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GLOBAL ASYMPTOTIC STABILITY OF OPTIMAL CONTROL SYSTEMS WITH APPLICATIONS TO THE THEORY OF ECONOMIC GROWTH*

by

William A. Brock and José A. Scheinkman†

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† Department of Economics, University of Chicago

"I wish that I could do something useful
Like planting a tree
on the bottom of the sea.
But I am just a guitar player."

Bob Dylan

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The qualitative study of optimal economic growth has attracted the attention of economic theorists for some number of years. One major focus of this research has been to find sufficient conditions on models of economic growth for the convergence of growth paths to a steady state. Two kinds of models have been studied: (1) descriptive growth models and (2) optimal growth models. The important entities common to (1) and (2) are (a) a vector of capital stocks, (b) a vector of prices of the capital stocks, (c) a set of differential equations that govern the rate of change of capital stocks and their prices in response to the current capital stock and price level.

The major differences between descriptive models and optimal growth models is that for the descriptive case conditions are imposed directly on the capital-price differential equations in order to obtain qualitative results, whereas for optimal growth models an optimizing process determines the capital-price differential equations. In the optimum growth case, the preferences of society and the production technology are studied carefully in order to determine qualitative results. Assumptions are placed directly on preferences and technology.

An optimal growth model generates a capital-price differential equation by using the Hestenes-Pontriagin ([16], [27]) Maximal Principle to write down necessary conditions for an optimal solution. This process generates a type of differential equation system that we will call "a Modified Hamiltonian Dynamical System." The adjective "modified" appears because it is a certain type of perturbation peculiar to economics of the standard Hamiltonian system.

Since the standard Hamiltonian case will be a special case of our problem, our results will be of independent interest to mathematicians working in the field of Hamiltonian dynamical systems.

In order to discuss what is new in our approach, we need a definition. A Modified Hamiltonian Dynamical System (call it an MHDS for short) is a differential equation system of the form

$$(1.1)\qquad \begin{aligned} \dot{q}_j &= \rho q_j - \frac{\partial H}{\partial k_j}(q,k) \\ \dot{k}_j &= \frac{\partial H}{\partial q_j}(q,k) \quad j = 1,2,\ldots,n \end{aligned}$$

Here $H: R^n \times R^n_+ \to R$, $\rho \in R_+$. In economics k_j is stock of capital good j and q_j is the demand price of capital good j. The function H is called a Hamiltonian and it is defined only on $R^n \times R^n_+$ for economic problems. H turns out to be the current value of national income evaluated at prices q. The number ρ is a discount factor on future welfare arising from the structure of social preferences. The first part of (1.1) states that capital gains plus yield must equal the common rate of return for all capital goods. The second equation of (1.1) just boils down to a "budget restraint" in economic problems. If preferences and technology are assumed to be concave and certain other economically plausible assumptions are satisfied then optimal growth problems generate a family of solutions of (1.1) having the following property:

(1.2) There exists a closed convex set $K \subseteq R_{+}^{n}$ such that for $k_0 \in K$, there exists $q_0 \in R^n$, such that the solution to (1.1) with initial conditions (q_0,k_0), $\phi_t(q_0,k_0)$, has the following properties.

(i) $\phi_t(q_0,k_0)$ is defined for all $t \geqq 0$, $\phi_t(q_0,k_0) \in R^n \times K$.

(ii) $\Pi_2(\phi_t(q_0,k_0)) \geqq \varepsilon > 0$ for some $\varepsilon(k_0) > 0$. Here, Π_2 denotes the projection on the k subspace.

(iii) $|\phi_t(q_0,k_0)| \leq M(k_0)$ for some $M(k_0) > 0$. Here, $|.|$ denotes any norm in R^{2n}.

Property (1.2) is a plausible requirement for many dynamic economic problems to be well posed. The reason follows. (i) requires that a solution exists for all t (what is, of course, a necessary condition for the existence of an optimal path) and that orbits starting in $R^n \times K$ remain there i.e. that $R^n \times K$ be "positively invariant." If $k_T = \Pi_2\ (\phi_T(q_0,k_0))$ and $q_T = \Pi_1\ (\phi_T(q_0,k_0))$ then $\phi_t(q_T,k_T)$ obviously satisfies (i), (ii) and (iii) if $\phi_t(q_0,k_0)$ does. Hence, the set of $k \in R_{+}^{n}$ that satisfies (i), (ii) and (iii) is clearly positively invariant. If every capital good is indispensible in the production of at least one good whose marginal utility goes to infinity as consumption goes to zero, then (ii) is satisfied. (iii) holds if one postulates the existence of a maximum reproducible capital stock. (cf. McKenzie's [24] assumption III and Scheinkman [34] proof of Lemma 5).

In the appendix, we show that in the one-sector model one can choose $K = \{k \in R_{+}^{n} / \varepsilon \leq k \leq \hat{k}\}$ for any $\varepsilon > 0$, small enough where $\hat{k}$ is the limit of the path of pure accumulation (i.e. the maximum reproducible stock).

Our theorems on convergence will be proved for $k_0 \in K$. In economic growth models, a capital stock $k_0 \notin K$ is one such that either the economy is not viable (for instance, $k_0 \equiv 0$), or such that the economy would be better off with a smaller initial capital stock (for instance, when $k_0 > \hat{k}$ in the one sector model). If free disposal is present and the economy is viable, then one can assume $k_0 \notin K$. If disposal is impossible, then there exists no optimal path for $k_0 \notin K$. When one introduces a positive, but finite, cost of disposal, then with the use of negative demand prices, one has $K \equiv \{k \in R/k \geq \varepsilon\}$ for any $\varepsilon > 0$, small enough in the one sector model.

The special interest on bounded paths is justified since by assuming concave preference and technology, one can prove that bounded paths are optimal. If uniqueness and property (1.2) holds, then for $k \in K$ optimal and bounded paths coincide.

The problem we shall address in this paper may now be defined.

<u>Problem 1:</u> Find sufficient conditions on solutions ϕ_t of (1.1) satisfying (1.2) such that $\phi_t \to (\bar{q},\bar{k})$, $t \to \infty$. Also, find sufficient conditions such that the steady state $(\bar{q},\bar{k})$ is independent of the initial condition (q_0,k_0).

The literature on Problem 1 has two main branches: (1) analysis of the local behavior of (1.1) in a neighborhood of a steady state, and (2) analysis of global behavior of solutions of (1.1).

The first branch of the literature is fairly complete. It studies the linear approximation of (1.1) in a neighborhood of a rest point. Eigenvalues have a well known symmetric structure that determines the local behavior. Since we have nothing new to contribute to this branch of study, therefore, we cite some representative references and move on. (Kurz [20], Samuelson [33], Levhari and Leviatan [22]). To the global problem we now turn.

The literature on Problem 1 is very large yet there are no general results on global stability. Representative literature on existing global stability results for paths satisfying property (1.2) follows. The case $n = 1$ (the one good optimal growth model) is well understood. See Burmeister and Dobell [6], Cass [7], Koopmans [18], Kurz [19]. The case $n = 2$ has received a lot of study since the pioneering work of Uzawa [35] in the two sector optimal growth literature. The two sector models form a subclass of systems of type (1.1) for $n = 2$. Ryder and Heal [31] analyze a case of (1.1) for $n = 2$. They generate a variety of examples of different qualitative behavior of property (1.2) paths. Burmeister and Graham [37] present an analysis of a model where there is a set S containing the steady state capital $\bar{k}$ such that for $k \notin S$ there is a unique q such that $\phi_t(q,k) \to (\bar{q},\bar{k})$, $t \to \infty$. See [37] Theorem 2, page 149.

The case $\rho = 0$ is the famous Ramsey problem studied first by F. Ramsey for the one good model then by Gale [12], McKenzie [24], McFadden [23], and Brock [3] for the n goods model in discrete time and by Rockafellar [28] for continuous time. These results state, roughly speaking: if $H(q,k)$ is strictly convex in q and strictly concave in k and $\rho = 0$, then all solution paths of (1.1) satisfying property (1.2) converge to a unique steady state $(\bar{q},\bar{k})$ as $t \to \infty$ independently of (q_0,k_0).

J. Scheinkman [34] has recently proven a result that shows that the qualitative behavior for $\rho = 0$ is preserved for small changes in ρ near $\rho = 0$.

Until very recently no general results on the convergence of solutions of (1.1) satisfying (1.2) were available. In fact, little was known about sufficient conditions for the uniqueness of steady states of (1.1). Recently a paper of W. Brock's [4] gave a fairly general set of sufficient conditions for uniqueness of the steady state. Thus, the uniqueness problem is fairly well understood. There was nothing done in the Brock paper on convergence, however.

In 1973, Cass and Shell [9], Rockafellar [30], and Brock and Scheinkman [5], [38], working independently, came up with sets of sufficient conditions on Problem 1. The Cass, Shell, Rockafellar (CSR) conditions are curvature conditions on the Hamiltonian of (1.1). The conditions in [5] stem from the stable manifold theory developed by Hirsh and Pugh [17], and are hard to interpret from an economic viewpoint. In [38], we proved results on the global stability of bounded trajectories of systems of type (1.1) that were stimulated by the basic paper of Hartman and Olech [15] on the global stability of differential equations. We also developed some thoughts on "dominant diagonal" type of results. The sequel is a revision of [38]. We will do the following in this paper.

First, in section 2, we will present a general method to obtain global asymptotic stability results for Hamiltonian Systems. The results proved there resembles a "Lyapounov Method." It contains as a corollary some well known theorems on global asymptotic stability (cf. [1], [13], [14]), and can be used to derive the convergence of bounded solutions of differential equations which are not globally asymptotically stable i.e. systems where the stable manifold $S = \{x_0 | \phi_t(x_0) \to 0,\ t \to \infty\}$ is not the whole space. In particular, the results in [38] are now simple corollaries. The proof which is inspired on the elegant proof of one of the Hartman and Olech results that was obtained by A. Mas-Collel [39] is much simpler than the one in [38].

Second, in section 3, we show how the results of [38] can be obtained from the general method and also discuss their economic interpretations. We will also derive certain results concerning the relation between the "shadow" prices and capital stocks. These theorems are useful when one interprets the prices as equilibrium prices in a decentralized framework.

Third, in section 4, we will try to develop some thoughts on "dominant diagonal" type of results. The reason for this is that we believe that both our "Q-Conditions" used in sections 2 and 3 and the CSR condition would be very sensitive to the value of ρ. For ρ small they are very natural, but as ρ increases more stringent conditions are placed on the eigenvalues of certain matrices. As an illustration, we prove that if the Modified Hamiltonian Dynamical System is upper triangular then convergence occurs independent of ρ. This suggests that if "off-effects" are small, convergence should still occur.

Finally, we shall close with a summary, conclusions and suggestions for further research in section 5.

Section 2

A GENERAL RESULT ON CONVERGENCE OF BOUNDED TRAJECTORIES

In this section, we shall present a general theorem that will generate Hartman and Olech's basic result [15, p.157, Theorem 2.3], our results [38] and many other results - all as simple corollaries. Furthermore, the general theorem will be stated and proved in such a way as to highlight a general Lyapounov method that is especially useful for the stability analysis of optimal paths generated by optimal control problems arising in capital theory.

Theorem: Let $f: R^m \to R^m$ be C^2. Assume there is $\bar{x}$ such that $f(\bar{x}) = 0$ (W.L.O.G. put $\bar{x} = 0$) such that there is $V: R^m \to R$ satisfying: (1) Every rest point $\tilde{x}$ of the differential equation system

(2.1) $\dot{x} = f(x)$

is L.A.S.[1/] on bounded trajectories in the sense that the hypothesis there is $\varepsilon_o > 0$ such that $x_o \in N_{\varepsilon_o}(\tilde{x})$, there exists a bound B such that $|\phi_t(x_o)| \leq B$ for all $t \geqq 0$ implies the conclusion: $\phi_t(x_o) \to \tilde{x}$, $t \to \infty$. I.E. every rest point has a neighborhood such that all bounded trajectories starting in this neighborhood converge to the rest point. We also assume[2/]

(2) (a) For all $x \neq 0$, $x \nabla^2 V(0) [J(0)x] < 0$

(b) For all $x \neq 0$, $\nabla V(0) [\frac{d}{d\lambda} J(\lambda x)]_{\lambda=0}\, x = 0$, $\nabla V(0) J(0)x = 0$

(c) For all $x \neq 0$, $\nabla V(x) f(x) = 0$ implies $x \nabla^2 V(x) f(x) = 0$

(d) For all $x \neq 0$, $x \nabla^2 V(x) f(x) = 0$ implies $\nabla V(x) J(x) x < 0$

Then

(1) $\nabla V(x) f(x) < 0$ for all $x \neq 0$

(2) All trajectories that remain bounded for $t \geqq 0$ converge to a rest point.

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[1] L.A.S. is an abbreviation for "locally asymptotically stable."

[2] The notation ∇V denotes the gradient of V, $\nabla^2 V$ is the matrix of second order partial derivatives of V, and J(x) denotes the Jacobian matrix of V evaluated at x. If x and y are vectors, we write xy for the dot product of x,y i.e. $xy \equiv \sum_{i=1}^{n} x_i y_i$. If A is a matrix, we write xAy for the bilinear form $\sum_i \sum_j x_i a_{ij} y_j$. We will write x^T, A^T for the transpose of the vector x and the matrix A when confusion is prevented thereby.

Proof: Let $x \neq 0$ and put

(2.2) $$g(\lambda) \equiv \nabla V(\lambda x)\ f(\lambda x)$$

We shall show that $g(1) < 0$ in order to obtain (1). We do this by showing that $g(0) = 0$, $g'(0) = 0$, $g''(0) < 0$, $g(\bar{\lambda}) = 0$ implies $g'(\bar{\lambda}) < 0$ for $\bar{\lambda} > 0$. (At this point, the reader will do well to draw a graph of $g(\lambda)$ in order to convince himself that the above statements imply $g(1) < 0$). Calculating we get

(2.3) $$g'(\lambda) = x\ \nabla^2\ V(\lambda x)\ f(\lambda x) + \nabla\ V(\lambda x)\ J(\lambda x)x$$

(2.4) $$g''(\lambda) = x[\frac{d}{d\lambda}\ \nabla^2\ V]\ f(\lambda x) + x\ \nabla^2\ V(\lambda x)\ [J(\lambda x)x] + x\ \nabla^2\ V(\lambda x)\ [J(\lambda x)x] + \nabla\ V(\lambda x)\ [\frac{d}{d\lambda}\ J(\lambda x)]x$$

Now $\lambda = 0$ implies $f(\lambda x) = 0$ so $g(0) = 0$. Also $g'(0) = 0$ from $f(0) = 0$ and (2b). Furthermore, $f(0) = 0$, (2b) imply

(2.5) $$g''(0) = 2\ x\ \nabla^2\ V(0)\ [J(0)x]$$

But this is negative by (2a). By continuity of g'' in λ, it must be true that there is $\varepsilon_o > 0$ such that $g(\lambda) < 0$ for $\lambda \varepsilon\ (0, \varepsilon_o]$. Suppose now that there is $\lambda > 0$ such that $g(\lambda) = 0$. Then there must be a smallest $\bar{\lambda} > 0$ such that $g(\bar{\lambda}) = 0$. Also, $g'(\bar{\lambda}) \geqq 0$. Let us calculate $g'(\bar{\lambda})$, show that $g'(\bar{\lambda}) < 0$, and get an immediate contradiction. From (2.3)

(2.6) $$g'(\bar{\lambda}) = x\ \nabla^2\ V(\bar{\lambda} x)\ f(\bar{\lambda} x) + \nabla\ V(\bar{\lambda} x)\ J(\bar{\lambda} x)x$$

Now $g(\bar{\lambda}) = 0$ implies $\nabla\ V(\bar{\lambda} x)\ f(\bar{\lambda} x) = 0$. But this, in turn, implies that $\bar{\lambda}\ x\ \nabla^2\ V(\bar{\lambda} x)\ f(\bar{\lambda} x) = 0$ by (2c). Finally, (2d) implies that $\nabla\ V(\bar{\lambda} x)\ J(\bar{\lambda} x)(\bar{\lambda} x) < 0$. Thus, $g'(\bar{\lambda}) < 0$ – contradiction to $g'(\bar{\lambda}) \geqq 0$. Thus,

(2.7) $$\nabla\ V(x)\ f(x) < 0 \quad \text{for all } x \neq 0.$$

It is easy to see that all trajectories that remain bounded for $t \geqq 0$ converge to a rest point. For let $\{\phi_t(x_o)\}_{t \geqq 0}$ be such a trajectory where x_o is the initial condition. Since $\nabla V(\phi_t(x_o))\, f(\phi_t(x_o)) < 0$, therefore, $\frac{dV}{dt} < 0$. Thus, $V(\phi_t(x_o))$ decreases in $t \geqq 0$. We claim that $\{\phi_t(x_o)\}_{t \geqq 0}$ clusters at the rest point $\bar{x} = 0$. If it does not cluster at the rest point $\bar{x} = 0$, then there is $\varepsilon_o > 0$ such that

(2.8) $$\frac{dV}{dt} \equiv \nabla V(\phi_t(x_o))\, f(\phi_t(x_o)) < -\varepsilon_o, \quad t \geqq 0$$

This follows from boundedness of $\{\phi_t(x_o)\}_{t \geqq 0}$, (2.7) and continuity. Therefore, V becomes unbounded as $t \to \infty$. Contradiction to boundedness of $\{\phi_t(x_o)\}_{t \geqq 0}$ and continuity of V.[3/] Since 0 is a locally asymptotically stable rest point for bounded trajectories, this ends the proof of the theorem.

Note that to get global asymptotic stability results for bounded trajectories all one needs to do is find a V that is monotone on bounded trajectories and assume that rest points are locally asymptotically stable for bounded trajectories. This result is important for global asymptotic stability analysis of optimal paths generated by control problems arising in Capital Theory. Also, Hartman-Olech [15] type results emerge as simple corollaries. Let us demonstrate the power of the theorem by extracting some corollaries.

<u>Corollary 2.1</u> Let $f: R^m \to R^m$. Consider the ordinary differential equations

$$\dot{x} = f(x), \quad f(0) = 0$$

If $J(x) + J^T(x)$ is negative definite for each x, then 0 is globally asymptotically stable.

<u>Proof</u>: Put $V = x^T x$. Then $\nabla V(x) = 2x \quad \nabla^2 V(x) = 2I$ where I is the n x n

[3]The reader may wonder if our theorem implies that $x = 0$ is the only rest point. It does. This is made clear by the assumptions 2a-2d. For 2a-2d imply that the trajectory derivative of V is negative for $x \neq 0$, and this implies that the rest point is unique.

identity matrix. Assumption (2a) becomes

$$x\, J(0)x < 0 \quad \text{for all } x \neq 0.$$

But this follows because

$$2\, x\, J(0)x = x\, (J(0) + J^T(0)\, x < 0.$$

Assumption (2b) trivially holds since $\nabla V(x) = 2x$. Assumption (2c) amounts to $2x\, f(x) = 0$ implies $x(2I)\, f(x) = 0$ which obviously holds. (2d) obviously holds because $2x\, J(x)x < 0$ for <u>all</u> $x \neq 0$. It is obvious that rest points are L.A.S. since $J(0) + J^T(0)$ is a negative definite matrix. Thus, all bounded trajectories converge to 0, as $t \to \infty$. It is easy to use $V = x^T x$ decreasing in t in order to show that all trajectories are bounded. This ends the proof.

The following corollary is a stronger result than Hartman and Olech [15] in one way and weaker in another. We will explain the difference in more detail below.

<u>Corollary 2.2</u> (A. Mas Collel [39]) Consider $\dot{x} = f(x)$, $f(0) = 0$. Assume that $x\, [J(0) + J^T(0)]\, x < 0$ for all $x \neq 0$ and

(2.9) $$x\, f(x) = 0 \text{ implies } x\, [J(x) + J^T(x)]\, x < 0 \quad \text{for all } x \neq 0.$$

Then 0 is globally asymptotically stable.

Proof: Let $V = x^T x$. We show that

(2.10) $$\frac{dV}{dt} = 2\, x^T\, f(x) < 0 \quad \text{for } x \neq 0$$

Assumptions 2a,b,c,d of the theorem are trivially verified. Therefore, $\frac{dV}{dt} < 0$, and the rest of the proof proceeds as in Corollary 2.1.

This type of result is reported in Hartman and Olech [15] and in Hartman's book [14]. In [14] and [15], 0 is assumed to be the only rest point and it is assumed to be locally asymptotically stable. On the one hand, Mas Collel puts the stronger assumption: $x\ [J(0) + J^T(0)]\ x < 0$ for $x \neq 0$ on the rest point. It is well known that negative real parts of the eigenvalues of $J(0)$ does not imply negative definiteness of $J(0) + J^T(0)$, but negative definiteness of $J(0) + J^T(0)$ does imply negative real parts for $J(0)$.

But on the other hand, Hartman and Olech [15] make the assumption: for all $x \neq 0$

$$w^T f(x) = 0 \text{ implies } w^T [J(x) + J^T(x)]\, w \leqq 0 \quad \text{for all vectors } w.$$

Note that Mas Collel only assumes $x^T f(x) = 0$ implies $x^T [J(x) + J^T(x)]\, x < 0$. So he places the restriction on a much smaller set of w, but he requires the strong inequality. Furthermore, the proof of the Mas Collel result is much simpler than that of Hartman and Olech.[4]

It is possible to obtain general results of Hartman and Olech type from the theorem. For example,

<u>Corollary 2.3</u> Let G be a positive definite symmetric matrix, and let 0 be the unique rest point of $\dot{x} = f(x)$.

Assume that

$$x \ (G + G^T)\, f(x) = 0 \text{ implies } x^T [(G + G^T)\, J(x)]\, x < 0 \quad \text{for all } x \neq 0$$

Then $x = 0$ is globally asymptotically stable for bounded trajectories.

Proof: Let $V(x) = x^T Gx$. Then

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[4] At this point, it is interesting to compare the "Hartman and Olech - Mas Collel" type of result with the more standard result of corollary 2.1. This is more easily seen if we assume that $f(x) = Dg(x)$ for some $g: R^n \to R^n$. In this case $J(x)$ is symmetric (being the second derivative of g) and hence corollary 2.1 becomes: If $J(x)$ is negative definite, then $\dot{x} = f(x)$ is G.A.S. This condition is equivalent to g being a concave function (except for borderline cases). On the other hand, Corollary 2.2 will read $x \cdot f(x) = 0 \Rightarrow x\, J(x)x < 0$ for all $x \neq 0$. A sufficient condition for this is that g be quasi-concave.

$$\nabla V(x) = x^T [G + G^T]$$

Also,

$$\nabla^2 V(x) = G + G^T$$

The rest of the proof is now routine.

Corollary 2.3 is closely related to Hartman and Olech's [15, Theorem 2.3, p.157] and Hartman's book [14, Theorem 1.4, p.549]. Hartman and Olech also treat the case of G depending on x. We have not been able to obtain their result for non constant G as a special case of our theorem. Thus, their different methods of proof yield theorems that our methods presented in this section are unable to obtain. This leads us to believe that the original method of proof developed in [38] is needed to develop Hartman and Olech type of generalizations for non constant G for Modified Hamiltonian Dynamical Systems. In fact, we obtain such results in [39]. We turn now to the study of Modified Hamiltonian Dynamical Systems.

Section 3

CONVERGENCE OF BOUNDED TRAJECTORIES OF MODIFIED HAMILTONIAN SYSTEMS

In this section, we apply the results obtained in Section 2 to systems of the form (1.1). We will assume that (1.1) has a singularity $(\bar{q},\bar{k})$ and rewrite it as

$$\dot{z}_1 = \rho\,(z_1 + q) - H_2(z) \equiv F_1(z) \tag{3.1}$$

$$\dot{z}_2 = H_1(z) \equiv F_2(z)$$

In [38] Brock and Scheinkman presented a "natural" generalization of the Hartman-Olech type of result to differential equation systems of the form (3.1). Our sufficient condition for global asymptotic stability of bounded trajectories of (3.1) was for all $z \neq 0$, $w_1^T F_2(z) + w_2^T F_1(z) = 0$ implies $w^T Q(z) w > 0$ for all $0 \neq w = (w_1, w_2) \in R^{2n}$. The proof in [38] was long and complicated. The theorem proved in Section 2 generates a much better result than that in [38] as a simple corollary.

Theorem 3.1: Let

$$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\ I \\ \rho/2\ I & -H_{22}(z) \end{bmatrix} \tag{3.2}$$

where I is the n x n identity matrix. Assume

(a) $0 = F(0)$ is the unique rest point of $\dot{z} = F(z)$

(b) there is $\varepsilon_0 > 0$ such that $|z_0| < \varepsilon_0$ and $\phi_t(z_0)$ bounded imply $\phi_t(z_0) \to 0,\ t \to \infty$

(c) for all $x \neq 0$

$$z_1^T F_2(z) + z_2^T F_1(z) = 0 \ \text{ implies } z\,Q(z)z > 0 \tag{3.3}$$

(d) for all $w \neq 0$ $wQ(0)w > 0$

then all trajectories that are bounded for $t \geqq 0$ converge to 0 as $t \to \infty$.

Proof: Let $V = z^T Az$ where $A = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Here I is the n x n identity matrix. Note $z^T Az = -2z_1^T z_2$. Assumption (1) of Theorem 2.1 is assumed in (3.1)(b). Since $\nabla^2 V(0) = 2A$ and $(wA)(J(0)w) = -w\, Q(0)w$, therefore, assumption 2(a) of Theorem 2.1 follows from (3.1)(d). Also $\nabla V(z) = z^T (A + A^T)$ so 2(b) of Theorem 2.1 follows trivially. Now, 2(c) of Theorem 2.1 amounts to $\nabla V(z)\, F(z) =$

$$z^T (A + A^T)\, F(z) = 0 \text{ implies } z\, \nabla^2 V(z)\, F(z) = z^T [(A + A^T)]\, F(z) = 0$$

which is trivially true. Furthermore, assumption 2(d) of Theorem 2.1 amounts to

(3.4) $$z^T (A + A^T)\, F(z) = 0 \text{ implies } z^T [(A + A^T)\, J(z)]\, z < 0$$

But (3.4) is identical to (3.3) as an easy calculation will immediately show. Thus, $\dot{V} < 0$ except at the rest point 0. The rest of the proof is routine by now.

Q.E.D.

The above result can be used for the study of optimal growth paths. Under fairly general assumptions an optimal growth problem generates an M.H.D.S. as (1.1) [cf. Kurz [20], Brock [4]]. If property (1.2) holds for a $K \subset R_+^n$ then for any $k_o \varepsilon K$, there exists q_o such that the solution of (1.1) with initial conditions (k_o, q_o) is bounded. If the hypothesis of Theorem (3.1) holds, such solution converges to the Optimal Steady State. Furthermore, if utility and production functions are assumed concave, bounded solutions are optimal. This implies that an optimal solution that converges exists for any $k_o \varepsilon K$. Also, if strict concavity holds, optimal solutions are unique.

The reader should note that in the special case where $\rho = 0$, the assumptions of Theorem (3.1) always hold for a M.H.D.S. derived from a problem of optimal growth in a multisector economy, with concave production sets and concave utility function. In fact, in this case the concavo-convexity of the Hamiltonian imply that Q is positive definite. Furthermore, the local stability of bounded trajectories (saddle-point property) is also obtained [cf. Samuelson [33]].[1/]

From the proof of the result, one can also notice that if the conditions hold for a convex, positively invariant set Z, then for any $z \varepsilon Z$ the conclusion follows. Hence, this result can be used to prove global asymptotic stability of optimum growth paths with initial conditions in a certain subset of all positive capital stocks.

The proof of Theorem 2 also yields that $z_1 z_2 = (q - \bar{q})(k - \bar{k})$ is monotonically increasing. Under stronger assumptions, one can also show that along any bounded path $\dot{q}\,\dot{k} = F_1(z)\,F_2(z)$ is negative. This establishes that the change in the capital stock (i.e. the net demand for new capital) has a negative inner product with the price change. This is proven in

Corollary 3.1 Let z_0 be such that $\phi_t(z_0)$ is bounded for $t \geqq 0$. Assume that for any $w \neq 0$, for any z.

(3.5) $w_1 F_2(z) + w_2 F_1(z) = 0$ implies $w\,Q(z)w > 0$ and that conditions (a) and (b) of Theorem (3.1) hold. Then $F_1(z_0)\,F_2(z_0) < 0$.

Proof: Let $z(t) \equiv \phi_t(z_0)$. Suppose there is a time $t_o > 0$ such that $F_1(z(t_0))$ $F_2(z(t_o)) = 0$. Then

$$\frac{d}{dt}(F_1 F_2) = F\,Q\,F > 0 \quad \text{at } t_o \text{ bt (3.5)}$$

We claim that $F_1 F_2 > 0$ for $t \geqq t_o$. For if not then let t_1 be the first time larger than t_o such that $F_1 F_2 = 0$. Now the slope $\frac{d}{dt}(F_1 F_2)$ must be non-positive at t_1. But this is a contradiction to (3.5). Hence, $F_1 F_2 > 0$

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[1] In the spirit of footnote 4 of Section 2, we can compare the result of Theorem 3.1 with the weaker result that can be obtained if one assumes Q(x) positive definite (cf. [38] Section 2). For the case $\rho = 0$, Q(x) is positive definite iff the Hamiltonian H is concave in k and convex in q. Our assumption 3.3 then is related to quasi-concavity of H in k and quasi-convexity of H in q (and not concavo-convexity). This is important in the study of certain models where one does not obtain the concavo-convexity of the Hamiltonian.

for $t \geqq t_o$. This cannot be. For Q is positive definite at 0 by (3.5). Thus, by continuity of Q in t, there is a neighborhood N of 0 such that there is $t_o > 0$ such that

$$F(z)\ Q(z)\ F(z) > \varepsilon_o\ (F^T(z))\ (F(z))$$

for $z \in N$. Now

$$\frac{d}{dt}(F_1\ F_2) = F^T\ Q(z)F > \varepsilon_o\ |F^T(z)|^2 \geqq \varepsilon_o\ |F_1(z)\ F_2(z)|$$

for $z \in N$. Since by Theorem (3.1), $\phi_t(z_0) \to 0$, $t \to \infty$, therefore, there is $T(z_0)$ such that $t \geqq T(z_0)$ implies that $\phi_t(z_0) \equiv z(t) \in N$. Thus, $F_1\ F_2$ becomes unbounded because $F_1\ F_2$ is positive for $t \geqq t_o$ and $\frac{d}{dt}(F_1\ F_2) > \varepsilon\ |F_1\ F_2|$ for t large. This is a contradiction to continuity of F in t and boundedness of z(t). Thus, the corollary follows. Q.E.D.

A further result on the relation between q and k can be obtained when the following is assumed

Assumption 3.1 - There exists C^1 function $g: K \to R^n$ such that k(t) with $k(0) = k_o$ is an optimal path iff $k(t) = \Pi_2(\ _t(g(k_o),k_0))$ where $\phi_t(g(k_o),k_o)$ is a solution of (1.1) satisfying assumption 1.2.

Corollary 3.2 If Q is positive definite, $J_g(k)$[2] is quasi-negative definite for any $k \in K$.

Proof: Let $z_2 = k - \bar{k}$. Then $z_1 = q - \bar{q} = g(k) - \bar{q} \equiv h(z_2) - \bar{q}$. Then $J_g(k) \equiv J_h(z_2)$ for $z_2 = k - \bar{k}$. Fix z_2 and write J(t) for the Jacobian matrix of the system (1.1) evaluated at $\phi_t(h(z_2),z_2)$. By Theorem (3.1), $\phi_t(h(z_2),z_2) \to 0$ as $t \to \infty$. For any $\mu \in R^n$, consider the system $\dot{y} = J(t)y$, $y(0) = u \equiv (J_h\ \mu,\ \mu)$ where $J_h = J_h(z_2)$. Since $u = (J_h\ u_2,\ u_2)$, we have $y(t) \to 0$ as $t \to \infty$.[3]

[2] J_g denotes the Jacobian matrix of g.

[3] This follows from the fact that since $\phi_t(h(z_2),z_2) \to 0, J(t) \to J(0)$ and from a theorem on stability of differential equations (cf. [10],p.316). A detailed proof can be found in [39].

However, $\frac{d}{dt}\ y_1(t)\ y_2(t) = y(t)\ Q\ (\phi_t(h(z_2),z_2)\ y(t) \geq \varepsilon\ |y_1(t)\ y_2(t)|$ for some $\varepsilon > 0$, since $\phi_t(h(z_2),z_2)$ is bounded and $Q(z)$ is positive definite. Hence, if $y_1(t)\ y_2(t) \geq 0$ for some t, $\lim_{t\to\infty} |y(t)| = \infty$. But this is a contradiction to $y(t) \to 0$ as $t \to \infty$. Hence, $y_1(t)\ y_2(t) < 0$ for all $t \geq 0$. In particular, $y_1(0)\ y_2(0) \equiv \mu\ J_g\ \mu < 0$.

Q.E.D.

Section 4

SOME PRELIMINARY RESULTS OF A "DOMINANT DIAGONAL" TYPE

In this section, we develop some thoughts on a "dominant diagonal" type of method for Hamiltonian systems. Lazer [21] establishes some results in this direction. The difficulty in applying such results, however, for Modified Hamiltonian systems that arise from economic problems is that the assumption of dominant diagonal is, in general, not fulfilled. Consider for instance the one sector model of growth analysed by Cass [7].

$$\max \int_0^\infty e^{-\rho t} u(c\ )\, dt$$

s.t.

$$\dot{k} + c = f(k)$$

k_0 given

The Hamiltonian system becomes

$$\dot{q} = \rho q - H_k = \rho q - qf'; \quad \dot{k} = H_q = f(k) - g(q)$$

where $q = u'(c)$ and $g = (u')^{-1}$

Hence,

$$J(t) = \begin{bmatrix} -f'(k) + \rho & -qf'' \\ -g' & f'(k) \end{bmatrix}$$

Note that at the steady state k^*, we have $-f'(k^*) + \rho = 0$ and hence the first diagonal element is zero.

This shows us that a simple "dominant diagonal" assumption is not natural. In economic parlance the "dominant diagonal" assumption is usually translated as "the own-effects dominate the off-effects." This suggests that a useful definition of dominant diagonal for our case would relate the sum of the absolute values of the diagonal term with the "own" off diagonal term to the sum of all other off diagonal terms e.g.: $|H_{q_ik_i}| + |H_{q_iq_i}| \geq \sum_{j\neq i} |H_{q_ik_j}| + \sum_{j\neq i} |H_{q_iq_j}|$. We have not yet been able to prove such a theorem. However, a weaker theorem can be stated. We start with some preliminary lemmas. As before, we will be considering a Modified Hamiltonian system like (1.1) of Section 1, or translated to the origin like (3.1) of Section 3. We will use assumption 1.2 and some of the assumptions made in Section 3 that we now summarize.

Assumption 4.1

(a) 0 is the unique rest point of $\dot{z} = F(z)$

(b) there exists $\varepsilon_0 > 0$ such that if $|z_0| < \varepsilon_0$ and $\phi_t(z_0)$ is bounded for $t \geq 0$, then $\phi_t(z_0) \to 0$ as $t \to \infty$

(c) there exists C^1 function $g\colon K \to R^n$ such that $k(t)$ with $k(0) = k_0$ is an optimal path iff $k_t = \Pi_2(\phi_t(g(k_0), k_0))$ where $\phi_t(g(k_0), k_0)$ is a solution of (1.1) satisfying assumption 1.2

Note that (a) and (b) were used in the proof of Theorem 3.1. 4.1 (c) was used to prove corollary 3.2. We will also take $|V| = \max_i V_i^2$

Assumption 4.2 For $(q,k) \in R^{2n}$ the following inequalities hold

(a) $$\left|-H_{k_iq_i} + \rho + H_{k_ik_i}\frac{\dot{k}_i}{\dot{q}_i}\right| > \sum_{j\neq i} \left|- H_{k_iq_j} + \rho\right| + \sum_{j\neq i} |H_{k_ik_j}| \text{ for } \dot{q}_i \neq 0$$

and

(b) $$\left|H_{q_iq_i}\frac{\dot{q}_i}{\dot{k}_i} + H_{q_ik_i}\right| > \sum_{j\neq i} |H_{q_iq_j}| + \sum_{j\neq i} |H_{q_ik_j}| \text{ for } \dot{k}_i \neq 0$$

Lemma 4.1 Suppose assumptions (1.2) and (4.1) hold and that assumption 4.2 holds for (q,k) in the set $\bar{B} = \{(q,k) \in R^{2n} | q = g(k),\ k \in K\}$, then $\frac{d}{dt} \,|\, (\dot{q},\dot{k}) \,|\, \neq 0$ for any $(\dot{q},\dot{k}) \neq 0$

Proof: Suppose there exists $(\bar{q},\bar{k})$ such that $\frac{d}{dt} \,|\, (\dot{q},\dot{k}) | = 0$. In particular there exists q_i (or k_i) such that $|\, (\dot{q},\dot{k}) \,| = |\dot{q}_i|^2$ (or $|\dot{k}_i|^2$). If $|\, (\dot{q},\dot{k}) | = |\dot{q}_i|^2$ then,

$$0 = \left| \frac{1}{2}\frac{d}{dt} \,|(\dot{q},\dot{k})|\right| = |\dot{q}_i \cdot \ddot{q}_i| = |\dot{q}_i \,(\sum_j (-H_{k_i q_j} + \rho)\, \dot{q}_j$$

$$+ \sum_j H_{k_i k_j}\, \dot{k}_j)| = |(-H_{k_i q_i} + \rho)\dot{q}_i^2 + \sum_{j \neq i} (-H_{k_i q_j} + \rho)\dot{q}_i\dot{q}_j$$

$$+ H_{k_i k_i} \frac{\dot{k}_i}{\dot{q}_i}\, \dot{q}_i^2 + \sum_{j\neq i} H_{k_i k_j}\, \dot{k}_j\dot{q}_i| \geq |(-H_{k_i q_i} + \rho)\, \dot{q}_i^2$$

$$+ H_{k_i k_i} \frac{\dot{k}_i}{\dot{q}_i}\, \dot{q}_i^2| - \sum_{j\neq i} |(-H_{k_i q_j} + \rho)|\ |\dot{q}_i\, \dot{q}_j|$$

$$- \sum_{j\neq i} |H_{k_i k_j}|\ |\dot{k}_j q_i| \geq |-H_{k_i q_i} + \rho + H_{k_i k_i} \frac{\dot{k}_i}{\dot{q}_i}|\ \dot{q}_i^2$$

$$- \sum_{j\neq i} |-H_{k_i q_j} + \rho|\ \dot{q}_i^2 - \sum_{j\neq i} |H_{k_i k_j}|\ \dot{q}_i^2 \quad (\text{since } |\dot{q}_i|^2 \geq |\dot{q}_j|^2$$

and $|\dot{q}_i|^2 \geq |\dot{k}_j|^2$).

Since $\dot{q}_i^2$ is maximum and $(\dot{q},\dot{k}) \neq 0$, therefore $\dot{q}_i^2 > 0$, and thus we have by inequality (a) of assumption 4.2 that the last expression is positive, which is a contradiction. The case where $|\dot{k}_i|^2 \geq \max_j (|\dot{q}_j|^2, |\dot{k}_j|^2)$ follows similarly using inequality 4.1(b).

Q.E.D.

We can now prove our

<u>Theorem 4.1</u> Under the assumptions of the preceeding lemma, if $\phi_t(q,k)$ is bounded then $\phi_t(q,k) \to (q^*, k^*)$ as $t \to \infty$.

Proof: Write $z = (q - q^*, k - k^*)$ and consider the variational system $\ddot{z} = J(t)\,\dot{z}$ where $J(t) = J(\phi_t(z_0))$, J the Jacobian of F, with initial condition,

(4.1) $\quad \dot{z}_0 = J(z_0)z_0$ for $z_0 \in B = \bar{B} - (q^*, k^*)$ of (3.1).

We write $\psi_t(\dot{z}_0)$ for the flow associated with (4.1).

By the preceeding lemma since $\phi_t(z_0) \in B$ for all t, $\frac{d}{dt}|\psi_t(\dot{z}_0)| \neq 0$ since 0 is the unique singularity of (3.1). Suppose $\frac{d}{dt}|\psi_0(\dot{z}_0)| > 0$. Then either there exists $\varepsilon > 0$ such that $\frac{d}{dt}|\psi_t(\dot{z}_0)| > \varepsilon$ for all $t \in [0,\infty)$, or there exists sequence t_k such that $\frac{d}{dt}|\psi_{t_k}(\dot{z}_0)| \to 0$ as $t_k \to \infty$. Suppose the last statement is true. Then since $\phi_t(z_0)$ is bounded, a subsequence k_ℓ exists such that

$\phi_{t_{k_\ell}}(z_0) \to z^* \in B$. By continuity, $\frac{d}{dt}|\psi_0(z^*)| = 0$ which is a contradiction.

If $\frac{d}{dt}|\psi_t(\dot{z}_0)| > \varepsilon$ then $|\psi_t(\dot{z}_0)|$ becomes unbounded which is also a contradiction since $\phi_t(z_0)$ is bounded and F is continuous. Hence, $\frac{d}{dt}|\psi_0(\dot{z}_0)| < 0$ and hence $\frac{d}{dt}|\psi_t(\dot{z}_0)| < 0$ for all $t \in [0,\infty)$. Also, suppose $\phi_t(z_0) \nrightarrow 0$. By assumption 4.1(b) we must have $|\phi_t(z_0)| > \delta$ for some $\delta > 0$, for all t. Since 0 is the only singularity of (**), we must have $|\psi_t(\dot{z}_0)| > \eta$ for some $\eta > 0$. Since $\frac{d}{dt}|\psi_t(\dot{z}_0)| < 0$, this implies $\frac{d}{dt}|\psi_t(\dot{z}_0)| \to 0$ as $t \to \infty$. As before, this implies that there exists $z^* \neq 0$ $z^* \in B$ such that $\frac{d}{dt}|\psi_0(z^*)| = 0$, which is a contradiction.

Q.E.D.

Let us now turn to a result for "upper triangular" M.H.D.S.'s.

A result by Markus and Yamabe [26, pp.314-316] on the global asymptotic stability of the solution $x \equiv 0$ of the system $\dot{x} = f(x)$ where $\dot{x}_i = f_i(x_i, x_{i+1}, \ldots, x_n)$ $i = 1,2,\ldots,n$, and $f_{ii} < 0$ for all x generalizes naturally to M.H.D.S.'s of the form

(4.2) $$\dot{q}_i - \rho q_i = -H_{k_i}(q_i, q_{i+1}, \ldots, q_n; k_i, k_{i+1}, \ldots, k_n)$$
$$\dot{k}_i = H_{q_i}(q_i, q_{i+1}, \ldots, q_n; k_i, k_{i+1}, \ldots, k_n), \quad i = 1,2,\ldots,n$$

Before we state a theorem and prove it, let us intuitively discuss the Markus and Yamabe result. They use $\dot{x}_n = f_n(x_n)$, $f_n(0) = 0$, $f_{nn}(x_n) < 0$ to prove that $x_n(t) \to 0$, $t \to \infty$. Then they examine $\dot{x}_{n-1} = f_{n-1}(x_{n-1}, x_n)$. First, they use $f_{n-1,n-1}(x_{n-1}, x_n) < 0$ to get $f_{n-1}(x_{n-1}, 0) = 0$ implies $x_{n-1} = 0$. They proceed in this manner by induction to show that $f(x) = 0$ implies $x_i = 0$, $i = 1,2,\ldots,n$. Now, $x_n(t) \to 0$, $t \to \infty$ independent of initial conditions. Look at

(4.3) $$\dot{x}_{n-1} = f_{n-1}(x_{n-1}, x_n)$$

the right hand side of this is converging to $f_{n-1}(x_{n-1}, 0)$ as $t \to \infty$. Thus it seems intuitive that since $\dot{y}_{n-1} = f_{n-1}(y_{n-1}, 0)$ is G.A.S. ($f_{n-1,n-1} < 0$, $f_{n-1}(0,0) = 0$!), therefore, (4.3) should be G.A.S. This is what is proved in [26]. Markus and Yamabe then proceed by induction to complete the proof. We will follow a somewhat similar procedure here.

Look at the usual one sector growth model which generates an MHDS of the form

(4.4) $$\dot{q} - \rho q = -H_k(q,k)$$
$$\dot{k} = H_q(q,k)$$

where $q \notin R_+$, $k \notin R_+$. Our Q condition is much too strong to impose on this model. The usual phase diagram analysis leads to very general sufficient conditions that are essentially independent of ρ for the convergence of a bounded trajectory of (4.4) to a unique steady state. Rather than restate these well known conditions, let us just assume that the one sector model is G.A.S. on bounded trajectories at the outset. For the one sector model a natural assumption to make is that $\dot{k} = H_q(g(k),k)$ is G.A.S. where k is the rest point and it is assumed to be unique. See Cass [7] for a complete discussion of the one sector model.

Theorem Suppose that (4.4) may be written in the form

$$(4.5) \quad \dot{q}_i - \rho\, q_i = -H_{k_i}(q_i, q_{i+1}, \ldots, q_n;\ k_i, k_{i+1}, \ldots, k_n)$$

$$(4.6) \quad \dot{k}_i = H_{q_i}(q_i, q_{i+1}, \ldots, q_n;\ k_i, k_{i+1}, \ldots, k_n)$$

$$(4.7) \quad \dot{q}_i = g(k_i, k_{i+1}, \ldots, k_n) \text{ along B and letting}$$

$${}_jq = (q_j, \ldots, q_n),\ {}_jk = (k_j, \ldots, k_n)$$

Write

$$(4.8) \quad \dot{k}_i = H_{q_i}(q_i, {}_{i+1}q;\ k_i, {}_{i+1}k) \equiv f_i(k_i, k_{i+1}, \ldots, k_n)$$

Assume that

$$\dot{k}_n = f_n(k_n)$$

is G.A.S. with the unique rest point $\bar{k}_n$;

$$\dot{k}_{n-1} = f_{n-1}(k_{n-1}, \bar{k}_n)$$

is G.A.S. with the unique rest point $\bar{k}_{n-1}$, and thus assume for $i = 2,3,\ldots,n+1$

$$\dot{k}_{i-1} = f_{i-1}(k_{i-1}, \bar{k}_i, \ldots, \bar{k}_n)$$

is G.A.S. with unique rest point $\bar{k}_{i-1}$.

Then every bounded trajectory of (4.5) and (4.6) converges to a unique rest point $(\bar{q}, \bar{k})$ as $t \to \infty$.

Proof: We proceed by induction. Look at (4.5) and (4.6) for $i = n$. It becomes

$$(4.9)\quad \dot{q}_n - \rho q_n = -H_{k_n}(q_n,k_n)$$
$$\dot{k}_n = H_{q_n}(q_n,k_n)$$

Since (q_n,k_n) is bounded $(q_n,k_n) \to (\bar{q}_n,\bar{k}_n)$, $t \to \infty$ by hypothesis. Look at n-1. Equations (4.5) and (4.6) become

$$(4.10)\quad \dot{q}_{n-1} = \rho q_{n-1} - H_{k_{n-1}}(q_{n-1},q_n;\, k_{n-1},k_n)$$
$$\dot{k}_{n-1} = H_{q_{n-1}}(q_{n-1},q_n;\, k_{n-1},k_n)$$
$$(4.11)\quad q_{n-1} = g(k_{n-1},k_n)$$

Now the set $\{(q_{n-1}(t),k_{n-1}(t))\}_{t \geqq 0}$ is bounded on a bounded trajectory. Suppose that $k_{n-1}(t)$ does not converge to a steady state i.e. if we write

$$(4.12)\quad \dot{k}_{n-1} = H_{q_{n-1}}(g_{n-1}(k_{n-1},k_n),\, g_n(k_n),k_{n-1},k_n) \equiv f_{n-1}(k_{n-1},k_n)$$

then

$$(4.13)\quad f_{n-1}(k_{n-1},k_n) \not\to 0,\ t \to \infty$$

Let $\tilde{k}_{n-1}$ be a cluster point of the bounded set $\{k_{n-1}(t)\}_{t \geqq 0}$. Since $k_n \to \bar{k}_n$, $t \to \infty$, therefore, by definition of a cluster point there is a sequence $\{t_j\}$ such that $k_{n-1}(t_j) \to \tilde{k}_{n-1}$, $j \to \infty$.

Thus

$$f_{n-1}(k_{n-1}(t_j),k_n(t_j)) \to f_{n-1}(\tilde{k}_{n-1},\bar{k}_n)$$

But by hypothesis $f_{n-1}(k,\bar{k}_n)$ is zero only at $k = \bar{k}_{n-1}$, and is positive for $k < \bar{k}_{n-1}$, negative for $k > \bar{k}_{n-1}$. Since the family of functions $f_{n-1}(.,k_n(t)) \to f_{n-1}(.,\bar{k}_n)$, $t \to \infty$, it follows that $k_{n-1}(t)$ cannot cluster at a point $\tilde{k}_{n-1}$ where $f_{n-1}(\tilde{k}_{n-1},\tilde{k}_n) \neq 0$. Thus, $k_{n-1}(t) \to \bar{k}_{n-1}$ along the bounded manifold.

Proceed in this manner for k_{n-2},k_{n-3}, etc. This ends the proof.

It is worthwhile to make a few comments about what type of model will generate M.H.D.S.'s that are covered by this theorem.

Consider the following growth model

$$(4.14) \quad \text{maximize } \int_0^\infty e^{-\rho t}\, u(c_1,\ldots,c_n)\,dt$$

$$\text{s.t. } c_i + \dot{k}_i = f_i(k_i, k_{i+1},\ldots,k_n),\ i = 1,2,\ldots,n$$

$$k(0) = k_0$$

where u is separable i.e. $u = \sum_{i=1}^{n} u_i(c_i)$. (The f_i here are not the same as the f_i in equation (4.13)). The usual necessary conditions for an optimum for (4.14) give an MHDS of the form (4.5), (4.6) and (4.7). This model is not meant to be realistic - it is only meant to be illustrative. We are presently working on applications and extensions of the "dominant diagonal" result proved above and of this result for "triangular M.H.D.S.'s". These results may be important leads to developing a valuable set of sufficient conditions for G.A.S. that are not so dependent on the size of ρ as are the Q conditions. After all the one sector model is G.A.S. on bounded trajectories with hardly any restriction on the size of ρ. Therefore, there should be a class of useful sufficient conditions for G.A.S. for n-sector models that do not depend upon the size of ρ. The results presented here are a beginning.

Section 5

SUMMARY

We have been quite successful at obtaining a fairly general set of sufficient conditions for G.A.S. of bounded trajectories of M.H.D.S.'s of the type arising in optimal accumulation problems. It is reasonable to exhibit diminishing returns to capital and labor, and satisfy $\frac{\partial u}{\partial c_i} = +\infty$ when $c_1 = 0$ for all consumption goods are almost always bounded for each initial stock vector k_0.

In section 3, we developed a set of sufficient conditions that were more likely to hold when ρ is small. Section 4 gave some preliminary results that are not so dependent upon small ρ's in order to be useful.

We did not develop applications in any detail in this paper. As we will show in a future paper, our sufficient conditions are applicable to a wide range of economic problems including optimal growth, adjustment cost models in the theory of the firm, optimal harvesting of several interacting animal species, and many more. In fact, just about any optimal control problem that arises in economics will generate a differential equation system of a type such that our results will be useful in its qualitative study.

It seems to us that in this paper we have come part way at least to doing for optimal accumulation problems with many goods what Arrow, Block and Hurwicz [1], and Arrow and Hurwicz [2] and the literature that followed did for the global qualitative study of the Walrasian tâtonnement. Since optimal accumulation problems are the heart of intertemporal economics, we believe that in time the CSR results and the results presented here will become useful tools for the study of dynamic economics.

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APPENDIX

In this appendix, we set up the one sector growth model to exemplify our assumptions. More details can be found in Cass [7] or Koopmans [18].

Here one studies the following problem

$$\text{Max} \int_0^\infty e^{-\rho t} u(c(t))\, dt$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t)$$

$$k(0) = k_0$$

The Hamiltonian system becomes

$$\begin{aligned} \dot{q} &= \rho q - H_k = \rho q - qf' \\ \dot{k} &= H_q = f(k) - g(q) \end{aligned} \qquad \text{(A.1)}$$

where $q = u'(c)$, $g = (u')^{-1}$

Hence, $$J(t) = \begin{bmatrix} -f'(k) + \rho & -qf'' \\ -g' & f'(k) \end{bmatrix}$$

and $$Q(t) = \begin{bmatrix} -qf'' & \rho/2 \\ \rho/2 & -g' \end{bmatrix}$$

If one assumes the so called Inada "conditions" i.e.

$$\text{I(a)}, \quad \lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) < 0, \quad f''(k) < 0$$

and

$$\text{I(b)}, \quad u'(c) > 0, \quad u''(c) < 0, \quad \lim_{c\to 0} u'(c) = \infty$$