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THE ARROW-DEBREU LEMMA ON ABSTRACT ECONOMIES WITH NONCOMPLETE AND NONTRANSITIVE PREFERENCES

by

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I. Introduction

The purpose of this paper is to present a generalization of the Arrow-Debreu lemma on the existence of equilibrium in a generalized N-person game (Debreu [3]), called here as in [1] an abstract economy. Applying standard techniques the result can be used to prove the existence of equilibrium in economies with interdependent preferences, price dependent preferences, and preferences which may be both nontransitive and noncomplete. Our work is inspired by recent analysis of the existence of competitive equilibrium in economies with nontransitive and noncomplete preferences, in particular the beautiful result of A. Mas-Colell [5].

An abstract economy $\mathcal{E} = (X_i, P_i, \mathcal{A}_i)_{i=1}^{i=N}$ is defined by, for each i,

a. a compact convex choice set $X_i \subset R^\ell$ (define $X = \prod_{j=1}^{j=N} X_j$);

b. a preference correspondence $P_i : X \twoheadrightarrow X_i$ such that

   b1) (continuity) P_i has an open graph in $X \times X_i$,

   b2) (convexity) for each $x \in X$, $P_i(x)$ is convex, and

   b3) (irreflexivity) for each $x \in X$, $x_i \notin P_i(x)$;

c. a constraint correspondence $\mathcal{A}_i: X \twoheadrightarrow X_i$ such that

c1) $\mathcal{A}_i$ is continuous, and

c2) for each $x \in X$, $\mathcal{A}_i(x)$ is nonempty and convex.

An equilibrium for $\mathcal{E}$ is an $\bar{x} \in X$ such that for each i,

d 1) $\bar{x}_i \in \mathcal{A}_i(\bar{x})$, and

d 2) $P_i(\bar{x}) \cap \mathcal{A}_i(\bar{x}) = \emptyset$.

II. Theorem

<u>Theorem</u> Every abstract economy $\mathcal{E} = (X_i, P_i, \mathcal{A}_i)_{i=1}^{i=N}$ has an equilibrium.

Proof: For each i let $U_i: X \times X_i \to R_+$ be a continuous function which is zero on the complement of the graph of P_i and positive elsewhere. For example, take $U_i(y, x_i)$ as the distance between (y, x_i) and the complement of the graph of P_i. Then by b1), $U_i(y, x_i) > 0$ if and only if $x_i \in P_i(y)$.

For each i define $F_i: X \twoheadrightarrow X_i$ by $x_i \in F_i(y)$ if x_i maximizes $U_i(y, \cdot)$ subject to $x_i \in \mathcal{A}_i(y)$. Then, since U_i is a continuous function and $\mathcal{A}_i$ is a continuous, nonempty, compact valued correspondence, $F_i(y)$ is nonempty and has a closed graph. Define $G: X \twoheadrightarrow X$ by $G(y) = \prod_{i=1}^{i=N} H(F_i(y))$, where $H(A)$ denotes the convex hull of A. Then G is a nonempty and convex valued correspondence with closed graph. By Kakutani's fixed point theorem there exists $\bar{x} \in X$ such that $\bar{x} \in G(\bar{x})$; that is, $\bar{x}_i \in H(F_i(\bar{x}))$ for all i. We will prove that $\bar{x}$ is an equilibrium for $\mathcal{E}$.

Now, $\bar{x}_i \in H(F_i(\bar{x}))$ means that $\bar{x}_i$ is a convex combination of a finite set of points $y_1, y_2, \ldots, y_m \in F_i(\bar{x})$. Since $y_j \in \mathcal{A}_i(\bar{x})$ for all j and $\mathcal{A}_i(\bar{x})$ is convex, we have $\bar{x}_i \in \mathcal{A}_i(\bar{x})$. It remains to prove d 2).

Observe that the value of the maximum problem which defines $F_i(\overline{x})$ is zero. If not, then $U_i(\overline{x}, y_j) > 0$ for all j, which implies $y_j \in P_i(\overline{x})$ for all j, and this together with b 2) yields $x_i \in P_i(\overline{x})$, a contradiction. Then, $z_i \in P_i(\overline{x})$ implies $z_i \notin \mathcal{a}_i(\overline{x})$, since $z_i \in P_i(\overline{x})$ implies $U_i(\overline{x}, z_i) > 0$, which completes the proof.

III. Remarks

R.1) The above theorem can be combined with a standard argument to prove the theorem of Mas-Colell mentioned in the introduction. First, replace $P_i(x)$ with the interior of the convex hull of $P_i(x) \cup \{x_i\}$. (This technique is employed by Gale and Mas-Colell [4] to eliminate the need for a local nonsatiation assumption.) Then, except for some minor changes in notation, follow the Arrow-Debreu proof of the existence of competitive equilibrium [1].

R.2) Our theorem is a generalization of a theorem proved in [4] in that it allows constraint sets to vary with points in X. It is this feature, absent in [4], which enables us to obtain the Mas-Colell result as an immediate corollary. It also suggests the existence of equilibrium for a class of non-competitive models.

R.3) A suitable reinterpretation of the Arrow-Debreu theorem on the existence of competitive equilibrium [1], proves the existence of equilibrium for economies in which individual preferences are dependent on the actions of others as well as on prices. The applications of our theorem similarly yields equilibrium with dependent (noncomplete, nontransitive) preferences.

R.4) The existence of equilibrium was proved by Sonnenschein [8] without transitive preferences. Shafer [7] extended his results, and the technique he employed is closely related to the one used in this paper. The existence of equilibrium was proved by Schmeidler [6] without complete preferences. Bergstrom [2] investigated the existence of equilibrium without either a transitivity or completeness assumption on preferences; however, he uses a convexity assumption which guarantees that the excess demand correspondence is convex valued (as does Weddepohl [9]). The papers of Mas-Colell [5] and Gale-Mas-Colell [4] are more general in that they allow for excess demand functions from which no convex valued selection exists with closed graph.

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