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CONSISTENCY OF DECISION PROCESSES

by

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Abstract. Often it is tacitly assumed for decision procedures that if the parts agree, then that is the outcome for the whole. If two subcommittees reach a common decision, then that should be the outcome when they join as a committee of the whole. If statistical tests conducted in two different locales reach the same conclusion, then that conclusion should hold for the aggregated data. It is shown, however, that this consistency property does not hold for large classes of standard statistical and decision procedures. Also, it is shown how the properties of this concept of "weak consistency" are related to questions of manipulability and to certain classes of decision paradoxes.

"The sum of the parts exceeds the whole." This often used cliche is intended to explain why certain procedures are successful. This success, of course, is predicated on a lack of conflict between the parts and the whole. Unfortunately, as I show here, such consistency need not accompany many standard decision procedures. Instead, even though each of the "parts" agrees with one another, the "whole" may disagree. For instance, it may be that each of two subcommittees prefers Ann for President, yet when they join as a committee of the whole, Kay is elected. It may be that when a statistical test is conducted in two different locales the results indicate that the new test drug is superior to the standard treatment, yet the aggregated data supports the superiority of the standard treatment. There are related manifestations of this phenomena. For instance, by voting for a candidate, we expect to improve her chances of winning. Yet, by voting against his top choice, or by staying home on election day, a voter may be rewarded with a personally more favorable outcome. Why? These and many other examples of the same kind assault our sense of fairness and common sense. We expect election and statistical procedures to be monotonic; we expect the direction of the whole to agree with that of the parts.

The purpose of this paper is to explain this conflict between the parts and the whole. I do so by defining a concept of "weak consistency" and establishing some of its surprisingly elementary but most useful properties. Then, these properties are used to identify those procedures that fall victim to the kinds of anti-monotonicity mentioned above. A major conclusion is that many decision processes fail to satisfy weak consistency. Indeed, by using the properties, it becomes easy to construct several new examples that violate the consistency between the parts and the whole.

Most of the results given here are illustrated with commonly used election procedures and with some simple statistical methods. The election procedures are used to illustrate how a specific theory can be created by combining the simple general properties developed here with the more specific structures of a given class of procedures. The statistical methods are used to indicate the generality of the approach. To introduce some of the procedures, as well as to provide concrete illustrations of the above comments, I start with examples.

Example 1. a. Suppose treatment "X" is being compared with the standard approach of chicken soup to cure the common cold. Both in Evanston and in Chicago, the data supports treatment X over chicken soup. Indeed, in Evanston, 33% (100 out of 300) of the "X" people regained health compared to only 30% (30 out of 100) of the chicken soup subjects. In Chicago, 50% (50 out of 100) of the X people regained health compared to the 46% (140 out of 300) of the chicken soup subjects. Although it appears that treatment X is superior, the aggregated data clearly supports the reversed outcome of chicken soup. (A higher fraction of the chicken soup people, 170/400, regained health than treatment X subjects, 150/400.) This disturbing feature is known as *Simpson's Paradox*. For a description of some of the history of this paradox, see Cohen [3] and the references he cites. For a description how to extend this paradox in several different ways and for a mathematical explanation that differs from the one offered here, see Saari [5,6].

b. Two subcommittees meet separately to recommend a new Dean from the final candidates of Alice (A), Barbara (B), and Cathy (C). Each committee uses a run-off election where, after a plurality vote is taken (each voter votes only for his top ranked candidate), the bottom ranked candidate is dropped. Of the two remaining top ranked candidates, the winner is the one who wins a majority vote. With this procedure, Alice is the top choice of both subcommittees. Yet, when the voters join together as a full committee that uses the same selection procedure, not only does Alice fail to win, but she does not even qualify for the run-off.

The rankings of the 13 voters in the first subcommittee are split in the following manner: 4 have the ranking A>B>C, and three each have the rankings B>A>C, C>A>B, C>B>A. For this subcommittee, A beats C in the run-off by a vote of 7 to 6. The rankings of the 13 voters in the second subcommittee have the following split: 4 have the ranking A>B>C while 3 each have C>A>B, B>C>A, and B>A>C. Here, A beats B in the run-off by a vote of 7 to 6. So, although A wins in both subcommittees, she is *bottom ranked* at the end of the first vote in the joint committee of all 26 voters. In the run-off, B is the decisive winner over C by a vote of 17 to 9.

Notice that the phenomenon illustrated by this example is not based on differences in subcommittee sizes or on wild disparities in the rankings. Both groups have the same number of voters with an almost identical split in the preference rankings. In fact the only difference is that the counterparts of the three members in subcommittee-one with the ranking C>B>A are the three members in subcommittee-two with the slightly different ranking B>C>A. Because of these strong similarities in the rankings, it is clear that the explanation for this behavior must rely upon the structure of the procedure.

c. A committee of three is formed to grant a tenured position to one of the three candidates Ann (A), Martha (M), and Lil (L). This committee uses a sequential vote where they first compare M and L, and the winner of the majority vote is advanced to be compared with A. The rankings of the three voters are L>M>A, M>A>L, and A>L>M. According to this procedure, the group's sincere winner is A. (L beats M in the first vote, and A beats L in the final vote.) However, during the first ballot, voter one votes *against* his top choice of L by voting for M. This forces the second ballot to be between M and A where M emerges as the winner. So, by voting against his top choice, voter 1 gains a personally more favorable outcome.

2. Weak Consistency

To provide a general model for the above kind of behavior, let $C^n = \{c_1,\ldots,c_n\}$ be a given set of $n\geq 2$ candidates (alternatives, etc.). Let $P(C^n)$ denote the set of all 2^n-1 non-empty subsets that can be constructed from C^n; $P(C^n)$ serves as the range for the procedures discussed here. The domain is represented by a set S on which there is defined a closed, associative, binary operation called "addition." An element of S is called a *profile*. The binary operation represents how profiles are combined.

Example 2. a. A typical choice of S is the positive orthant of R^k, denoted by R^k_+, where the binary operation "+" is vector addition. For instance, consider a voting problem with n candidates. There are n! different kinds of voters where each kind is determined by the voter's linear ranking (without ties) of the candidates. Let S be the lattice of integer points in $R^{n!}_+$, and let each coordinate axis of $R^{n!}_+$ be identified with one of the n! rankings of the candidates. In this way each component of a vector from $R^{n!}$ specifies the number of voters of that particular type. Vector addition characterizes the change in the number of voters.

b. For a statistical method, let S represent the integers in R^4_+. For $(x_1,x_2;x_3,x_4) \in R^4_+$, let x_1 and x_3 represent, respectively, the number of subjects returning to health using the new and the standard treatment, while x_2 and x_4 represent, respectively, the number of subjects that did not return to health. The vector sum corresponds to the aggregation of data from different experiments.

c. Let $S = R_+ x Si(k) = \{x = (u;x_1,\ldots,x_k) \in R_+ x R^k \mid x_i \geq 0; \Sigma x_i = 1$ and u a positive number$\}$. Thus Si(k) is the unit simplex in R^k_+. One can view $x \in S$ as being the l_1 polar coordinate representation of a vector $\mathbf{X} \in R^k_+$. The scalar u represents the l_1 length of **X** - the sum of the magnitudes of the components of **X** - while the Si(k) term represents the directional component - the vector **X**/u. (If **X** has integer components, as in part a, then u is a positive integer representing the number of voters. In this setting $S = Z_+ x Si(k)$.) The binary operation for the l_1 polar coordinates is defined to conform with vector addition in R^k_+. Namely, define the binary operation "+" of $\mathbf{x} = (u;x_1,..,x_k)$ $\mathbf{y} = (v;y_1,...,y_k)$ as the $(u+v;z_1,...,z_k)$ where $z_i = (u/(u+v))x_i + (v/(u+v))y_i$. Notice that the directional outcome is the convex combination of original two directions in Si(k) where the scalar multiples are determined by u and v. Also notice that because **p** ∈ Si(k) can be viewed as being a probability distribution, Si(k) can be treated as a discrete probability space.

Definition. A mapping

2.1 f:S --> $P(C^n)$ is called a *choice function*.[1] A choice function satisfies the *weak consistency condition* if

i. f is non-constant valued; i.e., the image of f contains at least two elements of $P(C^n)$, and

ii. if **p**, **p'** ∈ S are such that f(**p**) = f(**p'**), then f(**p**+**p'**) = f(**p**).

In other words, a choice function is weakly consistent if when the parts agree (f(**p**) = f(**p'**)), then this common conclusion is the outcome of the whole (f(**p** + **p'**) = f(**p**) = f(**p'**)). Consequently, this definition captures the class of choice functions that are spared the potential inefficiencies and indignities illustrated by Example 1. For many, if not most procedures, the potential of a conflict between the parts and the whole is sufficient reason to examine this concept. There exist, however, many other reasons; an important one is that weak consistency subsumes "strong versions" of several other concepts that have been widely studied. For instance, a choice function is *manipulable* if a voter can use a ranking that differs from his sincere one in order to change the outcome to a personally more favorable one. Call a choice function *strongly manipulable* with respect to a voter τ (or a small group of voters) if the following conditions hold: There exist α, β ∈ $P(C^n)$, α ≠ β, and profiles so that the group's profile

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1. I concentrate on choice functions only for convenience. Indeed, because the emphasis of my analysis is on the relationship between the binary operation and the geometry of sets in the domain S, one can replace the range set $P(C^n)$ with *any other set*; the basic ideas still hold. As one natural substitution, $P(C^n)$ could be replaced with the set of all linear rankings of the n candidates.

without voter τ is $\mathbf{p}$ where $f(\mathbf{p}) = \beta$. The sincere profile for voter τ (or for the small group) is $\mathbf{p}_1$ where $f(\mathbf{p}_1) = \alpha$. There exists another profile, $\mathbf{p}'$, for voter τ so that $f(\mathbf{p}') = \beta$ and $f(\mathbf{p}+\mathbf{p}') = \alpha$.

According to the definition, if f is strongly manipulable, then not only can the strategic voter manipulate the system (by using $\mathbf{p}'$ instead of the sincere $\mathbf{p}_1$) to attain his top choice of α, but, because $f(\mathbf{p}') = \beta$, he can do so in a manner that preserves the appearance of cooperating with the group choice of attaining β. It follows immediately from the definition that *if a procedure f is strongly manipulable, then f is not weakly consistent.* Conversely, if f is not weakly consistent, then it is reasonable to suspect that not only is f manipulable, but it is strongly manipulable with respect to some small group. (This is demonstrated in the examples of Section 3.)

Another interesting voting feature is the "Abstention Paradox"[2] where if a voter or a small group of voters abstain from voting, then outcome is more favorable to them. An extreme situation, *the strong abstention paradox*, is when the voter (or a small group of voters) does not vote, then the outcome is not only more favorable, but it is their top choice. To express this definition in a mathematical formulation, let $\mathbf{p}$ be the profile of the other voters, and let $\mathbf{p}_1$ be the profile of the single voter. For the strong abstention paradox to hold, it must be that $f(\mathbf{p})=f(\mathbf{p}_1)$ and (because when the group modeled by $\mathbf{p}_1$ votes they obtain a less satisfactory outcome) $f(\mathbf{p}+\mathbf{p}_1) \neq f(\mathbf{p})$. Therefore *if the strong abstention paradox is admitted by choice function f, then f does not satisfy the weak consistency condition.* Again, for many of the processes that are not weakly consistent, it is possible to create examples that exhibit the strong abstention paradox.

In much the same manner as used above, one can conceive of all sorts of other interpretations to describe consequences when the parts are in conflict with the whole. Weak consistency is a central unifying theme for this wide class of paradoxes.

I label weak consistency as a "weak" requirement because no conditions are imposed on what should happen if the "parts" are only in partial agreement. As such, weak consistency admits large classes of choice functions. In particular.

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2. This occurs in run-off elections ranked by the plurality vote. That this is true was noted by Smith [9] and developed by Brams and Fishburn [1]. This feature is related to other election behavior in Moulin [4]. In Saari [7,8], I extend these results to all voting methods and to many other procedures. In Theorem 3 of this current paper, the stronger result, related to weak consistency, is obtained.

it admits many of the statistical methods which would be excluded with a stricter definition. (For instance, differences in "confidence regions," "levels of significance," or "indifference regions" may preclude a procedure from satisfying a stronger consistency condition.) The three choice mappings described in the next example satisfy the weak consistency conditions, but each handles partial agreement in a different manner.

Example 3. **a.** A plurality election, where the top-ranked candidate(s) is selected, satisfies the weak consistency condition. Suppose that $f(\mathbf{p}) \neq f(\mathbf{p}')$, but that $f(\mathbf{p}) \cap f(\mathbf{p}') \neq \emptyset$. (For this to occur, the election outcome for at least one of these profiles had to end in a tie vote for the top place.) It is easy to show that $f(\mathbf{p}+\mathbf{p}') = f(\mathbf{p}) \cap f(\mathbf{p}')$. It is reasonable to use this condition to define a stronger form of consistency that specifies what should occur when the parts are only in partial agreement. However, as shown in Young [10], such a definition imposes severe restrictions on what choice functions are admissible. For additional discussion about this definition, see Moulin [4], where it is called the "Reinforcement Axiom."

b. Let $f(\mathbf{p})$ be the set of all candidates who are top ranked by at least one voter. For example, if $\mathbf{p}$ represents the two voters with the rankings $c_1 \succ c_2 \succ c_3$, $c_3 \succ c_1 \succ c_2$, then $f(\mathbf{p}) = \{c_1, c_3\}$. It is obvious that f satisfies the weak consistency condition, and that $f(\mathbf{p}+\mathbf{p}')$ is the union $f(\mathbf{p}) \cup f(\mathbf{p}')$.

c. Let $n \geq 5$, and let $f(\mathbf{p}) = \{c_i, c_j\} \in P(C^n)$ iff these two candidates are the two top ranked candidates for *each* voter represented in the profile $\mathbf{p}$; otherwise let $f(\mathbf{p}) = C^n$. This choice function is weakly consistent, but unless $f(\mathbf{p}) = f(\mathbf{p}')$, the union $f(\mathbf{p}) \cup f(\mathbf{p}')$ is a proper subset of $f(\mathbf{p}+\mathbf{p}')$.

Definition A set C in S is *algebraically closed* with respect to the binary operation if whenever $\mathbf{p}, \mathbf{p}' \in C$, then $\mathbf{p} + \mathbf{p}' \in C$.

Theorem 1. A choice function $f: S \longrightarrow P(C^n)$ satisfies the weak consistency condition iff for each $\alpha \in P(C^n)$, $f^{-1}(\alpha)$ is algebraically closed with respect to the binary operation.

The proof of this theorem is almost a tautology, so it is left to the reader. But, even though Theorem 1 is easy to prove, it has important consequences. As I show, this theorem completely explains the above examples.

The strength of Theorem 1 derives from the geometry of the algebraically closed sets. For instance, if B is a set in S then define the *cone of B*, to be $Co(B) = \{n\mathbf{p}_1+m\mathbf{p}_2 | n,m \in Z_+, \mathbf{p}_1, \mathbf{p}_2 \in B$, and $n\mathbf{p}$ is the n fold sum of $\mathbf{p}$ with itself$\}$. It is clear that *B is algebraically closed iff Co(B) = B.*[3] So, by emphasizing the geometry of the cones, one can determine whether a choice method is weakly consistent.

Simpson's Paradox.

To demonstrate how to use Theorem 1, I use it to explain Simpson's paradox. I do so with two different representations of the statistical model. One of them leads to (what appears to be new) necessary and sufficient conditions to avoid Simpson's paradox in the design of statistical experiments. A second feature of this discussion is to show how the choice of a mathematical representation significantly effects the analysis. For instance, the first representation admits a simple binary operation, so the emphasis is on the geometry of the inverse images of the choice function. The second representation admits simple inverse images, so the emphasis is on the geometry of the cones generated by the binary operation.

For the first representation let $S = \{\mathbf{x} = (x_1, x_2; x_3, x_4) \in R^4_+ | x_i$ is a non-negative integer$\}$ where the binary operation is vector addition. The range space is based on the two alternatives $C^2 = \{c_1, c_2\}$. As a typical example, c_1 represents a test group where x_1 and x_2 are, respectively, the number of successes and failures. Similarly, c_2 represents the control group where x_3 and x_4 denote, respectively, the number of successes and failures. The choice function, f, selects the group with the largest fraction of successes, so

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$$\begin{aligned} f(\mathbf{x}) &= c_1 \text{ if } x_1/(x_1+x_2) > x_3/(x_3+x_4), \\ f(\mathbf{x}) &= c_2 \text{ if } x_1/(x_1+x_2) < x_3/(x_3+x_4), \text{ and} \\ f(\mathbf{x}) &= \{c_1, c_2\} \text{ if } x_1/(x_1+x_2) = x_3/(x_3+x_4). \end{aligned}$$

According to Theorem 1, this procedure is weakly consistent iff each of the three sets $f^{-1}(c_1)$, $f^{-1}(c_2)$, and $f^{-1}(\{c_1, c_2\})$ are algebraically closed with respect to vector addition. To analyze these inverse sets, it suffices to examine the boundary set $f^{-1}(\{c_1, c_2\})$.

If f were defined over all of R^4_+, rather than just the integer lattice

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3. Suppose f is not weakly consistent, but there are choices of $\alpha \in P(C^n)$ so that $Co(f^{-1}(\alpha)) = f^{-1}(\alpha)$. This condition ensures that the conflict between the parts and the whole will not occur if the parts agree on α.

points, then $f^{-1}(\{c_1,c_2\})$ would be a three dimensional surface. As the binary operation is vector addition, a three dimensional surface is equal to its cone (i.e., it is algebraically closed) only if it is a portion of a three dimensional linear subspace. This is not the case for $f^{-1}(\{c_1,c_2\})$ as it is the non-linear surface $x_1x_4-x_2x_3 = 0$. Indeed, this highly non-linear surface is given by the union of the products of the two hyperbolas $x_2x_3 = c = x_1x_4$ as c varies over the positive values. Thus, it follows immediately that when restricted to the integer lattice points, this surface is not algebraically closed. Moreover, by using the "bulge" caused by the nonlinearity of the product of the hyperbolas, one can construct many different examples of this paradox. To do so, simply select two points that are on one side of the surface (so they are in the same $f^{-1}(c_j)$ set), but where the vector sum penetrates the bulge to emerge on the other side.

In my second representation of this problem, the focus of the analysis is on the geometry of the cone defined by the binary operation. This representation uses the l_1 polar decomposition, $(u;x,y)$, of the data from each sample group where $u \in Z_+$ is the total number of subject, $x \geq 0$ is the fraction of the u subjects judged successful, and $y = 1-x \geq 0$ is the fraction judged unsuccessful. With two groups, say a control and a test group, the data can be represented as $(u,v;x_1,x_2;x_3,x_4)$ where $(u;x_1,x_2)$ and $(v;x_3,x_4)$ represent, respectively, the data from each of the two groups. The three regions defined by $P(C^2)$ are in the unit square $U = [0,1]\times[0,1]$ defined by the (x_1,x_3) coordinates where $f^{-1}(c_1)$ is the set $x_1 > x_3$, $f^{-1}(c_2)$ is the set $x_1 < x_3$, and $f^{-1}(\{c_1,c_2\})$ is the line segment $x_1 = x_3$. These three convex regions in U are defined and separated by the diagonal $x_1=x_3$. It remains to show why these convex regions are *not* algebraically closed with respect to the binary operation.

The binary operation "+", defined by the l_1 polar coordinates, is $(u,v;x_1,x_2;x_3,x_4) + (u',v';x_1',x_2';x_3',x_4') = (u+u',v+v';sx_1+(1-s)x_1', sx_2+(1-s)x_2;tx_3+(1-t)x_3',tx_4+(1-t)x_4')$ where $s = u/(u+u')$ and $t = v/(v+v')$. Because s and t are determined by different independent variables, their values are independent of each other. This means that the cone defined by two points (x_1,x_3), $(x_1',x_3') \in U$ is the unique rectangle (with edges parallel to the coordinate axis) where these points serve as two of the vertices. (See Figure 1.) It immediately follows that for any set B, Co(B) is the union of all of the rectangles defined by pairs of points in B.

Using the above, it now follows that $Co(f^{-1}(c_1)) = Co(f^{-1}(c_2)) = Co(f^{-1}(\{c_1,c_2\}) = U$. Thus, these regions are not algebraically closed, so f is not weakly consistent -- Simpson's Paradox holds.

Creating and avoiding Simpson's Paradox.

Suppose no restrictions are imposed on the values of u and v -- the number of subjects assigned to each locale or group. Such a situation is typical when analyzing statistical data from cohort groups from different countries, or when it is morally wrong to assign subjects to a particular group as true for, say, certain medical experiments. In such situations there are no restrictions on the (rational) values of s and t in the unit interval (0,1). Therefore, for a given set B, the (rational) points in Co(B) define all attainable outcomes. Using this fact, it becomes easy to construct examples illustrating the paradox. To do so, select two points (x_1,x_3) and (x_1',x_3') in the same triangular region where the bulge of the rectangle $Co(\{(x_1,x_3),(x_1',x_3')\})$ extends into the other triangular region. (See Figure 1.) By choosing a point in the bulge that is in the second region, by finding the corresponding value of (s,t) for this point, and by translating the (s,t) value into (u,v) (u',v') values, an example is created. Moreover, it follows from this construction (and Figure 1) that examples can be created whereby both subgroups reach the same decision, but the reversed decision wins in the aggregate by as large of a fraction (smaller than unity) as one desires.

Conversely, there are many situations where it is appropriate to assign the values of (s,t). When such a situation applies, the (s,t) variables can be used as control parameters that are selected to avoid the paradox. The idea is to recognize that avoiding the paradox is equivalent to battling the bulge of the cone. A bulge does not occur iff the only attainable points are on the straight line connecting the two base points (x_1,x_3) and (x_1',x_3'). A necessary and sufficient condition for this to hold is that s = t; namely, there is a restriction on u and v so that the binary operations only admits convex combinations. Thus, if no restrictions are imposed on u+v, u'+v', then a *necessary and sufficient condition to avoid Simpson's paradox is to require* $u/(u+u') = v/(v+v')$; *e.g.*, $u'/u = v'/v$. This means that in both locales, if the same ratio of the subjects are in the test group, then the paradox cannot occur.

If the base points are bounded away from the line $x_1=x_3$ and if it is possible to limit the size of the bulge, one can derive alternative, weaker conditions to avoid this paradoxical behavior. For example, suppose there is an upper bound, M, on the magnitudes of u+v and u'+v'. This requires the denominators of the fractions in (x_1,x_3) and (x_1',x_3') to be no larger than M. If both points are in the same region, say $f^{-1}(c_1)$, then these points are bounded away from the line $x_1=x_3$ by a fixed value δ that depends on M. Thus, any condition limiting the bulge to be bounded by δ avoids Simpson's paradox. For instance, if the inequality $|s - t| < 1/(M-1)$ is satisfied, then the paradox does not occur. In turn, this leads to more relaxed bound

2.3 $$|(u'/u)-(v'/v)| < [1+(u'/u)][1+(v'/v)]/(M-1)$$

on the assignment ratios of subjects to groups.

Comment on statistical procedures

The second representation of Simpson's paradox explains why one must expect that many of the statistical procedures are not weakly consistent. After all, many statistical procedures are based on probability distributions where the combination of probabilities, according to Bayes' rule, is similar to the one given above. But this non-linear combination of probabilities defines the binary operation on S; an operation the leads to the bulge in the cone. The next example illustrates another class of non-linear operations of the kind found in statistics and in voting.

Generalized majority voting

Simple majority voting between a pair of candidates is where the winner is determined by which candidate receives the highest percentage of the votes cast. Majority voting is defined only for pairs of candidates, but there are ways to generalize this procedure in order to rank several candidates. For instance, to define the *pure majority voting scheme*, let $p_{i,j}$ be the fraction of all voters that prefer c_i to c_j, and assign to c_i the point total $\Sigma_k\ p_{i,k}$. The winning candidate is the one with the largest point total.

There may be reasons to assign points to candidates in a manner that differs from the above. For instance, one might wish to reward a candidate who receives more than 2/3 of the total vote in a pairwise comparison by assigning the candidate more than 2/3 points. One such criterion might assign zero points for $p_{i,j} < 1/3$, 1/2 points for $1/3 \le p_{i,j} \le 2/3$, and 1 point for $p_{i,j} > 2/3$. Here the assignment of points is established by a non-decreasing step function. A related scheme, developed to avoid some of the cyclic effects of pairwise voting, is described in Capplin and Nalebuff [2]. The following definition is a more general representation that includes most commonly used versions.

Definition. Let g be a non-constant, non-decreasing function from [0,1] to [0,1] such that

2.4 $g(p) + g(1-p) = 1.$

A *generalized majority vote* is where the number of points assigned to c_i is $\Sigma\, g(p_{i,j})$. The candidate with the largest point total is declared the winner. If $g(p) = cp$, $c>0$, the procedure is the *pure majority vote scheme.*

It is not overly difficult to show that a pure majority vote scheme is weakly consistent. Are there any other weakly consistent generalized majority voting methods?

Theorem 2. A generalized majority vote scheme satisfies weak consistency iff it is a pure majority vote scheme.

As shown in the outline of the proof (Section 4), the procedure satisfies weak consistency iff $\nabla\Sigma\, g(p_{i,j})$ is a constant vector. This requirement forces g to define the linear procedure of pure majority voting. So, as true for Simpson's paradox and many other statistical procedures, it is the nonlinearity that creates the gap between the parts and the whole. Incidentally, pure majority voting is the only generalized majority voting scheme that can be represented as a "positional voting procedure;" the class of "linear" voting methods described in the next section.

3. Convexity and Election Paradoxes.

To effectively illustrate Theorem 1, I now concentrate on a single topic. My goal is to demonstrate how the general principle of Theorem 1 can be combined with the structures of a specified class of procedures to characterize which of those procedures satisfy weak consistency. To do this, I emphasize those decision procedures based on the voters' positional voting rankings of the candidates. This large class includes run-off elections, sequential ballots, tournaments, etc. It is necessary to introduce some of the technical structures for voting in order to state the main result of this section. The main result, however, is quite easy to use. (See Example 5 to get an indication of the kinds of possible results.)

To get a loose but intuitive flavor of my main result, consider a run-off election for C^3 where the top two ranked candidates at the end of the first vote are advanced to the run-off. If this procedure satisfies weak consistency, then, according to Theorem 1, $f^{-1}(c_i) = Co(f^{-1}(c_i))$, $i = 1,2,3$. To determine the structure of this set, note that the possible ways c_1 could be elected are if at the end of the first stage [she and c_2 are the two top ranked candidates *and* she beats c_2 in a majority vote] *or* [she and c_3 are the two top ranked candidates and she beats c_3 in a majority vote]. Loosely speaking, it follows from Theorem 3 that a run-off is not weakly consistent because $f^{-1}(c_1)$, cannot be described without using the word "or." The connection "or" is what violates the algebraic closure of $f^{-1}(c_1)$.

Similarly, consider the sequential election where the majority winner of an election between c_1 and c_2 is advanced to be compared with c_3. Here $f^{-1}(c_1)$ has the description [c_1 beats c_2] *and* [c_1 beats c_3]. As I show, it is essentially because this description uses only the connection "and" and no "ors", that $Co(f^{-1}(c_1)) = f^{-1}(c_1)$. Consequently, at least for those **p**'s where $f(\mathbf{p}) = c_1$ (or c_2), if the parts agree, then this is the outcome of the whole. However, f is not weakly consistent because the description of $f^{-1}(c_3)$ is [c_1 beats c_2 *and* c_3 beats c_1] *or* [c_2 beats c_1 *and* c_3 beats c_2]. Again, the needed connection "or" causes weak consistency to be violated.

For the kinds of election procedures discussed in this section, a set of profiles is algebrically closed iff it is convex. It is well known that the union of two convex sets need not be convex. Thus, the connection "or", which corresponds to the union of sets, is what violates the algebraic closure for the above examples. A word of caution; one can construct weakly consistent examples where the outcome is defined with the word "or;" these examples are identified with situations where the union of sets is convex. (Most of these examples, however, admit other representations that avoid this conjunction.) The following technical description develops simple conditions to check this convexity condition.

Positional voting methods

To start I review some definitions and some recent developments about positional voting. Recall that a *positional voting method* for the n candidates C^n is defined by a *voting vector* $\mathbf{W} = (w_1,..,w_n)$ where, without loss of generality, $w_i \geq w_{i+1}$ for i = 1,..,n-1, and $w_1 > w_n = 0$. In the tabulation of each voter's ballot, w_i points are assigned to the i^{th} ranked candidate. The ranking of each candidate is determined by the sum of the points she receives. In this manner, the vector (1,0,...,0) corresponds to the plurality vote while $B_n = (n-1,n-2,..,0)$ defines the Borda Count (BC). (More generally, a BC vector is any voting vector where the differences between successive weights are a fixed constant value. The BC is equivalent to a pure majority voting scheme.)

The outcome for each of the procedures mentioned in the introductory paragraph of this section is based on the positional election rankings of several sets of candidates. Thus, for what follows, we need to understand what relationships exist among the same voters' sincere election rankings of different subsets of candidates. This relationship is characterized in Saari [8,9], and I repeat those basic results needed here. To start, note that to have an election we need at least two candidates. So, from $C^n = \{c_1,..,c_n\}$ list the $2^n-(n+1)$ subsets of two or more candidates in some order as $S_1,..,S_{2^n-(n+1)}$. For each subset S_i, assign a voting vector $\mathbf{W}_i$ that is to be used to tally the election for S_i, and let the *system vector* be $\mathbf{W}^n = (\mathbf{W}_1,...,\mathbf{W}_{2^n-(n+1)})$. Let B^n denote the special case where a BC is used to tally the elections for all sets of three or more candidates.

Let R_j be the space of all possible election rankings for S_j; namely, R_j is the collection of all possible linear rankings (including those with ties) admitted by the set S_j. (As an example, if $S_1 = \{c_1,c_2\}$, then $R_1 = \{c_1 \succ c_2,\ c_1=c_2,\ c_2 \succ c_1\}$. If S_j has three candidates, then R_j has 13 rankings; 3! are linear rankings with no ties, 3! are rankings with one tie, and one is the complete indifference ranking with a three way tie.) Let U^n, the *universal set*, be the cartesian product $R_1 x \ldots x R_{2^n-(n+1)}$. By definition, an element of U^n is a listing of rankings where the i^{th} ranking is for the set S_i. Notice that U^n contains all possible listings and that there is no assumption that any particular listing has anything to do with election outcomes.

As described in Example 2a, with n candidates, there are n! linear rankings without ties. Let each of these rankings be identified with a coordinate axis of $R^{n!}$. In the computation of an election outcome, we only need the fraction of all voters that are of a particular type. This suggests using the l_1 polar representation of the integer vectors where a profile, $(u,\mathbf{p}) \in Z_+ x Si(n!)$ and the vector $\mathbf{p}$ specifies the fraction of all voters that are of each type. The binary operation is as defined in Example 2c, so the direction component (the $Si(n!)$ component) of $(u,\mathbf{p})+(u',\mathbf{p}')$ is a convex combination of $\mathbf{p}$ and $\mathbf{p}'$. Once $\mathbf{p}$ and the system voting vector $\mathbf{W}^n$ are given, the *election mapping* $F(\mathbf{p};\mathbf{W}^n)$ is the listing in U^n that gives the sincere election ranking for each subset.

Example 4. Consider the profile $\mathbf{p}$ where 6 voters have the ranking $c_1 \succ c_2 \succ c_3$, five voters have the ranking $c_2 \succ c_1 \succ c_3$, and four voters have the ranking $c_3 \succ c_2 \succ c_1$. Let $S_1 = \{c_1,c_2\}$, $S_2 = \{c_1,c_3\}$, $S_3 = \{c_2,c_3\}$, $S_4 = \{c_1,c_2,c_3\}$, and let the system vector be $\mathbf{W}^3 = (1,0;1,0;1,0;2,1,0)$. This choice of a system vector determines that the rankings of the sets of two candidates by a majority vote (1,0), while the ranking for the set of all three candidates is determined by (2,1,0), the BC. A simple computation shows that $F(\mathbf{p};\mathbf{W}^3) = (c_2 \succ c_1,\ c_1 \succ c_3,\ c_2 \succ c_3,\ c_2 \succ c_1 \succ c_3)$.

The *dictionary* determined by a system vector $\mathbf{W}^n$ is the collection of all possible listings of election rankings that could ever occur with some profile. Namely, $D(\mathbf{W}^n) = \{F(\mathbf{p};\mathbf{W}^n) \mid \mathbf{p}$ is a profile of a finite number of voters$\}$. An element of a dictionary is called a *word*, and each of the $2^n-(n+1)$ rankings in a word is called a *symbol*. In Example 4, the specified election outcome is a word in $D(\mathbf{W}^3)$, while the ranking $c_2 \succ c_3$ is a symbol for S_3.

By definition, $D(\mathbf{W}^n)$ is a subset of U^n. The issue is to determine how large or small of a subset it is. For instance, if the dictionary is a "large" subset of U^n then the system admits a wide class of paradoxes; if it is a small set, then only a few unexpected outcomes can occur. My answer for this question involves the facts that $\mathbf{W}^n$ is a vector in the positive orthant of an Euclidean space and that an algebraic set is a small, lower dimensional subset of an Euclidean space determined by the zeros of a finite set of polynomials.

Theorem 3. (Saari [8,9]). a. Let $n \geq 3$. There exists an algebraic set, $\mathfrak{a}^n$, such that if $\mathbf{W}^n \notin \mathfrak{a}^n$, then $D(\mathbf{W}^n) = U^n$.

b. For all $n \geq 3$, $B^n \in \mathfrak{a}^n$. Indeed, for $n = 3$, $D(B^n)$ is the only dictionary that is a proper subset of U^3. For all $n \geq 3$, if at least one component of $\mathbf{W}^n$ is not a Borda vector, then $D(B^n)$ is a proper subset of $D(\mathbf{W}^n)$.

Part a of this theorem asserts that for almost all choices of a system vector, any listing from U^n can represent a sincere election outcome. This means that any imagined paradox actually can occur! As an extreme, this theorem permits one to use a random number generator to choose a ranking for each of the subsets of candidates. In this manner, a randomly generated listing from U^n is created. Even though the rankings for each subset may have absolutely nothing to do with one another, the theorem asserts that there is a profile so that the sincere election ranking for each set of candidates, as determined by $\mathbf{W}^n$, is the specified, randomly generated one.

Part b of the theorem means that the only method that could possibly provide any relief from permitting randomly generated election outcomes is the BC - the Borda Count. This theorem also means that if a word from the BC dictionary can be used to indicate a fault with the BC, then, because the exact same word is in all other dictionaries, the exact same criticism holds for all other system vectors.

According to my earlier definition, a choice function is a mapping from $Si(n!)$ to $P(C^n)$. I now consider those choice functions where the outcome is determined by the voters' positional vote rankings.

Definition. A *positional choice function* based on the system vector W^n is a mapping $f:Si(n!) \dashrightarrow P(C^n)$ that is that can be expressed as the composition $f = G \circ F$ where F is the election mapping and G is a non-constant mapping from $D(W^n)$ to $P(C^n)$.

So, a profile and a system voting vector are used to find the election ranking for all subsets of candidates, $F(p)$. The winning candidate is determined by these rankings as $G(F(p)) \in P(C^n)$. (In practice, not all subsets of candidates need to be ranked. Which sets need to be ranked depends upon the definition of G.) Notice that the G function is a formalized version of the usual definition of an positional choice function. This description explains what sets of candidates needs to be ranked based on what kinds of election rankings result for other subsets of candidates; the formal definition, mapping G, has the same description but in terms of the words of a dictionary. (Often such methods are described in terms of a tree structure.) For instance, consider a run-off election where the two top ranked candidates are advanced to the second stage. The G mapping uses only the symbols of a word describing the ranking of the set of all n candidates and the rankings of the pairs. The ranking of the set of all candidates dictates which symbol (which pair of candidates) G should use next. Namely, the two top ranked candidates for the total set determines which pair is to be examined. So, for the word $(c_1 > c_2 > c_3;\ c_1 > c_2;\ c_3 > c_1;\ c_2 > c_3)$, c_1 is the winner, where only the first and the second symbols are used. On the other hand, for the closely related word $(c_1 > c_3 > c_2;\ c_1 > c_2;\ c_3 > c_1;\ c_2 > c_3)$, c_3 is the winner and only the first and third symbols of this word are needed.

Positional choice functions include tournaments, sequential voting, and various kinds of run-off elections, so this large class includes many (if not most) of the commonly used choice functions. My concern is to characterize the positional choice functions that satisfy the weak consistency condition. This involves two steps. The first is to find the inverse images of G. This involves characterizing all words in the dictionary $D(\mathbf{W}^n)$ that lead to the same outcome in $P(C^n)$. Next, one must determine the properties of the set in Si(n!) that leads to these words; namely, one must characterize the sets

3.1 $F^{-1}(G^{-1}(\alpha))$ for $\alpha \in P(C^n)$.

Because the binary operation in Si(n!) defines a point on the convex combination between two points, *a necessary and sufficient condition for these sets to be algebraically closed is that they are convex.*

The analysis of which sets in Si(n!) are convex is greatly simplified by the facts that Si(n!) is part of the affine space and that the tallying procedure is a linear mapping. This leads to a natural decomposition of F in terms of an *election tally mapping*, T, and an *ordinal representation mapping*, Od, that converts the election tally to the ordinal election rankings. Both mappings are described in the following section.

A geometric representation for election tallies

Before defining the election tally mapping, the appropriate range space needs to be introduced. If there are n candidates, then candidate c_i can be identified with the coordinate axis x_i of R^n_+, i = 1,...,n. A binary ranking on R^n_+ can be defined by saying $c_i > c_j$ iff $x_i > x_j$, and $c_i = c_j$ iff $x_i = x_j$. In this manner, the n(n-1)/2 hyperplanes $x_i = x_j$ divide R^n_+ into cones, or *ranking regions*. For instance, if A denotes the ranking $c_1 > c_2 > \ldots > c_n$, it corresponds to the ranking region $x_1 > x_2 > \ldots > x_n$.

One advantage of this R^n_+ representation is that it permits the tally of ballots to be represented by vector sums. This is because the voting vector $\mathbf{W}$ is in the closure of the ranking region associated with A, and it can be identified with how an "A" voter's ballot would be tallied. In general, if $\pi(A)$ denotes a permutation of A, then there is a permutation of $\mathbf{W}$, $\mathbf{W}_{\pi(A)}$, to represent how a $\pi(A)$ voter's ballot would be tallied. Continuing, if $n_{\pi(A)}$ represents the fraction of all voters with the ranking $\pi(A)$, then the election outcome is given by the ranking region that contains the vector sum

3.2 $\Sigma_{\pi(A)} \; n_{\pi(A)} \; \mathbf{W}_{\pi(A)}$,

where the summation is over all n! permutations of A.

Because $\{n_{\pi(A)}\} \in$ Si(n!), the vector sum is a convex combination of the vectors $\{\mathbf{W}_{\pi(A)}\}$. Therefore this sum is on the simplex $\Sigma x_i = \Sigma w_i$. Without loss of generality, assume that $\Sigma w_i = 1$. This has the effect of making the baricentric division (given by the ranking regions) of Si(n) serve as the space of election outcomes. To identify Si(n) with the set C^n, denote this simplex as $Si(|C^n|)$. Extending the sum in 3.2 from the rational points in Si(n!) to all points leads to my representation of the *election tally mapping* $F(\mathbf{p};\mathbf{W},C^n)):Si(n!) \to Si(|C^n|)$ as

3.3 $$T(\mathbf{p};\mathbf{W},C^n) = \Sigma_{n(A)} \; p_{n(A)} \; \mathbf{W}_{n(A)}$$

where $\mathbf{p} = \{p_{n(A)}\}$.

The definition of $T(\mathbf{p};\mathbf{W},C^n)$ holds for any set S_j of two or more candidates to define $T(\mathbf{p};\mathbf{W}_j,S_j)$ as a mapping from Si(n!) to $Si(|S_j|)$. For a voting vector $\mathbf{W}^n = (\mathbf{W}_1,\ldots,\mathbf{W}_{2^n-(n+1)})$, define the *system tally mapping*

3.4 $$T(\mathbf{p};\mathbf{W}^n) = (T(\mathbf{p};\mathbf{W}_1,S_1), \ldots, T(\mathbf{p};\mathbf{W}_{2^n-(n+1)},S_{2^n-(n+1)}))$$

where $T: Si(n!) \to Si(|S_1|) \times \ldots \times Si(|S_{2^n-(n+1)}|)$. Let Ra represent the product range space. Notice that there is an mapping $Od:Ra \to U^n$, *the ordinal representation* mapping, that takes the product of ranking regions in Ra to a listing in U^n. This mapping converts the election tally for each of the sets to the ordinal election ranking for this set. Therefore, $F = Od^{\circ}T$.

To illustrate the definitions, note that in Example 4, $T(\mathbf{p})$ = ((6/15,9/15), (11/15,4/15), (11/15, 4/15), (17/35, 20/35, 8/35). So, for the $\{c_1,c_2\}$ outcome, the image of Od((6/15,9/15)) is $c_2 > c_1$. In general, $Od^{\circ}T$ has the outcome specified above.

Positional Choice Procedures

My first result characterizes all positional choice procedures that satisfy the weak consistency condition. After I give a series of elementary propositions and statements, I illustrate how to use this theorem.

Theorem 4. a. Let $n \geq 2$ and let $f=G^{\circ}F = G^{\circ}(Od^{\circ}T):Si(n!) \to P(C^n)$ be a positional choice method that is based on the system voting vector $\mathbf{W}^n \notin \alpha^n$. The choice procedure f is weakly consistent iff for each $\beta \in P(C^n)$ the set $[G^{\circ}Od]^{-1}(\beta)$ is a convex set in Ra.

b. If $\mathbf{W}^n \in \alpha^n$, then the image of T, $T(Si(n!);\mathbf{W}^n)$, is a lower dimensional affine hyperplane passing through the point of indifference of Ra. The assertion of part a holds if $[G^{\circ}Od]^{-1}(\beta) \cap T(Si(n!))$ is a convex set. Moreover, if a positional choice procedure f based on the system voting vector B^n is not weakly consistent, then f is not weakly consistent when it is based on any other choice of a system vector $\mathbf{W}^n$.

As asserted at the beginning of this section, it now becomes reasonably straightforward to determine whether a procedure is weakly consistent. Part a asserts that for most choices of $\mathbf{W}^n$, all that is required is to take the verbal description of a positional choice method and determine whether its geometric description in Ra is a convex set. According to part b, the general situation is handled by considering whether a choice function is weakly consistent when it is based on $\mathbf{B}^n$. Such an analysis involves using the characterization of the Borda Dictionary given in Saari [9].

Most procedures are defined in terms of what should be done if different election rankings occur. This means that the set of words in $D(\mathbf{W}^n)$ that lead to the outcome α, $G^{-1}(\alpha)$, usually is easy to determine for any given f. Moreover, the mapping Od^{-1} just identifies words in $D(\mathbf{W}^n)$ with the corresponding regions in Ra, so this inverse set also is easy to compute. Thus the power of this theorem is that it involves easily checked conditions in Ra rather than depending upon conditions for the far more complicated space Si(n!). Indeed, in this manner, the several new results given in Example 5 are easy to find. Example 5 is based on the following series of simple technical propositions.

Proposition a. The T image of Si(n!) into a set $Si(|S_j|)$ is a convex set with the complete indifference point is an interior point. For all $\mathbf{W}^n \notin \alpha^n$, the T image of the interior of Si(n!) to Ra is a convex open set where the point giving the ranking of complete indifference for all subsets is an interior point.

b. The Od inverse image of a symbol is a convex set in Ra.

c. As the intersection of convex sets yields a convex set, the Od inverse image of a word is a convex set.

d. Let K represent a subset of k candidates that are in S_j. The Od inverse image of all symbols (rankings) for S_j where the k top ranked candidates are from K is a convex set. Similarly, the Od inverse image of all symbols for S_j where the k bottom ranked candidates are from K is a convex set.

e. The direct product of two convex sets is a convex set. The product of two sets where one is not convex is not convex.

f. The union of two convex sets need not be convex. In particular, the Od inverse image of all words where the symbols for S_j, $|S_j| \geq 3$, have either one of two specified candidates as top ranked is not convex.

Example 5. a. Let S_j be a subset of at least three candidates, and let f be the choice procedure that selects the top ranked candidate(s) from a positional election. This choice of f is weakly consistent. This is because of part c of the Proposition. Likewise, if f selects the k top ranked or if it selects the k bottom ranked candidates from an positional election, f is weakly consistent. On the other hand, if f selects the second ranked candidate, then f is not weakly consistent. This is most easily seen for n = 3 by using Figure 2. Notice that $f^{-1}(c_2)$ are regions 1 and 4 from the figure; that is, the regions $c_1 > c_2 > c_3$ *or* $c_3 > c_2 > c_1$. Quite obviously, the union of these two regions is not convex.

b. Consider a sequential, or agenda election where the winner of a majority vote between c_1 and c_2 is advanced to be compared with c_3. The relevant space representing this procedure involves the rankings of three pairs of candidates. As Si(2), the space for a pair of candidates, is one dimensional, the product space can be identified with R^3. Here, let the positive x, y and z axis represent, respectively, the rankings $c_1 > c_2$, $c_2 > c_3$, and $c_3 > c_1$. The negative axes represent the reversed rankings of $c_2 > c_1$, $c_3 > c_2$, and $c_1 > c_3$. Now, $f^{-1}(c_3)$ is the union of the two regions $\{c_1 > c_2\} \cap \{c_3 > c_1\}$ and $\{c_2 > c_1\} \cap \{c_3 > c_2\}$. Although each of these regions is convex (according to the proposition), the union is not. To see this, note that the first region corresponds to the region $\{x>0, z>0\}$ while the second region corresponds to $\{x<0, y<0\}$. This union consists of 4 orthants of R^3. This union contains a "corner" where the positive x axis is the edge, so the union is, most clearly, not convex.

By mimicking the procedure used to design examples of Simpson's paradox, it is easy to create an example for this procedure to illustrate that weak consistency is not satisfied. Just choose two points **p** and **p'** - one in each of these two regions - where the convex combination is outside of the union. The l_1 radius (the number of voters) for each group is selected so that the convex sum is in a region outside of the union. (The convex hull of these regions meets all 8 orthants of R^3. Therefore, this construction can be achieved to allow any desired candidate to be the winner.) So, although each group chooses c_3 as the winner, when they combine into a single group, a different candidate wins. One of these profiles could be selected so that it is in a region $c_3 > c_2 > c_1$ $(x<0, y<0, z>0)$. Therefore, if **p'** is sufficiently close to the boundary of the union of the region, **p** could be selected to be a single voter with c_3 as the top ranked candidate. However, when **p** is combined with a group, **p'**, that prefers c_3, the outcome differs from c_3. If **p** represents a sincere ranking, then this illustrates the strong abstention paradox; if **p** represents a strategic ranking, then this illustrates a strongly manipulable situation with a single voter.

c. Consider a run-off for C^3 where the two top ranked candidates are advanced to compete in a majority vote election. the set $G^{-1}(c_1)$ is where [{c_1 and c_2 are the two top ranked candidates}$\cap${$c_1 > c_2$}] *or* [{c_1 and c_3 are the two top ranked candidates}$\cap$ ($c_1 > c_3$}]. Each bracketed region is the intersection of convex regions, so it is convex. However, the union is not convex, so a run-off is not weakly consistent. To see the non-convexity, notice that the region in the space for three candidates, Si(3), is [c_1 and c_2 are the two top ranked candidates] *or* [c_1 and c_3 are the two top ranked candidates]. In Figure 2, this is everything except regions 4 and 5 -- a non-convex set. Notice that the lack of convexity is manifested by admitting points **p** and **p'** whereby for each profile c_1 is one of the two top ranked candidates in an election, but the convex combination forces c_1 to bottom place. This is how Example 1b was created.

d. How does one create a weakly consistent procedure? To show how to use Theorem 4 to do this, I use the important choice procedure, f_1, that selects the Condorcet winner when one exists. (Recall, c_i is a Condorcet winner if c_i beats all other candidates in pairwise elections.) Can one extend the definition of f_1 so that it applies when a Condorcet winner does not exist? I show for n = 3 the kinds of restrictions that must be imposed on the extension if the extension is to be weakly consistent. To skip some steps, extend f_1 so that it selects c_i if c_i is a Condorcet winner, or if c_i is the only candidate that beats one of the candidates and ties with the remaining candidate.

Recall that the product space for the three pairs, Ra, can be represented by R^3 where the x, y, z axis represent, respectively, the rankings of {c_1,c_2}, {c_2,c_3}, {c_3,c_1} and where positive values indicate that the first listed candidate wins. To determine how to define f_1 when no Condorcet winner exists, it is necessary to compute what regions in R^3 remain unassigned to an outcome. These are the regions corresponding to cycles, such as {$c_1 > c_2$, $c_2 > c_3$, $c_3 > c_1$}, and to three situations where two candidates are tied and both beat the third candidate. Thus, after computing the sets $f^{-1}(c_j)$, j = 1,..,3, the remaining regions are given by the closure of the union of the positive and the negative orthant along with the union of portions from the three coordinate planes. The portion from each coordinate plane is that quadrant of the plane corresponding to where two candidates are tied, but both beat the remaining candidate. This defines five regions.

If these five regions are to be assigned to sets of $P(C^n)$ that differ from singletons, then five different sets from $P(C^n)$ are required. However, only four sets remain in $P(C^n)$. Consequently some of these regions must be combined into a single convex region. However, the union of any of these regions does not form a convex set. This means that at least one of these regions must be assigned to a singleton. Moreover, the assignment must allow the region associated with this singleton to be convex. The only way this can be done is if one of the portions of a coordinate plane is assigned to a singleton; for instance, the region corresponding to $c_1=c_2$, $c_1>c_3$ $c_2>c_3$ could be assigned either to c_1 or to c_2. (This flexibility in assignment is a general fact. Each of the two other quadrants of a plane also could be assigned to one of two possible singletons.) Now, because only four regions remain, each can be assigned to one of the remaining sets in $P(C^n)$. Alternatively, the two remaining quadrants of planes could be assigned to singletons, and each orthant to one of the four remaining sets of $P(C^n)$.

As the above construction shows, certain assignments of the regions of R^3 are forced upon us in order to achieve weak consistency. An unfortunate side effect is that whatever choice is made, the extension of f_1 must violate neutrality. Namely, for n = 3, *it is impossible to extend the Condorcet winner into a procedure that is both neutral and weakly consistent.* Here, neutrality means that if σ is some permutation of the names of the candidates, then $f(\sigma(\mathbf{p})) = \sigma(f(\mathbf{p}))$. In other words, the outcome does not depend upon the names of the candidates, but rather on how the voters rank the various candidates.

e. The extension of f_1 in part d may be objectionable for reasons other than neutrality. Therefore, it is reasonable to wonder what happens if a choice function is a combination of two weakly consistent procedures. For instance, define f to select the Condorcet winner if one exists. If no Condorcet winner exists, then the selected candidate(s) is the top ranked candidate(s) in a plurality election. Is f weakly consistent?

This choice of f is not weakly consistent. To see this, note that $G^{-1}(c_1)$ is given by the words {[($c_1>c_2$, $c_1>c_3$, -, -)], [(-,-,-;$c_1>c_2>c_3$), (-,-,-; $c_1>c_2>c_3$)]} where a "-" means that any symbol can be substituted for the missing ranking. The words in each of the brackets do define convex sets, but the union does not. Here the relevant geometry is in a five dimensional space, but one of the dimensions - the one for the ranking of $\{c_2,c_3\}$ - plays no role in the analysis. So, let (x,y;u,v,w), u+v+w =1, represent the variables in the remaining four dimensional space. The first bracket corresponds to the region (x>0,y>0,-,-,

-) while the second bracket defines the region (-,-;u>v,u>w). One way to see the lack of convexity in the region is to envision a three dimensional representation where the traditional "z" axis represents the two dimensional (u,v,w) space. The first set defines the "quarter space" passing through the first quadrant of the x-y plane. The second set is in a region in the upper half plane. Thus, the union forms a "corner" so it cannot be convex. From this description, it now is easy to construct examples where weak consistency is violated.

This construction reinforces a notion already suggested by the run-off example in part b. If a procedure involves rankings of several subsets of candidates, then the regions in Ra involve the product of regions from different subspaces. Because products of regions are used, the unions runs the risk of admitting "corners" that violate convexity. In both part b and the above, the corners arise because there are situations when the rankings of a particular subset of candidates are needed for the final outcome, and then there are other situations where the rankings of this subset are not used at all. (This is where a "-" occurs in the listing of rankings.) As the next statement asserts, procedures admitting such situations usually are not weakly consistent.

Corollary 4.1. **Suppose the outcome of a positional choice procedure is based on the ranking of one of at least two different subsets of candidates. Suppose that the selection of this final set is determined by the rankings of other subsets of candidates. Furthermore, suppose there is a candidate that could be chosen with more than one choice of the final set. The procedure is not weakly consistent.**

This corollary includes run-off elections, tournaments, sequential procedures and many other processes as special cases.

f. As my last example, I outline why for $n \geq 4$ one cannot extend Condorcet's method to define a weakly consistent choice method. This construction should be compared with the example given in Moulin [4]. For the stronger definition of consistency given in Example 3a, Moulin obtains a similar assertion. Therefore, the construction given here extends Moulin's theorem, it suggests how to construct examples that do not satisfy weak consistency, and it offers a geometric explanation why results of this kind must be expected.

I first outline the ideas for $n = 4$. Here, there are 6 pairs, so the representation of Ra is R^6. Each of the $2^6 = 64$ orthants corresponds to a particular strict ranking of pairs. The idea is to follow the lead of part d.

So, for f_1, find which orthants are assigned to the Condorcet winners. (In part d, most of the analysis involved the assignment to a set in $P(C^3)$ of the portion of the coordinate planes that did not define a Condorcet winner. I ignore this extra complication for $n \geq 4$ because the negative conclusion already is forced by the assignment problem for the orthants.) For n = 4, each Condorcet winner is determined by the appropriate symbols in a word. For instance, c_1 is a Condorcet winner for the words $(c_1 > c_2, c_1 > c_3, c_1 > c_4, -, \ldots, -)$. In each of the three blanks, any ranking of the appropriate pair of candidates can be used. Thus, associated with each Condorcet winner is the union of $2^3 = 8$ of the orthants of R^6. There are four possible choices for a Condorcet winner, so the Condorcet winners account for $4(2^3) = 32$ of the 64 orthants.

To extend the definition of f_1 to a weakly consistent procedure, each of the 32 remaining orthants must be assigned to a set in $P(C^4)$. It is not hard to show that these orthants cannot be assigned to a singleton without creating a set with corners -- hence the convexity of the inverse sets $f_1^{-1}(c_j)$ would be violated. Therefore, these extra orthants need to be assigned to non-singleton sets of $P(C^4)$. But, there are only 11 non-singleton sets in $P(C^4)$ - the 6 pairs, 4 triplets, and the full set of all 4 candidates. Consequently, the only way these 32 orthants can be assigned in a convex way to the 11 sets is by combining the orthants so that they define at most 11 convex units.

In order for the unions of the orthants to be convex, adjacent orthants must be combined. Moreover, to avoid introducing a non-convex corner into the union, the union must consist of 2^k orthants where k is a non-negative integer. To see why this is so, I start with the obvious situation where the union of three orthants cannot define a convex set. Note that one can think of each pair of adjacent orthants as defining a direction. (For instance, there is a unique way to place a unit cube in an orthant so that only one vertex is not on a coordinate plane. The direction defined by two orthants could be the vector difference between these vertex positions.) Therefore, a convex unit consisting of two orthants, denoted by 1 and 2, defines a direction. Adding a third orthant (3) that is adjacent to orthant 1 introduces a second direction. There is a unique orthant adjacent both to orthants 2 and 3 so that the direction defined by this new orthant and 2 is parallel to the second direction. If this new orthant is not included in the union, then a non-convex set arises. In general, to avoid corners, all orthants associated with an admitted direction and an existing orthant must be included in the union. This leads to the conclusion that 2^k

orthants are required.

By using the geometry of the above argument, it follows fairly easily that $k \le 2$. This is because if $k \ge 3$, the region must include some of the orthants already assigned to a Condorcet winner. Thus the geometric location of the orthants assigned to a Condorcet winner plays an important role. At the other extreme, if $k = 0,1$, then there are too many convex units for the assignment process -- the best one can do is to create 16 convex units. Consequently, if a weakly consistent procedure is to be defined, some of the units must involve the union of 4 orthants.

What remains is a simple combinatoric argument. There are two different ways to create units of four orthants. Just by keeping track of which orthants are used, and which orthants are assigned to the Condorcet winners, it follows that one cannot create fewer than 12 convex units from the 32 orthants. In other words, it is because of combinatorics that create a large number of orthants to be assigned to elements of $P(C^n)$ and because of the geometric positions of the orthants assigned to the Condorcet winners that weak consistency cannot occur.

For $n > 4$, the assignment procedure becomes more complex. This is because the space Ra is $R^{n(n-1)/2}$ with $2^{n(n-1)/2}$ orthants. The Condorcet winners account for $n2^{(n-1)(n-2)/2}$ of them. This leaves $2^{(n-1)(n-2)/2}(2^{n-1} - n)$ orthants to be assigned, in a convex manner, to the remaining $2^n-(n+1)$ entries of $P(C^n)$. It is this exponentially growing difference between the number of orthants of Ra and the number of sets in $P(C^n)$ that makes it impossible to extend f_1 to a weakly consistent procedure. An important contributing factor is the geometric positioning of the orthants assigned to the Condorcet winners.

From the above it becomes clear that the appealing, yet seemingly innocuous condition of "weak consistency" can be difficult to satisfy. A certain theme is common for the above kind of argument. When a choice function depends on the rankings of different sets - where the sets can change with the rankings - then the corresponding geometry in Ra will tend to have "corners." These corners are caused by the fact that the restrictions are in different components of the product space Ra. As these corners violate convexity, the procedure is not weakly consistent. Once more complicated positional choice procedures are used, an accompanying cost is the possibility of a conflict between the parts and the whole. Thus, one must accept the fact that it can be difficult to avoid problems of manipulability, abstention paradoxes and other behavior associated with the lack of weak consistency.

4. Proofs

Proof of Theorem 2

As true in Simpson's paradox, one can analyze the geometry of the boundary situation where, for instance, there is a tie vote between c_1 and c_2. The binary operation combining $\mathbf{p}_1$ and $\mathbf{p}_2$ is the l_1 vector addition where the outcome is on the line combining the two points. Therefore, this forces the procedure to be weakly consistent iff the boundaries of each region corresponding to the tie outcome is a portion of a linear subspace. I show that this condition holds only for the pure majority vote scheme.

First, consider the situation where g is a smooth function and n = 3. Using the notation suggested by Figure 2, let x_j denote the fraction of all voters with a ranking given by the j^{th} region, where a profile $\mathbf{p} = (x_1,..,x_6)$. A tie outcome between c_1 and c_2 is given by the equation

4.1 $F(\mathbf{p}) = \Sigma\, g(p_{1,j}) - \Sigma\, g(p_{2,j}) = 0.$

To analyze the geometry of the level set of F, it suffices to determine the properties of ∇F; in particular, in order for the level set to be part of an affine space, it must be true that $\nabla F(\mathbf{p})/|\nabla F(\mathbf{p})|$ is a constant vector on the boundary of the region $F(\mathbf{p}) = 0$. (If $g'(p) \neq 0$, then $F(\mathbf{p}) = 0$ is a codimension 1 surface. However, if g is constant valued over an interval, then it turns out that $F(\mathbf{p})$ contains an open set.) A computation yields

4.2 $\nabla F = \Sigma\, g'(p_{1,j})\, \nabla(p_{1,j}) - \Sigma\, g'(p_{2,j})\, \nabla(p_{2,j}) =$
$g'(p_{1,2})(1,1,1,0,0,0) + g'(p_{1,3})(1,1,0,0,0,1) - g'(p_{2,1})(0,0,0,1,1,1) - g'(p_{2,3})(1,0,0,0,1,1).$

As $p_{i,j} = 1 - p_{j,i}$ and as $g'(p) = g'(1-p)$ (see Eq. 2.4), Eq. 4.2 can be represented in the form

4.3 $\nabla F = (g'(p_{1,2})+g'(p_{1,3})-g'(p_{2,3}),\ g'(p_{1,2}) + g'(p_{1,3}),\ g'(p_{1,2}),\ -g'(p_{1,2}),\ -g'(p_{1,2})-g'(p_{2,3}),\ g'(p_{1,3})-g'(p_{1,2})-g'(p_{2,3})).$

The vector $\nabla F(\mathbf{p})/|\nabla F(\mathbf{p})|$ does not depend upon $\mathbf{p}$ iff each component of the vector in Eq. 4.3 is a scalar multiple of $|\nabla F(\mathbf{p})|$. In turn, this means that each component of $\nabla F(\mathbf{p})$ is a scalar multiple of each other component. So, by comparing the fourth and the fifth components, it follows that $g'(p_{1,2})$ must be a constant multiple of $g'(p_{2,3})$ for all $\mathbf{p}$ on the boundary of a region where $F(\mathbf{p}) = 0$. Similarly, it turns out that for each of the three variables, $g'(p_{1,2})$, $g'(p_{1,3})$, and $g'(p_{2,3})$ all are fixed scalar multiples of the others for all choices of **p** so that $F(\mathbf{p}) = 0$. As $p_{i,j}$ are independent variables, this last condition means that over the domain of definition of $p_{i,j}$ on a boundary where $F(\mathbf{p}) = 0$, g' must be a constant. However, this domain allows the variables to range over all values; thus the generalized majority vote is weakly consistent iff g'(p) is a constant. (To see this, start by selecting $x_1 = x_6 = 1/2$. This leads to a tie vote between c_1 and c_2 where $p_{2,3} = 1$. Now, by varying the example, all values of p are attained.)

If g'(p) is a constant, then ∇F is a constant vector, and the procedure is weakly consistent. However, g'(p) is a constant iff g(p) = p; namely, g'(p) is constant corresponds to the pure majority vote scheme. The argument for $n > 3$ and g a smooth function is much the same, where the only difference is that more $p_{i,j}$ variables are involved. Therefore, I now turn to the setting for $n = 3$ where g is not smooth.

With the appropriate changes, the proof where g(p) is not smooth is essentially the same as the one given above. Because g is monotonic, the derivative is defined at many points. Here, the above argument holds. The purpose of using the derivative is to obtain a linear approximation for the computation of the boundary of the set $F(\mathbf{p}) = 0$. If g' is not defined at such a boundary point, then either g has a discontinuity, or g is continuous, but g' has a discontinuity. A simple argument shows that in the first setting, the boundary is not continuous - it has a "jump" corresponding to the discontinuity of g. In the second setting, the above argument can be used where limits of g' are used.

Proof of Theorem 4

The binary operation in Si(n!) forces a set to be algebraically closed iff it is convex. Part a of the theorem is proved if it can be shown that the kinds of sets considered in Ra have the feature that they are convex iff their inverse image under T:Si(n!) --> Ra is a convex set in Si(n!). The proof strongly uses the fact that T is a linear mapping that maps the point of indifference $I \in Si(n!)$ to the point of indifference, I, in Ra. Also, T maps hyperspaces into hyperspaces.

If a set in Ra is defined by the intersection of planes or half spaces defined by planes passing through the point of indifference, then it is convex. As the inverse image of this set also can be described as a similar intersection, it too is convex. All ranking regions can be defined in this fashion, so the inverse image of any ranking region in Ra is a convex set in Si(n!). Suppose the union of ranking regions defines a set that is not convex. This means that there are two points $\mathbf{x}_1$ and $\mathbf{x}_2$ where some convex combination of them is outside of this union. As T is linear, this convex combination defines a line in Si(n!) that connects preimages of $\mathbf{x}_1$ and $\mathbf{x}_2$ where the line is not contained in the preimage of the union. Thus, the preimage is not convex. This completes the proof of part a. The same proof holds for the the first part of part b.

For part b, suppose that f is not weakly consistent when $\mathbf{B}^n$ is used. This means that the inverse image of $G^{-1}(\alpha)$, for some $\alpha \in P(C^n)$, defines a set of words from the Borda Dictionary which correspond to non-convex regions in $T(Si(n!);\mathbf{B}^n)$. But, according to the assertion from Theorem 3, any word that is in the Borda Dictionary also is a word in all other dictionaries. Therefore, this same set of words is in all other dictionaries.

Notice that (with the appropriate scalar changes - see Saari [8]) is a affine space that includes $T(Si(n!);\mathbf{B}^n)$. Therefore, if $G^{-1}(\alpha)$ is a convex set in $T(Si(n!);\mathbf{W}^n)$, then the section meeting $T(Si(n!);\mathbf{B}^n)$ must also be convex. This is not the case, so the proof is completed.

Proof of the Proposition

Part a. The assertion that T maps the interior of Si(n!) to an open set in Ra if $\mathbf{W}^n \notin \alpha^n$ comes from the fact that the linear map T has maximal rank. This is proved in Saari [7]. The rest of the statements are immediate.

Part b. This image can be described in terms of the convex intersections of convex regions (half spaces or planes). The conclusion follows immediately.

The rest of the proposition is standard material.

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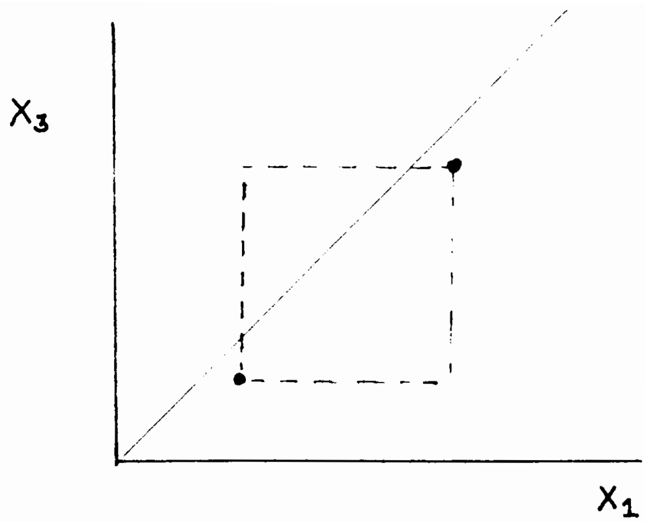

Fig. 1.

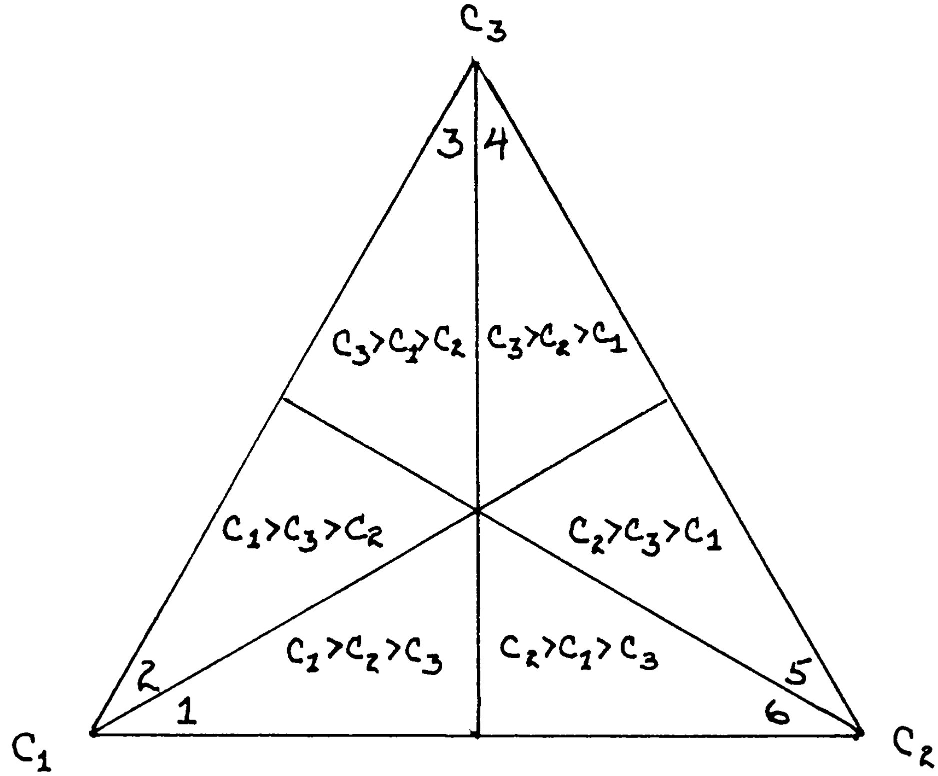

Fig. 2.