equilibrium. Thus our one-period approach does not lose the dynamic nature of the problem.

First we describe the set of pure strategy Nash equilibria of the endurance competition game.

Theorem 1. Assume condition (2.1). Let s be a pure strategy n-tuple, and let $S = \{i \in N : s_i = 0\}$. Then s is a Nash equilibrium (in mixed strategies) if and only if

$$(2.6)\qquad P[\beta(D - \sum_{i \in S} D_i)] \ \ge\ P[D] \ \ge\ P[\beta(D - \sum_{i \in S - \{j\}} D_i)] \ \text{ for all } j \in S.$$

Proof. Let s be a Nash equilibrium. If $s_i = 1$ for all $i \in N$, then each player has an incentive to change its strategy 1 to 0, since P[D] > P[βD] by (2.1), which is impossible. Hence $s_k = 0$ for some k. Since player k weakly prefers selling ($s_k = 0$) to waiting, we have $P[D] \geq P[\beta(D - \sum_{i \in S - \{k\}} D_i)]$. If $s_i = 0$ for all $i \in N$, then S = N, which implies $P[\beta(D - \sum_{i \in N} D_i)] = P[0] \geq P[D]$ by (2.1). Suppose $s_j = 1$ for some j. Since player j weakly prefers waiting ($s_j = 1$) to selling, we have $P[\beta(D - \sum_{i \in S} D_i)] \geq P[D]$.

Conversely, suppose (2.6). This together with (2.1) implies $S \neq \emptyset$. Then no player in S or in N-S has an incentive to change its strategy since P[D] $\geq$ P[$\beta(D - \sum_{i \in S - \{j\}} D_i)$] and P[$\beta(D - \sum_{i \in S} D_i)$] $\geq$ P[D], respectively. //

First we consider some implications of Theorem 1 under the assumption that

(2.7) P[d] is a decreasing function.

Under this assumption, condition (2.6) is equivalent to

$$(2.6^*) \qquad \beta(D - \sum_{i \in S} D_i) \leq D \leq \beta(D - \sum_{i \in S - \{j\}} D_i) \quad \text{for all } j \in S.$$

This states that in a pure strategy equilibrium, the remaining debt $\beta(D - \sum_{i \in S} D_i)$ with accrued interest differs from the original total debt D by at most $\min_{j \in S} \beta D_j$. The existence of a pure strategy equilibrium follows from (2.6*). Indeed, if $S = \emptyset$, $\beta(D - \sum_{i \in S} D_i) = \beta D > D$. Therefore we can find a minimal S with the property $\beta(D - \sum_{i \in S} D_i) \leq D$. This S satisfies the right inequality. Thus the pure strategy n-tuple s with $s_i = 0$ if $i \in S$ and $s_i = 1$ otherwise is a Nash equilibrium. Actually, there are many pure strategy Nash equilibria. Under the additional condition

$$(2.8) \qquad D > \beta \sum_{i \neq j} D_i \qquad \text{for all } j \in N,$$

– this condition could be true for a small number of banks – there are exactly n pure strategy Nash equilibria. Each of them is represneted as $s^i = (1, ..., 1, \overset{i}{0}, 1, ..., 1)$ for some i, since the set S in Theorem 1 is $\{i\}$. In the general case, the structure of the set of equilibria is more complicated.

We return to the general case. Since $D - \sum_{i \in S} D_i$ is represented as $\sum_{i \in N} s_i D_i$, condition (2.6) is represented as

$$(2.6^{**}) \qquad P[\beta(\sum_{i \in N} s_i D_i)] \geq P[D] \geq P[\beta(\sum_{i \in N} s_i D_i + D_j)] \qquad \text{for all } j \text{ with } s_j = 0.$$

This means that in a pure strategy equilibrium, the resulting price is not smaller than the present price, but the difference between them is bounded by $P[\beta \sum_{i \in N} s_i D_i] - P[\beta(\sum_{i \in N} s_i D_i + D_j)]$. If D_j is small, then the new price does not differ much from the present price. To state this observation more explicitely, we introduce a sequence $\{G^\nu\} = \{(N^\nu, \{D_i^\nu\})\}$ of the endurance competition games with

(2.9) $\quad |N^\nu| \to \infty \quad \text{as} \quad \nu \to \infty;$[5]

(2.10) $\quad$ for some K, $D_i^\nu \leq \dfrac{K}{|N^\nu|}$ $\qquad$ for all $i \in N^\nu$ and $\nu \geq 0$.

Denote $D^\nu = \sum_{i \in N^\nu} D_i^\nu$, and note that D^ν is bounded by K. Then the above observation is formulated as follows.

Theorem 2. Assume condition (2.1) for each G^ν. For any sequence $\{s^\nu\}$ where each s^ν is an arbitrary pure strategy Nash equilibrium in G^ν, we have

$$\lim_{\nu \to \infty} |P[\beta \sum_{i \in N} s_i^\nu D_i^\nu] - P[D^\nu]| = 0. \tag{2.11}$$

3. Behavior of Nash Equilibria for a Large Number of Banks

In Section 2, we proved that in pure strategy equilibria, the resulting secondary market price of debt remains almost the same as the present price when the number of banks is large. This section extends this result to mixed strategy equilibria. In Section 4, we will show that there are many mixed strategy Nash equilibria. Nevertheless, in any mixed strategy equilibria, the resulting price of debt remains almost the same as the present price with arbitrarily high probability for a large number of banks. In fact, we can state this result for any mixed and pure strategy equilibria.

To state the result, we introduce random variables describing the outcomes of strategies. For a strategy n-tuple $p = \{p_i\}_{i \in N}$, we define $X = \{X_i\}_{i \in N}$ by

$$X_i = \begin{cases} D_i & \text{if the realization of bank } i\text{'s mixed strategy } p_i^\nu \text{ is to wait;} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

[5] $|S|$ stands for the cardinality of S.

That is, $Pr(X_i = D_i) = p_i$ and $Pr(X_i = 0) = 1 - p_i$. These random variables are independent because of the basic assumption that the strategy choices are independent.[6] The resulting secondary market price of debt in the game G is given by $P[\beta \sum_{i \in N} X_i]$, which is also a random variable.

Now we can state the main result of the paper.

Theorem 3. Let $\{G^\nu\} = \{(N^\nu, \{D^\nu\})\}$ be a sequence of endurance games with conditions (2.1), (2.9) and (2.10), and let $\{p^\nu\}$ be a sequence of Nash equilibria for G^ν's. Let $\{X^\nu\}$ be the sequence of random variables where each X^ν is defined with p^ν by (3.1). Then for any $\varepsilon > 0$,

$$\lim_{\nu \to \infty} Pr\left(|P[\beta \sum_{i \in N^\nu} X_i^\nu] - P[D^\nu]| \leq \varepsilon\right) = 1. \tag{3.2}$$

In fact, this convergence is uniform on the choice of a Nash equilibrium p^ν for each G^ν. That is, it holds that for any $\varepsilon > 0$ and $\delta > 0$, there is a ν_0 such that $Pr\left(|P[\beta \sum_{i \in N^\nu} X_i(p)] - P[D^\nu]| \leq \varepsilon\right) \geq 1 - \delta$ for all Nash equilibria p of the game G^ν and all $\nu \geq \nu_0$. Here $\{X_i(p)\}_{i \in N^\nu}$ are random variables defined by (3.1) with a Nash equilibrium p.

This theorem states that the probability of the distance between the prices of the next and present periods to be less than or equal to ε converges to 1 when the number of banks becomes large. When every p^ν in $\{p^\nu\}$ is a pure strategy Nash equilibrium, then the assertion of Theorem 3 is equivalent to Theorem 2. Indeed, Theorem 2 clearly implies (3.2). Conversely, when every p^ν in $\{p^\nu\}$ is a pure strategy equilibrium, (3.2) means that for any $\varepsilon > 0$ there is a ν_0 such that $| P[\beta \sum_{i \in N^\nu} X_i^\nu] - P[D^\nu]| \leq \varepsilon$ for all $\nu \geq \nu_0$, which is (2.11). If every p^ν in $\{p^\nu\}$ is a mixed strategy equilibrium, i.e., $0 < p_i^\nu < 1$ for at least one player i, condition (2.1) is not necessary for Theorem 3 (see the proof of Theorem 3).

In Nash equilibria, the price of debt remains almost unchanged when the number of banks is large. On one hand, the total debt may increase through accrued interest, and, on the other hand, the total debt may decrease through the country's buyback of some debt. Theorem 3 says that

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[6] Each random variable X_i is a function from $S = S_1 \times \ldots \times S_n$ to $\{0, D_i\}$ with $X_i(s) = D_i$ if $s_i = 1$ and $X_i(s) = 0$ if $s_i = 0$. Also, the space S has the probability measure μ determined by $\mu(\{s\}) = (\prod_{s_i = 1} P_i)(\prod_{s_i = 0} (1 - p_i))$ for all $s \in S$. Since it is not necessary to return to this basic probability space in this paper, we work on the random variables without referring to the probability space.

these two effects balance and the total outstanding debt does not change much. The proportion of debt bought back is approximately

$$(3.3) \qquad (D^* - \frac{D^*}{\beta})/D^* = \frac{\beta - 1}{\beta}.$$

Since the interest factor is close to 1, this proportion is also close to the interest rate $\beta - 1$. The country succeeds in buying back a part of debt tantamount to the interest on the total outstanding debt discounted at the present secondary market price. Consequently, the total outstanding debt remains almost the same and *a fortiori* the secondary market price is almost unchanged.

When the distribution of loan exposures is relatively equal, the proportion of banks selling is also approximately $(\beta - 1)/\beta$. However, if the loan exposures are unequal, then we could expect a larger number of banks to sell their exposures. This follows from the next theorem which is proved under condition (2.7). The theorem states that in a mixed strategy equilibrium, a bank with a smaller loan exposure has a greater tendency to sell than one with a larger loan exposure.

Theorem 4. Suppose condition (2.7). Let $(\hat{p}_1, ..., \hat{p}_n)$ be a Nash equilibrium. Then it holds that for any i, j with $0 < \hat{p}_i,\ \hat{p}_j < 1$,

$$(3.5) \qquad D_i \geq D_j \text{ if and only if } \hat{p}_i \geq \hat{p}_j.$$

Since banks with smaller exposures have higher probabilities of selling, the number of banks selling has an increasing tendency when the loan exposures are more unequally distributed, since the proportion of debt bought back is almost constant. This phenomenon could be observed in the real world. For example, in the case of Bolivia, all American banks with larger debt exposures have kept their loans but some banks with small exposures have sold theirs, which caused the price increase from 6¢ in 1986 to 11¢ in 1988.[7]

The consequence of the almost unchaged price of debt is compatible with the functioning of the secondary market. The country buys back the almost constant portion of the total outstanding debt. This does not prevent the creditor banks and other banks from trading on the secondary market. For the secondary market to function, there must be (at least potentially) some trade of loan exposures. Since the present price is almost the same as the future price, keeping a loan ex-

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[7] cf. Bulow and Rogoff (1988).

posure is almost equally profitable (unprofitable) as keeping the money corresponding to the secondary market value of the exposure as a deposit on the money market. Therefore banks may extensively trade the loan exposures among themselves, and their trade does not affect the price. On the other hand, the trade between the creditor banks and the country affects the price of debt. Therefore the creditor banks as a whole sell to the country the proportion of debt approximately equal to (3.3) in equilibrium, to have the almost constant price of debt on the secondary market.

Unfortunately, the prediction that the secondary market prices are almost constant is not confirmed, looking at the data of prices in 1980s. For some countries the secondary market prices seemed to be constant, for some they looked increasing, and for some others decreasing. In totality the tendency of decreasing prices have been regarded as slightly dominant (cf. Hajivassiliou (1989)). In our game, we regard one period as short, i.e., one or two months. Therefore the corresponding decreasing tendency could also be small, which means that the disagreement is not significant. Nevertheless, for the long run there remains some discrepancy between our predection and the evidence.

Many factors could be thought of as causing the discrepancy between our prediction and the evidence. Some unexpected political events may be regarded as one of them. If we disregard the political disturbance, there is a subtle issue in the interpretation of the empirical evidence. When the decreasing tendency is dominant and is expected by each bank, no trade among banks occurs. Consequently, the secondary market does not function at all, which implies that the secondary market price cannot be quoted. However, this is not true in the real world. Thus the dominant tendency of decreasing prices is viewed as a result of some unexpected disturbances.

Proof of Theorem 3. We show the claim of Theorem 3 under the assumption that every p^ν has at least one player who plays a completely mixed strategy. If this is done, then this together with Theorem 2 implies Theorem 3. Indeed, suppose the claim is proved under this assumption. If only a finite number of p^ν in $\{p^\nu\}$ have mixed strategies, then Theorem 2 is applied and we obtain the claim. If only a finite number of p^ν in $\{p^\nu\}$ are pure strategy equilibria, then the above claim is applicable. Consider the case where the sequence $\{p^\nu\}$ is divided into subsequences $\{p^{\lambda_\nu}\}$ and $\{p^{\mu_\nu}\}$ so that p^{λ_ν} is a pure strategy Nash equilibrium for each ν and p^{μ_ν} is a mixed strategy Nash equilibrium for each ν. There is a ν_1 by Theorem 2 such that $Pr\Big(|P[\beta \sum_{i \in N^{\lambda_\nu}} s_i X_i^{\lambda_\nu}] - P[D^{\lambda_\nu}]| \le \varepsilon\Big) = 1$ for all $\nu \geq \nu_1$, and also for any $\delta > 0$ there is a ν_2 by the above claim such that $Pr\left(|P[\beta \sum_{i \in N^{\mu_\nu}} X_i^{\mu_\nu}] - P[D^{\mu_\nu}]| \leq \varepsilon\right) \geq 1 - \delta$ for all $\nu \geq \nu_2$. Therefore $Pr\left(|P[\beta \sum_{i \in N^\nu} s_i X_i^\nu] - P[D^\nu]| \leq \varepsilon\right)$ $\geq 1 - \delta$ for all $\nu \geq \max(\lambda^{\nu_1}, \mu^{\nu_2})$. This means $\lim_{\nu \to \infty} Pr\left(|P[\beta \sum_{i \in N^\nu} s_i X_i^\nu] - P[D^\nu]| \leq \varepsilon\right) = 1$.

In the following, we assume that for every ν, $0 < p^\nu < 1$ for some $i \in N^\nu$.

Let us prepare several notions and some lemmas. Since $Pr(X_i^\nu = D_i^\nu) = p_i^\nu$ and $Pr(X_i^\nu = 0) = 1 - p_i^\nu$, the mean and variance of X_i^ν are given as

$$(3.6) \qquad E(X_i^\nu) = p_i^\nu D_i^\nu \text{ ; and } V(X_i^\nu) = E((X_i^\nu - p_i^\nu D_i^\nu)^2) = (D_i^\nu)^2 p_i^\nu (1 - p_i^\nu) < \left(\frac{K}{|N^\nu|}\right)^2.$$

Denote $\sum_{i \in N^\nu} X_i^\nu$ by S^ν. Then

$$(3.7) \qquad E(S^\nu) = \sum_{i \in N^\nu} p_i^\nu D_i^\nu \text{ ; and } V(S^\nu) = \sum_{i \in N^\nu} V(X_i^\nu) < |N^\nu| \left(\frac{K}{|N^\nu|}\right)^2 = \frac{K^2}{|N^\nu|}.$$

Lemma 3.1. $\lim_{\nu \to \infty} Pr(|S^\nu - E(S^\nu)| \leq \varepsilon) = 1$ for any $\varepsilon > 0$,

Proof. Applying Chebyshev's inequality (Feller (1957, p.219)) to S^ν, we have

$$Pr\left(|S^\nu - E(S^\nu)| > t\sqrt{V(S^\nu)}\right) \leq \frac{1}{t^2} \text{ for any } t > 0.$$

It follows from (3.7) that

$$Pr\left(|S^\nu - E(S^\nu)| > t \frac{K}{\sqrt{|N^\nu|}}\right) \leq \frac{1}{t^2} \text{ for any } t > 0.$$

Putting $t = |N^\nu|^{1/4}$, we have

$$Pr\left(|S^\nu - E(S^\nu)| > \frac{K}{|N^\nu|^{1/4}}\right) \leq \frac{1}{\sqrt{|N^\nu|}}.$$

Choose a ν_0 so that $\frac{K}{|N^\nu|^{1/4}} < \varepsilon$ for all $\nu \geq \nu_0$. Then we have

$$Pr(|S^\nu - E(S^\nu)| > \varepsilon) < \frac{1}{\sqrt{|N^\nu|}} \text{ for all } \nu \geq \nu_0.$$

Therefore we have

$$Pr(|S^\nu - E(S^\nu)| \leq \varepsilon) \geq 1 - \frac{1}{\sqrt{|N^\nu|}} \to 1 \text{ as } \nu \to \infty. \qquad //$$

Lemma 3.2. For any $\varepsilon > 0$, $\lim_{\nu \to \infty} Pr(|P[\beta S^\nu] - P[\beta E(S^\nu)]| \leq \varepsilon) = 1$.

Proof. Since the price function P is continuous on $[0, +\infty)$, P is uniformly continuous on its relevant domain $[0, \beta K]$. Therefore there is a $\delta > 0$ such that $|S^\nu - E(S^\nu)| \leq \delta \Rightarrow |P[\beta S^\nu] - P[\beta E(S^\nu)]| \leq \varepsilon$. Thus it follows from Lemma 3.1 that $Pr(|P[\beta S^\nu] - P[\beta E(S^\nu)]| \leq \varepsilon)$ $\geq Pr(|S^\nu - E(S^\nu)| \leq \delta) \to 1$ as $\nu \to \infty$. $\qquad //$

Lemma 3.3. 1) $\lim_{v\to\infty} |E(P[\beta S^v]) - P[\beta E(S^v)]| = 0$; 2) $\lim_{v\to\infty} |E(P[\beta S^v]) - P[D^v]| = 0$;

3) $\lim_{v\to\infty} |P[\beta E(S^v)] - P[D^v]| = 0$.

Proof. 1) Let ε be an arbitrary positive number. Since P is a uniformly continuous function on the relevant domain $[0, \beta K]$, there is a $\delta > 0$ such that $|S^v - E(S^v)| \le \delta \Rightarrow |P[\beta S^v] - P[\beta E(S^v)]| \le \frac{\varepsilon}{2}$. From Lemma 3.1, there is a v_1 such that for all $v \ge v_1$,

$$Pr(|S^v - E(S^v)| > \delta) < \frac{\varepsilon}{2M}, \text{ where } M = \max_{0 \le d \le \beta K} \mathrm{P}[d].$$

The difference $|P[\beta S^v] - P[\beta E(S^v)]|$ is less than $\frac{\varepsilon}{2}$ if $|S^v - E(S^v)| \le \delta$ and is less than M if $|S^v - E(S^v)| \ge \delta$. Therefore we have

$$|E(P[\beta S^v]) - P[\beta E(S^v)]| = |E(P[\beta S^v] - P[\beta E(S^v)])|$$
$$\le (\frac{\varepsilon}{2})Pr(|S^v - E(S^v)| \le \delta) + MPr(|S^v - E(S^v)| > \delta)$$
$$\le \frac{\varepsilon}{2} + M \times \frac{\varepsilon}{2M} = \varepsilon \text{ for all } v \ge v_1.$$

2) Recall that in each equilibrium p^v, at least one player i plays a completely mixed strategy p_i^v $(0 < p_i^v < 1)$. We rewrite $E(P[\beta S^v])$ as

$$E(P[\beta S^v]) = \sum_{R \subseteq N} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{j \notin R} (1 - p_j^v)\right) P[\beta \sum_{j \in R} D_j^v] \tag{3.8}$$

$$= p_i^v \left[\sum_{R \subseteq N - \{i\}} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{\substack{j \notin R \\ j \ne i}} (1 - p_j^v)\right) P[\beta \sum_{j \in R \cup \{i\}} D_j^v] \right] + (1 - p_i^v) \left[\sum_{R \subseteq N - \{i\}} \left(\prod_{j \in R} p_j^v\right) \left(\prod_{\substack{j \notin R \\ j \ne i}} (1 - p_j^v)\right) P[\beta \sum_{j \in R} D_j^v] \right].$$

Notice that there is the one-one correspondence between the terms of the first and second brackets, and that each term of the first bracket differs from the corresponding term of the second in that the first has the additional βD_i^v in P[•]. Since $0 < p_i^v < 1$, the player i is indifferent between his choices of waiting and selling. Since the expected payoff from waiting is given as the first bracket and the expected payoff from selling is $\mathrm{P}[D^v]$, the first bracket of (3.8) is equal to $\mathrm{P}[D^v]$. Since $\max_{j \in N^v} D_j^v \le \frac{K}{|N^v|} \to 0$ as $v \to \infty$, the difference between the first and second brackets of (3.8) converges to 0 as $v \to \infty$. As was mentioned above, the first bracket is always the same as $P[D^v]$. This means $\lim_{v\to\infty} |E(P[\beta S^v]) - P[D^v]| = 0$.

3) It follows from 1) and 2) that for any $\varepsilon > 0$, there is a v_0 such that for all $v \ge v_0$,

$$|E(P[\beta S^v]) - P[\beta E(S^v)]| < \frac{\varepsilon}{2} \text{ and } |E(P[\beta S^v]) - P[D^v]| < \frac{\varepsilon}{2}.$$

Therefore we have $|P[\beta E(S^v)] - P[D^v]| \le |P[\beta E(S^v)] - E(P[\beta S^v])| + |E(P[\beta S^v]) - P[D^v]|$

$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $v \ge v_0$. //

Finally we prove the assertion of the theorem. From Lemma 3.3, for any $\varepsilon > 0$, there is a v_0 such that for all $v \geq v_0$, $|P[\beta E(S^v)] - P[D^v]| < \frac{\varepsilon}{2}$. Thus, using Lemma 3.2, we have,

$$Pr(|P[\beta S^v] - P[D^v]| \leq \varepsilon) \geq Pr(|P[\beta S^v] - P[\beta E(S^v)]| + |P[\beta E(S^v)] - P[D^v]| \leq \varepsilon)$$
$$\geq Pr\left(|P[\beta S^v] - P[\beta E(S^v)]| \leq \frac{\varepsilon}{2}\right) \to 1 \text{ as } v \to \infty. \quad //$$

Proof of Theorem 4. Since $0 < \hat{p}_i, \hat{p}_j < 1$, banks i and j are indeferrent between waiting and selling. Therefore the expected payoffs from waiting and selling are the same. This is expressed as

$$(3.9) \qquad \sum_{R \subseteq N - \{k\}} \left(\prod_{\substack{m \in N \\ m \notin R \cup \{k\}}} \hat{p}_m \right) \left(\prod_{m \in R} (1 - \hat{p}_m) \right) P[\beta(D - \sum_{m \in R} D_m)] = P[D] \text{ for } k = i, j.$$

The left hand side for $k = i$ is written as

$$\hat{p}_j \left[\sum_{R \subseteq N - \{i,j\}} \left(\prod_{\substack{m \notin N - R \\ m \neq i,j}} \hat{p}_m \right) \left(\prod_{m \in R} (1 - \hat{p}_m) \right) P[\beta(D - \sum_{m \in R} D_m)] \right]$$
$$+ (1 - \hat{p}_j) \left[\sum_{R \subseteq N - \{i,j\}} \left(\prod_{\substack{m \notin N - R \\ k \neq i,j}} \hat{p}_m \right) \left(\prod_{m \in R} (1 - \hat{p}_m) \right) P[\beta(D - \sum_{m \in R \cup \{j\}} D_m)] \right].$$

Each term in the second bracket corresponds to one in the first bracket with the same R, and each is bigger than the cooresponding one by (2.7). Therefore the whole sum in the second bracket is bigger than that in the first one. When $D_i = D_j$, the left hand side of (3.9) for bank j is obtained from the above formula by replacing $\hat{p}_j$ by $\hat{p}_i$. In this case, if $\hat{p}_i \neq \hat{p}_j$, then the value of the obtained formula for bank i is also different from the value of the formula for j. But these two must be the same as P[D]. Hence $D_i = D_j$ implies $\hat{p}_i = \hat{p}_j$.

Now it is sufficient to prove that $D_i > D_j$ implies $\hat{p}_i > \hat{p}_j$. When $D_i > D_j$, the left hand side of (3.9) for bank j is obtained from the above formula by replacing $\hat{p}_j$ by $\hat{p}_i$ and D_j by D_i in P$[D - \sum_{m \in R \cup \{j\}} D_m]$ in the second bracket. In this case, the value of the new second bracket is greater than that of the original one. Hence to keep the left hand side of (3.9) equal to P[D] for i, the probability coefficient $\hat{p}_i$ for the first bracket must be higher than $\hat{p}_j$. //

4. The Structure of the Set of Nash Equilibria.

To state and prove the limit theorem of the preceding section, we did not need to investigate the structure of the set of all Nash equilibria. The limit theorem holds independently of the choice of a Nash equilibrium and is proved without using specific equilibria. From this result one might expect that the number of mixed strategy Nash equilibria would be small. However, the fact is quite different from this expectation. A number of mixed strategy equilibria may exist in addition to the pure strategy Nash equilibria described by Theorem 1. The limit Theorem holds commonly for all Nash equilibria. In this section, we investigate the structure of the set of mixed strategy Nash equilibria.

First we have the following theorem.

Theorem 5. Suppose conditions (2.7) and (2.8). Let T be a subset of N with $|T| \geq 2$. Then there exists a Nash equilibrium $\hat{p}$ such that

$$0 < p_i < 1 \text{ for all } i \in T; \text{ and } p_i = 1 \text{ for all } i \in N - T. \tag{4.1}$$

Theorem 5 says that each choice T with $|T| \geq 2$ gives a mixed strategy equilibrium. Thus there are at least $2^n - n - 1$ mixed strategy equilibria, since condition $|T| \geq 2$ excludes the possibility of T being a singleton or empty. As we mentioned after Theorem 1, there are n pure strategy Nash equilibria under conditions (2.7) and (2.8). Thus the total number of equilibria is at least $2^n - 1$.

Condition (2.8) holds for a relatively small n. Thus Theorem 5 gives the structure of mixed strategy Nash equilibria in the case of a relatively small number of banks. The next theorem gives the structure of mixed strategy equilibria for any number of banks under the assumption of identical loan exposures, i.e.,

$$D_1 = D_2 = \ldots = D_n > 0. \tag{4.2}$$

Theorem 6. Suppose conditions (2.7) and (4.2). Let T be a subset of N with $D > \beta(D - \frac{|T| - 1}{n} D)$. Then there exists a unique Nash equilibrium $p = (p_1, \ldots, p_n)$ such that $0 < p_i < 1$ for all $i \in T$ and $p_i = 1$ for all $i \in N - T$.

Under condition (2.8), $D > \beta(D - \frac{|T| - 1}{n} D)$ implies $|T| \geq 2$. In this case, Theorem 6 implies that the total number of Nash equilibria becomes exactly $2^n - 1$.

Consider the case without condition (2.8). Let $t = |T|$ be the smallest integer with $D > \beta(D - \frac{|T| - 1}{n} D)$. Theorem 6 gives exactly $\sum_{k=t}^{n} \binom{n}{k}$ number of mixed strategy equilibria. Every subset T of N with $|T| = t - 1$ satisfies condition (2.6*), which implies that T gives one pure strategy Nash equlibria. Thus the total number of Nash equilibria is given as $\sum_{k=t-1}^{n} \binom{n}{k}$. This number is approximately 2^n for large n and $\beta < 2$. More formally,

Theorem 7. Under the assumption of Theorem 6 and $\beta < 2$,

$$\lim_{n \to \infty} \frac{|\text{the set of all Nash equilibria}|}{2^n} = 1.$$

The results of this section show that there are approximately 2^n number of Nash equilibria in the endurance competition games, independent of the total number n of banks.

Proof of Theorem 5. We define the following functions:

$$(4.3) \qquad \bar{f}_i(p) = \sum_{R \subseteq N - \{i\}} \Big(\prod_{j \notin R \cup \{i\}} p_j\Big) \Big(\prod_{j \in R}(1 - p_j)\Big) P[\beta(D - \sum_{j \in R} D_j)] \quad \text{and} \quad f_i(p) = \bar{f}_i(p) - P[D]$$

for $p \in [0, 1]^n$ and $i \in N$. Our objective is to find a point $\hat{p}$ in $[0, 1]^n$ such that $f_i(\hat{p}) = 0$, $0 < \hat{p}_i < 1$ for all $i \in T$ and $\hat{p}_j = 1$ for all $j \in N - T$. Suppose the existence of such a $\hat{p}$ is proved. In this case, each bank i in T is indifferent between waiting and selling, so $\hat{p}_i$ is a best response. By the following lemma, $f_j(\hat{p}) > 0$ for all j in N-T, which implies that $\hat{p}_j = 1$ is a (unique) best response to $\hat{p}$ for all $j \in N - T$. Hence $\hat{p}$ is a Nash equilibrium with the property (4.1).

Lemma 4.1. For any $p \in [0, 1]^n$ and i, j in N, if $p_i < p_j = 1$, then $f_i(p) < f_j(p)$.

Proof. The value $\bar{f}_j(p)$ is described as

$$\bar{f}_j(p) = p_i \left[\sum_{R \subseteq N - \{i,j\}} \Big(\prod_{\substack{k \in N - R \\ k \neq i,j}} p_k\Big) \Big(\prod_{k \in R}(1 - p_k)\Big) P[\beta(D - \sum_{k \in R} D_k)] \right]$$
$$+ (1 - p_i) \left[\sum_{R \subseteq N - \{i,j\}} \Big(\prod_{\substack{k \in N - R \\ k \neq i,j}} p_k\Big) \Big(\prod_{k \in R}(1 - p_k)\Big) P[\beta(D - \sum_{k \in R \cup \{i\}} D_k)] \right].$$

Each element in the second bracket corresponds to one element of the first bracket with respect to R, and each element in the second bracket is larger than the corresponding one. This implies that the second bracket has a greater value than the first one. The value $\bar{f}_i(p)$ is obtained from $\bar{f}_j(p)$ by replacing p_i by p_j, and D_i by D_j in the second bracket. However, since $p_j = 1$, the second term of

$\bar{f}_i(p)$ disappears, and only the small part, the first bracket, remains. This means that $\bar{f}_j(p) > \bar{f}_i(p)$. //

Now we prove the existence of a point $\hat{p}$ in $[0, 1]^n$ such that $f_i(\hat{p}) = 0$, $0 < \hat{p}_i < 1$ for all $i \in T$ and $\hat{p}_j = 1$ for all j in N-T. Since it is sufficient to look for such a point $\hat{p}$ in the restricted set $\{p \in [0, 1]^n : p_j = 1 \text{ for all } j \in N - T\}$, we do not worry about $p_j = 1$ for all $j \in N - T$. Therefore we assume for notational simplicity that T = N.

Now we consider the following mapping:

$$\phi_i(p) = p_i + f_{i-1}(p) \quad \text{for all i in N and} \quad p \in [0, 1]^n,$$

where $i - 1$ is interpreted as n if i = 1.[8] We would like to apply Brouwer's fixed point theorem to this mapping $\phi = (\phi_1, ..., \phi_n)$, but the image of ϕ may not be included in $[0, 1]^n$. Therefore we modify this mapping ϕ by the retraction mapping r: $R^n \to R^n$:

$$r_i(x) = \begin{cases} 0 & \text{if } x_i \leq 0 \\ x_i & \text{if } 0 \leq x_i \leq 1 \\ 1 & \text{if } 1 \leq x_i \end{cases}$$

for all $x \in R^n$ and i = 1, ..., n. Then the composite mapping $\psi = r \bullet \phi$ is a continuous mapping from $[0, 1]^n$ to $[0, 1]^n$. Now, by Brouwer's fixed point theorem, there exists a fixed point $\hat{p}$ in $[0, 1]^n$, i.e., $\psi(\hat{p}) = r \bullet \phi(\hat{p}) = \hat{p}$.

If $\hat{p}$ satisfies $0 < \hat{p}_i < 1$ for all i in N, then, by the definition of the retraction mapping r, $\psi(\hat{p}) = \hat{p}$, that is, $\hat{p}_i + f_{i-1}(\hat{p}) = \hat{p}_i$ for all i in N, which means that $\hat{p}$ satisfies condition (4.1). Now we prove that $0 < \hat{p}_i < 1$ for all i in N.

Claim 1. i) If $\hat{p}_i = 1$, then $f_{i-1}(\hat{p}) \geq 0$; and ii) if $\hat{p}_i < 1$, then $f_{i-1}(\hat{p}) \leq 0$

$\because$) i) Suppose $f_{i-1}(\hat{p}) < 0$. Then, by the definition of ϕ, $1 = \hat{p}_i = r_i \bullet \phi_i(\hat{p}) = r_i(\hat{p}_i + f_{i-1}(\hat{p})) < 1$, a contradiction.

ii) Similar. //

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8 If we define $\phi_i(p)$ by $p_i + f_i(p)$ for i = 1, ..., n, then we can only prove the existence of a Nash equilibrium, which cannot be guaranted to have the property required in Theorem 3. The mere existence of a Nash equilibrium was already obtained by Theorem 2.

Claim 2. $\hat{p} \neq (1,...,1)$ and $\hat{p} \neq (0,...,0)$.

$\because$) If $\hat{p} = (1,...,1)$, then $f_i(\hat{p}) = P[\beta D] - P[D] < 0$ for i = 1,...,n by (2.7), but $f_i(\hat{p}) \geq 0$ for i = 1,...,n by Claim 1, which is a contradiction. If $\hat{p} = (0,...,0)$, then $f_i(\hat{p}) = P[\beta D_i] - P[D] > 0$ for all i = 1,...,n by (2.8), but $f_i(p) \leq 0$ for i = 1,...,n by Claim 1, a contradiction. //

Claim 3. $0 < \hat{p}_i < 1$ for all i = 1,...,n.

$\because$) Suppose that $\hat{p}_i = 0$ for some i in N. Then, by Claim 1, $f_{i-1}(\hat{p}) \leq 0$ However it follows from (2.7), (2.8), (4.3) and $\hat{p}_i = 0$ that $\bar{f}_j(\hat{p}) \geq P[\beta(D - D_i)] > P[D]$, i.e., $f_j(\hat{p}) > 0$ for all $j \neq i$. This is a contradiction.

Finally we prove that $\hat{p}_i < 1$ for all $i \in N$. Suppose, on the contrary, that $\hat{p}_i < 1$, $\hat{p}_{i+1} = \hat{p}_{i+2} = ... = \hat{p}_{i+k} = 1$ and $\hat{p}_{i+k+1} < 1$ for some i in N.[9] By Claim 1 and $\hat{p}_{i+1} = 1$, $f_i(\hat{p}) \geq 0$. By applying Lemma 4.1 to i and i+k, we have $f_{i+k}(\hat{p}) > f_i(\hat{p}) \geq 0$. However it follows from Claim 1 and $\hat{p}_{i+k+1} < 1$ that $f_{i+k}(\hat{p}) \leq 0$, a contradiction. //

Proof of Theorem 6. Since the game is symmetric, we can assume without loss of generality that $T = \{1, 2, ..., t\}$. Suppose that $(\hat{p}_1, ..., \hat{p}_n)$ is a Nash equilibrium with the property $0 < \hat{p}_i < 1$ for all $i \in T$ and $\hat{p}_i = 1$ for all $i \in N - T$. Then $\hat{p}_i = \hat{\alpha}$ for all $i \in T$ by Theorem 4. Since each $i \in T$ is indifferent between waiting ($s_i = 1$) and selling ($s_i = 0$),

$$(4.4) \qquad \sum_{k=0}^{t-1}\binom{t-1}{k}\hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n}D)] = P[D].$$

Conversely, if (4.4) holds, then every bank $i \in T$ is indifferent between waiting and selling, and every bank $i \in N - T$ prefers waiting to selling, i.e., $\hat{p}_i = 1$, since

$$\begin{aligned}
&\sum_{k=0}^{t}\binom{t}{k}\hat{\alpha}^{t-k}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n}D)] \\
&= \hat{\alpha}\left[\sum_{k=0}^{t-1}\binom{t-1}{k}\hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k}{n}D)]\right] \\
&\quad + (1-\hat{\alpha})\left[\sum_{k=0}^{t-1}\binom{t-1}{k}\hat{\alpha}^{t-k-1}(1-\hat{\alpha})^k P[\beta(D - \frac{k+1}{n}D)]\right] > P[D].
\end{aligned}$$

Thus it satisfies to prove the unique existence of $\hat{\alpha}$ satisfying (4.4).

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[9] If i + m > n, then i + m is interpreted as i + m-n.

Let $g(\alpha) = \sum_{k=0}^{t-1} \binom{t-1}{k} \alpha^{t-k-1}(1-\alpha)^k P[\beta(D - \frac{k}{n} D)]$ for $\alpha \in [0, 1]$. Then $g(0) = P[\beta(D - \frac{t-1}{n} D)] > P[D]$ by the assumption of the theorem, and $g(1) = P[\beta D] < P[D]$. Therefore, by the Intermediate-Value Theorem, there is an $\hat{\alpha}$ in $(0, 1)$ such that $g(\hat{\alpha}) = 0$. This $\hat{\alpha}$ is a solution of (4.4).

The uniqueness of such an $\hat{\alpha}$ is verified by checking the negative sign of the derivative of $g(\alpha)$ for any α $(0 < \alpha < 1)$,

$$g'(\alpha) = \sum_{k=0}^{t-1} \binom{t-1}{k} [(t-k-1)\,\alpha^{t-k-2}(1-\alpha)^k - k\alpha^{t-k-1}(1-\alpha)^{k-1}]\, P[\beta(D - \frac{k}{n} D)]$$

$$\leq - \sum_{k=1}^{t-1} \frac{(t-1)!}{(t-k-1)!\,(k-1)!}\, \alpha^{t-k-1}(1-\alpha)^{k-1} P[\beta(D - \frac{k}{n} D)]$$

$$+ \sum_{k=0}^{t-2} \frac{(t-1)!}{(t-k-2)!\,k!}\, \alpha^{t-k-2}(1-\alpha)^k P[\beta(D - \frac{k}{n} D)]$$

$$= \sum_{k=1}^{t-1} \frac{(t-1)!}{(t-k-1)!\,(k-1)!}\, \alpha^{t-k-1}(1-\alpha)^{k-1}\Big[- P[\beta(D - \frac{k}{n} D)] + P[\beta(D - \frac{k-1}{n} D)]\Big] < 0.$$

Proof of Theorem 7. First note that $D > \beta(D - \frac{t-1}{n} D)$ is equivalent to $\frac{t-1}{n} > 1 - \frac{1}{\beta} \equiv \alpha$. Since $\beta < 2$, we have $\alpha < \frac{1}{2}$. Hence we have to show that

$$\lim_{n\to\infty} \frac{\sum_{k/n > \alpha} \binom{n}{k}}{2^n} \to 1.$$

Define a family of independent random variables $\{X_k^n\}_{k=1}^n$ by

$$X_k^n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

for $k = 1, 2, \ldots, n$, and define $S^n = \sum_{k=1}^{n} X_k^n / n$. Then $E(S^n) = \frac{1}{2}$ and $V(S^n) = \sum_{k=1}^{n} V(X_k^n)/n^2 = \frac{1}{4n}$. Let ε be a positive number with $\varepsilon \leq \frac{1}{2} - \alpha$. Then it follows from Chebyshev's inequality that

$$Pr\Big(|S^n - \frac{1}{2}| \geq \varepsilon\Big) \leq \frac{V(S^n)}{\varepsilon^2} = \frac{1}{4n\varepsilon^2},$$

that is,

$$Pr\Big(|S^n - \frac{1}{2}| < \varepsilon\Big) \geq 1 - \frac{1}{4n\varepsilon^2}.$$

Since $\frac{\sum_{k/n>\alpha}\binom{n}{k}}{2^n} = \sum_{k/n>\alpha}\binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = Pr(S^n > \alpha)$, and since $\alpha \leq \frac{1}{2} - \varepsilon$, we have

$$\frac{\sum_{k/n>\alpha}\binom{n}{k}}{2^n} = Pr(S^n > \alpha) \geq Pr\left(\frac{1}{2} - \varepsilon < S^n < \frac{1}{2} + \varepsilon\right) \geq 1 - \frac{1}{4n\varepsilon^2} \to 1 \quad \text{as } n \to \infty. \quad //$$

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