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ON THE EXISTENCE OF NONCOOPERATIVE EQUILIBRIA IN SOCIAL SYSTEMS

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In Section 1 we first formulate the notion of an abstract feasibility-choice system (1.1). The main idea in these systems is that, given a point x in a space X and a feasible region $F \subset X$, a subset $\gamma(x, F) \subset F$ must be chosen and each $y \in \gamma(x, F)$ determines, in turn, a new feasible region $\delta(y, F) \subset X$ by moving the old F. For these systems we define equilibrium notions and show (1.3) various sets of equilibria to be nonempty and compact under rather general topological and geometric assumptions. While this is all groundwork for our study of social systems in Section 2, we believe the notion of a feasibility-choice system and the equilibrium results (1.3) to be of interest in their own right.

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In Section 2, we start by defining a social system (2.1). This amounts to specifying a set of individuals, each of whom faces a certain feasibility dynamics moving about his feasible region as a function of his and others' behavior and of others' feasibilities, and each of whom has a preference amongst four-tuples with coordinates: own next behavior (to be chosen out of his present feasible region), own and others' behavior, and others' feasibilities. From such a system we derive a feasibility-choice system in which choice takes place according to individuals' preferences in a noncooperative way, so that the equilibria of this feasibility-choice system are the noncooperative equilibria of the underlying social system. In showing (2.4) the non-emptiness and compactness of the set of noncooperative equilibria of a social system, results of the previous section and a lemma (2.3) basic to optimization become our main tools.

Section 3 interprets Debreu's pioneering work [2] in terms of our present framework and compares his results with ours.

The Mathematical Appendix A aims to supply a minimal amount of information from a relatively new and inaccessible area of Mathematics, namely, topological semivector spaces and some of their fixed point theory which applies to certain hyperspaces of topological vector spaces. (This material, essential to the main body of the paper, was developed elsewhere [6, 7, 8, 9, 10, 11] by the authors with the present sort of application weighing heavily in their motivation.) This appendix is meant to serve also as a glossary for terms and notions used in the paper that may be unfamiliar to the reader.

0. STANDING NOTATION AND CONVENTIONS: R denotes the set of real numbers with the usual topology, and $R_+ = \{\lambda \in R \mid \lambda \geq 0\}$.

For any set Y, $\mathcal{P}[Y]$ denotes the power set of Y and $[Y]$ denotes the set of nonempty subsets of Y. When Y is a topological space, $\mathcal{C}[Y]$ denotes the set of closed nonempty subsets of Y, and $\mathcal{K}[Y]$ denotes the set of compact nonempty subsets of Y. When Y lies in a topological semivector space (see A.2.1), e.g., a real topological vector space, $\mathcal{KQ}[Y]$ denotes the set of compact and convex nonempty subsets of Y.

Projection into a set X is denoted by π_X, and the diagonal $\{(x, x) \mid x \in X\}$ is denoted by $D(X)$. Given maps $f: X \to Y$ and $g: Y \to Z$, $g \circ f: X \to Z$ denotes the composition defined by $g \circ f(x) = g(f(x))$ $(x \in X)$.

For a set-valued map $f: X \to [Y]$, by the graph of f we mean the set $\{(x, y) \mid x \in X, \ y \in f(x)\} \subset X \times Y$, denoting it by $G(f)$.

Let $\{Y_\alpha \mid \alpha \in A\}$ be a family of topological spaces. We denote

$$\mathcal{B}[\{Y_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \in [Y_\alpha] \text{ for each } \alpha \in A\},$$

$$\mathcal{BK}[\{Y_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \in \mathcal{K}[Y_\alpha] \text{ for each } \alpha \in A\}.$$

Let X be a set, and let $\{f_\alpha: X \to Y_\alpha \mid \alpha \in A\}$ and $\{g_\alpha: X \to [Y_\alpha] \mid \alpha \in A\}$ be families of maps. We define the maps $\{f_\alpha\}_A : X \to \prod_A Y_\alpha$ and $\prod g_\alpha : X \to \mathcal{B}[\{Y_\alpha\}_A] \subset [\prod_A Y_\alpha]$ by

$$\left.\begin{aligned} \{f_\alpha\}_A(x) &= \{f_\alpha(x)\}_A \\ \prod_A g_\alpha(x) &= \{\{y_\alpha\}_A \mid \alpha \in A \Rightarrow y_\alpha \in g_\alpha(x)\} \end{aligned}\right\} \qquad (x \in X).$$

1. FEASIBILITY-CHOICE SYSTEMS AND THEIR EQUILIBRIA

1.1 DEFINITION: A *feasibility-choice system* is an ordered quadruplet

1.1.0 $\Omega = (X, \mathcal{F}, \gamma, \delta)$,

where

1.1.1 X, called the *behavior space*, is a nonempty set, whose elements $x \in X$ are called *behaviors*;

1.1.2 $\mathcal{F} \subset [X]$, called the *feasibility space*, is a cover of X, whose elements $F \in \mathcal{F}$ are called *feasibilities*;

1.1.3 $\gamma: X \times \mathcal{F} \to [X]$, called *choice*, is a map such that $\gamma \subset \pi_{\mathcal{F}}$ (where $\pi_{\mathcal{F}}$ is the projection of $X \times \mathcal{F}$ into $\mathcal{F}$); and

1.1.4 $\delta: X \times \mathcal{F} \to [\mathcal{F}]$ is a map called the (*feasibility*) *dynamics*.

Given a feasibility-choice system Ω, the set

1.1.5 $E(\Omega) = \{(x, F) \in X \times \mathcal{F} \mid x \in \gamma(x, F), \quad F \in \delta(x, F)\}$ is called the *set of equilibria* of Ω.

1.2 DISCUSSION: A feasibility-choice system Ω may be regarded as formed by two more basic systems

1.2.1 $\Gamma = (X, \mathcal{F}, \gamma)$, a choice system, and

1.2.2 $\Delta = (X, \mathcal{F}, \delta)$, a feasibility system,

whose maps γ and δ both have domain $X \times \mathcal{F}$. Furthermore, defining the equilibrium sets

1.2.3 $E(\Gamma) = \{(x, F) \in X \times \mathcal{F} \mid x \in \gamma(x, F)\}$, and

1.2.4 $E(\Delta) = \{(x, F) \in X \times \mathcal{F} \mid F \in \delta(x, F)\}$, we see that

1.2.5 $E(\Omega) = E(\gamma) \cap E(\Delta)$.

Now, defining the maps $\gamma^*: X \times \mathcal{F} \to [X \times \mathcal{F}]$ and $\delta^*: X \times \mathcal{F} \to [X \times \mathcal{F}]$ through

1.2.6 $\gamma^* = \gamma \times \{\pi_{\mathcal{F}}\}$ (i.e., $\gamma^*(x, F) = \gamma(x, F) \times \{F\}$ for each $x \in X$ and $F \in \mathcal{F}$), and

1.2.7 $\delta^* = \{\pi_X\} \times \delta$ (i.e., $\delta^*(x, F) = \{x\} \times \delta(x, F)$ for each $x \in X$ and $F \in \mathcal{F}$),

we see that $E(\Gamma)$ and $E(\Delta)$ are simply the sets of all the fixed point of γ^* and δ^*, respectively. Also, the composed maps $\delta^* \circ \gamma^*$ and $\gamma^* \circ \delta^*$ both have precisely $E(\Omega)$ as their sets of fixed points. However, defining the map $\omega^*: X \times F \to [X \times F]$ through

1.2.8 $\omega^* = \gamma \times \delta$ (i.e., $\omega^*(x, F) = \gamma(x, F) \times \delta(x, F)$ for each $x \in X$ and $F \in \mathcal{F}$),

$E(\Omega)$ coincides also with the set of fixed points of ω^*.

1.3 THEOREM: Consider a feasibility-choice system $\Omega = (X, \mathcal{F}, \gamma, \delta)$, where X is compact and convex in a locally convex Hausdorff topological vector space and, giving $K[X]$ the finite (A.1.1) - or, equivalently (see A.1.2), the uniform (A.1.1) - topology, $\mathcal{F} \subset KQ[X]$ is closed and convex. If γ and δ map $X \times \mathcal{F}$ upper semi-continuously (see A.1.3) into $KQ[X]$ and $KQ[\mathcal{F}]$, respectively, then all of the following equilibrium sets are nonempty and compact:

1.3.1 (1) $E_F(\Gamma) = \{x \in F \mid x \in \gamma(x, F)\}$ for each $F \in \mathcal{F}$;

(2) $E(\Gamma) = \bigcup_{F \in \mathcal{F}} (E_F(\Gamma) \times \{F\})$;

1.3.2 (1) $E_x(\Delta) = \{F \in \mathcal{F} \mid F \in \delta(x, F)\}$ for each $x \in X$;

(2) $E(\Delta) = \bigcup_{x \in X} (\{x\} \times E_x(\Delta))$;

1.3.3 $E(\Omega) = E(\Gamma) \cap E(\Delta)$.

Proof: Assume that γ and δ are upper semi-continuous into $K\mathcal{Q}[X]$ and $K\mathcal{Q}[F]$, respectively.

(ad 1.3.1(1)): Let $F \varepsilon \mathcal{F}$. Then $E_F(\Gamma)$ is nothing but the set of fixed points of the restriction γ_F defined on F by $\gamma_F(x) = \gamma(x, F)$. Clearly, γ_F maps F into $K\mathcal{Q}[F]$ upper semi-continuously. Thus, by Fan's Fixed Point Theorem (A.2.5), $E_F(\Gamma) \neq \emptyset$. In fact, $E_F(\Gamma)$ is the projection into F of the intersection $G(\gamma_F) \cap D(F)$. As F is compact Hausdorff, by A.1.5(2), the graph $G(\gamma_F) \subset F \times F$ is closed, hence compact, and so is the diagonal $D(F)$. Thus, $E_F(\Gamma)$ is also compact.

(ad 1.3.1(2)): That $E(\Gamma) \neq \emptyset$ follows from 1.3.1(1). As γ is usc, so is γ^* (see A.1.4). Now, from A.2.8, $\mathcal{F}$ is compact Hausdorff. Thus, $X \times \mathcal{F}$ is compact, and so is $D(X \times \mathcal{F})$. By A.1.5(2), $G(\gamma^*)$ is also compact, whereby $E(\Gamma)$, being the projection of $G(\gamma^*) \cap D(X \times \mathcal{F})$ into $X \times \mathcal{F}$, is compact.

(ad 1.3.2(1)): Now let $x \varepsilon X$ and define the transformation δ_x on $\mathcal{F}$ by $\delta_x(F) = \delta(x, F)$ $(F \varepsilon \mathcal{F})$, so that $E_x(\Delta)$ is nothing but the set of fixed points of δ_x. As δ is upper semi-continuous into $KQ[F]$, so is δ_x. Furthermore, by A.2.6, the (nonempty convex) set F lies in a pointwise convex Hausdorff topological semivector space with singleton origin (A.2.1) and, by A.2.8, F is also compact. Moreover, F is 3^o locally convex (see A.2.9). Thus, A.2.3 applies and $E_x(\Delta) \neq \emptyset$.

(<u>ad</u> 1.3.2(2)): Imitate the proof of 1.3.1(2).

(<u>ad</u> 1.3.3): That $E(\Omega)$ is compact follows from 1.3.1(2) and 1.3.2(2). To see that $E(\Omega) \neq \emptyset$, first we note that, by A.1.4, $\omega^* = \gamma \times \delta$ is upper semi-continuous, and that $\omega^*(x, F) = \gamma(x, F) \times \delta(x, F)$ is nonempty, compact and convex for each $(x, F) \in X \times F$. Now, from A.2.7 and A.2.10, $X \times F$ is nonempty, compact, convex and 3^o locally convex in a pointwise convex topological semivector space with singleton origin. Thus, A.2.3 applies and the set $E(\Omega)$ of fixed points of ω^* is nonempty. ◊

2. SOCIAL SYSTEMS AND THEIR NONCOOPERATIVE EQUILIBRIA

2.1 DEFINITION: A social system is a family

2.1.0 $\mathcal{S} = \{(X_\alpha, \mathcal{F}_\alpha, \leq_\alpha, \delta_\alpha)\}_A$

of ordered quadruplets indexed by a set $A \neq \emptyset$ of "individuals," where, for each $\alpha \in A$,

2.1.1 $X_\alpha \neq \emptyset$ is a set called the behavior space of α, and we denote $X = \prod_A X_\alpha$;

2.1.2 $\mathcal{F}_\alpha \subset [X_\alpha]$, called the feasibility space of α, is a cover of X_α, and we denote $\mathcal{F} = \{F = \prod_A F_\alpha \mid F_\alpha \in \mathcal{F}_\alpha\}$, $\mathcal{F}^\alpha = \{F^\alpha = \prod_{A \setminus \{\alpha\}} F_\beta \mid F_\beta \in \mathcal{F}_\beta\}$;

2.1.3 $\leq_\alpha \subset (X \times \mathcal{F}^\alpha \times X_\alpha) \times (X \times \mathcal{F}^\alpha \times X_\alpha)$, called the preference of α, is a complete preorder on $X \times \mathcal{F}^\alpha \times X_\alpha$;

2.1.4 $\delta_\alpha : X \times \mathcal{F} \to [\mathcal{F}_\alpha]$ is a map called the feasibility dynamics of α, and we denote $\delta = \prod_A \delta_\alpha$.

Given a social system $\mathcal{S}$, for each $\alpha \in A$ we regard the noncooperative choice $\gamma_\alpha : X \times \mathcal{F} \to \mathcal{P}(X_\alpha)$ of α, where

2.1.5 $\gamma_\alpha(x, F) = \{y_\alpha \in F_\alpha \mid \{(x, F^\alpha)\} \times F_\alpha \leq_\alpha (x, F^\alpha, y_\alpha)\}$

$(x \in X, \quad F = F^\alpha \times F_\alpha, \quad F^\alpha \in \mathcal{F}^\alpha, \quad F_\alpha \in \mathcal{F}_\alpha)$ and, defining

2.1.6 $\gamma = \prod_{A} \gamma_\alpha,$

when γ is into $[X]$ we refer to

2.1.7 $\Omega(\mathcal{S}) = (X, \mathcal{F}, \gamma, \delta)$

as the <u>noncooperative</u> feasibility- choice system determined by $\mathcal{S}$. In that case, by the set of <u>noncooperative equilibria</u> of $\mathcal{S}$ we mean the set $E(\Omega(\mathcal{S}))$. (see 1.1.5).

<u>2.2</u> <u>REMARK</u>: Given a social system $\mathcal{S}$, two facts are clear: (1) When X_α is a topological space and $\leq_\alpha$ is upper semi-closed on X_α (see A.3.1) with F_α compact, then $\gamma_\alpha(x, F)$ is nonempty and compact (see A.3.3) for each $x \in X$ and $F \in \mathcal{F}$. (2) When X_α lies in a real vector space and $\leq_\alpha$ is upper semi-convex on X_α (see A.3.2) with F_α convex, $\gamma_\alpha(x, F)$ is convex for each $x \in X$ and $F \in \mathcal{F}$ (see A.3.3).

2.3 <u>LEMMA</u>: <u>Let</u> Y <u>be</u> <u>a</u> <u>Hausdorff</u> <u>space</u>.

(1) <u>The</u> <u>graph</u> $\Lambda = \{(K, x) \mid K \in \mathcal{K}[Y], \quad x \in K\} \subset \mathcal{K}[Y] \times Y$ <u>is</u> <u>closed</u> <u>when</u> $\mathcal{K}[Y]$ <u>carries</u> <u>the</u>

<u>the upper semi-finite topology</u> (<u>A.1.3</u>).

(2) <u>Let</u> $\leq \subset Y \times Y$ <u>be a closed preorder on</u> Y. <u>Then</u> $P = \{(K, x) | K \in \mathcal{K}[Y], \quad x \in Y, \quad K \leq x\} \subset \mathcal{K}[Y] \times Y$ <u>is closed when</u> $\mathcal{K}[Y]$ <u>carries the finite topology</u>.

(3) <u>If</u> $\leq$ <u>is also complete</u>, Y <u>is compact and</u> $\mathcal{K}[Y]$ <u>carries the finite topology, then the map "optimization"</u> $K \mapsto \hat{K} = \{x \in K | K \leq x\}$ <u>is upper semi-continuous on</u> $\mathcal{K}[Y]$ <u>with</u> $\hat{K}$ <u>nonempty and compact for each</u> $K \in \mathcal{K}[Y]$.

<u>Proof</u>: (<u>ad</u> (1)): Suppose $(K, x) \in (\mathcal{K}[Y] \times Y)\setminus\Lambda$. Then $x \notin K$ and, since Y is Hausdorff and $K \subset Y$ compact, there are disjoint open nbds $U, V \subset Y$ of K and x, respectively. Defining $\mathcal{U} = \mathcal{K}[U]$, $\mathcal{U} \subset \mathcal{K}[Y]$ is then an open nbd of K when $\mathcal{K}[Y]$ carries the upper semi-finite topology, in which case $\mathcal{U} \times V$ is an open nbd of (K, x) that is disjoint from Λ, showing that Λ is closed.

(<u>ad</u> (2)): Suppose $(K, x) \in (\mathcal{K}[Y] \times Y)\setminus P$. Then there is a $y \in K$ such that $(y, x) \in (Y \times Y)\setminus\leq$ and, since $\leq \subset Y \times Y$ is closed, there are open nbds $U, V \subset Y$ of x, y, respectively, with $(V \times U) \cap \leq = \emptyset$. Let W Y be any open set with $K \subset W$. Give $\mathcal{K}[Y]$ the finite topology. Defining $\mathcal{W} = [W, V] \cap \mathcal{K}|Y|$, $\mathcal{W} \times U \subset \mathcal{K}|Y| \times Y$ is then an open nbd of (K, x) with $(\mathcal{W} \times U) \cap P = \emptyset$, showing that $P \subset \mathcal{K}[Y] \times Y$ is closed.

(*ad* (3)): Note that the graph G of the map "optimization" is nothing but $\Lambda \cap P$. Give $\mathcal{K}[Y]$ the finite topology (which contains the upper semi-finite topology). By (1) and (2), Λ and P are then closed, and so is G. Assume $\leq$ complete. Now, by A.3.3., for each $K \in \mathcal{K}[Y]$, $\hat{K}$ is nonempty and compact. If Y is compact Hausdorff, then so is $\mathcal{K}[Y]$ (A.1.2), and A.1.5(1) applies, so that "optimization" is upper semi-continuous. ◊

<u>2.4</u> <u>THEOREM</u>: *Let $\mathcal{S}$ be a social system where, for each $\alpha \in \mathcal{A}$, X_α is a topological space. For each $\alpha \in \mathcal{A}$, give $\mathcal{K}[X_\alpha]$ the finite topology and assume that*

(1) *X_α is compact and convex in a locally convex Hausdorff topological vector space;*

(2) *$F_\alpha \subset \mathcal{K}\mathcal{Q}[X_\alpha]$ is closed and convex, and covers X_α;*

(3) *$\leq_\alpha$ is closed and on X_α it is upper semi-convex;*

(4) *δ_α is upper semi-continuous into $\mathcal{K}\mathcal{Q}[F_\alpha]$.*

Then $\Omega(\mathcal{S})$ is a (noncooperative) feasibility-choice system and the set $E(\Omega(\mathcal{S}))$ of noncooperative equilibria of $\mathcal{S}$ is nonempty and compact.

<u>Proof</u>: As it is clear from our hypothesis that $\Omega(S)$ satisfies all other requirements of 1.1, to see that $\Omega(S)$ is a feasibility-choice system, we need only check that $\gamma(x, F) \neq \emptyset$ for each $(x, F) \in X \times F$. This we do by noting that, in view of 2.2 and A.3.1, for each $\alpha \in A$, the conjunction of (2) and (3) implies $\gamma_\alpha(x, F) \in KQ[X_\alpha]$ for each $(x, F) \in X \times F$, so that, in fact, $\gamma(X \times F) \subset KQ[X] \subset [X]$. Now to show that $E(\Omega(S))$ is nonempty and compact, we simply check that Theorem 1.3 applies.

First, by (1), X is compact and convex in a locally convex Hausdorff topological vector space.

Second, from (2), $F \subset KQ[X]$ is convex and covers X. Give $K[X]$ the finite topology. We show that F $KQ[X]$ is also closed. By A.1.2 and A.2.8, (1) implies that, for each $\alpha \in A$, $KQ[X_\alpha]$ is compact Hausdorff. Similarly, $KQ[X]$ is also compact Hausdorff. Since $F_\alpha \subset KQ[X_\alpha]$ is closed (see (2)), it is compact, so that $\prod_A F_\alpha$ is compact. Now, by A.1.6, F is homeomorphic to $\prod_A F_\alpha$, so that F is compact, whereby $F \subset KQ[X]$ is closed.

Third, by taking $Y = X \times F^\alpha \times X_\alpha$ and $K = \{(x, F^\alpha)\} \times F_\alpha$ in 2.3(3) for each $\alpha \in A$, we see that γ_α is upper semi-continuous. Thus, by A.1.4, the map γ (already seen to be into $KQ[X]$) is also upper semi-continuous.

Finally, again by A.1.4, (4) implies that δ is upper semi-continuous; clearly, (4) also implies $\delta(X \times F)$ $\subset KQ[F]$.

Thus, Theorem 1.3 applies, so that $E(\Omega(S))$ is non-empty and compact. ◊

3. COMPARISON WITH DEBREU'S SOCIAL EQUILIBRIUM EXISTENCE THEOREM

The last three decades have seen a flurry of activity in the existence theory of equilibria in games, economies and, in general, social systems. This being no survey article, for bibliographical purposes we refer to the most recent book of Hildenbrand [4] and focus our attention in this literature on Debreu's Social Equilibrium Existence Theorem [2], which despite its rather early date, not only preserves its landmark nature, but also continues to stand apart in its simplicity. Debreu's mentioned work actually forms a base for the celebrated Arrow and Debreu Existence Theorem [1] for competitive equilibria in economies - the type of result which we expect to be derivable along similar lines as in [1] from our existence theorem 2.4. His notion of a social system is, moreover, a forerunner of our own. For all these reasons we find it useful to interpret Debreu's mentioned study in terms of our present framework.

In the terminology of 2.1, Debreu's social system is characterized by a finite set A of individuals (his "agents") α, whose behavior spaces X_α are contractible polyhedra in a Euclidean space, whose feasibility spaces F_α may be taken as the spaces $KT[X_\alpha]$ of contractible compacta in X_α, whose feasibility dynamics are maps $\delta_\alpha : X^\alpha = \prod_{A \setminus \{\alpha\}} X_\beta \to F_\alpha$ with closed graph $G_\alpha \subset X$, and whose preferences $\leq_\alpha \subset G_\alpha \times G_\alpha$ are determined by continuous "payoff" functions $f_\alpha : G_\alpha \to [0, 1]$ for which $\bar{f}_\alpha : X^\alpha \to [0, 1]$, defined by $\bar{f}_\alpha(x^\alpha) = \sup_{\{x^\alpha\} \times \delta_\alpha(x^\alpha)} f_\alpha$, is also continuous. (Thus, γ^α depends on x^α and F_α.) Debreu's theorem then says that the social system has a non-cooperative equilibrium if the points y_α in $\delta_\alpha(x^\alpha)$ with $f_\alpha(x^\alpha, y_\alpha) = \bar{f}_\alpha(x^\alpha)$ form a contractible set for each $x^\alpha \in X^\alpha$ and $\alpha \in A$.

To compare Debreu's result with ours, first note that, since Debreu has already taken X compact Hausdorff and $G_\alpha \subset X$ closed, by Tietze's characterization of normality, the function f_α can be extended to a continuous f^*_α on the whole of X, extending $\leq_\alpha$ to a closed complete preorder $\leq^*_\alpha \subset X \times X$ on X - without, however, affecting the map γ_α $(\alpha \in A)$, and hence leaving the equilibrium set unchanged. Now, Debreu does not require the behavior spaces and feasibilities to be convex, and neither does he require the preferences $\leq_\alpha$ to be upper semi-convex on X_α (equivalent, in his case, to f_α being quasi-concave on $\{x^\alpha\} \times X_\alpha$ for each $x^\alpha \in X^\alpha$). Therefore, our theorem (2.4) is not a strict generalization of Debreu's. In a number of ways, however, 2.4 extends Debreu's result substantially.

First, we do not restrict the number of individuals to be finite, and this should be important for any application of 2.4 to showing the existence of competitive equilibria in economies, since one really needs large economies (see [4]) to speak of competitive equilibria properly.

Second, we work in locally convex Hausdorff topological vector spaces - the natural habitat of probability functions - which should allow one economic applications incorporating stochastic behavior or infinite dimensional commodity space.

Third, we do not assume individuals' preferences to be representable by continuous real-valued functions. Apart from this, however, a significant aspect of the type of preferences we allow an individual is that it yields the individual's choice sensitive, in general, to many variables which Debreu's agents ignore in their choice.

Namely, as a fourth point, while in Debreu's model the generic individual α's choice γ_α depends only on "others last behavior" x^α and α's "own feasibility" F_α, our model allows it to depend on (i) α's "own last behavior" and (ii) "others' feasibility" F^α as well. The significance of allowing the additional variable (i) is quite clear: it permits sequential dependence on "own behavior," such as one associates with "learning," "addiction" - or, what is the opposite, "withdrawal" - or simply choosing to eat dessert rather than appetizers after the main course.

Our inclusion of variable (ii) may appear somewhat unusual upon first sight, but there are many common scenarios in which its role will be immediately recognized: (a) An individual possesses an <u>exceptional talent</u> the exercise of which - by reason of its rareness! - gets promoted in his ranking of alternative occupations; (b) "What are the enemy's various strike <u>capabilities</u>? For on that depends what guns (versus "butter") we want;" (c) "He's a bully: he'll pick on you if you can't hit back;" (d) "He's a noble man: he would spare a helpless soul." (Common ?)

Last, but not least, we must contrast Debreu's individual feasibility dynamics δ_α with ours. First compare the domains of the maps. In Debreu's case, the individual's feasibility $F_\alpha = \delta_\alpha(x^\alpha)$ is completely determined by others' last behavior x^α. In contrast, we view the feasibility dynamics as a "deformation" process: surely, the productive capability F'_α an economy α has today depends on the productive capability F_α it had yesterday, and the dependence is through domestic and foreign economic behavior (investment), i.e., x_α and x^α, feasible and chosen yesterday. This is why we have felt it necessary to include x_α and F_α among the arguments of δ_α in our model. Why have we included also others' last feasibility F^α among its arguments? Directing our imagination to a rather different realm of social life, consider the idea of <u>precedence</u> in jurisprudence. Roughly, this says that, if individual β was allowed the option of, say, conscientious objection yesterday, then this option is henceforth available to all - and this is <u>regardless</u> of whether β exercised his option. Now even all these variables, x^{α}, x_{α}, F^{α}, F_{α}, may in reality fail to determine a <u>unique</u> next feasibility F'_{α}, simply because some determining factors may have been left out. For this reason we specify the range of δ_{α} to lie not in F_{α}, but in $[F_{\alpha}]$.

A. MATHEMATICAL APPENDIX

For the convenience of the reader, this appendix collects mathematical facts and notions used in the main body of the paper.

A.1 HYPERSPACES AND SET-VALUED MAPS:

A.1.1 Let Y be a topological space. The finite topology for $[Y]$ is the topology generated by declaring as a basis for open collections in $[Y]$ collections of the form $[U_i | i \in M] = \{P \in [Y] | P \subset \bigcup_M U_i$ and $P \cap U_i \neq \emptyset$ for each $i \in M\}$ with M a finite set and $U_i \subset Y$ open for each $i \in M$ [5, 1.7]. When Y is a uniform space with fundamental system of symmetric entourages E, defining $\mathcal{E} = \{(P, Q) \in [Y] \times [Y] | P \subset E(Q)$ and $Q \subset E(P)\}$, the entourages $\mathcal{E}$ form a fundamental system for a uniform structure on $[Y]$, and the topology so determined for $[Y]$ is called its uniform topology [5, 1.6]. By the finite [resp., uniform] topology of a hyperspace $\mathcal{H} \subset [Y]$ of a topological [resp., uniform] space Y is meant the relative topology of $\mathcal{H}$ as a subspace of $[Y]$ when $[Y]$ is given the finite [resp., the uniform] topology.

<u>A.1.2</u> When Y is a uniform T_1 space, the finite topology and the uniform topology agree on $K[Y]$ [5, 3.3]. For Y a T_1 space, $K[Y]$ with the finite topology is (compact) Hausdorff iff Y is (compact) Hausdorff [5, 4.9.8 and 4.9.12].

<u>A.1.3</u> Given topological spaces X and Y and a mapping $f: X \to [Y]$, f is said to be <u>upper semi-continuous</u> iff, for each $x \in X$ and each open set $V \subset Y$ with $f(x) \subset V$, there is a nbd U of x such that $f(x') \subset V$ for every $x' \in U$. This is equivalent to f being continuous when $[Y]$ is given the so-called <u>upper semi-finite topology</u> [5, pp. 179], i.e., the topology generated by declaring collections $[V]$ open for all open $V \subset Y$. Thus, the composition of upper semi-continuous maps is upper semi-continuous. Also, given a family $\{f_\alpha: X \to [Y_\alpha] \mid \alpha \in A\}$ of maps, the map $\{f_\alpha\}_A$ is upper semi-continuous iff each f_α is upper semi-continuous.

<u>A.1.4</u> <u>PROPOSITION</u>: <u>Let</u> X <u>be a topological space</u>, $\{Y_\alpha\}_A$ <u>a family of topological spaces with</u> $Y = \prod_A Y_\alpha$, <u>and</u> $\{f_\alpha: X \to K[Y_\alpha] \mid \alpha \in A\}$ <u>a family of maps. The map</u> $f = \prod_A f_\alpha: X \to K[Y]$ <u>is upper semi-continuous iff</u> f_α <u>is upper semi-continuous for each</u> $\alpha \in A$.

Proof: (ad "if"): Assume f_α upper semi-continuous for each $\alpha \in A$, take any $x \in X$, and let $V \subset Y$ be any open set with $f(x) \subset V$. Now V contains an open tube $(\prod_M V_\alpha) \times (\prod_{A \setminus M} Y_\alpha) \supset f(x)$ with $M \subset A$ finite and $V_\alpha \subset Y_\alpha$ open for each $\alpha \in M$, since $f(x) = \prod_A f_\alpha(x)$ is a compact box [9, 2]. Using the upper semi-continuity of f_α, for each $\alpha \in M$, there is a nbd U_α of x with $f_\alpha(x') \subset V_\alpha$ for each $x' \in U_\alpha$. Writing $U = \bigcap_M U_\alpha$, U is thus a nbd of x and $f(x') \subset V$ for every $x' \in U$, showing that f is upper semi-continuous.

(ad "only if"): If f is upper semi-continuous, then, for each $\alpha \in A$, the projection $\pi_{Y_\alpha} \circ f$ is also upper semi-continuous, and evidently $\pi_{Y_\alpha} \circ f = f_\alpha$. ◇

A.1.5 PROPOSITION: *Let X and Y be topological spaces and $f: X \to [Y]$ a map.*

(1) *Assume that the graph $G(f) \subset X \times Y$ is closed. Then f is into $C[Y]$. If, furthermore, X is Hausdorff, then f is upper semi-continuous whenever Y or $G(f)$ is compact.*

(2) *If f is upper semi-continuous into $C[Y]$, then the graph $G(f) \subset X \times Y$ is closed whenever Y is regular.*

<u>Proof</u>: (<u>ad</u> (1)): Take any $x \varepsilon X$. For each $y \varepsilon Y \backslash f(x)$, $(x, y) \varepsilon (X \times Y) \backslash G(f)$ and, since $G(f)$ is closed, there is a nbd $U \subset X$ of x and a nbd $V \subset Y$ of y with $(U \times V) \cap G(f) = \emptyset$, so that $V \cap f(x) = \emptyset$, showing $f(x)$ to be closed. Now let $W \subset Y$ be an open set with $f(x) \subset W$ so that $W^c = Y \backslash W$ is closed. Then $G(f) \cap (X \times W^c)$ is compact whenever Y is compact or $G(f)$ is compact, in which case its projection P into X is compact. Thus, when X is Hausdorff $P \subset X$ is closed and $P^c = X \backslash P$ is a nbd of x with $f(x') \subset W$ for each $x' \varepsilon P^c$, whereby f is upper semi-continuous.

(<u>ad</u> (2)): Assume Y regular and f upper semi-continuous into $C[Y]$, and take any $(x, y) \varepsilon (X \times Y) \backslash G(f)$. Then there are disjoint open sets $V, W \subset Y$ with $y \varepsilon V$ and $f(x) \subset W$. As f is upper semi-continuous, there is also an open nbd $U \subset X$ of x with $f(x') \subset W$ for each $x' \varepsilon U$. Now $U \times V \subset (X \times Y) \backslash G(f)$ is an open nbd of (x, y), showing that $G(f)$ is closed. ◊

<u>A.1.6</u> THEOREM [<u>9</u>, 3]: <u>Let $\{Y_\alpha\}_A$ be a family of topological spaces and, for each $\alpha \varepsilon A$, give $K[Y_\alpha]$ its respective finite topology. Also, give $K[\prod_A Y_\alpha]$ its finite topology. Now the Cartesian product map Π, defined by</u>

$$\Pi(\{K_\alpha\}_A) = \prod_A K_\alpha \quad (K_\alpha \in K[Y_\alpha],\ \alpha \in A),$$

is a homeomorphism of $\prod_A K[Y_\alpha]$ _onto the space_ (_of compact boxes_) $BK[\{Y_\alpha\}_A] \subset K[\prod_A Y_\alpha]$.

A.2 TOPOLOGICAL SEMIVECTOR SPACES AND SOME FIXED POINT THEORY:

A.2.1 A topological semivector space is a nonempty Hausdorff space S equipped with two continuous maps $\oplus: S \times S \to S$ ("semivector addition") and $\psi: R_+ \times S \to S$ ("scalar multiplication") such that, denoting $\oplus(s, t) = s \oplus t$ and $\psi(\lambda, s) = \lambda s$,

(1)	$r \oplus (s \oplus t) = (r \oplus s) \oplus t$	(associativity)
(2)	$s \oplus t = t \oplus s$	(commutativity)
(3)	$\lambda(s \oplus t) = \lambda s \oplus \lambda t$	(homomorphism)
(4)	$1s = s$	(unitariness)
(5)	$\lambda(\mu s) = (\lambda.\mu)s$	(action)

($r, s, t \in S$: $\lambda, \mu \in R_+$) [8, 1.1]. Given a topological semivector space S, the set $OS = \{Os | s \in S\}$ is called the origin of S. A set X S is called convex iff $\lambda x \oplus \lambda' x' \in X$ whenever $x, x' \in X$ and $\lambda = (1-\lambda') \in [0, 1]$ (see 2.1 of 8); and S is called pointwise convex iff each singleton $\{s\} \subset S$ is convex [8, 2.3].

A.2.2 A convex subset $X \subset S$ is called 3^o locally convex iff its relative topology admits a uniformity with a fundamental system $E = \{E_\alpha \subset X \times X | \alpha \in A\}$ of convex entourages [8, 3.0.3]. A weaker [8, 3.1] property for a subset $X \subset S$ is 2^o local convexity, i.e., that the relative topology of X

admits a quasi-uniformity $\mathcal{E} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ such that, for each $\alpha \in A$, there is a $\beta \in A$ with $E_\beta \subset E_\alpha$ closed and $E_\beta(K)$ convex whenever $K \subset X$ is compact and convex [8, 3.0.2].

A.2.3 FIXED POINT THEOREM [8, 4.6]: Let S be a topological semivector space, and let $X \subset S$ be a non-empty, compact and convex subset with OX singleton. If X is pointwise convex and 2° l.c., then X has the fixed point property for upper semi-continuous transformations $f: X \to CQ[X]$.

A.2.4 COROLLARY [8, 4.7]: Let $\{X_\alpha \subset S_\alpha \mid \alpha \in A\}$ be a nonempty family, where, for each $\alpha \in A$, S_α and X_α satisfy the hypothesis of A.2.3; and let $\{f_a: X \to CQ[X_\alpha] \mid \alpha \in A\}$ be a family of upper semi-continuous transformations, where $X = \prod_A X_\alpha$. Define $f: X \to CQ[X]$ by $f(x) = \prod_A f_\alpha(x)$. Then there exists a (fixed) point $x \in X$ such that $x^* \in f(x^*)$.

A.2.5 COROLLARY [8, 4.8] (Fan's Fixed Point Theorem [3]): Let X be nonempty, compact and convex in a locally convex Hausdorff topological vector space, and let $f: X \to CQ[X]$ be an upper semi-continuous transformation. Then there exists a (fixed) point $x^* \in X$

such that $x^* \varepsilon f(x^*)$.

The rest of this section provides examples of spaces for which the fixed point theory of A.2.3-4 applies.

A.2.6 THEOREM [10, 2.1]: Let L be a real Hausdorff topological vector space, give $K\mathcal{Q}[L]$ the finite topology, and equip it with the operations $\oplus$ and ψ as follows:

$$\left.\begin{array}{l} A \oplus B = \{a + b \mid a \varepsilon A,\ b \varepsilon B\} \\ \lambda A = \{\lambda a \mid a \varepsilon A\} \end{array}\right\} \quad \begin{array}{l} A, B \varepsilon K\mathcal{Q}[L]; \\ \lambda \varepsilon R_+ \end{array}$$

$K\mathcal{Q}[L]$ is then a pointwise convex Hausdorff topological semivector space with singleton origin.

A.2.7 COROLLARY: With all as in A.2.6, equipping $L \times K\mathcal{Q}[L]$ with coordinatewise addition and scalar multiplication, one obtains again a pointwise convex Hausdorff topological semivector space with singleton origin.

A.2.8 THEOREM [10, 2.3]: Given L as in A.2.6, and taking any compact and convex $X \subset L$, the set $K\mathcal{Q}[X] \subset K\mathcal{Q}[L]$ is compact and convex.

A.2.9 PROPOSITION [7, 4.4]: In A.2.8, assume that L is locally convex. Then $KQ[L]$ is 3° locally convex (and so is its every convex subset, e.g., $KQ[X]$).

Proof: Let $\{W_\alpha \subset L \mid \alpha \in A\}$ be a fundamental system of convex nbds of the identity element $e \in L$, so that, defining $E_\alpha = \{(x, y) \in L \times L \mid x \in y + W_\alpha, \quad y \in x + W_\alpha\}$, $\{E_\alpha \mid \alpha \in A\}$ is a fundamental system of convex entourages for (the uniform space) L. Thus, writing $\mathcal{E}_\alpha = \{(P, Q) \in KQ[L] \times KQ[L] \mid P \subset E_\alpha(Q), \quad Q_\alpha \subset E_\alpha(P)\}$, $\{\mathcal{E}_\alpha \mid \alpha \in A\}$ is a fundamental system of entourages imparting the uniform (equivalently - see A.1.2 - the finite) topology to $KQ[L]$. Let $\alpha \in A$. To see that $\mathcal{E}_\alpha$ is convex, let (A, B), $(A', B') \in \mathcal{E}_\alpha$ and, taking any $\lambda = (1-\lambda') \in [0, 1]$, define $\bar{A} = A \oplus \lambda' A'$ and $\bar{B} = \lambda B \oplus \lambda' B'$. Now,

$$\begin{array}{l} \lambda A \subset \lambda E_\alpha(B) = \lambda (B + W_\alpha) = \lambda B + \lambda W_\alpha \\ \oplus\ \lambda' A' \subset \lambda' E_\alpha(B') = \lambda'(B' + W_\alpha) = \lambda' B' + \lambda' W_\alpha \\ \hline \bar{A} \subset \qquad\qquad\qquad\qquad \bar{B} + W = E_\alpha(\bar{B}). \end{array}$$

Similarly, $\bar{B} \subset E_\alpha(\bar{A})$. Thus, $(\bar{A}, \bar{B}) \in \mathcal{E}_\alpha$ and $\mathcal{E}_\alpha$ is convex, showing that $KQ[L]$ is 3° locally convex. ◇

A.2.10 COROLLARY: With all as in A.2.9, and given any closed and convex collection $F \subset KQ[X]$, $X \times F$ is 3° locally convex, compact and convex in $L \times KQ[L]$.

A.3 PREORDERS:

A.3.1 Let Y be a topological space and $\leq \subset Y \times Y$ a preorder (i.e., a transitive, reflexive binary relation) on Y. We say that $\leq$ is upper [resp., lower] semi-closed iff $\mathcal{U}(x) = \{y \in Y | \ x \leq y\}$ [resp., $\ell(x) = \{y \in Y | \ y \leq x\}$] is closed for each $x \in Y$. We say that $\leq$ is semi-closed iff it is both upper and lower semi-closed. Clearly, if $\leq \subset Y \times Y$ is closed, then it is also semi-closed. When Y is a product space $Y = Y_1 \times Y_2$, we say that $\leq$ is (upper, lower) semi-closed on Y_2 iff the restriction $\leq \cap (\{y_1\} \times Y_2) \times (\{y_1\} \times Y_2)$ of $\leq$ to each "slice" $\{y_1\} \times Y_2$ with $y_1 \in Y_1$ is (upper, lower) semi-closed.

A.3.2 When Y lies in a topological semivector space S, we refer to $\leq$ as upper semi-convex iff $\mathcal{U}(x)$ is convex for each $x \in Y$.

A.3.3 For each $A \subset Y$, denote $\hat{A} = (\bigcap_{A} \mathcal{U}(a)) \cap A$. Thus, $\hat{A}$ is closed [resp., convex] whenever A is closed [resp., convex] and $\leq$ is upper semi-closed [resp., upper semi-convex]. When A is compact Hausdorff and $\leq$ upper semi-closed and complete on A (i.e., for each pair $a, b \in A$, $a \leq b$ or $b \leq a$ obtains), then the family $\{\mathcal{U}(a) \cap A | \ a \in A\}$ of closed sets has the finite intersection property, so that $\hat{A} \neq \emptyset$.