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PLAYERS' OBSERVATION, DEDUCTIVE KNOWLEDGE AND INFORMATION PARTITIONS

by

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Abstract: In this paper, we develop a framework in which players' knowledge and their information partitions are constructed based on the players' perception and their logical deduction. A player knows something if it is observed or logically deduced from the observed facts. A player's information partition is derived from the player's indistinguishability defined as follows: Two states of the world are indistinguishable for a player if there is no difference in his observation at each of those two states. It is shown that the players' knowledge so defined is equivalent to their knowledge defined in the manner given in Aumann (1976) using the information partitions constructed in our framework. As an application, we consider the characterization of information partitions. We obtain a necessary and sufficient condition for one player's information partition to be a refinement of the other's. We also discuss the properties of information partitions caused by a player's particular observing abilities.

1. Introduction

1.1. Motivation

In game theory, information partitions (partitions of the state space) are widely used to express players' knowledge. A set of information partitions indicates each player's knowledge of an event (a subset of the state space) as well as each player's knowledge of mutual knowledge of an event. Based on this knowledge, denoted by the given information partitions, the game theorist can analyze a game. For example, the notion of common knowledge[2] was firstly formulated by Aumann (1976) using the players' information partitions. And several important properties of the players' believes and/or actions are derived by that formulation (e.g., Aumann (1976), Kobayashi (1980) and Milgrom and Stokey (1982)).

But, how are the players' information partitions determined ? The information partitions are usually assumed to exist by the game theorist without any reasons why they are so determined. The basic structure which explains the reason why the players have those information partitions is completely hidden behind the mathematical formulation.

In the real world, people get information through their perceptions and logical deduction. People know something from watching a fact, hearing someone's announcement and applying logical deduction to what we watch and hear. The players' information partitions should be determined to denote what the players know from such activity. In other words, the information partitions are the accumulation of the information which the players finally

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[2] An event is informally said to be common knowledge between players 1 and 2 if 1 knows it, 2 knows it, 1 knows that 2 knows it, 2 knows that 1 knows it, 1 knows that 2 that 1 knows it, 2 knows that 1 knows that 2 knows it, and so on.

obtained from their mental activity.

Indeed, when we are interested in the players' knowledge as a given parameter of a game, this accumulation is a benefit to us. However, if we must investigate in detail how the players know something, and how the players come to know that another player knows something, then this accumulation property of information partitions represents an obstacle to our investigation.

For example, let us consider the situation in which a player communicates a part of his private information to another player. In such case, if the first player has incentive to communicate it, he will make effort to show that he really knows it to let the second player believe him. His effort will succeed if he can show objective evidence and what he knows is logically deduced from it. But, if he cannot show enough evidence, it becomes hard for him to communicate it, and futhermore, it is much more difficult for him to convince the second player to really believe it. To consider rigidly this situation, we will need the foundational structure which specifies how a player knows, how a player knows what another player knows, etc.

In this paper, we develop a basic framework which explicitly describes the players' perception and their logical deduction. In the framework, we will formulate naively players' knowledge and their information partitions. To show the consistency of our framework, we prove that the players' knowledge which we define is equivalent to their knowledge induced from their information partitions constructed in our framework. As the first step of the application of the framework, we will characterize the information partitions in some special situations of the players' perceptions.

1.2. Review of the framework

We consider two players in our framework. In this paper, we use the term observation to denote the players' perception.

We describe the objective statements in the world by propositions. We call them objective propositions. An objective statement stands for the statement which describes the world but not any players' cognizance. For example, "The temperature of New York City on August 1, 1989 is 100 degrees Fahrenheit." is an objective statement. (Of course it may or may not be true.) We have a set of objective propositions in our framework.

Each player can observe the truth values of some set of objective propositions. Precisely, each player knows that some objective propositions are true and some are false by directly observing the propositions.

We consider that the difference in players' knowledge arises from their observing abilities. To specify the players' observing abilities, we introduce a hypothetical concept called an observation device. An observation device (abbreviated as O.D.) is an imaginative machine through which the player who owns it observes the world. It is assumed that each player can observe the other player's O.D.

We assume that there are various types of O.D's. What each player can observe is determined by the type of his O.D. For each type of O.D., there exist

i) a set of objective propositions which the player can observe, and

ii) a set of types of the other player's O.D. which are considered as possible.

Given the type of O.D. of each player, a player's knowledge is defined as follows:

i) Player 1 knows that an objective proposition p is true if it is observed to be true or logically deduced (by modus ponens) from the true objective propositions which he observes.

ii) Player 1 knows that player 2 knows that p is true if, for any type of 2's O.D. which 1 considers as possible, p is observed to be true or logically deduced from the objective propositions which are observed by the type and which 1 knows to be true.

(We will also define the cases in which 1 knows that 2 knows that 1 knows that p is true, 1 knows that 2 knows that 1 knows that 2 knows that p is true,... But, to avoid complexity, we omit the verbal description of the definitions of such cases here.)

Next, let us see how we formulate the players' information partitions. In our framework, a state of the world is defined to be an assignment rule of the true propositions and the true type of each player's O.D. We say two states are indistinguishable for a player if, the following three conditions hold:

i) The set of objective propositions which can be observed are identical at each of the two states;

ii) The truth values of the objective propositions which can be observed are identical at each of the two states;

iii) The possible types of the other's O.D. are identical at each of the two states.

For each player, his indistinguishability is verified to be an equivalence relation on the set of the states. We define the information partition of a player as the equivalence class under his indistiguishability.

To make sure of the consistency between the players' knowledge and their information partitions, we establish that the players' knowledge defined in our framework is equivalent to their knowledge defined in the manner given by Aumann (1976) using the information partitions constructed in our framework (Theorem 1). Precisely, we prove that, at any state, player i knows that j knows that i knows that ... that an objective proposition p is true if and only if, in Aumann (1976)'s sense, i knows that j knows that i knows that ... knows the event that p is true.

As an application of our framework, we will consider the characterization of the information partitions. We present a necessary and sufficient condition for one player's information partition to be a refinement of the other's (Theorem 2). We also consider the following three cases: a player can observe all the truth values of the objective propositions and the true type of other's O.D. (the case of complete observation); a player observes nothing about the true type of the other's O.D. (the case of purely private observation); a player observes nothing about the truth value of the propositions nor the true type of other's O.D. (the case of null observation). We examine the properties of the information partitions corresponding to these cases.

This paper is organized as follows: In Section 2, we formulate our framework. Section 3 presents the definition of player's knowledge. In Section 4, we define the information partition for each player, and prove Theorem 1. Section 5 is devoted to the application. We prove Theorem 2, and examine some characterization of the information partitions.

2. The Observation Device Model

In this section, we formulate our model named the Observation Device Model. The Observation Device Model is given by the eight tuple $< P, A_1, A_2, f_1, f_2, g_1, g_2, \Omega >$, where P and A_i (i=1,2) are countable sets ($P \cap A_i = \emptyset$ (i=1,2), $A_1 \cap A_2 = \emptyset$), f_i (i=1,2) is a mapping $A_i \to 2^P$, g_i (i=1,2) is a mapping $A_j \times A_i \to 2^{A_j}$ ($j \in \{1,2\}$, $i \neq j$) and Ω is a set of the mapping ω: $P \cup A_1 \cup A_2 \to \{0,1\}$.

We interpret as follows: P is a set of the propositions which stand for objective statements of the world. We call $p \in P$ an <u>objective proposition</u> to distinguish the propositions which we will define later to denote player's knowledge. In this paper, we use the propositional connectives ~ (negation), $\wedge$ (conjunction), $\to$ (conditional) and v (disjunction). P is assumed to be the set of the propositions generated from the set of the atomic propositions $P_0 = \{p_1, p_2, p_3 \ldots\}$ by those propositional operations. Formally, let $P_n = \{ \sim p: p \in P_{n-1} \} \cup$ $\{ p \wedge p', p \to p' \text{ and } p \vee p': p, p' \in P_{n-1} \}$ for n = 1,2,..., and $P = \cup_{n=0}^{\infty} P_n$.

Each $\alpha_i \in A_i$ denotes a type of player i's <u>observation device</u>. A_i is the set of all the possible types of player i's O.D. Given $\alpha_i \in A_i$ as the true type of i's O.D., we interpret that i can see the truth value of every $p \in f_i(\alpha_i)$.

For each $\alpha_i \in A_i$ and $\alpha_j \in A_j$ ($i,j \in \{1,2\}$, $i \neq j$), $g_i(\alpha_j, \alpha_i)$ stands for the set of the types of player j's ($j \neq i$) O.D. which player i considers as possible when α_i when α_j are respectively the true types of i,j's O.D.'s.

Each $\omega \in \Omega$ is interpreted as a truth assignment on $P \cup A_1 \cup A_2$. For each $r \in P \cup A_1 \cup A_2$, we interpret $\omega(r) = 1$ (0) to mean that r is true (false). Below, we say, for economy of expression, that <u>i's (true) O.D. is</u>

$\underline{\alpha_i}$ instead of saying that <u>the true type of i's O.D. is α_i</u>. We call each $\omega \in \Omega$ a <u>state of the world</u>. (Or, a <u>state</u> in short.) We assume that Ω is the set of all ω's which satisfies the standard rule: For p, p' $\in$ P, $\omega(p) = 1$ iff $\omega(\sim p) = 0$; $\omega(p \wedge p') = 1$ iff $\omega(p) = \omega(p') = 1$; $\omega(p \rightarrow p') = 1$ iff $\omega(p) = 0$ or $\omega(p \wedge p') = 1$; $\omega(p \vee p') = 0$ iff $\omega(p) = \omega(p') = 0$.

On this basic framework, we put the following five assumptions.

Assumption 1.

For any $\omega \in \Omega$ and i=1,2, there is a unique $\alpha_i \in A_i$ such that $\omega(\alpha_i) = 1$.

This assumption states that the enumeration of the type of O.D. of each player is exclusive and exhausting. By $\alpha_i(\omega)$ we denote the unique α_i, i.e., player i's true O.D. at ω.

Assumption 2.

For any $\omega \in \Omega$ and i,j (i,j $\in$ {1,2}, i$\neq$j), $\alpha_j(\omega) \in g_i(\alpha_j(\omega), \alpha_i(\omega))$.

Assumption 2 says that each player's true O.D. is counted to be possible by the other player.

Assumption 3.

For any $\alpha_i \in A_i$ (i=1,2) and $\alpha_j^1, \alpha_j^2 \in A_j$ (j$\in${1,2}, j$\neq$i), if $\alpha_j^1 \in g_i(\alpha_j^2, \alpha_i)$, then $g_i(\alpha_j^1, \alpha_i) = g_i(\alpha_j^2, \alpha_i)$.

The meaning of this assumption is as follows. Suppose that a player i thinks that a group of the types of j ($\neq$i)'s O.D. are possible. Then, player i will think that this group of the types of j's O.D. will be possible whichever type in that group may be true. It seems natural that if a matter A occurs, and matters A and B look possible to occur for a person, then A and B look possible for him if B really occurs.

Assumption 4.

For any $\alpha_i^1, \alpha_i^2 \in A_i$ and any $\alpha_j \in A_j$ $(i,j\in\{1,2\},\ i\neq j)$, if $f_i(\alpha_i^1) = f_i(\alpha_i^2)$ and $g_i(\alpha_j,\alpha_i^1) = g_i(\alpha_j,\alpha_i^2)$, then $g_j(\alpha_i^1,\alpha_j) = g_j(\alpha_i^2,\alpha_j)$.

Assumption 4 states that if player i cannot recognize any difference in his observation by α_i^1 and α_i^2 when j's O.D. is α_j, then j should also observe that both α_i^1 and α_i^2 are equally possible. Roughly, it means that when a player cannot distinguish two types of his O.D. by his own observation, then the other player cannot distinguish those two types.

Assumption 5.

For each i=1,2, and any $\alpha_i \in A_i$, $f_i(\alpha_i)$ is a finite set.

This assumption means that a player cannot perceive the truth values of infinitely many propositions at a time. Its adequacy seems to be controversial. We have this assumption only for technical reasons in this paper. Precisely, it is needed in the proof of Theorem 1.

We implicitly assume that the Observation Device Model is <u>common knowledge</u> between the players.

In the rest of this section, we present a simple example of the Observation Device Model.

Example 1 (Watching a coin with eyes).

Let P be the set of the propositions generated from $P_0 = \{p_1\}$;

$A_1 = \{a_1, a_2\}$;

$A_2 = \{b_1, b_2\}$;

$f_1(a_1) = f_2(b_1) = \{p_1\}$;

$f_1(a_2) = f_2(b_2) = \emptyset$;

$g_1(b_i, a_1) = \{b_i\}$ $(i=1,2)$;

$g_1(b_i, a_2) = \{b_1, b_2\}$ $(i=1,2)$;

$g_2(a_i, b_1) = \{a_i\}$ $(i=1,2)$;

$g_2(a_i, b_2) = \{a_1, a_2\}$ $(i=1,2)$.

There are eight states of the world in this model:

ω	p_1	a_1	a_2	b_1	b_2
ω_1	1	1	0	1	0
ω_2	1	1	0	0	1
ω_3	1	0	1	1	0
ω_4	1	0	1	0	1
ω_5	0	1	0	1	0
ω_6	0	1	0	0	1
ω_7	0	0	1	1	0
ω_8	0	0	1	0	1

The intended interpretation of this model is that: p_1 stands for the proposition " The coin appears head."; a_1 (b_1) means that player 1 (2) opens his eyes; a_2 (b_2) means that player 1 (2) closes his eyes. (We forbid half-closed eyes.)

3. Deductive Knowledge

In this section, for any objective proposition p, we introduce the propositions denoting the statements such as "1 knows that p is true.", "1 knows that 2 knows that p is true.", "1 knows that 2 knows that 1 knows that p is true.", etc. We extend the domain of each $\omega \in \Omega$ to include those propositions, and consider the reasonable truth assignment of them.

Let K_i (i=1,2) be a logical symbol. Let a formula $K_{i_1} K_{i_2} \ldots K_{i_n} p$ be a proposition, if $i_k \in \{1,2\}$ (k=1,2...,n), $i_k \neq i_{k+1}$ (k = 1,2,...,n-1) and $p \in P$. For any $p \in P$, $K_{i_1} K_{i_2} \ldots K_{i_n} p$ is intended to mean that "i_1 knows that i_2 knows that ... that i_n knows that p is true."

We assume that each player has the ability of logical deduction. In this paper, we divide it in the following three abilities:

A1) a player can calculate the truth values of propositions following the rule of ω ($\in \Omega$) on the connectives ~, $\wedge$, $\rightarrow$ and v,

A2) a player can recognize that any tautology is true, and

A3) a player can make successive operations of modus ponens for any finitely many times.

Specifically, A1) means the ability to assign the truth value for the propositions by the rule:

for any $p \in P$, ~p is true iff p is false;

for any p_1 and $p_2 \in P$, $p_1 \wedge p_2$ is true iff both p_1 and p_2 are true;

$p_1 \rightarrow p_2$ is true iff either p_1 is false or both p_1 and p_2 are true;

$p_1 \vee p_2$ is true iff either p_1 or p_2 is true. A tautology is a propositions which is true for any truth assignment. For example, for any p, $q \in P$, $p \vee \sim p$, $p \rightarrow (q \rightarrow p)$, $p \rightarrow (q \rightarrow (p \wedge q))$ are tautologies. A player can verify that a proposition is a tautology if he has the ability of A1) and he can list up all the possible truth assignment for the propositions which are included in the proposition in question. A2) means this ability. A3) means that if p, $p \rightarrow q$ are given to a player as true, then the player can conclude that q is true (modus ponens). A3) further means that the player can repeat arbitrarily many times this inference for the propositions which have already been obtained as true.

Given a set of propositions, we say a proposition is deduced from the set if it is obtained as true by A1) through A3) from the set of propositions. We call a player's knowledge acquired through these logical ability the deductive knowledge. We implicitly assume that the logical ability of each player is common knowledge between the players.

The deduction defined here is essentially the same as that of propositional calculus. However, our definition is redundant in comparison with it. Strictly speaking, the ability A1) is induced from A2) and A3). Moreover, a part of A2) is also redundant. However, I chose the ability A1) through A3) in this paper to make the meaning of the players' logical ability clear and to formulate simply a player's inference of the other's knowledge. We will discuss about this subject at the end of this section.

To formulate a player's knowledge, let us define the following notations i), ii) and iii).

i) For any $S \subset P$, define B(S) as follows. Let $B_0(S) = S \cup \{ \sim p: p \in S \}$, and for $n = 1,2,\ldots$, let $B_n(S) = \{ q \wedge r, q \rightarrow r, q \vee r, \sim(q \wedge r), \sim(q \rightarrow r)$ or $\sim(q \vee r): q, r \in B_{n-1}(S) \}$. And let $B(S) = \cup_{n=0}^{\infty} B_n(S)$. B(S) stands for the set of all the propositions generated from S by the connectives $\sim$, $\wedge$, $\rightarrow$ and $\vee$.

ii) For any $S \subset P$ and any $\omega \in \Omega$, define $T_\omega(S) = \{ q: q \in S \text{ and } \omega(q) = 1 \}$. Further, define $T = \{ q \in P: \omega'(q) = 1 \ \ \forall\omega' \in \Omega \}$, i.e., T is the set of all the <u>tautologies</u>. $T_\omega(B(S)) \cup T$ denotes the set of all the propositions which the player concludes to be true from S with abilities A1) and A2).

iii) For any $S \subset P$, define M(S) as follows: Let $M_0(S) = S$ and, for $n=1,2,\ldots$, let $M_n(S) = M_{n-1}(S) \cup \{q \in P: \exists\ q' \in P;\ q',\ (q' \rightarrow q) \in M_{n-1}(S)\ \}$. And let $M(S) = \cup_{n=0}^{\infty} M_n(S)$. M(S) indicates the set of the propositions which can be obtained from S by the finitely many successive operations of modus ponens.

Now we are in a position to define the truth assignment of the proposition $K_{i_1} K_{i_2} \ldots K_{i_n} p$ at given $\omega \in \Omega$. Before the general definition, let us consider the truth values of K_1p, K_1K_2p and $K_1K_2K_1p$ to clarify how the players' logical abilities work on the truth assignment.

First we consider first K_1p. Let $\hat{\alpha}_1 = \alpha_1(\omega)$. Then, player 1 observes the truth value of the proposition in $f_1(\hat{\alpha}_1)$. From A1), he can calculate the truth value of each proposition in $B(f_1(\hat{\alpha}_1))$. On the other hand, by A2), he knows that each element of T is true. Hence, he obtains the set of true propositions $T_\omega(B(f_1(\hat{\alpha}_1))) \cup T$. Then, by A3), he comes to know that the propositions contained in $M(T_\omega(B(f_1(\hat{\alpha}_1))) \cup T)$ are true.[3] Therefore, we should define $\omega(K_1p) = 1$ iff $p \in M(T_\omega(B(f_1(\hat{\alpha}_1))) \cup T)$.

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[3] He does not have to extend the true propositions using A1) again here. It can be shown that the true propositions produced by connecting propositions in $M(T_\omega(B(f_i(\alpha_i))) \cup T\ )$ with $\sim$, $\wedge$, $\rightarrow$ and $\vee$ have already been contained in $M(T_\omega(B(f_i(\alpha_i))) \cup T\ )$. (See Proposition A.1 in Appendix 1.)

Next we consider K_1K_2p. Let $M(T_\omega(B(f_1(\hat{\alpha}_1))) = D_\omega(\hat{\alpha}_1)$. Now we implicitly assume that player 1 knows that player 2 has the logical ability. So, we can consider that player 1 knows that player 2's knowledge is constructed in the same way as above. In our framework, player 1 knows that the types of 2's O.D. in $g_1(\alpha_2(\omega),\hat{\alpha}_1)$ are possible. Take $\hat{\alpha}_2$ arbitrarily from $g_1(\alpha_2(\omega),\hat{\alpha}_1)$. Suppose that 1 assumes that 2's O.D. is $\hat{\alpha}_2$. Then 1 thinks that 2 can calculate the truth value of the propositions in $B(f_2(\hat{\alpha}_2))$ based on 2's own observation. Here, 1 does not know the true state of the world, so that 1 cannot say which is the true proposition in this set. 1 can say only that 2 recognizes that the propositions contained in $D_\omega(\hat{\alpha}_1) \cap B(f_2(\hat{\alpha}_2)) \cup T$ are true. Hence, 1 knows that if $\hat{\alpha}_2$ is 2's true O.D., then 2 knows that the propositions contained in $M(D_\omega(\hat{\alpha}_1) \cap B(f_2(\hat{\alpha}_2)) \cup T)$ are true. Let $M(D_\omega(\hat{\alpha}_1) \cap B(f_2(\hat{\alpha}_2)) \cup T) = D_\omega(\hat{\alpha}_1,\hat{\alpha}_2)$. Here, $\hat{\alpha}_2$ is not the unique possible type of 2's O.D. Hence, at ω, 1 can say that 2 knows that p is true if, $\forall\hat{\alpha}_2 \in g_1(\alpha_2(\omega),\hat{\alpha}_1)$, $p \in D_\omega(\hat{\alpha}_1,\hat{\alpha}_2)$. Thus we should define $\omega(K_1K_2p) = 1$ iff $p \in D_\omega(\hat{\alpha}_1,\hat{\alpha}_2)$ $\forall\hat{\alpha}_2 \in g_1(\alpha_2(\omega),\hat{\alpha}_1)$.

For $K_1K_2K_1p$, the argument is almost the repetition of the above one. Suppose that 1 assumes that 2's true O.D. is $\hat{\alpha}_2 \in g_1(\alpha_2(\omega),\hat{\alpha}_1)$. Then, at ω, 1 can say that 2 knows that 1 knows that p is true iff $\forall\ \tilde{\alpha}_1 \in g_2(\hat{\alpha}_1,\hat{\alpha}_2)$, $p \in M(D_\omega(\hat{\alpha}_1,\hat{\alpha}_2) \cap B(f_1(\tilde{\alpha}_1)) \cup T)$, since 1 knows that 2 considers $\tilde{\alpha}_1 \in g_2(\hat{\alpha}_1,\hat{\alpha}_2)$ as possible if 2's O.D. is $\hat{\alpha}_2$. Let $M(D_\omega(\hat{\alpha}_1,\hat{\alpha}_2) \cap B(f_1(\tilde{\alpha}_1)) \cup T) = D_\omega(\hat{\alpha}_1,\hat{\alpha}_2,\tilde{\alpha}_1)$. Then, we can define that $\omega(K_1K_2K_1p) = 1$ iff $p \in D_\omega(\hat{\alpha}_1,\hat{\alpha}_2,\tilde{\alpha}_1)$ $\forall\hat{\alpha}_2 \in g_1(\alpha_2(\omega),\hat{\alpha}_1)$, $\forall\tilde{\alpha}_1 \in g_2(\hat{\alpha}_1,\hat{\alpha}_2)$.

For the general definition, let us define the following notations. We say that $(i_1,i_2,\ldots,i_n)$ is a proper order if $i_k \in \{1,2\}$ $(k=1,2,\ldots,n)$ and $i_k \neq i_{k+1}$ $(k=1,2,\ldots,n-1)$. For any $\omega \in \Omega$ and any proper order $\Theta = (i_1,i_2,\ldots,i_n)$, we say that $(\alpha^1,\alpha^2,\ldots,\alpha^n)$ is an α-sequence with Θ at ω if $\alpha^1 = \alpha_{i_1}(\omega)$, $\alpha^2 \in g_{i_1}(\alpha_{i_2}(\omega),\alpha^1)$, and $\alpha^k \in g_{i_{k-1}}(\alpha^{k-2},\alpha^{k-1})$ for $k=3,4,\ldots,n$. For any $\omega \in \Omega$, any proper order $\Theta = (i_1,i_2,\ldots,i_n)$, and any α-sequence $(\alpha^1,\alpha^2,\ldots,\alpha^n)$ with Θ at ω, define $D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^n)$ inductively as follows: Let $D_\omega(\alpha^1) = M(T_\omega(B(f_{i_1}(\alpha^1))) \cup T)$; Suppose $D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^k)$ is defined, then let $D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^{k+1}) = M(D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^k) \cap B(f_{i_{k+1}}(\alpha^{k+1})) \cup T)$.

The concept of proper order is defined to specify the order of the inference between the players. In our framework, $K_1K_2K_2p$, e.g., is not regarded as a proposition. The meaning of "proper" reflects this restriction. An α-sequence stands for the inference chain of possible types of O.D. between the players with a given proper order Θ at $\omega \in \Omega$.

Definition 1.

For any $p \in P$, any $\omega \in \Omega$ and any proper sequence $\Theta = (i_1,i_2,\ldots,i_n)$, we define $\omega(K_{i_1}K_{i_2}\ldots K_{i_n}p) = 1$ iff $p \in D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^n)$ for all α-sequence $(\alpha^1,\alpha^2,\ldots,\alpha^n)$ with Θ at ω.

To see the player's deductive knowledge concretely, we cite Example 2.

Example 2 (Watching white and red balls with eyes).

Suppose that three persons 1, 2 and 3 are standing around a table. There is an empty urn, a red ball and a white ball on the table. At first, 3 shows that the urn is empty and puts the two balls in the urn. Next, he picks up one ball from the urn and puts it on the table.

Let p_1 stand for the proposition "The ball on the table is red.", and let p_2 stand for the proposition "The ball in the urn is red.". For the sake of simplicity, we assume that 1 and 2 have their eyes open and see each other. Then we can consider the following Observation Device Model for this situation.

$P = B(\{p_1, p_2\})$;

$A_1 = \{a_1\}$, $A_2 = \{b_2\}$;

$f_1(a_1) = f_2(b_1) = \{ p_1 \vee p_2, \sim(p_1 \wedge p_2), p_1 \}$;

$g_1(b_1, a_1) = b_1$, $g_2(a_1, b_1) = a_1$.

There are the following four states of world:

ω	p_1	p_2	a_1	b_1
ω_1	1	1	1	1
ω_2	1	0	1	1
ω_3	0	1	1	1
ω_4	0	0	1	1

Of course, the states ω_1 and ω_4 do not occur. They exist only as the result of logical enumeration. Supposing now that ω_3 is true, I will verify that K_1p_2 and $K_1K_2p_2$ are true at ω_3.

Now, $f_1(a_1) = \{ p_1 \vee p_2, \sim(p_1 \wedge p_2), p_1 \}$. We can easily verify that $((p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2)) \rightarrow (\sim p_1 \rightarrow p_2)$ is a tautology. From the definition of $B(.)$, $(p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2) \in B(f_1(a_1)))$, and it is a true proposition at ω_3. Therefore, $(p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2)$ and

$((p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2)) \rightarrow (\sim p_1 \rightarrow p_2) \in T_{\omega_3}(B(f_1(a_1))) \cup T$. Thus, by modus ponens, we have $\sim p_1 \rightarrow p_2 \in M(T_{\omega_3}(B(f_1(a_1)))) = D_{\omega_3}(a_1)$. Futhermore, $\sim p_1 \in B(f_1(a_1))$ and $\omega_3(\sim p_1) = 1$. Hence, we have $\sim p_1 \in D_{\omega_3}(a_1)$. Thus, by modus ponens, we obtain $p_2 \in D_{\omega_3}(a_1)$, which means that $\omega_3(K_1 p_2) = 1$ by Definition 1.

Let us consider $K_1K_2p_2$. Since $f_2(b_1) = f_1(a_1)$, from the above argument, $(p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2)$, $((p_1 \vee p_2) \wedge \sim(p_1 \wedge p_2)) \rightarrow (\sim p_1 \rightarrow p_2)$ and $\sim p_1$ are contained in $B(f_2(b_1)) \cup T$. Therefore, these propositions are contained in $D_{\omega_3}(a_1) \cap B(f_2(b_1)) \cup T$. Hence, by applying modus ponens twice, we have $p_2 \in M(D_{\omega_3}(a_1) \cap B(f_2(b_1)) \cup T) = D_{\omega_3}(a_1, b_2)$, meaning $\omega_3(K_1K_2p_2) = 1$.

We close this section by stating the relationship between the deduction system defined in this paper and that of propositional calculus.

Let $A = \{\ p \in P\colon \exists q,\ r \text{ and } s \in P; \quad p = q \rightarrow (r \rightarrow q),$
$(q \rightarrow (r \rightarrow s)) \rightarrow ((q \rightarrow r) \rightarrow (r \rightarrow s) \text{ or } (\sim q \rightarrow \sim r)) \rightarrow ((\sim q \rightarrow r) \rightarrow q)\ \}$.
We can easily verify that all elements in A are tautologies. Using our notation, it is defined in propositional calculus that a proposition p is deduced from S ($\subset P$) if $p \in M(S \cup A)$. Therefore, if we make faithful deduction to propositional calculus, the set of deduced propositions when player i (i=1,2) observes the truth value of the propositions in $f_i(\alpha_i)$, should be $M_1 \equiv M(\ \{p\colon p \in f_i(\alpha_i) \text{ and } \omega(p) = 1\ \} \cup \{\sim p\colon p \in f_i(\alpha_i) \text{ and } \omega(p) = 0\ \} \cup A\ )$. However, this set is eventually equivalent to $M_2 \equiv M(T_\omega(B(f_i(\alpha_i))) \cup T\ )$. Let's verify it here. Since $A \subset T$ and $\{p\colon p \in f_i(\alpha_i) \text{ and } \omega(p) = 1\} \cup \{\sim p\colon p \in f_i(\alpha_i) \text{ and } \omega(p) = 0\} \subset B_0(f_i(\alpha_i)) \subset B(f_i(\alpha_i))$, we have $M_1 \subset M_2$. On the other hand, there is a

following theorem.

Completeness Theorem.

M(A) = T.

(See, e.g., Mendelson (1979).)

Therefore, $T \subset M_1$. Hence, $M_1 = M_1' \equiv M(\ \{p: p \in f_i(\alpha_i) \text{ and } \omega(p) = 1\ \} \cup \{\sim p: p \in f_i(\alpha_i)) \text{ and } \omega(p) = 0\ \} \cup T\ )$. Furthermore, we can show that $T_\omega(B(f_i(\alpha_i))) \subset M_1'$. (See Proposition A.2 in Appendix 1.) Thus, we have $M_1 \supset M_2$ so that $M_1 = M_2$, i.e., it can be said that our definition of deduction is essentially the same as that of propositional calculus.

The author quotes here important theorems established in propositional calculus. Those theorems will play significant roles in the proof of Theorem 1 in the next section. Let's define the following terminologies. We say S ($\subset$P) is <u>inconsistent</u> if there exists $p \in P$ such that both $p \in M(S \cup A)$ and $\sim p \in M(S \cup A)$ holds at a same time. S is said to be <u>inconsistent</u> iff S is not consistent. We also say that a state ω <u>satisfies</u> S ($\subset$P) if $\omega(q) = 1$ for all $q \in S$.

Generalized Completeness Theorem 1.

S ($\subset$P) is consistent iff there exists a state which satisfies S.

Generalized Completeness Theorem 2.

For any $p \in P$ and any $S \subset P$, $p \in M(S \cup A)$ iff $\omega(p) = 1$ for all ω $(\in \Omega)$ that satisfy S.

(Proof of these two theorems are in Appendix 2.)[4]

As we stated above, $M(S \cup A) = M(S \cup T)$ $\forall S \subset P$. Hence, we could write the definition of (in)consistency and Generalized Completeness Theorem 1 and 2 by replacing A by T. In the following part of this paper, we replace A and T in M(.) as occasion demands without comment. Such replacement will cause no confusion.

We abbreviate Generalized Completeness Theorem 1 and 2 as G.C.T.1 and G.C.T.2, below.

4. Information Partition

In this section, we define the indistinguishability of the states for each player, and induce their information partitions. We establish the equivalence theorem on the players' knowledge.

We define a player's indistinguishability of two states as follows:

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[4] Since it is difficult for us to find the proof in the literatures, I state them in Appendix 2. The proof is originally given in Fukuyama (1980).

Definition 2.

For any ω, $\omega' \in \Omega$, we say ω and ω' are <u>indistinguishable</u> for player i (i=1,2) and write $\omega \approx_i \omega'$ if

i) $f_i(\alpha_i(\omega)) = f_i(\alpha_i(\omega'))$,

ii) $\omega(p) = \omega'(p) \quad \forall p \in f_i(\alpha_i(\omega))$, and

iii) $g_i(\alpha_j(\omega),\alpha_i(\omega)) = g_i(\alpha_j(\omega'),\alpha_i(\omega'))$.

The intended interpretation of $\omega \approx_i \omega'$ is that there is no difference in player i's observation for ω and ω' with respect to:

i) the set of the observable objective propositions;

ii) the truth value of the observable propositions; and

iii) the possible types of the other player's O.D.

It can be easily shown that $\approx_i$ is an equivalence relation on Ω. Let Π_i (i=1,2) be the equivalence class under $\approx_i$. Then Π_i (i=1,2) is a partition of Ω.

Definition 3.

Let Π_i be the equivalence class under equivalence relation $\approx_i$ which is defined in Definition 2. Then, we call Π_i the <u>information partition</u> of player i.

Let us consider the information partitions for the Observation Device Model given in Example 1. There are the following eight states of the world:

ω	p_1	a_1	a_2	b_1	b_2
ω_1	1	1	0	1	0
ω_2	1	1	0	0	1
ω_3	1	0	1	1	0
ω_4	1	0	1	0	1
ω_5	0	1	0	1	0
ω_6	0	1	0	0	1
ω_7	0	0	1	1	0
ω_8	0	0	1	0	1

At ω_1, the type of player 1's O.D. is a_1, hence, $f_1(\alpha_1(\omega_1)) = \{p_1\}$ and $g_1(\alpha_2(\omega_1),\alpha_1(\omega_1)) = \{b_1\}$. This means that $\omega_1 \approx_1 \omega$ only for $\omega = \omega_1$. Because, $\omega(p_1) \neq \omega_1(p_1)$ for $\omega = \omega_i$ $(i \geq 5)$, $f_1(\alpha_1(\omega)) = \{b_1,b_2\} \neq f_1(\alpha_1(\omega_1))$ for $\omega = \omega_3$ or ω_4, and $f_1(\alpha_1(\omega_2)) = \{b_2\} \neq f_1(\alpha_1(\omega_1))$. Similarly, each of ω_2, ω_5 and ω_6 is indistinguishable with only itself for player 1. However, $\omega_3 \approx_1 \omega$ for $\omega = \omega_4$, ω_7 and ω_8, since $f_1(a_2) = \emptyset$ and $g_1(b_i,a_2) = \{b_1,b_2\}$ for i=1,2. We have Π_1 as follows:

$$\Pi_1 = \{ \{\omega_1\}, \{\omega_2\}, \{\omega_3,\omega_4,\omega_7,\omega_8\}, \{\omega_5\}, \{\omega_6\} \}.$$

In the same way, we can easily verify that:

$$\Pi_2 = \{ \{\omega_1\}, \{\omega_2,\omega_4,\omega_6,\omega_8\}, \{\omega_3\}, \{\omega_5\}, \{\omega_7\} \}.$$

Aumann (1976) gave the two different but equivalent definitions of common knowledge in terms of the players' information partitions. One of the definitions is as follows: An event $E \subset \Omega$ is said to be common knowledge at a state ω if E contains all $\omega' \in \Omega$ such that there is a sequence $\pi^1, \pi^2, \ldots, \pi^n$ satisfying that $\omega \in \pi^1$, $\omega' \in \pi^n$, π^k (k=1,2,...,n) belongs to either Π_1 or Π_2 and $\pi^k \cap \pi^{k+1} \neq \emptyset$ (k=1,2,...,n-1).

Here, let n be a fixed positive integer, and let us consider the following statement:

(*) $E \subset \Omega$ contains all $\omega' \in \Omega$ for which there is a sequence $\pi^1, \pi^2, \ldots, \pi^n$ such that $\omega \in \pi^1$, $\omega' \in \pi^n$, $\pi^k \in \Pi_{i_k}$ (k=1,2,...,n), where $i_k \in \{1,2\}$ (k=1,2,...,n) and $i_k \neq i_{k+1}$ (k=1,2,...,n-1)

The basic interpretation of the information partition of player i is that, for any $\pi \in \Pi_i$, if $\omega \in \pi$ is the true state, then i knows that the true state is in π, but he does not knows which $\omega' \in \pi$ is the true one. According to this interpretation, we can say that, at ω, i_1 knows that i_2 knows that ... that i_n knows E (occurs) iff (*) holds. This is the reason why the above definition of common knowledge can be considered to reflect the intuitive meaning of common knowledge. (See Footnote 2.)

In this paper, we adopt (*) as the formulation of a player's knowledge in terms of the information partitions. For the formal definition, we define the following notations: For any proper order $\Theta = (i_1, i_2, \ldots, i_n)$ and any $\omega \in \Omega$, we say $(\omega_1, \omega_2, \ldots, \omega_n)$ is an <u>ω-sequence with Θ at ω</u>, if $\omega_k \in \Omega$ (k=1,2,...,n) and $\omega_k \approx_{i_{k+1}} \omega_{k+1}$ (k=0,2,...,n-1), where $\omega_0 = \omega$.

Definition 4.

For any $\omega \in \Omega$, any proper order $\Theta = (i_1, i_2, \ldots, i_n)$, and any $p \in P$, we define that $K(i_1, i_2, \ldots, i_n)p$ holds at ω if for any ω-sequence $(\omega_1, \omega_2, \ldots, \omega_n)$ with Θ at ω, $\omega_n(p) = 1$.

Remarks.

i) The intended interpretation of $K(i_1,i_2,\ldots,i_n)p$ is, of course, that i_1 knows that i_2 knows that ... that i_n knows the event that p is true (occurs).

ii) Note that Π_i is the equivalent class under $\approx_i$. Then, it is clear that the "if" part of Definition 4 is equivalent to (*) for $E = \{ \omega'\in\Omega\colon \omega'(p)=1 \}$.

The next theorem establishes the equivalence of our two definitions of player's knowledge. It shows the consistency of our definitions of players' deductive knowledge and their information partitions.

Theorem 1.

For any $\omega \in \Omega$, any proper sequence $\Theta = (i_1,i_2,\ldots,i_n)$ and any $p \in P$, the following two statements are equivalent:

i) $\omega(K_{i_1}K_{i_2}\ldots K_{i_n}p) = 1$;

ii) $K(i_1,i_2,\ldots,i_n)p$ holds at ω.

For the sake of notational simplicity, we have the following convention here and in Appendix 3: For any $\alpha_i \in A_i$ (i=1,2) and any $\omega \in \Omega$, we write $f(\alpha_i)$ in stead of $f_i(\alpha_i)$, and write $T_\omega[\alpha_i]$ instead of $T_\omega(B(f_i(\alpha_i)))$ provided that it is clear whose O.D. is indicated by α_i and α_j.

Proof of i) → ii). We prove by induction. For n=1, $\Theta = (i_1)$ $(i_1\in\{1,2\})$. Hence, $(\alpha_{i_1}(\omega))$ is the unique α-sequence with Θ at ω, and the ω-sequence with Θ at ω is (ω_1), where ω_1 is a state satisfying $\omega_1 \approx_{i_1} \omega$. Suppose that

$\omega(K_{i_1}p) = 1$, i.e., $p \in D_\omega(\alpha_{i_1}(\omega))$ $(= M(T_\omega[\alpha_{i_1}(\omega)] \cup T))$. Then, by the G.C.T.2 (in Section 3), $\omega'(p) = 1$ for all ω' $(\in \Omega)$ that satisfy $T_\omega[\alpha_{i_1}(\omega)]$. Let us show that ω_1 satisfies $T_\omega[\alpha_{i_1}(\omega)]$. By the definition of $\approx_i$ and B(.), $\omega_1 \approx_{i_1} \omega$ means that $\omega_1(q) = \omega(q)$ $\forall\, q \in B(f(\alpha_{i_1}(\omega)))$. Hence, by the definition of $T_\omega(.)$, $\omega_1(q) = 1$ $\forall\, q \in T_\omega[\alpha_{i_1}(\omega)]$, i.e., ω_1 satisfies $T_\omega[\alpha_{i_1}(\omega)]$. Thus, we get $\omega_1(p) = 1$, namely, $K(i_1)p$ holds at ω.

Suppose that the assertion holds for $n = k-1$ $(k \geq 2)$. Let $\Theta = (i_1, i_2, \ldots, i_k)$ be a proper order and assume that $\omega(K_{i_1}K_{i_1}\ldots K_{i_k}p) = 1$, i.e., for any α-sequence $(\alpha^1, \alpha^2, \ldots, \alpha^k)$ with Θ at ω, $p \in D_\omega(\alpha^1, \alpha^2, \ldots, \alpha^k)$. Take any ω-sequence $(\omega_1, \omega_2, \ldots, \omega_k)$ with Θ at ω. Let us show that $\omega_k(p) = 1$. Here, we need the following lemma.

Lemma 1.

Let $\Theta = (i_1, i_2, \ldots, i_k)$ be any proper order and $(\omega_1, \omega_2, \ldots, \omega_k)$ be any ω-sequence with Θ at ω, then $(\alpha_{i_1}(\omega_0), \alpha_{i_2}(\omega_1), \alpha_{i_3}(\omega_2), \ldots, \alpha_{i_k}(\omega_{k-1}))$ is an α-sequence with Θ at $\omega = \omega_0$. (Proof is in Appendix 3.)

Let $(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^k) = (\alpha_{i_1}(\omega_0), \alpha_{i_2}(\omega_1), \ldots, \alpha_{i_k}(\omega_{k-1}))$, where $\omega_0 = \omega$. By lemma 1, $(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^k)$ is an α-sequence with Θ at ω. Hence, by assumption, we have $p \in D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^k)$. Therefore, by G.C.T.2, $\omega'(p) = 1$ for all ω' $(\in \Omega)$ that satisfy $D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1}) \cap B(f(\hat{\alpha}^k))$. Thus, in order to establish that $\omega_k(p) = 1$, it suffices to show that ω_k satisfies $D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1}) \cap B(f(\hat{\alpha}^k))$. Note that $(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1})$ is an α-sequence with proper order $(i_1, i_2, \ldots, i_{k-1})$ at ω. Hence, by the induction hypothesis, if $q \in D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1})$, then $\omega_{k-1}(q) = 1$. On the other hand, by the definition of ω-sequence, $\omega_{k-1} \approx_{i_k} \omega_k$, hence, $f(\alpha_{i_k}(\omega_{k-1})) =$

$f(\hat{\alpha}^k) = f(\alpha_{i_k}(\omega_k))$ and $\omega_{k-1}(q) = \omega_k(q) \ \ \forall\ q \in f(\hat{\alpha}^k)$. Therefore, $\omega_{k-1}(q) = \omega_k(q)\ \forall\ q \in B(f(\hat{\alpha}^k))$. Thus, we obtain $\omega_k(q) = 1 \ \ \forall\ q \in D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1}) \cap B(f(\hat{\alpha}^k))$, i.e., ω_k satisfies $D_\omega(\hat{\alpha}^1, \hat{\alpha}^2, \ldots, \hat{\alpha}^{k-1}) \cap B(f(\hat{\alpha}^k))$. Q.E.D.

Proof of ii) → i). We show the contrapositive. For n = 1. Suppose that there exist $\omega \in \Omega$, a proper order $\Theta = (i_1)$ and $p \in P$ such that $\omega(K_{i_1} p) = 0$, i.e., $p \notin D_\omega(\alpha_{i_1}(\omega))$. Then, by G.C.T.2, there exists $\omega' \in \Omega$ which satisfies $T_\omega[\alpha_{i_1}(\omega)]$ but $\omega'(p) = 0$. Define $\omega_1 \in \Omega$ by: $\omega_1(q) = \omega'(q) \ \ \forall\ q \in P$ and $\alpha_j(\omega_1) = \alpha_j(\omega)$ (j=1,2). Then, $\omega_1 \approx_{i_1} \omega$, while $\omega_1(p) = 0$, meaning $K(i_1)p$ does not hold at ω. For $n \geq 2$, suppose that there exist $\omega \in \Omega$, a proper order $\Theta = (i_1, i_2, \ldots, i_n)$, $p \in P$ and an α-sequence $(\alpha^1, \alpha^2, \ldots, \alpha^n)$ with Θ at ω such that $p \notin D_\omega(\alpha^1, \alpha^2, \ldots, \alpha^n)$. Here, the following Lemma 2 and Lemma 3 are needed to construct the ω-sequence $(\omega_1, \omega_2, \ldots, \omega_n)$ with Θ at ω, where $\omega_n(p) = 0$.

Lemma 2.

For any $p \in P$, any α-sequence $(\alpha^1, \alpha^2, \ldots, \alpha^{k+1})$ with a proper order $(i_1, i_2, \ldots, i_{k+1})$ $(k \geq 2)$ and any ω, $\omega' \in \Omega$, if ω' satisfies $D_\omega(\alpha^1, \alpha^2, \ldots, \alpha^k) \cap B(f(\alpha^{k+1}))$, then there exists an ω'' $(\in \Omega)$ such that $\omega''(q) = \omega'(q) \ \ \forall\ q \in f(\alpha^{k+1})$ and ω'' satisfies $D_\omega(\alpha^1, \alpha^2, \ldots, \alpha^{k-1}) \cap B(f(\alpha^k))$.

(Proof is in Appendix 3.)

Lemma 3.

For any $p \in P$, any α-sequence (α^1, α^2) with a proper order (i_1, i_2) and any ω, $\omega' \in \Omega$, if ω' satisfies $D_\omega(\alpha^1) \cap B(f(\alpha^2))$, then there exists an $\omega'' \in \Omega$ such that $\omega''(q) = \omega'(q) \ \ \forall\ q \in f(\alpha^2)$ and ω'' satisfies $T_\omega[\alpha^1]$.

(Proof is in Appendix 3.)

Since $p \notin D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^n)$, by G.C.T.2, there exists an ω^* which satisfies $D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^{n-1}) \cap B(f(\alpha^n))$ but $\omega^*(p) = 0$. Let ω_n be the state such that $\omega_n(q) = \omega^*(q)\ \ \forall\ q \in P$, $\alpha_{i_n}(\omega_n) = \alpha^n$ and $\alpha_{i_{n-1}}(\omega_n) = \alpha^{n-1}$. Define ω_k $(k=n-1,n-2,\ldots,2,1)$ inductively as follows:

for $2 \leq k \leq n-1$,

$$\omega_k(q) = \omega''(q) \quad \forall\ q \in P,$$
$$\alpha_{i_k}(\omega_k) = \alpha^k, \text{ and}$$
$$\alpha_{i_{k+1}}(\omega_k) = \alpha^{k+1},$$

where ω'' is the state referred in Lemma 2 for $\omega' = \omega_{k+1}$; and

$$\omega_1(q) = \omega''(q) \quad \forall\ q \in P,$$
$$\alpha_{i_1}(\omega_1) = \alpha^1, \text{ and}$$
$$\alpha_{i_2}(\omega_1) = \alpha^2,$$

where ω'' is the state referred in Lemma 3 for $\omega' = \omega_2$.

Then, $\omega_k \in \Omega$ $(k=1,2,3,\ldots,n-1)$, and we can verify that $\omega \approx_{i_1} \omega_1 \approx_{i_2} \omega_2 \approx_3 \cdots \approx_{i_{n-1}} \omega_{n-1} \approx_{i_n} \omega_n$. By the definition of α-sequence, $\alpha_{i_1}(\omega_1) = \alpha^1 = \alpha_{i_1}(\omega)$, thus, $f(\alpha_{i_1}(\omega)) = f(\alpha_{i_1}(\omega_1))$. Since, by definition, ω_1 satisfies $T_\omega[\alpha^1]$, we have $\omega_1(q) = \omega(q)\ \forall\ q \in f(\alpha_{i_1}(\omega_1))$. Further, by the definition of α-sequence, $\alpha_{i_2}(\omega_1) = \alpha^2 \in g_{i_1}(\alpha_{i_2}(\omega),\alpha_{i_1}(\omega))$. Hence, by Assumption 3 and $\alpha_{i_1}(\omega_1) = \alpha_{i_1}(\omega)$, we get $g_{i_1}(\alpha_{i_2}(\omega_1),\alpha_{i_1}(\omega_1)) = g_{i_1}(\alpha_{i_2}(\omega),\alpha_{i_1}(\omega))$. Therefore $\omega \approx_{i_1} \omega_1$. Consider $k = 2,3,\ldots,n-1$. By definition, $\alpha_{i_k}(\omega_{k-1}) = \alpha^k = \alpha_{i_k}(\omega_k)$ (hence, $f(\alpha_{i_k}(\omega_{k-1})) = f(\alpha_{i_k}(\omega_k))$ and $\omega_k(q) = \omega_{k-1}(q)\ \ \forall\ q \in f(\alpha_{i_k}(\omega_k))$. By the definition of α-sequence, $\alpha^{k+1} \in g_{i_k}(\alpha^{k-1},\alpha^k)$. Hence, by Assumption 3, we get $g_{i_k}(\alpha^{k+1},\alpha^k) = g_{i_k}(\alpha^{k-1},\alpha^k)$. By definition, $\alpha^{k+1} = \alpha_{i_{k+1}}(\omega_k)$, $\alpha^k = \alpha_{i_k}(\omega_k)$, $\alpha^{k-1} = \alpha_{i_{k-1}}(\omega_{k-1})$ and $\alpha^k = \alpha_{i_k}(\omega_{k-1})$. Thus we have $g_{i_k}(\alpha_{i_{k+1}}(\omega_k),\alpha_{i_k}(\omega_k)) = g_{i_k}(\alpha_{i_{k-1}}(\omega_{k-1}),\alpha_{i_k}(\omega_{k-1}))$. Therefore $\omega_{k-1} \approx_{i_k} \omega_k$. We can easily verify that $\omega_{n-1} \approx_{i_n} \omega_n$. Thus we obtain the ω-sequence $(\omega_1,\omega_2,\ldots,\omega_n)$ with Θ at ω, where $\omega_n(p) = 0$. Q.E.D.

5. Characterization of Information Partition

In this section, we characterize the information partitions in terms of the structure of the Observation Device Model. First, we consider the condition on $f_i(.)$ and $g_i(.,.)$ (i=1,2) for one player's information partition to be a refinement of the other's. Next, we see some cases in which a player has special observing abilities, and examine the properties of the information partitions corresponding to those cases.

The concept of the coarsening or refinement[5] of one information partition to another represents one player's informational superiority. Proposition 1 seeks the reason of this informational asymmetry.

Theorem 2.

For i, j $\in$ {1,2} i$\neq$j, Π_i is a refinement of Π_j iff

a) $\forall\alpha_i \in A_i$ and $\forall\alpha_j \in A_j$, $f_i(\alpha_i) \supset f_j(\alpha_j)$;

and b) $\forall\alpha_i \in A_i$ and $\forall\alpha_j', \alpha_j'' \in A_j$, $g_i(\alpha_j',\alpha_i) = g_i(\alpha_j'',\alpha_i)$ implies that $g_j(\alpha_i,\alpha_j') = g_j(\alpha_i,\alpha_j'')$ and $f_j(\alpha_j') = f_j(\alpha_j'')$.

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[5] For any partitions Π and Π' of Ω, we say Π is a refinement of Π' (equivalently, Π' is a coarsening of Π) if $\forall\pi \in \Pi$, $\exists\pi' \in \Pi'$, $\pi \subset \pi'$.

The first condition states that, at any types of the players' O.D., the set of i's observable objective propositions includes that of j's. The second condition states that, at any type of player i's O.D., if i cannot distinguish two types of j's O.D., then j cannot recognize any difference in his observations from each of the two types.

Proof of if - part. It suffices to show that, for any ω and $\omega' \in \Omega$, if $\omega \approx_i \omega'$, then $\omega \approx_j \omega'$. By definition, it is written as follows:

If i) $f_i(\alpha_i(\omega)) = f_i(\alpha_i(\omega'))$,

ii) $\omega(p) = \omega'(p) \quad \forall\ p \in f_i(\alpha_i(\omega))$, and

iii) $g_i(\alpha_j(\omega),\alpha_i(\omega)) = g_i(\alpha_j(\omega'),\alpha_i(\omega'))$,

then

iv) $f_j(\alpha_j(\omega)) = f_j(\alpha_j(\omega'))$,

v) $\omega(p) = \omega'(p) \quad \forall\ p \in f_j(\alpha_j(\omega))$, and

vi) $g_j(\alpha_i(\omega),\alpha_j(\omega)) = g_j(\alpha_i(\omega'),\alpha_j(\omega'))$.

From condition a), i) and ii) implies v). Further, Assumption 2 and iii) means that $\alpha_j(\omega') \in g_i(\alpha_j(\omega),\alpha_i(\omega))$, so that we have $g_i(\alpha_j(\omega),\alpha_i(\omega)) = g_i(\alpha_j(\omega'),\alpha_i(\omega))$ by Assumption 3. Hence, by condition b), we get

$$(1) \quad g_j(\alpha_i(\omega),\alpha_j(\omega)) = g_j(\alpha_i(\omega),\alpha_j(\omega'))$$

and $f_j(\alpha_j(\omega)) = f_j(\alpha_j(\omega'))$. Thus we have iv). On the other hand, from the equality $g_i(\alpha_j(\omega),\alpha_i(\omega)) = g_i(\alpha_j(\omega'),\alpha_i(\omega))$ we obtained above and iii), we get $g_i(\alpha_j(\omega'),\alpha_i(\omega)) = g_i(\alpha_j(\omega'),\alpha_i(\omega'))$. Hence, from i) and Assumption

4, we have

$$(2) \quad g_j(\alpha_i(\omega),\alpha_j(\omega')) = g_j(\alpha_i(\omega'),\alpha_j(\omega')).$$

Summing up (1) and (2), we obtain vi). Q.E.D.

Proof of only if - part. Suppose that condition a) does not hold. Then there exist $\alpha_i \in A_i$ and $\alpha_j \in A_j$ such that $f_i(\alpha_i) \not\supset f_j(\alpha_j)$. Let $p \in f_j(\alpha_j) - f_i(\alpha_i)$. Let ω and ω' be the two states which assign the same truth value on $P - \{p\}$, A_i and A_j but not for p. Then $\omega \approx_j \omega'$ does not hold, while $\omega \approx_i \omega$ does. Next suppose that condition b) does not holds. Then, there exist $\alpha_i \in A_i$ and α_2', $\alpha_2'' \in A_j$ such that $g_i(\alpha_j',\alpha_i) = g_i(\alpha_j'',\alpha_i)$ but $g_j(\alpha_i,\alpha_j') \neq g_j(\alpha_i,\alpha_j'')$ or $f_j(\alpha_j') \neq f_j(\alpha_j'')$. Let ω and ω' be the two states such that $\omega(p) = \omega'(p)$ $\forall p \in P$, $\omega(\alpha_i) = \omega'(\alpha_i) = 1$ but $\omega(\alpha_2') = 1$ and $\omega'(\alpha_2'') = 1$. Then $\omega \approx_i \omega'$, however, $\omega \approx_j \omega'$ does not hold. Q.E.D.

In our framework, we denote a player (say i)'s observing ability in terms of f_i and g_i. Hence, by using special f_i and g_i, we can formulate the situations in which a player (i) has special observing abilities. It will be worthwhile to examine the properties of information partitions corresponding to those situations. We consider here the following three cases.

5.1. Complete Observation

Let $f_i(\alpha_i) = P$ and let $g_i(\alpha_j,\alpha_i) = \{\alpha_j\}$ for all $\alpha_i \in A_i$ and $\alpha_j \in A_j$ $(i,j \in \{1,2\},\ j \neq i)$. Then player i will have complete observation. He observes all the truth value of the propositions, and he knows the true type of the other player's O.D.

In this case, player i's information partition is essentially the point wise partition of Ω, i.e., the partition whose elements consist of a single state.

Suppose that there exists an element of player i's information partition which contains more than one state, say ω and ω' ($\omega \neq \omega'$). Then, since $\omega \approx_i \omega'$ and player i has complete observation, ω and ω' assign the same truth values on P and A_j ($j \neq i$). Let $\alpha_j \in A_j$, α_i, $\alpha'_i \in A_i$ be respectively the types of j and i's O.D. such that $\omega(\alpha_j) = \omega'(\alpha_j) = 1$, $\omega(\alpha_i) = 1$ and $\omega'(\alpha'_i) = 1$. Then, since $f_i(\alpha_i) = f_i(\alpha'_i) = P$, $\omega(q) = \omega'(q)\ \ \forall\ q \in P$. And, furthermore, since $g_i(\alpha_j, \alpha_i) = \{\alpha_j\} = g_i(\alpha_j, \alpha'_i)$, by Assumption 4, $g_j(\alpha_i, \alpha_j) = g_j(\alpha'_i, \alpha_j)$. Therefore, $\omega \approx_j \omega'$.

Eventually, the difference between ω and ω' is only the "names" of the types of i's O.D. associated with them. The two types causes no difference on i's actual observation. Further, player j cannot distinguish the difference of the two types, and ω and ω' assigns the same truth values on P. That is, the two types are essentially the same.

In Example 1, if we erase a_2, then player 1 has complete observation. In that case, the states ω_3, ω_4, ω_7 and ω_8 vanish, and players 1 and 2 's information partitions are as follows:

$$\Pi_1 = \{\ \{\omega_1\},\ \{\omega_2\},\ \{\omega_5\},\ \{\omega_6\}\ \},$$

$$\Pi_2 = \{\ \{\omega_1\},\ \{\omega_2, \omega_6\},\ \{\omega_5\}\ \}.$$

5.2. Purely Private Observation

Let $g_i(\alpha_j,\alpha_i) = A_j$ for any $\alpha_i \in A_i$ and any $\alpha_j \in A_j$ $(i,j \in \{1,2\}, i \neq j)$. Then player i has purely private observation since he observes nothing about j's O.D.

In this case, if there exists a type $\alpha_j^* \in A_j$ which stands for no observation, i.e., $f_j(\alpha_j^*) = \emptyset$ and $g_j(\alpha_i,\alpha_j^*) = A_i \ \forall \ \alpha_i \in A_i$, then there exists an element in j's information partition which intersects all the elements of i's information partition. The reason is as follows: Since i has no information on j's O.D., every elements of i's information partition contains a state at which α_j^* is the type of j's O.D. Such states are indistinguishable for j because of no observation at α_j^*.

In Example 1, if $g_1(b_i,a_1) = g_1(b_i,a_2) = \{b_1,b_2\}$ for $i = 1,2$, then player 1 has purely private observation. It will be such a situation in which 1 knows that 2 is standing there, but 2 wears sun glasses, so 1 cannot know whether 2 opens his eyes or not. Π_1 is as follows:

$$\Pi_1 = \{ \ \{\omega_1,\omega_2\}, \ \{\omega_3,\omega_4,\omega_7,\omega_8\}, \ \{\omega_5,\omega_6\} \ \}.$$

It will be worthwhile to recall Π_2:

$$\Pi_2 = \{ \ \{\omega_1\}, \ \{\omega_2,\omega_4,\omega_6,\omega_8\}, \ \{\omega_3\}, \ \{\omega_5\}, \ \{\omega_7\} \ \}.$$

At states ω_2, ω_4, ω_6 and ω_8, player 2's O.D. is b_2 which stands for no observation.

5.3. Null Observation

Let $f_i(\alpha_i) = \emptyset$ and $g_i(\alpha_j,\alpha_i) = A_j$ for all $\alpha_i \in A_i$ and $\alpha_j \in A_j$ ($i,j \in \{1,2\}$, $i \neq j$). Then player i has no observation with any $\alpha_i \in A_i$.

In this case, i's information partition is $\{ \Omega \}$.

In the intended situation of Example 1, player 1 will be in this case if he locked in a box out of which he cannot see.

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Appendix 1.

We will prove the two facts which were stated in Section 3 without proof. The next lemma is the essence of them.

Lemma A.1

For any Q, R $\subset$ P, if $T_\omega(B_0(Q)) \subset M(R \cup T)$, then $T_\omega(B(Q)) \subset M(R \cup T)$.

Proof. We show that $T_\omega(B_n(Q))) \subset M(R \cup T)$ for $n = 0,1,2,...$ The case in which $n = 0$ is the assumption. Suppose that $T_\omega(B_k(Q)) \subset M(R \cup T)$. By the definition of B(.) (in Section 3), if $p \in T_\omega(B_{k+1}(Q))$, then there exist q, r $\in B_k(Q)$ and we can write $p = q \wedge r$, $q \rightarrow r$, $q \vee r$, $\sim(q \wedge r)$, $\sim(q \rightarrow r)$, or $\sim(q \vee r)$. If $p = q \wedge r$, then $\omega(p) = 1$ means that $\omega(q) = \omega(r) = 1$, i.e., $q, r \in T_\omega(B_k(Q))$. Then, by the induction hypothesis, $q, r \in M(R \cup T)$. On the other hand, $q \rightarrow (r \rightarrow (q \wedge r))$ is a tautology. Applying modus ponens twice, we have $p = q \wedge r \in M(R \cup T)$. We can prove the other cases in the same way. The truth values of q, r and the tautologies used in those cases are as follows. For $p = q \rightarrow r$, $\omega(q) = 0$ (hence, $\sim q \in T_\omega(B_k(Q))$) or $\omega(q) = \omega(r) = 1$, and the tautologies are $\sim q \rightarrow (q \rightarrow r)$ and $q \rightarrow (r \rightarrow (q \rightarrow r))$, respectively. For $p = q \vee r$, $\omega(q) = 1$ or $\omega(r) = 1$ and the tautologies are $q \rightarrow (q \vee r)$ and $r \rightarrow (q \vee r)$, respectively. For $p = \sim(q \wedge r)$, $\omega(q) = 0$ or $\omega(r) = 0$, and the tautologies are $\sim q \rightarrow \sim(q \wedge r)$ and $\sim r \rightarrow \sim(q \wedge r)$, respectively. For $p = \sim(q \rightarrow r)$, $\omega(q) = 1$ and $\omega(r) = 0$, and the tautology is $q \rightarrow (\sim r \rightarrow (\sim(q \rightarrow r)))$. For $p = \sim(q \vee r)$, $\omega(q) = \omega(r) = 0$, and the tautology is $\sim q \rightarrow (\sim r \rightarrow (\sim(q \vee r)))$. Q.E.D.

Proposition A.1.

For any $S \subset P$, $T_\omega(B(M(T_\omega(S) \cup T))) = M(T_\omega(S) \cup T)$.

Proof. By definition, $B_0(M(T_\omega(S) \cup T)) = M(T_\omega(S) \cup T) \cup \{\sim p: p \in M(T_\omega(S) \cup T)\}$. In modus ponens, if $\omega(p) = 1$ and $\omega(p \to q) = 1$, then $\omega(q) = 1$. So, $\omega(p) = 1 \;\; \forall\, p \in M(T_\omega(S) \cup T)$. Hence, $T_\omega(B_0(M(T_\omega(S) \cup T))) = M(T_\omega(S) \cup T)$. Therefore, $T_\omega(B(M(T_\omega(S) \cup T))) \supset M(T_\omega(S) \cup T)$. Further, $T_\omega(B_0(M(T_\omega(S) \cup T))) = (\subset)\ M(T_\omega(S) \cup T)$ leads the converse inclusion by Lemma A.1. Q.E.D.

Proposition A.2.

For any $S \subset P$,

$T_\omega(B(S)) \subset M(\{p: p \in S \text{ and } \omega(p) = 1\} \cup \{\sim p: p \in S \text{ and } \omega(p) = 0\} \cup T)$.

Proof. Since $B_0(S) = S \cup \{\sim p: p \in S\}$, $T_\omega(B_0(S)) = \{p: p \in S \text{ and } \omega(p) = 1\} \cup \{\sim p: p \in S \text{ and } \omega(p) = 0\}$. Hence, $T_\omega(B_0(S))$ is included by $M(\{p: p \in S \text{ and } \omega(p) = 1\} \cup \{\sim p: p \in S \text{ and } \omega(p) = 0\} \cup T)$.
By Lemma A.1, we complete the proof. Q.E.D.

Appendix 2.

We need the following Lemma A.2 through Lemma A.6 and a famous theorem called Deduction Theorem to prove Generalized Completeness Theorem 1 and 2.

Lemma A.2.

i) If ω ($\in \Omega$) satisfies S ($\subset$ P) and $p \in M(S \cup A)$, then $\omega(p) = 1$.

ii) S ($\subset$ P) is consistent if there exists ω ($\in \Omega$) which satisfies S.

Proof of i). By the definition of M(.), it suffices to show that $\omega(q) = 1$ $\forall\ q \in M_n(S \cup A)$ for n = 0,1,2,... For n = 0, we get it by the assumption. Suppose that $\omega(q) = 1$ $\forall\ q \in M_k(S \cup A)$. Let $q \in M_{k+1}(S \cup A)$. Then, $q \in M_k(S \cup A)$ or there exists $r \in P$ such that $r, r \to q \in M_k(S \cup A)$. In the latter case, by the induction hypothesis, $\omega(r) = 1$ and $\omega(r \to q) = 1$, so that $\omega(q) = 1$. Q.E.D.

Proof of ii). Suppose that S is inconsistent, then there exists $q \in P$ such that $q, \sim q \in M(S \cup A)$. By i), $\omega(q) = \omega(\sim q) = 1$, a contradiction. Q.E.D.

Lemma A.3.

S ($\subset$ P) is consistent iff any finite set S′ ($\subset$ S) is consistent.

Proof. We can obtain the contrapositive of the only if part from the fact that $S' \subset S$ implies $M(S' \cup A) \subset M(S \cup A)$. Let's prove the if - part. Suppose that S is inconsistent. Then there exists $p \in P$ such that $p, \sim p \in M(S \cup A)$. Hence $p, \sim p \in M_n(S \cup A)$ for some sufficiently large integer $n \geq 0$. Here, let us show that, for n = 0,1,2,..., if $q \in M_n(S \cup A)$, then there exists a finite set S' ($\subset S$) such that $q \in M_n(S' \cup A)$. For $n = 0$, this assertion holds for $S' = \{q\}$. Suppose that the assertion holds for $n = k$. For $n = k + 1$, let $q \in M_{k+1}(S \cup A)$. Then, $q \in M_k(S \cup A)$ or there exists $r \in P$ such that $r, r \to q \in M_k(S \cup A)$. In the latter case, by the induction hypothesis, there exist finite sets S', S'' ($\subset S$) such that $r \in M_k(S' \cup A)$ and $r \to q \in M_k(S'' \cup A)$. Then, $S' \cup S''$ is a finite subset of S, and $r, r \to q \in M_k(S' \cup S'' \cup A)$, which completes the induction. Hence, if $p, \sim p \in M_n(S \cup A)$, then there exist finite sets S_1, S_2 ($\subset S$) such that $p \in M_n(S_1 \cup A)$ and $\sim p \in M_n(S_2 \cup A)$. Therefore, we have $p, \sim p \in M_n(S_1 \cup S_2 \cup A)$, i.e., there exists a finite set $S_1 \cup S_2$ ($\subset S$) which is inconsistent. Q.E.D.

Lemma A.4.

Let $S \subset P$. If, for any finite set S' ($\subset S$), there exists a state which satisfies S', then the following two assertions a) and b) hold.

a) There exist functions $f_n: \{ p_i \in P_0: i<n \} \to \{1,0\}$ $(n = 1,2,3,\dots)$ each of which satisfies:

i) if $m < n$, then $f_m(p_i) = f_n(p_i)$ $\forall\ i < m$;

ii) for any finite set S' ($\subset S$), there exists an ω ($\in \Omega$) such that it satisfies S' and $\omega(p_i) = f_n(p_i)$ $\forall\ i < n$.

b) Define ω_0 as $\omega_0(p_i) = f_{i+1}(p_i)$. Then ω_0 satisfies S.

Proof of a). Suppose that the assertion holds for $n = k$. For $n = k + 1$, let $f_{k+1}(p_i) = f_k(p_i)$ for $i = 0,1,2,\dots,k-1$. And let $f_{k+1}(p_k) = 1$. If ii) holds for this f_{k+1}, then the assertion holds for $n = k + 1$. If not, then let $f_{k+1}(p_k) = 0$. We can show that ii) holds for this f_{k+1}. Now, ii) does not hold for f_{k+1} which takes 1 for p_k. It means that there exists a finite set S_0 ($\subset S$) which is not satisfied by any ω ($\in \Omega$) such that $\omega(p_i) = f_{k+1}(p_i)$ $\forall$ $i < k$ and $\omega(p_k) = 1$. Take any finite set S' ($\subset S$). Since $S_0 \cup S'$ is a finite subset of S, by the induction hypothesis, ii) holds for f_k. That is, there exists an ω ($\in \Omega$) such that it satisfies $S_0 \cup S'$ and $\omega(p_i) = f_k(p_i)$ $\forall$ $i < k$. From the condition of S_0, that ω must be $\omega(p_k) = 0$. Hence, the ω satisfies S' and $\omega(p_i) = f_{k+1}(p_i)$ $\forall$ $i < k+1$. That is, ii) holds for f_{k+1}. For $n = 1$, we can show the assertion in the same way as f_{k+1} by using the assumption of this lemma in stead of the induction hypothesis. Q.E.D.

Proof of b). Take any $p \in S$. By the definition of P, p consists of finitely many atomic propositions. Let n be large enough that all those atomic propositions are contained in $\{p_i \in P_0 : 0 \le i \le n-1\}$. Note that, for any ω ($\in \Omega$), the value of $\omega(p)$ is determined by the value of $\omega(p_0)$ through $\omega(p_{n-1})$. Since, $\{p\}$ is a finite subset of S, by ii) of a), there exists an ω ($\in \Omega$) such that $\omega(p) = 1$ and $\omega(p_i) = f_n(p_i)$ $\forall$ $i < n$. By the definition of ω_0, we have $\omega_0(p_i) = \omega(p_i)$ $\forall$ $i < n$, so that $\omega_0(p) = \omega(p) = 1$.

Q.E.D.

Lemma A.5.

Let $S \subset P$. The necessary and sufficient condition of that there exists a state which satisfies S is that, for any finite set S' ($\subset S$), there exists a state which satisfies S'.

Proof. Necessity is trivial. Sufficiency is established by constructing ω_0 stated in Lemma A.4 for this S. Q.E.D.

Lemma A.6.

For any $p, q_1, q_2, \ldots, q_n \in P$, the following three statements are equivalent.

i) $p \in M(\{q_1, q_2, \ldots, q_n\} \cup A)$.

ii) $q_1 \to (q_2 \to \ldots (q_n \to p) \ldots) \in M(A)$.

iii) For any state ω which satisfies $\{q_1, q_2, \ldots, q_n\}$, $\omega(p) = 1$.

Proof. i) → ii) is given from i) of Lemma A.1. Let us prove that iii) → ii). Let $r = q_1 \to (q_2 \to \ldots (q_n \to p) \ldots)$. By Completeness Theorem (Section 3), it suffices to show that r is a tautology. Take any $\omega \in \Omega$. Suppose that there exists an j $(1 \le j \le n)$ such that $\omega(q_j) = 0$. Let k be the smallest one of all such j's. Then $\omega(q_i) = 1$ for $i = 1,2,\ldots,k\text{-}1.$ and $\omega(q_k \to (q_{k+1} \to \ldots (q_n \to p) \ldots)) = 1$. Hence $\omega(r) = 1$. Suppose that $\omega(q_i) = 1 \ \forall \ 1 \le i \le n$, then, of course, $\omega(r) = 1$. Thus r is a tautology. ii) → i) is verified as follows: if ii) holds, then $q_1 \to (q_2 \to \ldots (q_n \to p) \ldots) \in M(\{q_1, q_2, \ldots, q_n\} \cup A)$; applying modus ponens n times, we have i). Q.E.D.

Deduction Theorem.

For any $p, q \in P$ and $S \subset P$, if $q \in M(S \cup \{p\} \cup A)$, then $p \to q \in M(S \cup A)$. (See, e.g., Mendelson (1979).)

Proof of Generalized Completeness Theorem 1.

First we prove the case in which S is finite. The if - part of this theorem has already been established in ii) of Lemma A.2. Let us show the only if - part. Suppose that there is no state which satisfies S. Then, taking any $q_0 \in P$, both of the following two statements are true.

For any ω ($\in \Omega$) which satisfies S, $\omega(q_0) = 1$;

For any ω ($\in \Omega$) which satisfies S, $\omega(\sim q_0) = 1$.

Therefore, by iii) → i) in Lemma A.6, we have q_0, $\sim q_0 \in M(S \cup A)$, i.e., S is inconsistent. Let us prove the theorem for infinite set S ($\subset$ P). We can show that the following i) through iv) are equivalent among each others.

i) S is consistent.

ii) Any finite set S′ ($\subset$ S) is consistent.

iii) For any finite set S′ ($\subset$ S), there exists a state which satisfies S′.

iv) There is a state which satisfies S.

i) ⇔ ii) is shown in Lemma A.3. We have proved ii) ⇔ iii) above. iii) ⇔ iv) is given in Lemma A.5. Thus we completes the proof. Q.E.D.

Proof of Generalized Completeness Theorem 2.

The only if - part of this theorem has already been proved in ii) of Lemma A.2. We show the contrapositive of the if - part. First we show that if $p \notin M(S \cup A)$, then $S \cup \{\sim p\}$ is consistent. Suppose on the contrary, i.e., $S \cup \{\sim p\}$ is inconsistent, then there exists $q \in P$ such that $q, \sim q \in M(S \cup \{\sim p\} \cup A)$. By Deduction Theorem, we have $\sim p \to q$, $\sim p \to \sim q \in M(S \cup A)$. It can easily verified that $(\sim p \to q) \to ((\sim p \to \sim q) \to q)$ is a tautology. Hence, we have $p \in M(S \cup A)$. Therefore, if $p \notin M(S \cup A)$, then $S \cup \{\sim p\}$ is consistent. If $S \cup \{\sim p\}$ is consistent, by G.C.T.1, there exists an ω ($\in \Omega$) which satisfies $S \cup \{\sim p\}$. This ω satisfies S, but $\omega(p) = 0$, which completes the proof. Q.E.D.

Appendix 3.

We need the following three lemmas to establish Lemma 1.

Lemma A.7.

For any ω', $\omega^* \in \Omega$, if $\omega' \approx_i \omega^*$ ($i \in \{1,2\}$), then, for j ($j \neq i$, $j \in \{1,2\}$),

$$g_j(\alpha_i(\omega'),\alpha_j(\omega')) = g_j(\alpha_i(\omega^*),\alpha_j(\omega')).$$

Proof of Lemma A.7. In the light of Assumption 4, it suffices to show that $f_i(\alpha_i(\omega')) = f_i(\alpha_i(\omega^*))$ and $g_i(\alpha_j(\omega'),\alpha_i(\omega')) = g_i(\alpha_j(\omega'),\alpha_i(\omega^*))$. The first equality is directly obtained by the assumption that $\omega' \approx_i \omega^*$. Further, from the definition of $\approx_i$, we get

$$\text{(a.1)} \quad g_i(\alpha_j(\omega'),\alpha_i(\omega')) = g_i(\alpha_j(\omega^*),\alpha_i(\omega^*)).$$

By Assumption 2, $\alpha_j(\omega')$ is contained in the left hand side so that we have $\alpha_j(\omega') \in g_i(\alpha_j(\omega^*),\alpha_i(\omega^*))$. Hence, from Assumption 3, we get $g_i(\alpha_j(\omega^*),\alpha_i(\omega^*)) = g_i(\alpha_j(\omega'),\alpha_i(\omega^*))$. Thus, by (a.1), we obtain $g_i(\alpha_j(\omega'),\alpha_i(\omega')) = g_i(\alpha_j(\omega'),\alpha_i(\omega^*))$. Q.E.D.

Lemma A.8.

Let k be an integer ≥ 3. If $(i_{k-3},i_{k-2},i_{k-1})$ is a proper order and ω_{k-3}, ω_{k-2} be the states such that $\omega_{k-3} \approx_{i_{k-2}} \omega_{k-2}$, then

$$g_{i_{k-1}}(\alpha_{i_{k-2}}(\omega_{k-2}),\alpha_{i_{k-1}}(\omega_{k-2})) = g_{i_{k-1}}(\alpha_{i_{k-2}}(\omega_{k-3}),\alpha_{i_{k-1}}(\omega_{k-2})).$$

Proof of Lemma A.8. Since, $\omega_{k-3} \approx_{i_{k-2}} \omega_{k-2}$, by Lemma A.2., we have $g_{i_{k-3}}(\alpha_{i_{k-2}}(\omega_{k-2}),\alpha_{i_{k-3}}(\omega_{k-2})) = g_{i_{k-3}}(\alpha_{i_{k-2}}(\omega_{k-3}),\alpha_{i_{k-3}}(\omega_{k-2}))$. Since,

$(i_{k-3}, i_{k-2}, i_{k-1})$ is a proper order, we can replace i_{k-3} by i_{k-1}, and obtain the equality in question. Q.E.D.

Lemma A.9.

Let k be an integer ≥ 3. If (i_{k-2}, i_{k-1}, i_k) is a proper order and ω_{k-3}, ω_{k-2}, ω_{k-1} and ω_k are the states such that $\omega_{k-3} \approx_{i_{k-2}} \omega_{k-2} \approx_{i_{k-1}} \omega_{k-1}$, then $\alpha_{i_k}(\omega_{k-1}) \in g_{i_{k-1}}(\alpha_{i_{k-2}}(\omega_{k-3}), \alpha_{i_{k-1}}(\omega_{k-2}))$.

Proof of Lemma A.9. By Assumption 3, we get $\alpha_{i_k}(\omega_{k-1}) \in g_{i_{k-1}}(\alpha_{i_k}(\omega_{k-1}), \alpha_{i_{k-1}}(\omega_{k-1}))$. Since $\omega_{k-2} \approx_{i_{k-1}} \omega_{k-1}$, the right hand side is equal to $g_{i_{k-1}}(\alpha_{i_k}(\omega_{k-2}), \alpha_{i_{k-1}}(\omega_{k-2}))$. Thus, we have $\alpha_{i_k}(\omega_{k-1}) \in g_{i_{k-1}}(\alpha_{i_k}(\omega_{k-2}), \alpha_{i_{k-1}}(\omega_{k-2}))$. From the fact that $i_k = i_{k-2}$ (by the definition of proper order) and Lemma A.8, we have

$g_{i_{k-1}}(\alpha_{i_k}(\omega_{k-2}), \alpha_{i_{k-1}}(\omega_{k-2})) = g_{i_{k-1}}(\alpha_{i_{k-2}}(\omega_{k-3}), \alpha_{i_{k-1}}(\omega_{k-2}))$, which completes the proof. Q.E.D.

Proof of Lemma 1. For k = 1, by definition $(\alpha_{i_1}(\omega_0))$ is the unique ω-sequence with (i_1) at $\omega = \omega_0$. Let us consider the case k = 2. By Lemma A.7, we get $g_{i_1}(\alpha_{i_2}(\omega_0), \alpha_{i_1}(\omega_0)) = g_{i_1}(\alpha_{i_2}(\omega_1), \alpha_{i_1}(\omega_0))$. By Assumption 2, $\alpha_{i_2}(\omega_1)$ is contained in the right hand side of this equation. Hence, $\alpha_{i_2}(\omega_1) \in g_{i_1}(\alpha_{i_2}(\omega_0), \alpha_{i_1}(\omega_0))$, which means that $(\alpha_{i_1}(\omega_0), \alpha_{i_2}(\omega_1))$ is an α-sequence with (i_1, i_2) at $\omega = \omega_0$. Suppose that, for k ≥ 3, $(\alpha_{i_1}(\omega_0), \alpha_{i_2}(\omega_1), \ldots, \alpha_{i_{k-1}}(\omega_{k-2}))$ is an α-sequence with $(i_1, i_2, \ldots, i_{k-1})$ at

ω. Since, $\omega_{k-3} \approx_{i_{k-2}} \omega_{k-2} \approx_{i_{k-1}} \omega_{k-1}$, by Lemma A.9, we have $\alpha_{i_k}(\omega_{k-1}) \in g_{i_{k-1}}(\alpha_{i_{k-2}}(\omega_{k-3}),\alpha_{i_{k-1}}(\omega_{k-2}))$. Hence, by the induction hypothesis and the definition of the α-sequence,

$(\alpha_{i_1}(\omega_0),\alpha_{i_2}(\omega_1),\ldots,\alpha_{i_{k-1}}(\omega_{k-2}),\alpha_{i_k}(\omega_{k-1}))$ is an α-sequence with $(i_1,i_2,\ldots,i_k)$ at ω, which completes the induction. Q.E.D.

We need more two lemmas to establish Lemma 2 and Lemma 3.

Lemma A.10.

For any $S \subset P$ and any $q \in P$, if $S \cup \{q\}$ is inconsistent, then $\sim q \in M(S \cup T)$.

Proof of Lemma A.10. Suppose that $\sim q \notin M(S \cup T)$. Then, by G.C.T.2, there exists an $\omega' \in \Omega$ which satisfies S but $\omega'(\sim q) = 0$. It means that this ω' satisfies $S \cup \{q\}$. Hence, by G.C.T.1, $S \cup \{q\}$ is consistent. Q.E.D.

Lemma A.11.

Suppose that $S_1 \subset P$ and that $S_2 = \{r_1,r_2,\ldots,r_m\}$, where $r_i \in P$ $(i=1,2,\ldots,m)$. Let $r' = r_1 \wedge r_2 \wedge\ldots\wedge r_m$.
Then $M(S_1 \cup S_2 \cup T) = M(S_1 \cup \{r'\} \cup T)$.

Proof of Lemma A.11. By G.C.T.2, $q \in M(S_1 \cup S_2 \cup T)$ iff $\omega(p) = 1$ for all ω $(\in \Omega)$ that satisfy $S_1 \cup S_2$, and $q \in M(S_1 \cup \{r'\} \cup T)$ iff $\omega(q) = 1$ for all ω $(\in \Omega)$ that satisfy $S_1 \cup \{r'\}$. Here, for any $\omega \in \Omega$, $\omega(r_i) = 1 \ \forall \ r_i \in S_2$ iff $\omega(r') = 1$. Therefore, $\{ \omega \in \Omega\colon \omega$ satisfies $S_1 \cup S_2 \} = \{ \omega \in \Omega\colon \omega$ satisfies $S_1 \cup \{r'\} \}$, completing the proof. Q.E.D.

Proof of Lemma 2. Throughout of this proof, we denote $D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^{k-1}) \cap B(f(\alpha^k))$ by D. Suppose that there exists no $\omega'' \in \Omega$ such that $\omega''(q) = \omega'(q)$ $\forall$ $q \in f(\alpha^{k+1})$ and ω'' satisfies D. Then, there exists no ω'' $(\in \Omega)$ which satisfies $D \cup T_{\omega'}(f(\alpha^{k+1}) \cup \{\sim p: p \in f(\alpha^{k+1})\})$, since $\omega''(q) = \omega'(q)$ $\forall$ $q \in f(\alpha^{k+1})$ is equivalent to that $\omega''(q) = 1$ $\forall$ $q \in T_{\omega'}(f(\alpha^{k+1}) \cup \{\sim p: p \in f(\alpha^{k+1})\})$. Hence, by G.C.T.1, $D \cup T_{\omega'}(f(\alpha^{k+1}) \cup \{\sim p: p \in f(\alpha^{k+1})\})$ is inconsistent. Here, from Assumption 5, $f(\alpha^{k+1})$ is a finite set, so we can choose $r_1,r_2,\ldots,r_m \in P$ such that $T_{\omega'}(f(\alpha^{k+1}) \cup \{\sim p: p \in f(\alpha^{k+1})\}) = \{r_1,r_2,\ldots,r_m\}$. Let $r' = r_1 \wedge r_2 \wedge\ldots\wedge r_m$. Then, $\omega'(\sim r') = 0$ and $\sim r' \in B(f(\alpha^{k+1}))$. Furthermore, by Lemma A.11, we have $M(D \cup T_{\omega'}(f(\alpha^{k+1}) \cup \{\sim p: p \in f(\alpha^{k+1})\}) \cup T) = M(D \cup \{r'\} \cup T)$. Hence, $D \cup \{r'\}$ is inconsistent, which means, by Lemma A.10, that $\sim r' \in M(D \cup T)$. That is, $\sim r' \in M(D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^{k-1}) \cap B(f(\alpha^k)) \cup T) = D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^k)$. Consequently, we have $\omega'(\sim r') = 0$ and $\sim r' \in D_\omega(\alpha^1,\alpha^2,\ldots,\alpha^k) \cap B(f(\alpha^{k+1}))$, which is a contradiction to the assumption of ω' in Lemma 2. Q.E.D.

Proof of Lemma 3. Suppose that there exists no $\omega'' \in \Omega$ such that $\omega''(q) = \omega'(q)$ $\forall$ $q \in f(\alpha^2)$ and ω'' satisfies $T_\omega[\alpha^1]$. It means that there exists no ω'' $(\in \Omega)$ which satisfies $T_\omega[\alpha^1] \cup T_{\omega'}(f(\alpha^2) \cup \{\sim p: p \in f(\alpha^2)\})$. Let $D = T_\omega[\alpha^1]$. Let $T_{\omega'}(f(\alpha^2) \cup \{\sim p: p \in f(\alpha^2)\}) = \{r_1,r_2,\ldots,r_m\}$ and let $r' = r_1 \wedge r_2 \wedge\ldots\wedge r_m$, then in the same way as the proof of Lemma 2, we have $\omega'(\sim r') = 0$, $\sim r' \in B(f(\alpha^2))$ and $\sim r' \in M(T_\omega[\alpha^1] \cup T) = D_\omega(\alpha^1)$. That is $\omega'(\sim r') = 0$ and $\sim r' \in D_\omega(\alpha^1) \cap B(f(\alpha^2))$, which is a contradiction to the assumption of ω' of Lemma 3. Q.E.D.