

point R , the root of T . Denote by T_i the tree that consists of all the vertices and arcs of T connected to the vertex $\beta(e_i)$, where $\beta(e_i)$ denotes the initial point of the arc e_i . The tree T_i has length at most L . Denote by σ_i the state of T_i that is the restriction of σ to T_i .

Denote by $C_{(i)}$ the restriction of C to the tree T_i . It is easy to see that the function $C\#_{(i)}$ is a transformation of the states $S(T_i; R^n)$. Therefore, it follows by the inductive hypothesis that for each $j > L$,

$$\{C_{(i)} \cdot I(a)\}^j \sigma_i = \{C_{(i)} \cdot I(a)\}^{j+1} \sigma_i.$$

Let

$$[\{C_{(i)} \cdot I(a)\}^j \sigma](\beta(e_i)) = b_i.$$

Then

$$\begin{aligned} & [\{C \cdot I(a)\}^{L+1} \sigma](R) = \\ & [(\{C \cdot I(a)\} \{C \cdot I(a)\}^L \sigma)](R) = \\ & C\#(R; b_1, \dots, b_{r'}). \end{aligned}$$

Furthermore, for each j larger than $L+1$, the $C_{(j)} \cdot I(a) \sigma$ assigns the value b_i to $\beta(e_i)$ $1 \leq i \leq r'$ and therefore for each j larger than $L+1$, $\{C \cdot I(a)\}^j \sigma$ assigns to R the value

$$C\#(R; b_1, \dots, b_{r'}).$$

Theorem C.1. Let C be a (r, d) -network with a leaf-free input-output digraph G and let t be a positive integer. Suppose that G has a single output

vertex OV . There is an in-tree $T(t)$ with root R and a mapping of digraphs $\theta(t)$ from $T(t)$ to G that satisfy the following conditions:

- (i) Each walk from a leaf of $T(t)$ to R has length t ,
- (ii) for each walk W of length t in G that connects a vertex w of G to the vertex OV , there is a leaf $\Lambda(W)$ of $T(t)$,
- (iii) if W is a walk of length t in G from a vertex w to the vertex OV , then there is a unique directed path $P(W)$ of $T(t)$ of length t that connects $\Lambda(W)$ to R ,
- (iv) $\theta(t)$ carries R to OV and $\theta(t)$ carries $P(W)$ onto W .
- (v) for each vertex v of $T(t)$,
 $\text{in-degree}(v) \leq \text{in-degree}(\theta(t)v)$.
- (vi) for each $t \geq 1$, $T(t-1)$ is a subgraph of $T(t)$ and the restriction of the map $\theta(t)$ to the subgraph $T(t-1)$ coincides with $\theta(t-1)$.

Proof. The proof is an induction on the lengths of the walks t . Suppose $t=1$. Because the digraph G is leaf-free, the vertex OV is the end point of at least one arc. Assume there are arcs e_1, \dots, e_{r^*} that end in OV , where $r^* \leq r$ is the in-degree of OV . Denote the initial vertex of the arc e_i by v_i , $1 \leq i \leq r^*$. The walks

of G of length 1 must each have as an initial point one of the vertices v_i and each vertex v_i is the initial point of exactly one walk W_i , $1 \leq i \leq r^*$, of length 1 that connects v_i to OV . The vertices v_i are not necessarily distinct. Construct a tree $T(1)$ and a map $\theta(1)$ as follows. $T(1)$ has root R , and leaves

$$\Lambda(W_1) = V_1, \dots, \Lambda(W_{r^*}) = V_{r^*}.$$

Each leaf V_i , $1 \leq i \leq r^*$, is connected to R by a single arc E_i that has initial point V_i and end point R . The map $\theta(1)$ carries R to OV , $\theta(1)$ carries V_i to v_i , and $\theta(1)$ carries the arc E_i to the arc e_i . Because the leaves V_i are all distinct, the underlying abstract mixed graph of $T(1)$ is a tree. Furthermore, the number of arcs that end at R is precisely r^* . Therefore

$$\text{in-degree}(R) \leq \text{in-degree}(OV) = \text{in-degree}(\theta(1)R).$$

Each walk W of length 1 in G starts at one of the v_i and exactly one path, $(\Lambda(W), E_i, R)$, is mapped by $\theta(1)$ to the walk W . Because the in-degree of a leaf is 0, this completes the argument for $t=1$.

Assume now that the value of t is L , and that the assertion of the lemma is true for all networks with a single output vertex and for all walks of lengths $t < L$.

Because of the inductive hypothesis, we can construct a directed tree $T(L-1)$ and a map $\theta(L-1)$ from $T(L-1)$ to G such that each walk of length $L-1$ in G that ends at OV is the image under $\theta(L-1)$ of a directed path

of length $L-1$ and such that for each vertex v of $T(L-1)$,

$$\text{in-degree}(v) \leq \text{in-degree}(\theta(L-1)v).$$

Each leaf of $T(L-1)$ is uniquely associated to a walk in G of length $L-1$ that ends in OV . These leaves are mapped by $\theta(L-1)$ to vertices in G .

By inductive hypothesis (v), for each vertex v of $T(L-1)$,

$$\text{in-degree}(v) \leq \text{in-degree}(\theta(L-1)v).$$

If W is a walk in G of length $L-1$ that ends in OV , then there is, by inductive hypothesis (iii), a unique path $P(W)$ in $T(L-1)$ that starts from a leaf $\Lambda(W)$, ends at the vertex R , and (because of assumption (iv)) it is mapped by $\theta(L-1)$ onto W . If $(B_0, A_1, \dots, A_n, B_n)$ is a walk with $n \geq 1$, then B_1 is the second vertex of W .

Suppose now that W_1, \dots, W_z are the walks in G of length L that end at OV . For $1 \leq i \leq z$, W_i is a walk

$(B_{i0}, A_{i1}, B_{i1}, A_{i2}, \dots, A_{iL}, B_{iL})$, where each B_{ij} is a vertex of G and each A_{ij} is an arc of G . For each

$1 \leq i \leq z$, the sequence of vertices and arcs

$(B_{i1}, A_{i2}, \dots, A_{iL}, B_{iL})$ is a walk $W_i\#$ in G of length $L-1$.

(Note that even if W_i and W_j are distinct, the walks $W_i\#$ and $W_j\#$ may not be distinct walks.) Therefore,

there is a unique path $P(W_i\#)$ in $T(L-1)$ that begins with the leaf $\Lambda(W_i\#)$, that ends at R , and is such

that the path $P(W_i\#)$ is mapped by $\theta(L-1)$ onto the walk

$W_i\#$. Construct the tree $T(L)$ and the map $\theta(L)$ as follows. Form a new set of vertices $\Lambda(W_i)$, $1 \leq i \leq z$, that are to be leaves of the tree $T(L)$. For each $1 \leq i \leq z$, connect the vertex $\Lambda(W_i)$ to the vertex $\Lambda(W_i\#)$ of $T(L-1)$ by an arc a_i . Note that only one arc now connects $\Lambda(W_i)$ to a vertex of $T(L-1)$, and that arc has as end point the vertex $\Lambda(W_i\#)$ of $T(L-1)$. We define the map $\theta(L)$ so that it coincides with $\theta(L-1)$ on the vertices and arcs of $T(L-1)$, while $\theta(L)_V(\Lambda(W_i)) = B_{i0}$ and $\theta(L)_A(a_i) = A_{i1}$.

Suppose that W is a walk in $T(L)$ from a leaf of $T(L)$ to R . Such a walk must start at some leaf $\Lambda(W_i)$. Because $L > 1$, the leaf $\Lambda(W_i)$ is not R . Therefore, for the walk W to end at R , the initial arc of W must be the initial arc of W_i . Such an arc has, by the construction carried out, an end point in $T(L-1)$ and that endpoint is a leaf $\Lambda(W^*)$ where W^* is a walk in $T(L-1)$. If $W\#$ is the walk that starts at the second vertex of W and otherwise coincides with the walk W , then the walk $W\#$ begins at a leaf of $T(L-1)$ and ends at R . Therefore, by (i), the walk $W\#$ has length $L-1$. It follows that the walk W has length L , because W has one more vertex and arc than $W\#$. This establishes condition (i). Condition (ii) is clearly satisfied, by construction.

We turn now to the proof of (iii). Suppose that W

is a walk of length L in G from a vertex v in G to the vertex OV . Use the same notation as that in the previous paragraph. Denote by $W\#$ the walk that starts with the second vertex of W and otherwise coincides with W . The walk $W\#$ is a walk of length $L-1$ that is entirely a walk in $T(L-1)$, because the only vertices of $T(L)$ that are not in $T(L-1)$ are leaves. The inductive hypothesis (iii) states that there is a unique directed path $P(W\#)$ in $T(L-1)$ of length $L-1$ that connects $\Lambda(W\#)$ to R , and hypothesis (iv) states that the map $\theta(L-1)$ carries $P(W\#)$ to $W\#$. The path $P(W\#)$ is a path from the leaf $\Lambda(W\#)$ of $T(L-1)$ to R . In $T(L)$ the leaf $\Lambda(W)$ is connected by an arc A to the vertex $\Lambda(W\#)$ in $T(L-1)$. By construction, the map $\theta(L)$ carries A to the initial arc of W . The path $P=(\Lambda(W), A, P(W\#))$ is carried by $\theta(L)$ onto W . To prove (iii) we need to establish that the path P is the unique directed path of $T(t)$ that connects $\Lambda(W)$ to R . If P' is a second such path, since it begins at $\Lambda(W)$ and ends at R , and since $R \neq \Lambda(W)$, it follows that the first arc of P' must be the same as the first arc of P because, by construction, there is only one arc A from $\Lambda(W)$. If the two paths P and P' are different, then the two paths that start from the second vertices of P and P' must be different. However, each of these new paths is a path in $T(L-1)$ that maps from the second vertex of P

to R, and (iii) for $T(L-1)$ asserts that there is only one such path. Therefore (iii) is established for $t=L$. The tree $T(L)$ has been constructed so that $T(L-1)$ is a subgraph of the tree $T(L)$ and the maps $\theta(L)$ and $\theta(L-1)$ coincide on $T(L-1)$, therefore (vi) is satisfied.

Finally, we need to show that (v) is true. We need only establish this for those vertices of $T(L-1)$ that are the leaves (in $T(L-1)$), $\Lambda(W_i\#)$, because they are the only vertices for which the in-degree is changed. For each $1 \leq i \leq z$, the in-degree of $\Lambda(W_i\#)$ is the number of the walks among the $\{W_j\}$ that have as second vertex the initial vertex of $W_i\#$. If $\Lambda(W_a)$ and $\Lambda(W_b)$ are to be connected to the same $\Lambda(W_a\#)$, the initial arcs of W_a and W_b must be different, because they must start from different leaves of $T(L)$. On the other hand, these initial arcs must have the same end point which is the initial point of the walk $W_a\#$. Therefore, the number of arcs that end in $\Lambda(W_a\#)$ is at most the number of arcs that are connected to the initial point of $W_a\#$. However, $\theta(L)(\Lambda(W_a\#))$ is the initial point of $W_a\#$. Therefore,

$$\text{in-degree}(\Lambda(W_a\#)) \leq \text{in-degree}(\theta(L)(\Lambda(W_a\#))). \quad \text{✠}$$

The following corollary of Theorem C.1 allows us to conclude that if an (r,d) -network computes a

function F in time t , then there is an (r,d) -network with input-output graph that is a tree of length t that also computes the function F in time t . The proof is easy and it is not included.

Corollary C.1. Suppose that C is an (r,d) -network with input-output digraph G that computes a function $F(x_1, \dots, x_n)$ in time t . There is an (r,d) -network C' with input-output digraph a directed tree T of length t that also computes F in time t . Furthermore, if v is a vertex of the input-output digraph T , then the function $C'_{\#}(v; \cdot)$ associated to that vertex is either one of the functions associated to a vertex of the input-output digraph of F , or the function is an identity function.

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