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ON THE ORDER OF ELIMINATING DOMINATED STRATEGIES

by

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Abstract

Different orders of eliminating dominated strategies in bimatrix games may yield different reduced games. However, in the case of elimination of strictly dominated strategies, and for the elimination of weakly dominated strategies in zero-sum games, the order is irrelevant. This follows from a general theorem about the uniqueness of the reduced game which unifies the above two cases and illustrates other families of games in which the order of elimination is unimportant.

A two person (bimatrix) game (a game for short) is a tuple $G = (I, J, U)$ where I and J are finite nonempty sets of actions (strategies) available to player 1 and 2 and $U = I \times J \to R^2$ is the payoff function. Specifically, if player 1 chooses action $i \in I$ and player 2 (simultaneously) chooses action $j \in J$, then the payoff to the players are $U_1(i,j)$ and $U_2(i,j)$, respectively. It is common to describe U_1 and U_2 by a bimatrix (a matrix with entries from R^2).

A strategy $i \in I$ of player 1 is said to weakly dominate strategy $k \in I$ if $U_1(i,j) \geq U_1(k,j)$ for every $j \in J$. The strategy is said to dominate k if, in addition, $U_1(i,j) > U_1(k,j)$ for at least one index $j \in J$. The strategy i strictly dominates k if $U_1(i,j) > U_1(k,j)$ for all $j \in J$. Similar definition applies to strategies of player 2 using the function $U_2(\bullet,\bullet)$.

Common sense seems to indicate that players would not use dominated strategies. Thus, one may wish to eliminate at the outset such strategies. Clearly, the elimination of dominated strategies for player 1 may create new dominations for player 2, etc. Continuing in this manner, one finally arrives at a reduced game, i.e., one which does not contain any further dominations. We refer the reader to Luce and Raiffa [LR] for general sequential elimination and to Owen [O] for the zero-sum case. Also, recently, Knuth, Papadimitriou and Tsitsiklis [KPT] studied the computational complexity of this elimination process.

It has been known to game theorists, contrary to an observation in [KPT], that different sequences of elimination of dominated strategies may yield completely different reduced games (see, for example, Myerson [M], and Examples 1 and 2 below). This dependence on the sequence casts some doubts on the validity of the elimination process. However, we give conditions

(Theorem 1) which ensure that the final reduced game is independent of the sequence (up to a permutation). The conditions of Theorem 1 cover the important special case of strict domination for any bimatrix game (in this case the final reduced game is actually unique), and the case of weak domination applied to zero sum games.

2. Examples

We first show that the order in which elimination takes place may affect the final result. In the following examples the strategies of player 1 are represented by rows and those of player 2 by columns. The first entry in each cell corresponds to U_1, the second to U_2.

Example 1 (weak domination). Let

$$U = \left|\begin{array}{c|c} 10, b_1 & 7, b_1 \\ \hline 9, b_2 & 8, b_2 \end{array}\right|$$

Then we can eliminate in the first stage either the first or second column. These choices lead after the obvious further eliminations to single cell bimatrices containing the entries $(8, b_2)$ in the one case, and $(10, b_1)$ in the other.

Example 2 (regular domination): Let

$$U = \begin{vmatrix} 0,b_1 & 0,0 & 1,0 \\ \hline 0,b_2 & 1,0 & 0,0 \end{vmatrix}$$

with $b_1,b_2 > 0$, $b_1 \neq b_2$. Then eliminating the rightmost column first we can then eliminate the top row and then the middle column, resulting with $(0,b_2)$. On the other hand, if we eliminate the middle column first, then the bottom row, then the right column, we are left with $(0,b_1)$.

3. Sufficient Conditions for Uniqueness of the Reduced Game

We now study conditions which ensure that the reduced game is unique. We present our result in terms of abstract dominance relations between strategies. We use the terms idom and jdom to represent, respectively, such domination among the strategies of players 1 and 2.

Let $G = (I,J,U)$ be fixed. A subgame of G is a game $\bar{G} = (\bar{I},\bar{J},U)$ where $\bar{I} \subseteq I$, $\bar{J} \subseteq J$, and U is the restriction of the original function U to $\bar{I} \times \bar{J}$. Two subgames $\bar{G}$ and G' are called equivalent if they can be obtained from each other by permutation of rows and columns, i.e., if there exist one-to-one onto correspondences π_I: $\bar{I} \rightarrow I'$ and π_J: $\bar{J} \rightarrow J'$ with $U_p(i,j) = U_p(\pi_I(i),\pi_J(i))$ for $p = 1,2$ and for all $i \in \bar{I}$ and $j \in \bar{J}$.

For every subgame $\bar{G}$ consider a relation $\mathrm{idom}_{\bar{G}}$ on $\bar{I}$ satisfying the following four conditions:

C1. $\mathrm{idom}_{\bar{G}}$ is transitive.

C2. $\mathrm{idom}_{\bar{G}}$ is defined by player 1's payoff, i.e.,

$$\text{if } i_1,i_2 \in \bar{I} \text{ and for every } j \in \bar{J},\ U_1(i_1,j) = U_1(i_2,j)$$

$$\text{then } i_1 \ \mathrm{idom}_{\bar{G}}\ i \iff i_2 \ \mathrm{idom}_{\bar{G}}\ i$$

and i idom$_{\bar{G}}$ i_1 <=> i idom$_{\bar{G}}$ i_2

for every $i \in \bar{I}$.

C3. idom is inherited to subgames, i.e.,

if $\hat{G}$ is a subgame of $\bar{G}$ and $i_1, i_2 \in \hat{I}$

then i_1 idom$_{\bar{G}}$ i_2 => i_1 idom$_{\hat{G}}$ i_2.

C4. idom$_{\bar{G}}$ is <u>essentially strict</u>, i.e.,

if i_1 idom$_{\bar{G}}$ i_2 and i_2 idom$_{\bar{G}}$ i_1

then for every $j \in \bar{J}$ $U_p(i_1, j) = U_p(i_2, j)$ for $p = 1,2$.

Similarly, consider relations jdom$_{\bar{G}}$ on the sets $\bar{J}$ which satisfy the symmetric conditions for player 2.

For two subgames $\bar{G}$ and $\hat{G}$ we say that $\hat{G}$ is a <u>one-step reduction</u> of $\bar{G}$ if either:

(i) $\hat{J} = \bar{J}$, and there are distinct elements $i_1, i_2 \in \bar{I}$ with i_2 idom$_{\bar{G}}$ i_1 and $\hat{I} = \bar{I} \setminus \{i_1\}$, or

(ii) the similar one-element reduction as in (i) is done in $\bar{J}$ and $\bar{I}$ remains unchanged.

A subgame $\bar{G}$ is a <u>reduction</u> of G if it can be obtained through successive one-step reductions. A reduction $\bar{G}$ is called <u>maximal</u> if it contains no dominated strategies.

<u>Theorem</u>: If the domination relations satisfy conditions C1-C4, then any two maximal reductions of G are equivalent.

<u>Proof</u>: The proof is by induction on the total number of strategies $|I| + |J|$. The case $|I| + |J| = 2$ is obvious. If $|I| + |J| > 2$ but there

is no dominated strategy $i \in I$ or $j \in J$ then G is the only maximal reduction of itself. Thus we are left with the case of a game with at least one dominated row or column. We assume without loss of generality that row $\hat{i}$ dominates row $\tilde{i}$ and $\hat{i} \neq \tilde{i}$. Let $\tilde{G} = (\tilde{I}, J, U)$ with $\tilde{I} = I \setminus \{\tilde{i}\}$. It suffices to show (because of the induction hypothesis) that for any maximal reduction $\bar{G}$ of G, there is an equivalent maximal reduction of $\tilde{G}$.

A reduction $\bar{G}$ of G is obtained by a series of one-step reductions which can be described by a typical sequence as follows:

$$S = (i_1, i_1^*)(i_2, i_2^*)(j_1, j_1^*)(i_3, i_3^*)(i_4, i_4^*)(j_2, j_2^*). \ . \ .$$

where i_1 is eliminated first using i_1^*, i_2 is eliminated second using i_2^*, etc.

We will first show that we can assume, without loss of generality, that one of the i_k's is $\tilde{i}$. If this is not the case then $\tilde{i} \in \bar{I}$ and $\hat{i} \notin \bar{I}$ by the maximality of the reduction. This means that for some i_k in the reduction sequence S, $\hat{i} = i_k$. Now consider the sequence $\hat{i}$, $\hat{i}^*, \hat{i}^{**}, \ldots$ obtained through the eliminations of S. After a finite number of steps, one of these elements belongs to $\bar{I}$ and is not eliminated, say $\hat{i}^{***} \in \bar{I}$. But from the maximality of the reduction we conclude that $\hat{i}^{***} = \tilde{i}$. Consider the entry $(\hat{i}^{**}, \hat{i}^{***})$ in S. From this entry and on (to the right) we can switch the rolls of $\hat{i}^{**}$ with $\hat{i}^{***} = \tilde{i}$, obtaining an equivalent reduction to $\bar{G}$ in which $\tilde{i}$ is eliminated.

Now, assuming that $\tilde{i} = i_k$ for some i_k of S, we modify S so that no $i_t^* = \tilde{i}$. Let t be the first index with the property $i_t^* = \tilde{i}$. Clearly $t < k$ (otherwise $i_t^* \neq \tilde{i}$ for all t). If $\hat{i}$ was not eliminated before i_t then we can replace $i_t^* = \tilde{i}$ by $i_t^* = \hat{i}$. Otherwise consider the finite sequence $i_{r_1}, i_{r_2}, \ldots$ generated by the following inductive rules: $i_{r_1} = \hat{i}$ and $i_{r_{n+1}} = i^*_{r_n}$. Let r_m be the largest index in the sequence with the property $r_m \leq t$. If $r_m < t$ we replace $i_t^* = \tilde{i}$ by $i_t^* = i^*_{r_m}$. If $r_m = t$, then we replace the rolls of i_{r_m} with $\tilde{i}$ from the entry (i_t, i_t^*) in S and on to the right. Under any of these replacements the resulting elimination sequence yields an equivalent reduced form. Following the above replacement procedure repeatedly yields an "equivalent" elimination sequence with $\tilde{i} \in i_k$ for some k and $\tilde{i} \neq i_t^*$ for all t. Now, after removing the entry (i_k, i_k^*) from the obtained "equivalent" procedure, the resulting elimination procedure can be used in $\tilde{G}$ to obtain a maximal reduction equivalent to the one obtained in G.

4. Special Cases

Corollary 1: Let G be an arbitrary game and consider strict dominance. Then there exists a unique maximal reduction of G.

Proof: Strict dominance is easily seen to satisfy C1-C4. In fact, in the proof of the theorem above, $\tilde{i}$ never belongs to $\bar{I}$. Also, $r_m < t$ in the proof of the above theorem. This implies that the maximally reduced subgame is unique (rather than unique up to equivalence).

Corollary 2: Let G be a zero-sum game and consider weak dominance. Then all maximally reduced subgames of G are equivalent.

Proof: Weak dominance satisfies C1-C3. For zero-sum games, it also satisfies C4.

In fact, weak dominance satisfies Condition C4 for a larger class of games, namely games for which

$$U_1(i,j) = U_1(\bar{i},\bar{j}) \iff U_2(i,j) = U_2(\bar{i},\bar{j}).$$

We call games which satisfy this property games with *jointly varying payoffs*. An interesting example, opposite to zero-sum games, are games with identical interests, $U_1(i,j) = U_2(i,j)$.

Corollary 3: Let G be a game with jointly varying payoffs, and consider weak dominance. Then all maximally reduced subgames of G are equivalent.

References

[KPT] D. E. Knuth, C. H. Papadimitriou and J. N. Tsitsiklis, "A Note on Strategy Elimination in Bimatrix Games," O.R. Letters, 1988, 103-108.

[LR] R. D. Luce and H. Raiffa, Games and Decisions: Introduction and Critical Survey, New York: John Wiley and Sons (1957), p. 100.

[M] R. B. Myerson, Game Theory: The Analysis of Conflict, Harvard University Press, 1990.

[O] G. Owen, Game Theory, Second Edition, San Diego: Academic Press Inc. (1982), pp. 23-24.