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Cournot-Nash Equilibrium Distributions for Games with Differential Information

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Abstract: In this paper we report results on the existence and upper hemicontinuity of Cournot-Nash equilibrium distributions for large anonymous games with differential information.

Keywords: Cournot-Nash equilibrium distribution, compact metric space, expected utility, Borel measure, conditional expectations, measurable selection, sub-σ-algebra, Borel σ-algebra, topologies on information, closed graph, topology of convergence in measure.

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1 Introduction

In recent work, Khan-Rustichini (1989) extend Mas-Colell's (1984) formulation of Cournot-Nash equilibria of "large" anonymous games to a setting with uncertainty and imperfect information. However, they assume that the imperfect information is commonly (publicly) held. In this paper, we show that their results extend to a version of one-shot games in which each agent is allowed to possess information which is particular to him or her.

There are two essential ideas underlying this extension. The first is to make information part of the characteristics of each agent. This is, of course, the motivation for the work of Allen (1983) and Cotter (1986) on the study of topologies on information. The second idea is to make essential and intensive use of the topology of convergence in measure. Since we assume that the set of actions has no linear structure, we are naturally led to this topology on the set of strategies. We also work with this topology on the set of payoffs; in particular, in our treatment of state dependent felicity functions.

The basic conceptual difficulty in our work is that in the presence of differential information each player's maximization problem must be phrased in terms of her information and, in particular, each player's optimal strategy ought to be measurable in terms of this particular information. However, the lack of a linear, and therefore convex, structure on the action set and hence on the space of strategies does not allow us to take conditional expectations of these strategies. We respond to this difficulty by exploiting the fact that the action set, being compact metric, is homeomorphic to a subset of the Hilbert cube. This allows us to work there whenever linear operations are required. We also condition the payoffs with respect to individual information but since these are assumed to be continuous functions on the action set, we can rely on the theory of conditional expectations of Banach space valued random variables.

In contrast to the earlier work of Khan-Rustichini (1989), we work with an uncertainty space whose cardinality is not necessarily countably infinite. This leads to deeper results from a measure-theoretic point of view. However, this generalization is obtained at the cost of restricting a player's payoff to be one generated by expected utility with respect to a common prior. Since our primary motivation is differential information in the context of games with, in principle, a continuum of traders, such a trade-off in terms of formulation seems a reasonable one. However, we leave it to the reader to use the techniques of this paper and extend for herself our results to the set-up of Khan-Rustichini with more general payoffs but with a countably infinite set of states of nature.

In terms of technicalities, our results involve the following considerations. Firstly, we assume compact strategy spaces in the topology of convergence in measure. This is unlike Khan-Rustichini for whom this followed as a consequence on account of their reliance on the topology of pointwise convergence. Secondly, we rely on the observation that Cotter's (1986) pointwise topology on the space of sub-σ-algebras is metrizable when the basic uncertainty space is one for which its corresponding space of random variables with finite mean is norm separable. It may be worth stating in this context that we make essential use of metric structures throughout our work. Thirdly, we make no equicontinuity type assumptions on the payoffs; in particular, on the space from which the felicity functions are chosen. This is in contrast to the work of Milgrom-Weber (1985). Indeed, the fact that our existence theorem could be proved without such a similarity assumption on the space of felicity functions is somewhat surprising and leads to considerable additional complication in the analysis.

Section 2 presents the model and results, Section 3 collects some mathematical preliminaries relating principally to the topology of convergence in measure and Section 4 is devoted to the proofs. It may be worth making the general remark that our proofs, as indeed the formulation of the problem, bring together topology and measure in a way that all the hypotheses of the Fan-Glicksberg 1952 fixed point theorem are satisfied.

2 The Model and Results

We begin with a broad overview of our results. In our formulation of a "large," anonymous game, a player is viewed as a pair consisting of a sub-σ-algebra and a random variable; a game as a probability distribution over this product space of sub-σ-algebras and random variables; and an equilibrium of this game as a suitable probability distribution over the joint space of players and players' strategies. We show the existence of such an equilibrium and furthermore, that the equilibrium correspondence is upper hemicontinuous (upper semicontinuous in the terminology of Berge (1963)) with respect to perturbations in the game. In the remainder of this section, we make these ideas precise.

Let $(\Omega, \mathcal{F}, \mathbf{Pr})$ be an abstract probability space. It formalizes uncertainty. Ω is the space of (countable or uncountable) states of nature, with a particular state of nature being denoted by ω. $\mathcal{F}$ formalizes full information and the probability measure $\mathbf{Pr}$ on $\mathcal{F}$ will do triple duty: it is used to define the metric underlying the topology of convergence in measure as well as that underlying Cotter's topology on sub-σ-algebras; to calculate expected utilities; and to formalize our assumption of uniformly bounded payoffs. We shall make the following assumption on $(\Omega, \mathcal{F}, \mathbf{Pr})$.

Assumption 1 *The Banach space $L_1(\mathbf{Pr}, \mathbf{R})$ is separable.*

It is well known that a sufficient condition for this is that the σ-algebra is countably generated.

Let A be a compact metric space with metric d_A. A is the basic space of actions available to each of the agents. It is assumed to be common to all agents and, in particular, not assumed to be convex.

Let $\mathbf{F}$ be the set of equivalence classes of sub-σ-algebras of $\mathcal{F}$ and any particular sub-σ-algebra $\mathcal{G} \in \mathbf{F}$ denotes incomplete information. We endow the set $\mathbf{F}$ with a topology proposed by Cotter (1986) under which

$$\mathcal{F}^{\nu} \longrightarrow \mathcal{F} \Longleftrightarrow E_{\mathcal{F}^{\nu}} f \longrightarrow E_{\mathcal{F}} f \text{ for all } f \in L_1(\mathbf{Pr}, \mathbb{R}),$$

where $\{\mathcal{F}^{\nu}\}$ is a net chosen from $\mathbf{F}$. A different topology on information has been proposed by Allen (1983) but Cotter shows that his topology of "pointwise convergence" is coarser and hence by working with it, our results also apply to Allen's topology. Cotter shows that his topology is metrizable under Assumption 1.

Let $Meas(\Omega, \mathbf{Pr}; A)$ be the space of measurable functions from $(\Omega, \mathcal{F}, \mathbf{Pr})$ to A and endowed with the topology of convergence in measure. We shall be working throughout with an equivalence class of such functions; this should be particularly kept in mind when we deal with almost-everywhere convergence. Note, for example from Choquet (1969, p. 35), that the metrizability of A implies the metrizability of $Meas(\Omega, \mathbf{Pr}; A)$ with metric given by

$$d(f, g) = inf\{r \in \mathbb{R}_+ : \mathbf{Pr}\{\omega \in \Omega : d_A(f, g) > r\} < r\}.$$

We assume that each player is constrained to choose her strategy (a measurable A-valued function) from a compact subset of this space, to be denoted $\mathcal{A}$. The fact that every player is constrained to $\mathcal{A}$ is common knowledge. Let $A^{\mathcal{F}} \equiv \mathcal{A} \cap Meas(\Omega, \mathbf{Pr}; A)$. Since $\mathcal{A}$ is compact, certainly $A^{\mathcal{F}}$ is compact. However, for emphasis we shall state this as an assumption on $A^{\mathcal{F}}$.

Assumption 2 *The set $A^{\mathcal{F}}$ is compact in the topology of convergence in measure.*

A player also faces another set of constraints derived from the extent of her imperfect information $\mathcal{G}$. When we consider a subset of $\mathcal{G}$-measurable functions, $\mathcal{G} \in \mathcal{F}$, of $A^{\mathcal{F}}$, we shall abbreviate this set to $A^{\mathcal{G}}$. Since $A^{\mathcal{G}}$ is a closed subset of $Meas(\Omega, \mathbf{Pr}; A)$ and hence of $A^{\mathcal{F}}$, it is also compact.

Since we are formulating Cournot-Nash equilibria, we need to specify how the response of any particular agent depends on the actions of the others. Towards this end, we shall assume that each player responds to the distribution over the space of strategies. Let $\mathcal{M}_+^1(A^{\mathcal{F}})$ be the space of Borel probability measures on $A^{\mathcal{F}}$ endowed with the weak*topology. Since $A^{\mathcal{F}}$ is a compact metric space, so is the space $\mathcal{M}_+^1(A^{\mathcal{F}})$.

Next, we turn to the space of payoffs. Let $C(A \times \mathcal{M}_+^1(A^{\mathcal{F}}))$ be the space of continuous functions from $A \times \mathcal{M}_+^1(A^{\mathcal{F}})$ to the real line $\mathbb{R}$ and endowed with the sup-norm topology. This specification takes into account the fact that the individual payoffs depend on actions and externalities. It does not take into account the uncertainty and the extent of imperfect information. Towards this end, we let

$Meas(\Omega, \mathbf{Pr}; C(A \times \mathcal{M}^1_+(A^{\mathcal{F}})))$ be the space of measurable functions from $(\Omega, \mathcal{F}, \mathbf{Pr})$ to $C(A \times \mathcal{M}^1_+(A^{\mathcal{F}}))$ and endowed with the topology of convergence in measure. We shall abbreviate this space to $\mathcal{U}_A$. Here again the metrizability of $C(A \times \mathcal{M}^1_+(A^{\mathcal{F}}))$ implies the metrizability of $\mathcal{U}_A$ with metric given by

$$d(f, g) = inf\{r \in \mathbf{R}_+ : \mathbf{Pr}\{\| f - g \| > r\} < r\}$$

for all f, g in $\mathcal{U}_A$ and with $\| \cdot \|$ denoting the norm on $C(A \times \mathcal{M}^1_+(A^{\mathcal{F}}))$.

All that remains to be discussed is conditioning due to imperfect information $\mathcal{G} \in \mathbf{F}$ which is available to an individual player. We simply assume that each player takes the conditional expectation with respect to $\mathcal{G}$ of u in $\mathcal{U}_A$ and maximizes expected utility of the resulting random variable with respect to the common probability measure $\mathbf{Pr}$. In other words, we assume that a player with payoff u and imperfect information $\mathcal{G}$ chooses as her strategy set $\mathcal{G}$-measurable functions $a \in A^{\mathcal{G}}$ and maximizes, for a given $\rho \in \mathcal{M}^1_+(A^{\mathcal{F}})$ for the other players, the function

$$I : A^{\mathcal{F}} \times \mathcal{U}_A \times \mathbf{F} \times \mathcal{M}^1_+(A^{\mathcal{F}}) \longrightarrow \mathbf{R} \text{ with } I(a, u, \mathcal{G}, \rho) = \int_{\Omega} (E_{\mathcal{G}} u)(\omega)(a(\omega), \rho) d\mathbf{Pr}.$$

A player then is an element $(u, \mathcal{G})$ of the space $(\mathcal{U}_A \times \mathbf{F})$. We shall denote this space by $\mathcal{P}_m$, m for measurable functions. From what has been said so far, certainly the space of players is a topological space.

A game is a distribution on the space of players. This formalizes that we are in a set-up with such a "large" number of agents that their individual identities are of no consequence but only the distribution of these identities. More formally, we can now present our formulation of an anonymous game with uncertainty and imperfect and differential information.

Definition 1 *A game μ with imperfect and differential information is a Borel probability measure on $\mathcal{P}_m$.*

A Cournot-Nash equilibrium of such a game is a distribution on the joint space of strategies and characteristics such that

(i) the marginal of this distribution on players is identical to the given distribution of players;

(ii) it gives full measure to payoff maximizing strategies when the payoffs are conditioned by the marginal of the distribution on strategies and by the extent of imperfect information.

The first requirement simply forces us to restrict attention to the particular game that is given. The second requirement is the essence of Cournot's original idea that one acts on the basis of one's prediction of others' actions and this act leads to the fulfillment of the prediction.

We can now present

Definition 2 *A Borel probability measure τ on $A^{\mathcal{F}} \times \mathcal{P}_m$ is a Cournot-Nash equilibrium distribution of a game μ if (i) $\tau_{\mathcal{P}_m} = \mu$ and (ii) $\tau(B_\tau) = 1$, where subscripts on τ denote marginals and where*

$$B_\tau \equiv \{(a, (u, \mathcal{G})) \in A^{\mathcal{G}} \times \mathcal{P}_m : I(a, u, \mathcal{G}, \tau_{A^{\mathcal{F}}}) \geq I(a', u, \mathcal{G}, \tau_{A^{\mathcal{F}}}) \; \forall a' \in A^{\mathcal{G}}\}.$$

Assumption 3 *A game μ is said to have uniformly bounded payoffs if there exists a real valued Lebesgue integrable function g on $(\Omega, \mathcal{F}, \mathbf{Pr})$ such that for any $u \in supp\, \mu_1$,*

$$\| u(\omega) \| \leq g(\omega) \text{ almost every } \omega \in \Omega.$$

We can now present our first result.

Theorem 1 *For any game satisfying Assumptions 1 to 3, there exists a Cournot-Nash equilibrium.*

Remark 1: Theorem 1 is valid if we imbed each player's strategies in a smaller space derived from a suitable union of the variety of information available in a given game; i.e., in the smallest space of information containing all individual information. In terms of a formal treatment, for any game μ, let $supp_2\, \mu$ be the projection of the support of μ on the second coordinate of $(\mathcal{U}_A \times \mathbf{F})$, namely on $\mathbf{F}$. Let $\mathcal{H}$ be the smallest σ-algebra that contains $\bigcup_{\mathcal{G} \in supp_2 \mu} \mathcal{G}$; certainly it belongs to $\mathbf{F}$. Now modify Definition 2 by substituting $A^{\mathcal{H}}$ in place of $A^{\mathcal{F}}$. The payoffs are defined, in part, on $\mathcal{M}^1_+(A^{\mathcal{F}})$ but a player can be more specific and focus on elements of $\mathcal{M}^1_+(A^{\mathcal{H}})$. Since $\mathcal{M}^1_+(A^{\mathcal{H}}) \subseteq \mathcal{M}^1_+(A^{\mathcal{F}})$, everything is well-defined and Theorem 2 is true with this modification.

Remark 2: Note that the earlier work of Khan (1986) and Khan-Sun (1987) is deterministic in an essential way. In that work, purely topological structures, in particular the property of complete regularity of the set of players, drive the proofs. This is no longer the case in the existence result presented here; Assumptions 1 to 3all involve measure-theoretic structures.

Remark 3: We do not assume any linear structure on the action set A but if such a structure is available, then we have available to us several topologies, other than the topology of convergence in measure, with which we can conduct the analysis. An investigation of these structures may be of interest but is outside our scope here.

We now turn to the behavior of the set of equilibria with respect to changes in the underlying game. Towards this end, we define the correspondence Γ which associates to every game the set of its Cournot-Nash equilibria. Formally, we have

$$\Gamma : \mathcal{M}^1_+(\mathcal{P}_m) \longrightarrow 2^{\mathcal{M}^1_+(A^{\mathcal{F}} \times Meas(\Omega, \mathbf{Pr}; C(A \times \mathcal{M}^1_+(A^{\mathcal{F}}))))}$$

where for any game μ, $\Gamma(\mu)$ is its Cournot-Nash equilibrium. We can now present

Theorem 2 *For any net (μ^ν, τ^ν) tending to (μ, τ) with $\tau^\nu \in \Gamma(\mu^\nu)$, $\tau \in \Gamma(\mu)$. Moreover, if $Meas(\Omega, \mathbf{Pr}; C(A \times \mathcal{M}^1_+(A^{\mathcal{F}})))$ is restricted to be compact, then Γ is upper hemicontinuous.*

3 Mathematical Preliminaries

In this section, we collect some results which constitute essential steps in the proofs of our Theorems 1 and 2. Since these results may have independent interest, we state them in a form that is self-contained and in somewhat more generality that is needed for our purposes.

We begin with a result which shows that convergence of a measure on a product space implies convergence of the marginals. Khan (1986, Theorem 2.5) states this theorem for a measure defined on the product of two completely regular spaces but the following more general statement is true.

Lemma 1 *Let S and T be two regular spaces and $\{\tau^n\}$ be a net chosen from $\mathcal{M}_+^1(S \times T)$ and such that it converges to τ in $\mathcal{M}_+^1(S \times T)$; then τ_i^ν converges to τ_i, for $i = S, T$.*

Our next result is a alternative characterization of Cotter's topology of pointwise convergence.

Lemma 2 *For any Banach space,*

$$\mathcal{G}^\nu \longrightarrow \mathcal{G} \Longleftrightarrow E_{\mathcal{G}^\nu} f \longrightarrow E_{\mathcal{G}} f \text{ for all } f \in L_1(\mathbf{Pr}, X),$$

where $\{\mathcal{G}^\nu\}$ is a net chosen from $\mathbf{F}$.

Our next six lemmata concern the properties of the topology of convergence in measure. Note that when we consider a product of some or all of these spaces we shall endow this product with the product topology.

Lemma 3 *Let $\{u^n, \mathcal{G}^n\}$ be a sequence chosen from $\mathcal{U}_A \times \mathbf{F}$ and converging to $(u, \mathcal{G})$. Then there exists a subsequence $\{u^k, \mathcal{G}^k\}$ of $\{u^n, \mathcal{G}^n\}$ such that $E_{\mathcal{G}^k} u^k$ converges pointwise $\mathbf{Pr}$-almost everywhere, and hence in measure, to $E_{\mathcal{G}} u$.*

Our next lemma is a far-reaching generalization of Exercise 4.1 in Chung (1968, Chapter 10). Y is a separable metric space with metric d_Y; $C(A; Y)$ is the space of continuous functions from A to Y and endowed with the compact-open topology; and $Meas(\Omega, \mathbf{Pr}; C(A; Y))$ has the obvious meaning.

Lemma 4 *Let $\{a^n\}$ be a sequence chosen from $Meas(\Omega, \mathbf{Pr}; A)$ and converging to a and u an element of $Meas(\Omega, \mathbf{Pr}; C(A; Y))$. Then the function $f^n : \Omega \longrightarrow Y$ with $f^n(\omega) = u(\omega, a^n(\omega))$ is a measurable function for each n and the sequence of functions $\{f^n\}$ converges in measure to f.*

Lemma 5 *Let $\{a^n, u^n, \rho^n\}$ be a sequence chosen from $A^{\mathcal{F}} \times \mathcal{U}_A \times \mathcal{M}_+^1(A^{\mathcal{F}})$ and converging to (a, u, ρ). Then the function $f^n : \Omega \longrightarrow \mathbf{R}$ with $f^n(\omega) = u^n(\omega)(a^n(\omega), \rho^n)$ is a real random variable for each n and the sequence of random variables $\{f^n\}$ converges in measure to f.*

Lemma 6 *For any* $(u, \mathcal{G}, \rho)$ *in* $\mathcal{U}_A \times \mathbf{F} \times \mathcal{M}^1_+(A^{\mathcal{F}})$,

$$I : A^{\mathcal{F}} \times \mathcal{U}_A \times \mathbf{F} \times \mathcal{M}^1_+(A^{\mathcal{F}}) \longrightarrow \mathbf{R} \text{ with } I(a, u, \mathcal{G}, \rho) = \int_\Omega (E_{\mathcal{G}}u)(\omega)(a(\omega), \rho)d\mathbf{Pr}$$

is an upper semicontinuous function on $A^{\mathcal{F}}$.

Lemma 7 *For any sequence* $\{a^n, u^n, \mathcal{G}^n, \rho^n\}$ *chosen from* $A^{\mathcal{F}} \times \mathcal{U}_A \times \mathbf{F} \times \mathcal{M}^1_+(A^{\mathcal{F}})$ *and converging to* $(a, u, \mathcal{G}, \rho)$, *there exists a subsequence* $\{a^k, u^k, \mathcal{G}^k, \rho^k\}$ *such that*

$$I(a^k, u^k, \mathcal{G}^k, \rho^k) = \int_\Omega (E_{\mathcal{G}^k}u^k)(\omega)(a^k(\omega), \rho^k)d\mathbf{Pr} \longrightarrow I(a, u, \mathcal{G}, \rho) = \int_\Omega (E_{\mathcal{G}}u)(\omega)(a(\omega), \rho)d\mathbf{Pr}.$$

Lemma 8 $A^{\mathcal{G}}$ *is a lower-hemicontinuous correspondence of* $\mathcal{G}$.

We shall also need a result that shows that pointwise convergence of functions measurable with respect to differing σ-algebras implies measurability of the limit function with respect to the limit σ-algebra.

Lemma 9 *Let* $\{a_n, \mathcal{G}^n\}_{n=1}$ *be a sequence chosen from* $A^{\mathcal{F}} \times \mathbf{F}$ *with* $a_n \in A^{\mathcal{G}^n}$ *and with* a_n *converging pointwise* $\mathbf{Pr}$*-almost everywhere to a* $\mathcal{G}^n$ *converging to* $\mathcal{G}$ *in Cotter's topology. Then a is* $\mathcal{G}-$*measurable.*

Finally, we shall need the following theorem in the proof of upper hemicontinuity of the equilibrium correspondence.

Lemma 10 *Let* Π *and* Ξ *be two topological spaces and* $\mathcal{T} : \Pi \longrightarrow 2^{\Xi}$ *and* $Q(\pi) : \mathcal{T}(\pi) \longrightarrow 2^{\mathcal{T}(\pi)}$ *be two correspondences. For each* π, *let* $\Gamma(\pi)$ *be the set of fixed points of* $Q(\pi)$, *i.e.,*

$$\Gamma(\pi) = \{\tau \in \Xi : \tau \in Q(\pi)(\tau)\}.$$

If the correspondences $\mathcal{T}$ *and* Q *have closed graphs, the latter in the space* $(\Pi \times \Xi \times \Xi)$, *then the correspondence* Γ *has a closed graph.*

4 Proofs

The proofs of Theorems 1 and 2 are modelled after those in Khan-Rustichini (1989), but unlike them, our uncertainty space Ω is not countably infinite. This necessitates the use of the topology of convergence in measure and of the measurable selection theorem in the proof of Theorem 1.

4.1 Proofs of Lemmata

Proof of Lemma 1: The proof relies on Lemma 5.1 in Hoffman-Jorgenson (1970) which is stated for completely regular spaces but is easily seen to hold for regular spaces. ∎

Proof of Lemma 2: ($\Longleftarrow$). For any $a \in X, a \neq 0$, consider the map $T_a : L_1(\mathbf{Pr},\mathbb{R}) \longrightarrow L_1(\mathbf{Pr}, X)$ such that for any $f \in L_1(\mathbf{Pr},\mathbb{R})$, and any $\omega \in \Omega$, $T_a f(\omega) = af(\omega)$. It is easy to check that T_a is a well-defined, continuous, linear operator.

For any $f \in L_1(\mathbf{Pr},\mathbb{R})$, we want to show that $E_{\mathcal{G}^n} f \overset{norm}{\longrightarrow} E_{\mathcal{G}} f$. Under our hypothesis, we know that $E_{\mathcal{G}^n}(T_a f) \overset{norm}{\longrightarrow} E_{\mathcal{G}}(T_a f)$. Since T_a is a closed, linear operator, we can apply Diestel-Uhl (1977; Theorem 6, page 47) to assert that $E_{\mathcal{G}^n}(T_a f) = T_a(E_{\mathcal{G}^n} f)$. But now, on writing out the norms and on noting the linearity of the operator T_a, we obtain

$$\begin{aligned} \| E_{\mathcal{G}^n}(T_a f) - E_{\mathcal{G}}(T_a f) \| &= \| T_a(E_{\mathcal{G}^n} f - E_{\mathcal{G}} f) \| \\ &= \int_\Omega \| T_a(E_{\mathcal{G}^n} f - E_{\mathcal{G}} f) \| (\omega) d\mathbf{Pr} \\ &= \int_\Omega \| a \| \, |(E_{\mathcal{G}^n} f - E_{\mathcal{G}} f)(\omega)| \, d\mathbf{Pr} \\ &= \| a \| \int_\Omega |(E_{\mathcal{G}^n} f - E_{\mathcal{G}} f)(\omega)| \, d\mathbf{Pr} \\ &= \| a \| \, \| E_{\mathcal{G}^n} f - E_{\mathcal{G}} f \| . \end{aligned}$$

Since the left hand side converges to zero, the proof is complete.

($\Longrightarrow$) Suppose $\mathcal{F}^\nu \longrightarrow \mathcal{G}$. We have to show that

$$f \in L_1(\mathbf{Pr}, X) \implies E_{\mathcal{G}^n} f \overset{norm}{\longrightarrow} E_{\mathcal{G}} f.$$

We first consider the case when f is a simple function. In this case, there exists an integer k, and $(x_i, A_i) \in (X, \mathcal{G})$, $i = 1, \cdots, k$, such that $f = \sum_{i=1}^k x_i \chi_{A_i}$, χ_{A_i} the characteristic function of the set A_i. But now it is easy to see that

$$\begin{aligned} \| E_{\mathcal{G}^n} f - E_{\mathcal{G}} f \| &= \| E_{\mathcal{G}^n}(\sum_{i=1}^k x_i \chi_{A_i}) - E_{\mathcal{G}}(\sum_{i=1}^k x_i \chi_{A_i}) \| \\ &= \| \sum_{i=1}^k x_i (E_{\mathcal{G}^n} \chi_{A_i} - E_{\mathcal{G}} \chi_{A_i}) \| \\ &\leq \sum_{i=1}^k \| x_i \| \, \| E_{\mathcal{G}^n} \chi_{A_i} - E_{\mathcal{G}} \chi_{A_i} \| . \end{aligned}$$

Since under our hypothesis, the last term converges to zero, we are done.

Next, we turn to the general case when f is not a simple function. Pick any arbitrary $\epsilon > 0$. Since $f \in L_1(\mathbf{Pr}, X)$, there exists a simple function f_ϵ such that $\| f - f_\epsilon \| < \epsilon/3$; see, for example Diestel-Uhl

(1977; page 44). From the argument above, we can find a n_0 such that

$$\| E_{\mathcal{G}^n} f_\epsilon - E_{\mathcal{G}} f_\epsilon \| < \epsilon/3 \text{ for all } n \geq n_0.$$

Using the fact that the conditional expectation operator is a contraction (see, for example, Diestel-Uhl (1977; Lemma 3, page 122)), we can now write

$$\begin{aligned} \| E_{\mathcal{G}^n} f - E_{\mathcal{G}} f \| &= \| E_{\mathcal{G}^n} f - E_{\mathcal{G}^n} f_\epsilon + E_{\mathcal{G}^n} f_\epsilon - E_{\mathcal{G}} f_\epsilon + E_{\mathcal{G}} f_\epsilon - E_{\mathcal{G}} f \| \\ &= \| E_{\mathcal{G}^n} \| \| f - f_\epsilon \| + \| E_{\mathcal{G}^n} f - E_{\mathcal{G}} f \| + \| E_{\mathcal{G}} \| \| f - f_\epsilon \| \\ &\leq 2 \| f - f_\epsilon \| + \| E_{\mathcal{G}^n} f_\epsilon - E_{\mathcal{G}} f_\epsilon \| \leq \epsilon. \end{aligned}$$

Since ϵ was chosen arbitrarily, we are done. ∎

Proof of Lemma 3: We can now assert that $E_{\mathcal{G}^\nu} u^n \overset{norm}{\longrightarrow} E_{\mathcal{G}} u$. To see this, note

$$\begin{aligned} \| E_{\mathcal{G}^\nu} u^n - E_{\mathcal{G}} u \| &= \| E_{\mathcal{G}^\nu} u^n - E_{\mathcal{G}^\nu} u + E_{\mathcal{G}^\nu} u - E_{\mathcal{G}} u \| \\ &\leq \| E_{\mathcal{G}^\nu} \| \| u^n - u \| + \| E_{\mathcal{G}^\nu} u - E_{\mathcal{G}} u \| \\ &\leq \| u^n - u \| + \| E_{\mathcal{G}^\nu} u - E_{\mathcal{G}} u \| . \end{aligned}$$

The last line follows from the fact that the conditional expectation operator is a contraction (see, for example, Diestel-Uhl (1977; Lemma 3, page 122)). But now the proof of our assertion can be completed by using Proposition 1. ∎

Proof of Lemma 4: We first show that for any n, f^n is a $\mathcal{F} - \mathcal{B}(Y)$ measurable function. Towards this end, let

$ev : (A \times C(A;Y)) \longrightarrow Y$ with $ev(a, u) = u(a)$

$\Psi^n = (a^n, u) : \Omega \longrightarrow (A \times C(A;Y)$ with $\Psi^n(\omega) = (a^n(\omega), u(\omega))$.

It is clear that $f^n = ev \circ \Psi^n$. Certainly, Ψ^n is measurable by hypothesis. Furthermore, since the evaluation map ev is jointly continuous when $C(A;Y)$ is endowed with the compact- open topology (see, for example, Dugundji (1966; Chapter XII, Theorem 2.4, p. 60)), we have proved our claim.

Next, we show that f^n converges to f in measure. Towards this end, we shall work with a function $\Delta : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ such that for each $\omega \in \Omega$ and $\epsilon > 0$

$$\Delta(\omega, \epsilon) = sup\{\delta : d_A(\xi, a(\omega)) < \delta \Longrightarrow d_Y(u(\omega, \xi), u(\omega, a(\omega)) < \epsilon\}.$$

The fact that $\Delta(\omega, \epsilon) > 0$ follows from the continuity of u and of d_Y. The fact that it is bounded follows from the compactness of A. We shall also show that for every $\epsilon > 0$, $\omega \longrightarrow \Delta(\omega, \epsilon)$ is a measurable function.

Let $\{x_n\}$ be a dense subset of A and define for each integer n, the function $\hat{x}_n : \Omega \longrightarrow A$ as

$$\hat{x}_n(\omega) = \begin{cases} x_n & \text{if } d_A(x_n, a(\omega)) < \delta \\ a(\omega) & \text{if } d_A(x_n, a(\omega)) \geq \delta \end{cases}$$

Since a is measurable and d_A is continuous, $\hat{x}_n$ is a measurable function. This allows us to deduce that for any integer n, the function $\omega \longrightarrow d_Y(u(\omega, \hat{x}_n(\omega)), u(\omega, a(\omega)))$ is a measurable function of ω.

Next, consider for any positive δ and any $\omega \in \Omega$ the following set

$$G(\omega, \delta) \equiv sup\{d_Y(u(\omega, \xi), u(\omega, a(\omega)) : d_A(\xi, a(\omega) < \delta\}.$$

It is now easy to check that

$$G(\omega, \delta) = sup_n d_Y(u(\omega, \hat{x}_n(\omega)), u(\omega, a(\omega)))$$

and therefore for any fixed δ, $G(\cdot, \delta)$ is measurable.

Let $\{\delta_n\}$ be the sequence of rationals in $(0, 1)$. For each integer n, let

$$\hat{\delta}_n(\omega) = \begin{cases} \delta_n & \text{if } G(\omega, \delta_n) < \epsilon \\ 0 & \text{if } G(\omega, \delta_n) \geq \epsilon \end{cases}$$

Since $G(\cdot, \delta_n)$ is a measurable function, so is $\hat{\delta}_n(\cdot)$. It is now easy to check that

$$\Delta(\cdot, \epsilon) = sup_n\{\hat{\delta}_n(\cdot)\}.$$

But this allows us to assert that $\Delta(\cdot, \epsilon)$ is a measurable function.

Now we shall show that for any positive real numbers ϵ, η, there exists an integer n_o such that

$$\mathbf{Pr}\{\omega \in \Omega : d_Y(f^n(\omega), f(\omega)) \geq \epsilon\} < \eta \text{ for all } n \geq n_o.$$

Note that for any integer i,

$$\begin{aligned} &\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) < \Delta(\omega, \epsilon)\} \\ &\quad \supseteq \{\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) < 1/i\} \bigcap \{\omega \in \Omega : \Delta(\omega, \epsilon) > 1/i\}\} \end{aligned}$$

and hence

$$\begin{aligned} &\mathbf{Pr}\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) \geq \Delta(\omega, \epsilon)\} \\ &\quad \leq \mathbf{Pr}\{(\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) < 1/i\} \bigcap \{\omega \in \Omega : \Delta(\omega, \epsilon) > 1/i\})^c\} \\ &\quad \leq \mathbf{Pr}\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) \geq 1/i\} + \mathbf{Pr}\{\omega \in \Omega : \Delta(\omega, \epsilon) \leq 1/i\}. \end{aligned} \tag{1}$$

Let $\Omega_i = \{\omega \in \Omega : \Gamma(\omega, \epsilon) \in]1/(i+1], 1/i)\}$. Since $\Gamma(\cdot, \epsilon)$ is measurable function, Ω_i is a measurable set for each integer i. Furthermore, $\Omega = \bigcup_i \Omega_i$. Since $(\Omega, \mathcal{F}, \mathbf{Pr})$ is a probability space, we can find an integer i_o such that

$$\mathbf{Pr}(\bigcup_{i=i_o}^{\infty} \Omega_i) < \eta/3. \tag{2}$$

Since a^n converges in measure to a, we can find an integer n_o such that

$$\mathbf{Pr}\{\omega \in \Omega : d_A(a^n(\omega), a(\omega)) \geq 1/i_o\} < \eta/3 \text{ for all } n \geq n_o. \tag{3}$$

On substituting (2) and (3) in (1) and given the definition of Γ, we have a proof of our claim. ∎

Proof of Lemma 5: To show that for any n, f^n is a measurable real valued function, simply note that the sup-norm topology coincides with the compact-open topology and follow the proof of the first claim of Lemma 4.

Next, we show that f^n converges to f in measure. Observe that we can ignore the dependence of $u^n(\omega)$ on ρ^ω without any loss of generality. Choose an arbitrary $\epsilon > 0$ and consider for any integer n the set

$$\begin{aligned}
&\{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\} \\
&\quad = \{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a^n(\omega)) + u(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\} \\
&\quad \subseteq \{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a^n(\omega)) \mid + \mid u(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\} \\
&\quad \subseteq \{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a^n(\omega)) \mid > \epsilon\} \bigcup \{\omega \in \Omega :\mid u(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\}.
\end{aligned}$$

By the definition of the topology of convergence in measure, there exists an integer n_1 such that for all $n \geq n_1$,

$$\mathbf{Pr}\{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a^n(\omega)) \mid > \epsilon\} < \epsilon/2.$$

Since a^n converges to a in the topology of convergence in measure, we can appeal to Lemma 4 and assert the existence of an integer n_2 such that for all $n \geq n_2$,

$$\mathbf{Pr}\{\omega \in \Omega :\mid u(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\} < \epsilon/2.$$

On putting these facts together, we have shown the existence of an integer $\bar{n} = Max(n_1, n_2)$ such that for all $n \geq \bar{n}$,

$$\mathbf{Pr}\{\omega \in \Omega :\mid u^n(\omega)(a^n(\omega)) - u(\omega)(a(\omega)) \mid > \epsilon\} < \epsilon.$$

Since ϵ was chosen arbitrarily, we are done. ∎

Proof of Lemma 6: We have to show that $H \equiv \{(a, \alpha) \in A^{\mathcal{F}} \times \mathbf{R} : I(a, u.\mathcal{G}, \rho) \geq \alpha\}$ is a closed set. Towards this end, pick any sequence $\{a^n, \alpha^n\}$ from H and converging to (a, α). We have to show that $(a, \alpha) \in H$. Since a^n converges to a in the topology of convergence in measure, we can extract a subsequence $\{a^k\}$ of $\{a^n\}$ which converges pointwise $\mathbf{Pr}$-almost everywhere to a. Since $u \in \mathcal{U}_A$, $u(\omega)(a^k(\omega, \rho))$ converges to $u(\omega)(a(\omega, \rho))$ for $\mathbf{Pr}$-almost every $\omega \in \Omega$. On appealing to the uniform integrability hypothesis in Assumption 3, we can invoke the dominated convergence theorem to assert that

$$I(a^k, u, \mathcal{G}, \rho) = \int_\Omega (E_\mathcal{G} u)(\omega)(a^k(\omega), \rho) d\mathbf{Pr} \longrightarrow I(a, u, \mathcal{G}, \rho) = \int_\Omega (E_\mathcal{G} u)(\omega)(a(\omega), \rho) d\mathbf{Pr}.$$

The proof is complete. ∎

Proof of Lemma 7: This is now a straightforward consequence of Lemmata 2, 3, 4 and 5. ∎

Proof of Lemma 8: Let $\{\mathcal{G}^n\}$ be a sequence chosen from $\mathbf{F}$ converging to $\mathcal{G}$ and $a \in A^{\mathcal{G}}$. We have to manufacture a sequence $\{a_n\}$, $a_n \in A^{\mathcal{G}^n}$ which converges in measure to a.

Since A is a compact metric space, we can appeal to Dugundji (1966; p.195; Cor.3.2) to assert the existence of a subset $\tilde{A}$ of the Hilbert cube I_∞ and a homeomorphism $h : A \longrightarrow \tilde{A}$. Let $\{x_n\}$ be a countable dense subset of $\tilde{A}$, $\|\cdot\|_{\ell^2}$ the I_∞ norm and $d_{\ell^2}(\cdot,\cdot)$ the metric based on this norm.

Certainly, $\omega \longrightarrow h \circ a(\omega)$ is a $\mathcal{G}$-measurable function taking values in I_∞. We can now appeal to Diestel-Uhl (1977; Chapter V.1) to assert the existence of $\omega \longrightarrow E_{\mathcal{G}^n}(h \circ a)(\omega)$. Since we are not guaranteed that the values of this function lie in $\tilde{A}$, consider the real valued function

$$\omega \longrightarrow d_{\ell^2}(E_{\mathcal{G}^n}(h \circ a)(\omega), \tilde{A}) \equiv inf_{a \in \tilde{A}} \| E_{\mathcal{G}^n}(h \circ a)(\omega) - a \| = inf_{x \in \{x_k\}} \| E_{\mathcal{G}^n}(h \circ a)(\omega) - x \| .$$

This function is $\mathcal{G}^n$-measurable for every integer n because infima of a countable set of measurable functions are measurable. Furthermore, since $h \circ a$ takes values in $\tilde{A}$,

$$d_{\ell^2}(E_{\mathcal{G}^n}(h \circ a)(\omega), \tilde{A}) \leq \| E_{\mathcal{G}^n}(h \circ a)(\omega) - h \circ a(\omega) \|_{\ell^2} \;\; \mathbf{Pr} \text{ a.e. } \omega.$$

Now define $\tilde{a}_n : \Omega \longrightarrow \tilde{A}$ as

$$\tilde{a}_n(\omega) = \{x_k \in \tilde{A} : x_k \in \Lambda(\omega),\; x_j \notin \Lambda(\omega) \text{ for } j < k\},$$

$$\Lambda(\omega) \equiv \{x \in \{x_k\} : \| E_{\mathcal{G}^n}(h \circ a)(\omega) - x \|_{\ell^2} \leq d_{\ell^2}(E_{\mathcal{G}^n}(h \circ a)(\omega), \tilde{A}) + 1/n\}.$$

On appealing to Castaing-Valadier (1977; Theorem III.14), we can assert that $\Lambda(\cdot)$ is a correspondence with a $\mathcal{G}^n \otimes \mathcal{B}(\tilde{A})$ measurable graph. This implies that $\tilde{a}$ is also $\mathcal{G}^n$-measurable. To see this, observe that $\tilde{a}_n = \sum_k x_k \chi_{\Omega^k}$ where $\Omega^k = \Omega \backslash (\bigcup_{j=1}^{k-1} \Omega^j)$, $k \geq 2$, and $\Omega^1 = \{\omega \in \Omega : x_1 \in \Lambda(\omega)\}$. Now for $\mathbf{Pr}$-almost every ω

$$\begin{aligned} \| \tilde{a}_n(\omega) - h \circ a(\omega) \|_{\ell^2} &\leq \| \tilde{a}_n(\omega) - E_{\mathcal{G}^n}(h \circ a)(\omega) \|_{\ell^2} + \| E_{\mathcal{G}^n}(h \circ a)(\omega) - h \circ a(\omega) \|_{\ell^2} \\ &\leq d_{\ell^2}(E_{\mathcal{G}^n}(h \circ a)(\omega), \tilde{A}) + 1/n + \| E_{\mathcal{G}^n}(h \circ a)(\omega) - h \circ a(\omega) \|_{\ell^2} \\ &\leq 1/n + 2 \| E_{\mathcal{G}^n}(h \circ a)(\omega) - h \circ a(\omega) \|_{\ell^2} . \end{aligned}$$

Since $\mathcal{G}^n$ converges to $\mathcal{G}$, $E_{\mathcal{G}^n} f \longrightarrow E_{\mathcal{G}} f$ in $L_1(\mathbf{Pr}, \ell^2)$, and hence we obtain for $\mathbf{Pr}$-almost every ω, $E_{\mathcal{G}^n} f(\omega) \longrightarrow E_{\mathcal{G}} f(\omega)$ in $\|\cdot\|_{\ell^2}$ norm. Thus we can conclude that

$$\| \tilde{a}_n(\omega) - h \circ a(\omega) \|_{\ell^2} \longrightarrow 0 \text{ for } \mathbf{P}\text{-almost every } \omega.$$

Certainly, $h^{-1}\tilde{a}_n$ is a $\mathcal{G}^n$-measurable function taking values in A and converging pointwise, and hence by Schwartz (1973; Proposition 1, p. 248) in measure, to a. ∎

Proof of Lemma 9: Consider an arbitrary open subset U of A. We want to show that $a^{-1} \equiv E \in \mathcal{G}$. On recalling that A is a metric space, we can write U as $U = \bigcup_{m=1}^{\infty} F_m$ with F_m closed and $F_m \subseteq F_{m+1}$ for every integer m. Then $E = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (a^n)^{-1}(F_m)$. From this it follows that $E \in \mathcal{G}^n$ for every n. Equivalently, that χ_E is a $\mathcal{G}^n$-measurable function for every n. Now from the assumption that $\mathcal{G}^n \longrightarrow \mathcal{G}$ in Cotter's topology, we have that $\chi_E = E_{\mathcal{G}^n}\chi_E \longrightarrow E_{\mathcal{G}}\chi_E$ in $L_1(\mathbf{Pr}, \mathbb{R})$. This leads us to conclude that $E_{\mathcal{G}}\chi_E = \chi_E$, i.e., $E \in \mathcal{G}$ as claimed. ∎

Proof of Lemma 10: Straightforward. A net $\{\pi^\nu, \tau^\nu\}$ in the graph of Γ and converging to (π, τ) will satisfy $\tau^\nu \in \mathcal{T}(\pi^\nu)$ and $\tau^\nu \in Q(\pi^\nu)(\tau^\nu)$, and therefore $\tau \in \mathcal{T}(\pi)$ and $\tau \in Q(\pi)(\tau)$. By the definition of Γ, $\tau \in \Gamma(\pi)$. ∎

4.2 Proof of Theorem 1

Let B_τ be the same as in Definition 2. Furthermore, let

$\mathcal{T} = \{\tau \in \mathcal{M}_+^1(A^{\mathcal{F}} \times \mathcal{P}_m) : \tau_{\mathcal{P}_m} = \mu\}$;

$Q : \mathcal{T} \longrightarrow 2^{\mathcal{T}}$ such that $Q(\tau) = \{\rho \in \mathcal{T} : \rho(B_\tau) = 1\}$.

The proof consists of three main steps: $\mathcal{T}$ is nonempty, convex and compact; the correspondence $\tau \longrightarrow B_\tau$ is nonempty valued and has a closed graph; the correspondence Q is nonempty and convex valued and has a closed graph. The proof is then completed by an application of the Fan (1952) Glicksberg (1952) fixed point theorem to the correspondence Q defined above. We shall use Glicksberg's version of the fixed point theorem. We now turn to the proofs of the three steps.

Step 1: That $\mathcal{T}$ is nonempty follows from Schwartz (1973, Theorem 17, p. 63) as described in Khan (1986). Also see Khan (1986) for the proof that $\mathcal{T}$ is compact. $\mathcal{T}$ is clearly convex.

Step 2: We begin with the claim that for any $\tau \in \mathcal{T}$, $B_\tau \neq \emptyset$. For any τ, and any pair $(u, \mathcal{G})$, we need to show the existence of an $\hat{a} \in A^{\mathcal{F}}$ which maximizes the function $I(\cdot, u, \mathcal{G}, \tau_A)$. This follows from Assumption 2 on the compactness of $A^{\mathcal{F}}$ and from the Lemma 6 on the upper semicontinuity of I on $A^{\mathcal{F}}$.

Next we turn to the fact that the correspondence $\tau \longrightarrow B_\tau$ has a closed graph. But this follows directly from Berge's (1963) Maximum theorem once we have Lemmata 1, 7, 8 and 9.

Step 3: We first prove that for any $\tau \in \mathcal{T}$, $Q(\tau) \neq \emptyset$. Towards this end, consider the correspondence

$$\Phi(u, \mathcal{G}) \equiv \{a \in A^{\mathcal{F}} : (a, (u, \mathcal{G})) \in B_\tau\}.$$

From the upper semicontinuity of $I(\cdot, u, \mathcal{G}, \tau_{A^{\mathcal{F}}})$ and the compactness of $A^{\mathcal{F}}$ $\Phi(u, \mathcal{G}) \neq \emptyset$ for every pair $(u, \mathcal{G})$. Since B_τ is a closed subset of $A^{\mathcal{F}} \times \mathcal{P}_m$, Φ has a measurable graph. We can now apply Aumann's measurable selection theorem (see Castaing-Valadier (1977, Theorem 3.22)), to obtain a measurable selection $h : \mathcal{P}_m \longrightarrow A^{\mathcal{F}}$. Now define $f : \mathcal{P}_m \longrightarrow A^{\mathcal{F}} \times \mathcal{P}_m$ by $f((u, \mathcal{G})) = (h(u, \mathcal{G}), (u, \mathcal{G}))$. On letting $\rho = f\mu$, we complete the demonstration by checking that $\rho \in \mathcal{T}$, $\rho(B_\tau) = 1$, and $\rho_{\mathcal{P}_m} = \mu$.

The second assertion that Q has a closed graph follows as in the proof of Claim 8 in the Proof of Theorem 3.1 in Khan (1986). ∎

4.3 Proof of Theorem 2

We first prove that the equilibrium correspondence Γ has a closed graph. We define the two correspondences:

$\mathcal{T} : \mathcal{M}^1_+(\mathcal{P}_m) \longrightarrow \mathcal{M}^1_+(A^{\mathcal{F}} \times \mathcal{P}_m)$ with $\mathcal{T}(\pi) = \{\tau \in \mathcal{M}^1_+(A^{\mathcal{F}} \times \mathcal{P}_m) : \tau_{\mathcal{P}_m} = \pi\}, \pi \in \mathcal{M}^1_+(\mathcal{P}_m)$,

$Q(\pi) : \mathcal{T}(\pi) \longrightarrow 2^{\mathcal{T}(\pi)}$ with $Q(\pi)(\tau) = \{\rho \in \mathcal{T}(\pi) : \rho(B_\tau) = 1\}$.

Note that these correspondences differ from the constant correspondences $\mathcal{T}$ and Q previously defined only in their explicit dependence on the measure π which is now a changing parameter. Thanks to Lemma 8, we have only to prove that $\mathcal{T}$ and Q have closed graphs.

We first consider the correspondence Q. Consider a net $\{\pi^\nu, \tau^\nu; \rho^\nu\}$ in the graph of Q and which tends to $(\pi, \tau; \rho)$. By definition of Q and the fact that $\rho^\nu \in Q(\pi^\nu)(\tau^\nu)$, we obtain that $\rho^\nu(B_{\tau^\nu}) = 1$ for every ν. From the proof of Theorem 1, we can derive $limsup_\nu B_{\tau^\nu} \subseteq B_\tau$. Now, as in Lemma 2 in the proof of Theorem 3.1 in Khan (1986), we obtain $\rho(limsup_\nu B_{\tau^\nu}) \geq \rho^\nu(B_{\tau^\nu})$. From the monotonicity properties of the measure ρ, $\rho(limsup_\nu B_{\tau^\nu}) \leq \rho(B_\tau)$, and hence $\rho(B_\tau) = 1$. This proves that $\rho \in Q(\pi)(\tau)$ and our claim.

We next turn to the correspondence $\mathcal{T}$. Let $\{\pi^\nu, \tau^\nu\}$ be a net in the graph of $\mathcal{T}$ and let it converge to (π, τ). Consider any π continuity subset V; see Topsoe (1970, p.40) for a definition. Recall that both $A^{\mathcal{F}}$ and $\mathcal{P}_m$ are metric spaces and hence completely regular. By convergence of $\{\pi^\nu, \tau^\nu\}$, $\pi^\nu(V) \longrightarrow \pi(V)$, and hence

$$\tau^\nu(A^{\mathcal{F}} \times V) \equiv \tau^\nu_{\mathcal{P}_m}(V) \longrightarrow \tau(A^{\mathcal{F}} \times V) \equiv \tau_{\mathcal{P}_m}(V).$$

Hence, we obtain that $\tau_{\mathcal{P}_m}(V) = \pi(V)$. On applying Topsoe (1970, Theorem 8.1), we conclude $\pi = \tau_{\mathcal{P}_m}$, and that therefore $\mathcal{T}$ has a closed graph.

The proof of the theorem is complete. ∎

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