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DIVIDING A CAKE BY MAJORITY: THE SIMPLEST EQUILIBRIA*

by

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Abstract

In a stochastic game of dividing a cake by majority, the simplest equilibria are the Baron-Ferejohn (1989) ones. The formal definition of simplicity and the computational methods of the equilibria make use of an automaton measure of complexity adopted for stochastic games.

1. Introduction

This paper applies complexity analysis to select and compute simple equilibria in a political stochastic game that has an infinite number of subgame perfect Nash equilibria. The game involves dividing a cake by majority rule, and Baron and Ferejohn (1989) have shown that all divisions can be supported as subgame perfect Nash equilibria if there are at least five players and the discount factor is sufficiently high. The equilibrium strategies that support these divisions, however, can be quite complex, so it is natural to ask if there is a focal point for the set of equilibria. A focal point is an outcome to which all players naturally can direct their attention. One focal concept is simplicity, since if players understand that there are an infinite number of equilibria, it seems natural for them to begin by contemplating simple rather than complex ones. If there is a unique simplest equilibrium, then it has a natural appeal as a focal point. This paper demonstrates that there is a unique (in payoffs) simplest equilibrium for the game of dividing a cake by majority, where "simplest" is defined in terms of the number of states of the automaton required to play the game.

Our approach to selecting simple equilibria differs from that proposed by Rubinstein (1986) and Abreu-Rubinstein (1988). Their approach led to the selection of an equilibrium which is as simple as possible from the point of view of an individual player, taking as given the behavior of the other players. This will be referred to as individual complexity. Our approach assumes that players facing a set of equilibria, many of which are appealing, will select the equilibrium that is simplest. Thus, we develop a

concept of equilibrium complexity that applies to the equilibrium itself rather than to the strategies of an individual player taking the strategies of the other players as given. Equilibrium complexity then can be used to characterize the simplest equilibria, and those serve as a focal point (see Schelling (1960)) for players to choose their strategies. Those strategies then will themselves be simple. By serving as a focal point, equilibrium simplicity then selects certain equilibria from the set of equilibria, and for the game of dividing a cake by majority, the equilibrium selected is unique.

We see evidence of the use of simple focal points in many examples, the most familiar of which is perhaps the repeated prisoners' dilemma game. When one is first introduced to this game, one tries the constant-cooperate and constant-defect strategies, which are the simplest strategies for this repeated game. After discovering that these solutions are not satisfactory (total cooperation is not at equilibrium and total defection is not Pareto efficient), one searches for the next simplest equilibrium strategies: tit-for-tat and trigger strategies. Often the analysis stops here, since these are sufficient to achieve the best outcome and there is thus no need to look for more complicated ones.

To illustrate the difference between the approach proposed here and the one of Abreu-Rubinstein, consider the repeated play of a two-person, pure coordination game (both players are paid 1 if the combinations top (T)-left (L) or bottom (B)-right (R) are played, and 0 otherwise). The game is played at discrete times, $t = 1,2,3,\ldots$, and let T denote an infinite set of fairly irregular times. For example, suppose T is the set of times which are prime-number integers. Consider the following equilibrium strategies.

At times $t \in T$, the players play (T,L) and at times $t \notin T$ they play (B,R). It is easy to see that this is an equilibrium strategy in which each player chooses the simplest possible best response strategy, given the choice of his opponent (these are actually unique best responses). Thus, this equilibrium satisfies the individual complexity criterion of Abreu-Rubinstein for players with low complexity costs (lexicographically considering complexity costs after "real" payoffs).

However, in the above example, it is difficult to imagine that the players will choose such a complex equilibrium. It seems most likely that they will settle on always playing (T,L) or always playing (B,R). These strategies also yield Pareto optimal payoffs, but they are much less complex. Our concept of equilibrium complexity identifies these two equilbiria as the simplest of the set of equilibria.

Of course, if the payoffs of (T,L) and (B,R) were not symmetric, then a further restriction on the criterion could be used, which may yield a focal point. For example, in the battle of the sexes game, the players may select an equilibrium alternating between (T,L) and (B,R), which is the simplest among the symmetric Pareto efficient ones. This then serves as a natural focal point.

We apply the approach of equilibrium simplicity to a game of dividing a cake by majority. The specific measure of complexity (simplicity) we use is that defined by finite automata as suggested by Aumann (1981), and used by Ben-Porath (1986), Neyman (1985), Rubinsetin (1986), and others (see Kalai (1987) and Sorin (1988) for surveys of this literature). However, in order to allow for subgame perfection, we use the modified notion of automata as defined by Kalai-Stanford (1988). These automata allow as part of the input the actions of the automaton user himself so that subgame perfection can be addressed. Also, unlike the above mentioned papers, the automata discussed in this paper play a stochastic game (see Shapley (1953)) rather than a repeated game (see Kalai-Samet-Stanford (1986) for automata playing stochastic games).

The principal results of the analysis are that the simplest equilibria are unique up to which players constitute the majority and that these equilibria are the ones suggested by Baron and Ferejohn (1989), which seem natural from a number of intuitive considerations.

A by-product of this analysis is a demonstration of how complexity analysis can be used in computing equilibria of infinite games. While backwards induction cannot be used for infinite games, finite complexity can replace it by reducing the problem to one of solving a system with a finite number of equalities and inequalities. This approach combined with simplicity is equivalent to Baron and Ferejohn's approach to characterizing stationary eguilibria. As in Baron-Ferejohn, we use the concept of subgame perfect Nash equilibrium refined to exclude the use of dominated strategies in any stage game.

2. The Game and Equilibrium

Our game consists of n players wishing to divide a unit cake. It takes a "majority" of s players, $0 < s < n$, to force a division and the rules of the division game are as follows. In the first stage, nature selects one of the n players at random as a proposer. The proposer, whose identity is made public, chooses a proposed division $r = (r_1, r_2, \ldots, r_n)$ consisting of n nonnegative rational numbers summing to no more than one, where r_i represents the proposed share for players $i = 1,2,\ldots,n$, respectively. After r is made public, simultaneously each player votes yes (Y) or no (N) on the proposed division. If at least s players vote Y the game ends, and the payoffs are r. If fewer than s people vote Y, then a new, identical stage game is started with a new independent draw of a proposer, and so on. The division game therefore either ends with some division r or goes on forever without reaching an agreement.

If the game ended by an acceptance of a proposal r, the overall payoffs of the game are given by the vector r, or if the game goes on forever, the payoffs are the vector $(0,0,\ldots,0)$. Discounting could be used in the evaluation of payoffs but the analysis would be the same except for added non-enlightening symbols and computations.

In this game we allow the use of behavioral (mixed) strategies. Thus, a player's strategy consists of a probability distribution over his pure choices following every feasible history of past pure actions. When behavioral strategies are used, payoffs are evaluated by their expected values.

As Baron and Ferejohn have shown, this game has an infinite number of subgame perfect Nash equilibria. Indeed, if the number of players is at least five and the discount factor is sufficiently high, all divisions of the cake are subgame perfect Nash equilibria. Some of the equilibria are very non-intuitive, such as the one in which each proposer in each stage proposes that player 1 receive the whole cake and everyone votes for that division in every stage. This equilibrium is supported by infinitely-nested punishment strategies with the property that no player prefers to deviate from equilibrium play because that player will be punished by receiving zero in the subsequent play of the game, both on and off the equilibrium path.

To avoid peculiar equilibria allowed by majority voting, we will restrict our attention to stage undominated equilibria, i.e., those with the property that at any induced stage game none of the players use dominated voting strategies. More precisely, starting with a strategy configuration f, we consider stage games as described above, and with payoffs assigned to each terminal node according to the expected payoff in the subgame following the terminal node of the stage game under consideration. Thus, if a stage game ends with a majority "yes" vote, the induced payoff in the stage game is the division just accepted. If a stage game ends without a final division, the associated payoffs will be the expected payoffs in the original game following that node when the strategies of f are followed. We say that f is an equilibrium with stage undominated strategies if the strategies it induces in every stage game are not dominated.

3. Automata and Complexity of Strategies

An automaton describing a strategy of a player is a triple $((M,m_0),B,T)$. M is a set of states of mind of the player with $m_0 \in M$ denoting the initial state. The behavior function B chooses a probability distribution over actions, B(m), for every state of mind, m. In this case, actions consist of proposed divisions, Y and N votes, or rest (no action; because it is the time for another player to act). The transition function T chooses a new state of mind, $T(m,\sigma)$, for each previous state of mind m and an input message σ. Input messages here can consist of the names of selected proposers, proposed vectors of divisions, or vectors consisting of Y or N votes.

We restrict ourselves to automata that play coherently, i.e., generate "legal actions" upon receiving "legal messages." Thus, the automata "know" and "follow" the rules of the game. To illustrate the representation by an automaton, Figure 1 presents an automaton description of a certain prima donna strategy for player one in a three player division game. In this figure, the circles represent states, and the entries inside the circles describe the distributions over actions chosen by the player at the corresponding states. Lines with arrows describe the transition function, and rest corresponds to when another player has a move as proposer, or when nature moves.

Player 1 is a prima donna because he is insulted forever (moves into the boxed area) if player 3 were ever chosen to propose or if player 2 were to offer him less than 90 percent of the cake. In the boxed area, he votes N with probability 1 and if called upon to propose, he proposes that he take everything. Before entering this absorbing mood, this player will vote almost surely Y (probability .99) on proposals giving him at least 90 percent and on any of his own proposals. His own proposals will give him either 100 or 90 percent, sometimes offering 10 percent of the cake to player 2.

Following an argument similar to the one in Kalai-Stanford (1988) (see also Kalai-Samet-Stanford (1986)), we know that every strategy can be described by an automaton (possibly one with infinitely many states). We define the <u>complexity of a strategy</u> to be the minimal number of states of an automaton describing it. Our objective now is to characterize the <u>simplest equilibria</u>--more precisely, the equilibria in which the complexity of the most complex player is minimal.

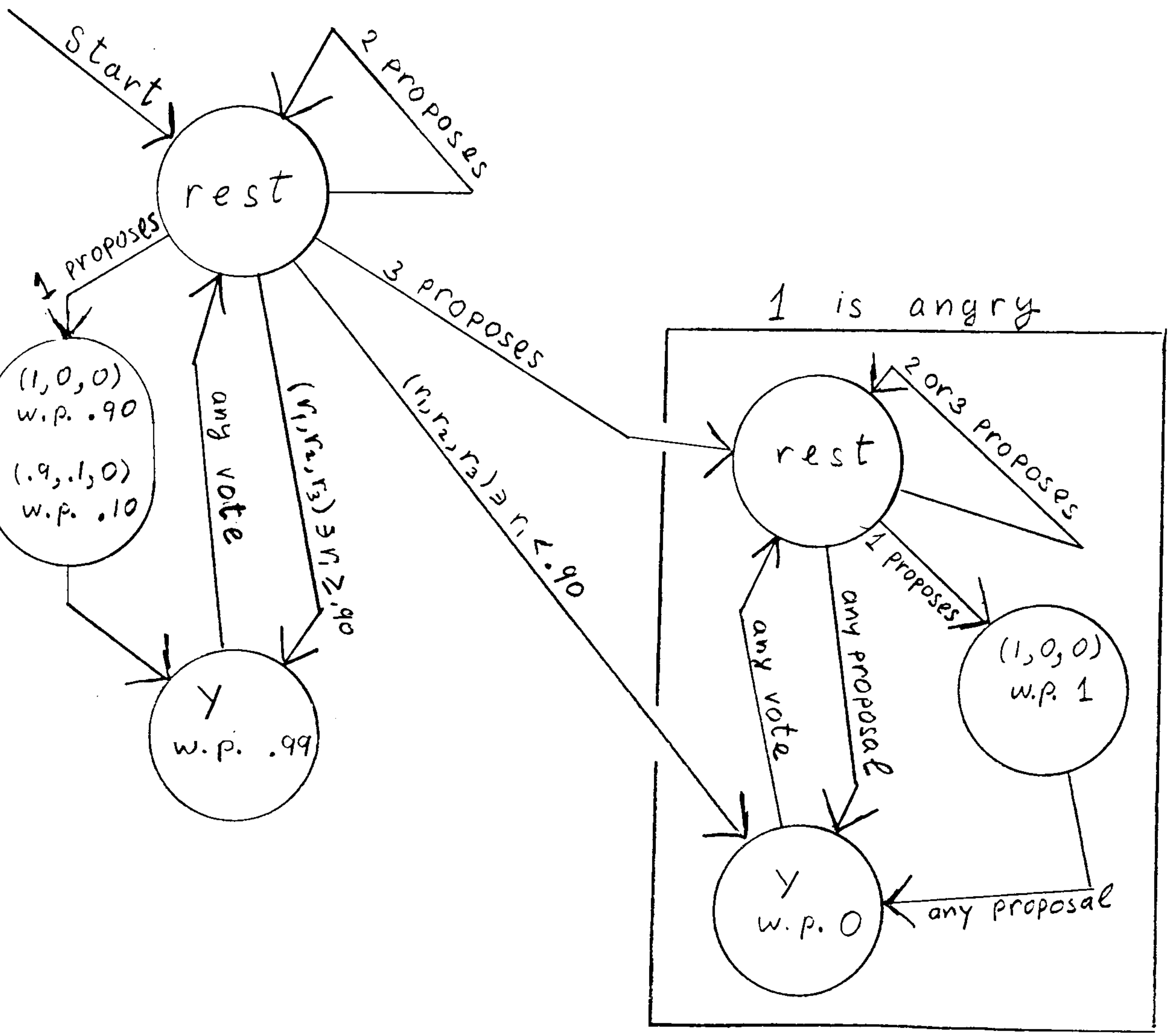
Start
rest
2 proposes
1 proposes
3 proposes
(1,0,0) w.p. .90
(.9,.1,0) w.p. .10
any vote
$(r_1,r_2,r_3) \ni r_1 \geq .90$
$(r_1,r_2,r_3) \ni r_1 < .90$
Y w.p. .99
1 is angry
rest
2 or 3 proposes
1 proposes
any vote
any proposal
(1,0,0) w.p. 1
Y w.p. 0
any proposal

Figure 1

We will first describe the family of Baron-Ferejohn equilibria. These are subgame perfect equilibria with stage undominated strategies in which each player uses strategies of complexity four. Then we will show that every subgame perfect equilibrium with stage undominated strategies in which each player uses strategies of complexity four or less is from this family. Thus, these are exactly the simplest subgame perfect equilibria with stage undominated strategies.

4. The Baron-Ferejohn Equilibria

We first define a Baron-Ferejohn (BF) strategy for player i by an automaton as in Figure 2.

The collection $(d^i_q, p^i_q)_{q \in Q^i}$ consists of a finite set of pairs. Each d^i_q is a proposed division in which s - 1 players other than i are offered 1/n, i is offer [n - (s - 1)]/n, and all other players are offered 0. P^i_q is the probability that i will propose the vector d^i_q and thus $\sum_{q \in Q^i} p^i_q = 1$.

Notice that at a BF strategy a player selected to propose chooses at random a minimal winning coalition (which includes himself). He offers each member of the coalition the average amount 1/n, and he proposes that he take the rest.

A vector $(f_1, f_2, \ldots, f_n)$ of BF strategies is said to be balanced if all the players have equal probability of being included in minimal winning coalitions when added up over all proposing players (recall that all the proposers are chosen with equal probabilities). It is easy to check the following.

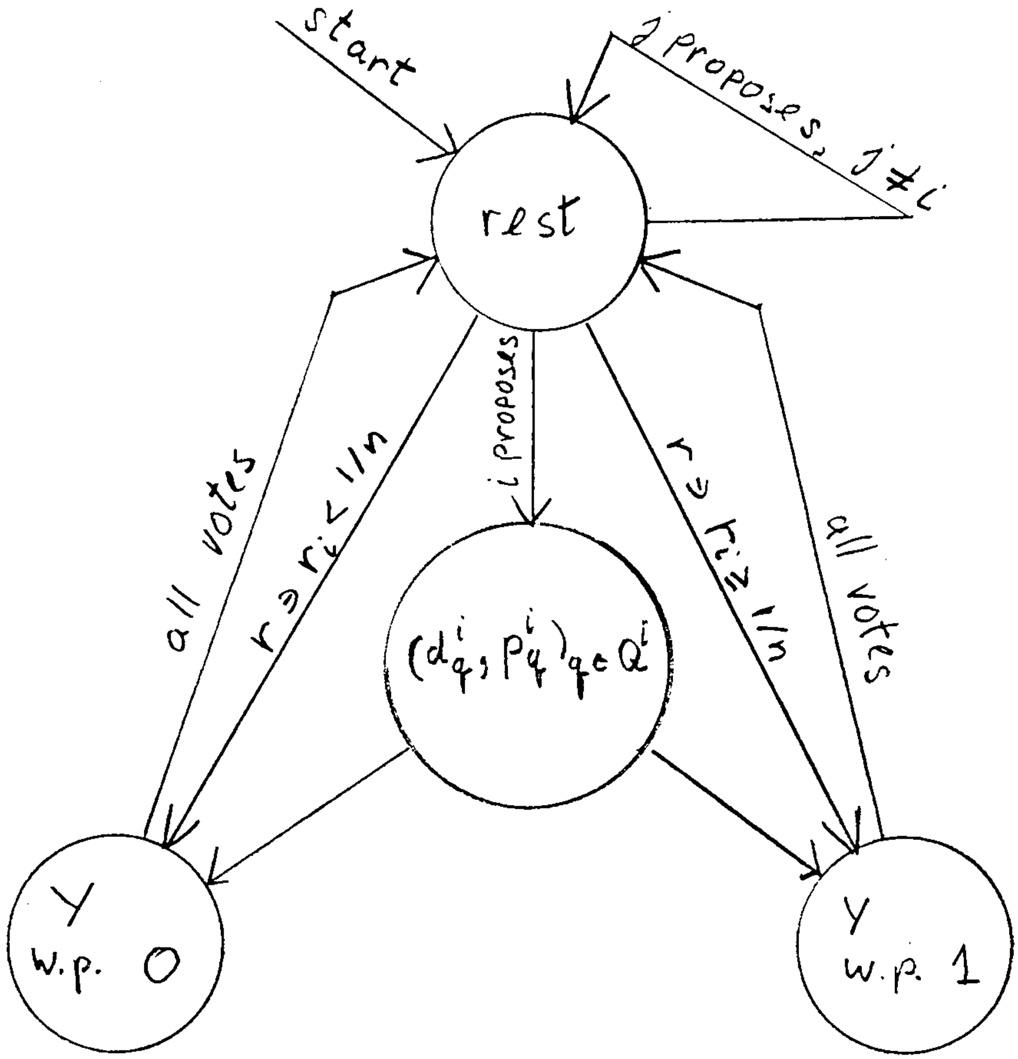

Figure 2

Each d_q^i consists of s - 1 players receiving 1/n and i receiving (n - s + 1)/n. P_q^i is the probability that i propose d_q^i. Thus, $\Sigma_{kq\in Q^i} P_q^i = 1$.

Observation: A balanced vector of BF strategies is a subgame prefect equilibrium with stage undominated strategies.

From now on we will refer to a balanced vector of BF strategies as BF-equilibrum.

5. Computation of the Simplest Equilibria

Theorem: f is a simplest subgame perfect equilibrium with stage undominated strategies if and only if f is a BF-equilibrium.

Proof: We have already discussed in the previous section that BF-equilibria are subgame prefect equilibria with stage undominated strategies. Since in these equilibria each player uses a strategy of complexity four, we conclude that: at a simplest subgame perfect equilibrium with stage undominated strategies each player must use a strategy of complexity four or less.

So we assume for the rest of this section that f is a fixed simplest subgame perfect equilibrium with stage undominated strategies. Thus, complexity of $f_i \leq 4$ for $i = 1, 2, \ldots, n$. We will show that f must be a BF-equilibrium.

Each automaton for f_i must have at most 4 states with one state in which it rests, one state to make proposals, and one state for voting. If the automaton had two resting states, then its behavior in all other states is fixed and one can combine the two resting states into one to obtain an equivalent 3 state automaton. So we assume without loss of generality that each player has exactly one resting state.

It follows that at the beginning of every stage game (when nature draws a proposer at random), all n players are in their unique resting state. Thus, the induced strategies of all the players are the same at every subgame starting with nature's move. We therefore make the following claims.

<u>Claim A</u>: There exists a fixed vector of "continuation values" $(V_1, V_2, \ldots, V_n)$, with $V_i \geq 0$ and $\sum_{i=1}^{n} V_i \leq 1$, representing the expected payoffs to the n players at the beginning of each stage game (starting with nature's move).

<u>Claim B</u>: In every stage game, if a proposal d is being voted on, then player i votes Y with probability one if $d_i > V_i$, and N with probability one if $d_i < V_i$.

Claim B follows immediately from the fact that the equilibrium strategies are not dominated in every stage game.

<u>Claim C</u>: $V_i > 0$ for $i = 1, 2, \ldots n$.

If $V_i = 0$, then there are players $j_1, j_2, \ldots, j_{s-1}$, all distinct and distinct from i with $V_i + \sum V_{j_r} < 1$. Thus, when player i is selected to propose he can guarantee himself (given the strategies of his opponents) a positive amount by giving each player j_r strictly more than V_{j_r}, giving himself more than zero and have all s players voting Y. Since he has a positive probability of being selected, his expected payoff must be positive.

Claim D: Every proposer proposes to a "least expensive" coalition of size $s - 1$, i.e., to a set of players $j_1, j_2, \ldots, j_{s-1}$ with minimal sum of continuation values (ΣV_{j_r}). In such a coalition he proposes V_{j_r} to each participating member and the rest, $1 - \Sigma V_{j_r}$, to himself.

This follows from Claim B since it imples that he can guarantee himself $1 - \min_{\{j_r\}} \Sigma V_{j_r} - \varepsilon$ for all ε. Thus, since he is best responding he must pay himself at least $1 - \Sigma V_{j_r} - \varepsilon$ for every $\varepsilon > 0$, i.e., at least $1 - \Sigma V_{j_r}$. Since the only way for him to obtain this payoff is by having his proposal accepted, it must be of the type described, and

Claim E: Every stage game ends with an accepted proposal and $\Sigma V_i = 1$.

Claim F: $V_1 = V_2 = \ldots = V_n = 1/n$.

Suppose the above equalities do not hold. Then, without loss of generality, by rearranging the names of the players, we can assume that for some j,

$$V_1 \leq V_2 \leq \ldots \leq V_{j-1} < V_j = V_{j+1} = \ldots = V_n.$$

Case 1. $s < j$.

In this case, player n is never proposed any positive amount by any player other than himself. Since his probability of proposing is $1/n$ and 1 is an upper bound on what he can get when he proposes, $1/n$ is an upper bound on V_n. But this contradicts the fact that he has the largest V_i and not all

V_i's are equal.

Case 2. $s \geq j$.

In this case, every proposer must propose at least V_1 to player one. Moreover, when player one proposes, he receives strictly more than V_1. Thus, we obtain that $V_1 > V_1$, a contradiction. Hence, from Cases 1 and 2, $V_i = 1/n$ for all i.

Since each player's $V_i = 1/n$, he must accept proposals giving him more than 1/n and reject any giving him less than 1/n. Therefore, every player must have two states for voting, one for resting, and one for proposing.

Moreover, he must accept proposals giving him exactly 1/n, since otherwise his V_i would be strictly smaller than 1/n. Also, each player proposes 1/n to all the members of a randomly selected coalition of s - 1 players with the remaining [n - (s - 1)]/n to himself. Thus, each player uses a BF strategy.

Since under the strategies described above all V_i's equal 1/n only if the proposals are balanced, we conclude that the players must be playing a BF-equilibrium.

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