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A MODEL OF A PROJECT ACTIVITY*

by

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ABSTRACT

This paper presents a simple model of a project activity in which the objective is to complete a given task at minimum cost. The problem is formulated as a decision problem with an uncertain number of stages. The optimal solution is found for the time-invariant case and the implications for the design of activity control systems are discussed.

A MODEL FOR PROJECT ACTIVITIES

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1. Introduction

This paper discusses a simple model of a project activity. It will be assumed that a given task must be completed at minimum cost. The time taken to complete the task is not specified beforehand and will not be known exactly until after the task has been finished. The expected duration of the activities may not be long enough to allow them to be described by stochastic processes which have achieved a steady state. Thus the activities have a "project" rather than a "process" orientation. When the task has been completed, the organization which performed the work either disbands or goes on to perform another task. In the mathematical model the system will have to move from an initial state to a final state; attainment of the latter will represent completion of the task. The problem will be stated as a single-person multi-stage decision problem under uncertainty. It will be assumed that the state, x_t, of the system at time t is a scalar variable representing the amount of work remaining to be completed. The decision at time t, $a_t \in R_+^m$, specifies the levels of m different resources which are to be used at time t.

Examples of economic activities which might be modelled in this way are: (i) (simple) construction projects in which the total amount of work involved can be aggregated and represented by a scalar quantity, (ii) a single activity from a PERT or CPM network, or (iii) a single production run from a job shop. The objective of the paper is to study the design of management control systems for this type of activity.

The model described here differs from the usual models in the management science literature as follows. In the literature on PERT and CPM the problem of controlling individual activities is not explicitly considered. There, each activity is described either by a given probability distribution of finishing times as in PERT or by a given set of deterministic cost-time trade-off curves as in "CPM cost" [8]. One possible use of the type of model developed in this paper would be to provide a rational method of developing data concerning the characteristics of individual activities for inclusion in these netowrk models. The management science literature concerning production activities has usually assumed either finite time horizons of infinite time horizons. A typical example is the aggregate production planning and smoothing model of Holt et al [4]. On the other hand, the model in this paper involves an uncertain time horizon. The present model is however restricted because the costs are not assumed to be functions of the state of the system and the state of the system is assumed to be a scalar quantity.

In section 2, the activity control system design problem is described. Section 3 analyzes problems where the decision stages are discrete and Section 4 analyzes a similar model in which it is assumed that the actions can be adjusted continuously over time. The optimal solution for the continuous case has a very simple and convenient form. The results for the discrete case approximate those for the continuous case for activities of long expected duration. Section 5 states solutions of the activity control problem for some commonly used cost and production functions. Section 6 uses the results of the previous sections to discuss the general problem of designing control systems for activities of random duration.

2. The Activity Control System Problem

Knowledge of the technology of the activity will be described by a sequence of cost functions, $c_t(a_t)$ and production functions, $f_t(a_t)$, $t = 0,1,2,...$ In general, undertainty will exist concerning these functions. For example, uncertainty about future factor prices will prevent exact specification of the function, c_t, and uncertainty with respect to such factors as the quality of the work force, quality of material inputs, and future weather conditions will prevent exact specification of the production function, f_t. These uncertainties are modelled by including additive random disturbance terms, γ_t and ξ_t, in the cost and production functions as shown in (1) below. The functions c_t and f_t are themselves assumed to be deterministic and continuous. Uncertainty will also exist with respect to the total quantity of work, x^0, involved in the task. In a construction context this uncertainty occurs for example, because estimates of the quantity of work involved are obtained from blueprints which may be based on only approximate data concerning actual topological and geological conditions. In a production setting, x^0 might represent the total orders outstanding for a product at the beginning of the production run. Uncertainty here might be due to inaccuracies or delays in the information system. Although in general the states x_t cannot be observed exactly, an assumption of perfect observation will be made throughout this paper. It will be shown that this assumption is not of great importance in that the expected value of perfect information will usually be small for the problems analyzed.

The objective of the activity manager is to choose actions, $a_t \in A_t$, $t = 0,1,2,...,$ which will minimize the expected cost of the activity. The action possibility set, $A_t \subseteq R^m_+$, defines a constraint on the actions available at time t. It is assumed that $f_t(a_t) + \xi_t$, $a_t \in A_t$, is always non-negative, or in other words, that the amount of work remaining to be completed decreases monotonically over time. Information, concerning the current level of x_t, becomes available at time t and an action, $a_t \in A_t$, is selected according to a decision rule, α_t. The decision rules can be functions of the history of prior observations, $x^t = (x_0, x_1, \ldots, x_t)$ and actions, $a^{t-1} = (a_0, a_1, \ldots, a_{t-1})$. Thus the period t action is given, in general, by $a_t = \alpha_t(x^t, a^{t-1})$. A policy, α, is a collection of decision rules, $(\alpha_0, \alpha_1, \alpha_2, \ldots)$. The state of the system is a random variable with a probability distribution which depends on the policy, α, chosen. Sometimes this dependence on α will be recognized explicitly by denoting the state at time t by x_t^α. The activity model can now be stated as follows. Find the policy, $\hat{\alpha}$, which solves

$$(1) \qquad V(\hat{\alpha}) = \min E[\sum_{t=0}^{T-2} (c_t(a_t) + \gamma_t) + \frac{(c_{T-1}(a_{T-1}) + \gamma_{T-1})x_{T-1}}{f(a_{T-1}) + \xi_{T-1}}$$

Subject to:

(a) Initial condition: $x_0 = x^0$

(b) Dynamics: $x_{t+1} = x_t - (f_t(a_t) + \xi_t), \qquad t = 0,1,2,\ldots$

(c) Final condition: $0 \leq x_{T-1} \leq f_{T-1}(a_{T-1}) + \xi_{T-1}$

(d) Admissible actions: $a_t = \alpha_t(x^t, a^{t-1}), \qquad t = 0,1,2,\ldots$

In (1) the expectation is taken with respect to $x^0, \xi_0, \xi_1, \ldots, \gamma_0, \gamma_1, \ldots$ The time of the last decision, T-1, is a random variable. It is assumed that the output, $f_t(a_t) + \xi_t$, and cost, $c_t(a_t) + \gamma_t$, occur uniformly over time. The random variable defined by the ratio, $\frac{x_{T-1}}{f_{T-1}(a_{T-1}) + \xi_{T-1}}$, in the objective function is therefore the fraction of the last time period in which work takes place and the term, $\frac{(c_{T-1}(a_{T-1})+\gamma_{T-1})}{f_{T-1}(a_{T-1})+\xi_{T-1}} x_{T-1}$, in (1) is the cost incurred in the last time period. In the following discussion the activity model (1) will be specialized to the time-invariant case where $c_t=c$, $f_t=f$, $A_t=A$, $t=0,1,2,\ldots$ and $\{\gamma_t, t=0,1,2,\ldots\}$ and $\{\xi_t, t=0,1,2,\ldots\}$ are each assumed to be identically distributed sequences of random variables. It is also assumed that x^0, $\xi_0,\xi_1,\ldots,\gamma_0,\gamma_1,\ldots$ are independent.

As stated above, the possibility of imperfect information concerning the states of the system is not considered in this model. However it is worth noting that the general problem of activity control system design would modify (1) to allow for imperfect observation and would explicitly take into account the cost of generating information concerning the system states. The modified model would then be solved to find the expected cost of completing the activity for each available information system and finally, the optimal information system would be chosen (see [6]).

3. Free-End Time Problems With Discrete Decision Stages

A deterministic time-invariant free-end time problem can be obtained from (1) by omitting the random disturbance terms:

$$(2) \qquad V(\hat{\alpha}) = \min_{a_t \in A} \left\{ \sum_{t=0}^{T-2} c(a_t) + \frac{c(a_{T-1})}{f(a_{T-1})} x^{\alpha}_{T-1} \right\}$$

subject to:

(a) Initial condition: $x^{\alpha}_0 = x^0 > 0$

(b) Dynamics: $x^{\alpha}_{t+1} = x^{\alpha}_t - f(a_t), \quad t=0,1,2,\ldots$

(c) Final condition: $0 \leq x^{\alpha}_{T-1} \leq f(a_{T-1})$

(d) Admissible actions: $a_t = \alpha_t(x^{\alpha}_t)$, $t=0,1,2,\ldots$

Note that the time, T-1, of the last decision is determined implicitly by the chosen policy and the constraint (2c). Define the time worked during the last period under policy α by

$$(3) \qquad m(\alpha) = \frac{x^{\alpha}_{T(\alpha)-1}}{f(\alpha_{T(\alpha)-1}(x^{\alpha}_{T(\alpha)-1}))}$$

where the dependence of T on α has been made explicit. Let α be the constant policy, $\alpha_t \equiv a$, t=0,1,... From (2c) and (3) and the assumption that work is completed at a uniform pace during each time period: $T(\alpha)-1+m(\alpha)=\frac{x_0}{f(a)}$.

Hence:

$$V(\alpha) = \left(\sum_{t=0}^{T(\alpha)-2} c(a)\right) + m(\alpha)c(a)$$

$$= (T(\alpha) - 1 + m(\alpha))c(a) = \frac{c(a)}{f(a)} x_0 \quad .$$

Let $\hat{a} \in R^m$ be a solution to $\hat{c} = \frac{c(\hat{a})}{f(\hat{a})} = \min_{a \in A} \frac{c(a)}{f(a)}$. The cost of the optimal constant policy, $\hat{\alpha}_t \equiv \hat{a}$; $t \geq 0$, is given by $V(\hat{\alpha}) = \hat{c}x_0$. Let β be any other admissible policy, b_t the action taken at time t, and $T(\beta) - 1 + m(\beta)$ the activity duration. The cost of policy, β, is

$$V(\beta) = \sum_{t=0}^{T(\beta)-2} c(b_t) + m(\beta)c(b_{T-1}) \quad .$$

Now, $c(b_t) \geq \frac{c(\hat{a})}{f(\hat{a})} f(b_t)$, $t \geq 0$, so

$$V(\beta) \geq \hat{c}\left(\sum_{t=0}^{T(\beta)-2} f(b_t) + m(\beta)f(b_{T-1})\right)$$

$$= \hat{c}x_0 = V(\hat{\alpha}) \ .$$

Hence $\hat{\alpha}$ is the optimal policy, the minimum cost of completing the activity is $V(\hat{\alpha}) = \hat{c}x^0$ and the optimal completion is $\hat{T} = \frac{x^0}{f(\hat{a})}$.

These results are similar to those which will be obtained in Section 4 for the stochastic continuous time problem. However, the stochastic discrete stages problem is not quite so straightforward even under the time invariance assumption. In this problem, the time, T-1, at which the last decision is made is a random variable with a probability distribution which depends on the chosen policy. The final condition, (1c), is equivalent to the definition of the last decision stage:

$$T = \min \{ s \geq 1 \mid \sum_{t=0}^{s-1} (f(a_t) + \xi_t) \geq x^0 \} .$$

Now a_t is a function of $x^0, \xi_0, \xi_1, \ldots$ and the event $\{ T \leq i \}$ is equivalent to the event $\{ \sum_{t=0}^{i-1} (f(a_t) + \xi_t) \geq x^0 \}$. Hence T is a stopping time for the dynamic process defined by (1b). It follows from the Wald identity [7, p.38] that:

$$E [\sum_{t=0}^{T-1} \xi_t] = E [T] \cdot E [\xi_0] \tag{4}$$

The constant action case will be considered first. Let α be any constant policy, $\alpha_t \equiv a$, $t=0,1,2,\ldots$ By definition:

$$\sum_{t=0}^{T-1} (f(a) + \xi_t) - (f(a) + \xi_{T-1}) + x_{T-1} = x^0.$$

Taking expectations and using (4):

$$E [T] (f(a) + E [\xi_0]) - f(a) - E [\xi_{T-1}] + E [x_{T-1}] = E [x^0],$$

or

$$E [T-1] (f(a) + E [\xi_0]) + f(a) + E [\xi_0] - f(a) - E [\xi_{T-1}] + E [x_{T-1}] = E [x^0].$$

Rearranging, and defining $\mu_0 = E [x^0]$:

$$E [T-1] = \frac{\mu_0 - E [x_{T-1}] - E [\xi_0] + E [\xi_{T-1}]}{f(a) + E [\xi_0]} \tag{5}$$

From (1) the expected cost of policy α is given by

$$V(\alpha) = E\left[\sum_{t=0}^{T-2} (c(a) + \gamma_t) + \frac{(c(a) + \gamma_{T-1})x_{T-1}}{f(a) + \xi_{T-1}}\right].$$

Now, T-1 is determined by x^0 and $\xi_0, \xi_1, \ldots, \xi_{T-1}$ and by assumption, $\gamma_0, \gamma_1, \gamma_2, \ldots$ are independent of $x^0, \xi_0, \xi_1, \ldots$ Hence it follows that T-2 is independent of $\gamma_0, \gamma_1, \ldots, \gamma_{T-2}$, that γ_{T-1} is independent of ξ_{T-1} and x_{T-1} and that $E[\gamma_{T-1}] = E[\gamma_0]$. Therefore from (5):

$$\begin{aligned}(6)\qquad V(\alpha) &= E[T-1](c(a) + E[\gamma_0]) + (c(a) + E[\gamma_0])E\left[\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right] \\ &= \mu_0 \frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]} \\ &\quad + (c(a) + E[\gamma_0])\left\{E\left[\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right] + \frac{E[\xi_{T-1}]}{f(a) + E[\xi_0]}\right. \\ &\qquad \left. - \frac{E[x_{T-1}]}{f(a) + E[\xi_0]} - \frac{E[\xi_0]}{f(a) + E[\xi_0]}\right\}.\end{aligned}$$

Let a^* be the solution to

$$(7)\qquad c^* = \frac{c(a^*) + E[\gamma_0]}{f(a^*) + E[\xi_0]} = \min_{a \in A}\left\{\frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]}\right\}$$

and $\alpha_t^* \equiv a^*$, t=0,1,... The policy α^* is the 'certainty equivalent' policy obtained from the optimal deterministic policy by replacing the random disturbance terms by their expectations. Let α^s be the optimal constant policy, $\alpha_t^s \equiv a^s$, $t \geq 0$. This policy must minimize the value of $V(\alpha)$ given by (6). Because of the last term in (6), a^s depends on the distributions of $x^0, \xi_0, \xi_1, \ldots$ and not just on their mean values.

Following the approach adopted for the deterministic problem, let β be any admissible policy and $b_t \in R^m$ the action actually taken at time t.

Since the action can be any function of the past history of the process, b_t is a random vector. Let $S - 1$ be the random variable denoting the last time at which a decision is made. The duration of the activity under this policy is the random variable, $S - 1 + \frac{x_{S-1}}{f(b_{S-1}) + \xi_{S-1}}$. The expected cost is given by:

$$V(\beta) = E\left[\sum_{t=0}^{S-2} (c(b_t) + \gamma_t) + x_{S-1} \frac{(c(b_{S-1}) + \gamma_{S-1})}{(f(b_{S-1}) + \xi_{S-1})}\right]$$

$$= E\left[\sum_{t=0}^{S-2} (c(b_t) + E[\gamma_0]) + x_{S-1} \frac{(c(b_{S-1}) + E[\gamma_0])}{f(b_{S-1}) + \xi_{S-1}}\right]$$

where the second line follows since $S-2$ depends only on $x^0, \xi_0, \xi_1, \ldots,$ γ_{S-1} is independent of x_{S-1}, and $E[\gamma_{S-1}] = E[\gamma_0]$. Now for $t=0,1,\ldots,$ $c(b_t) + E[\gamma_0] \geq c^*(f(b_t) + E[\xi_0])$, so:

$$V(\beta) \geq c^* E\left[\sum_{t=0}^{S-2} (f(b_t) + E[\xi_0]) + \frac{x_{S-1}(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right] \tag{8}$$

$$= c^* K.$$

By definition, $x^0 = \sum_{t=0}^{S-2} (f(b_t) + \xi_t) + x_{S-1}$. Taking expectations and using (4) and (8):

$$K - \mu_0 = E[S-1]E[\xi_0] - E[\sum_{t=0}^{S-2} \xi_t] + E\left[x_{S-1} \frac{(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}} - x_{S-1}\right]$$

$$= E[\xi_{S-1}] - E[\xi_0] + E\left[\frac{x_{S-1}(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}} - x_{S-1}\right]$$

$$= E\left[\xi_{S-1}\right] - E\left[\xi_0\right] - E\left[\frac{x_{S-1}(\xi_{S-1}-E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right]$$

> 0 with probability 1.

where the inequality follows since, from equation, (1b), and the definition of S, $0 \leq \dfrac{x_{S-1}}{f(b_{S-1}) + \xi_{S-1}} \leq 1$ with probability 1 and therefore

$$\left| E\left[\frac{x_{S-1}(\xi_{S-1} - E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right]\right| < \left| E\left[\xi_{S-1} - E[\xi_0]\right]\right|$$

with probability 1.

Hence $V(\beta) > c^{*}\mu_0$ for any admissible policy β. Evidently, the certainty equivalent policy, α^{*}, the optimal constant policy, α^{S}, and the optimal admissible policy, $\hat{\alpha}$, satisfy:

(9) $$V(\alpha^{*}) > V(\alpha^{S}) \geq V(\hat{\alpha}) \geq c^{*}\mu_0.$$

Temporarily, let x^0 be a known constant. For a constant action the dynamics of the time invariant random duration control problem with the stated independence assumptions define a renewal process (in the "amount of work completed" rather than in "time" as in the usual interpretation of renewal processes). In fact, the problem reduces to the usual definition of a "renewal reward process," [7], except for the terminating condition (1c) and the assumption that costs are incurred, and progress of work is achieved, uniformly over time.

From (6) $V(\alpha^*) = E[T-1](c(a^*) + E[\gamma_0]) + (c(a^*) + E[\gamma_0])E[\frac{x_{T-1}}{f(a)+\xi_{T-1}}]$,

hence using (7):

$$(10) \quad V(\alpha^*) - c^* x^0 = \{E[T-1] - \frac{x^0}{f(a^*) + E[\xi_0]} + E\frac{x_{T-1}}{f(a) + \xi_{T-1}}\}(c(a^*) + E[\gamma_0]).$$

Now $E[T-1]$ can be regarded as a function of x^0 (the "renewal function") and has the following property [2,p.366]:

$$E[T-1] - \frac{x^0}{f(a^*) + E[\xi_0]} \to \frac{E[(f(a^*) + \xi_0)^2]}{2(f(a^*) + E[\xi_0])^2} - 1 \quad \text{as } x^0 \to \infty.$$

Using the "Key Renewal Theorem" [7,p.42] it can be shown that $E\left[\frac{x_{T-1}}{f(a^*)+\xi_{T-1}}\right] \to \frac{1}{2}$, as $x^0 \to \infty$. Hence substituting in (10):

$$(11) \qquad (V(\alpha^*) - c^* x^0) \to L = \frac{\text{var}[\xi_0](c(a^*) + E[\gamma_0])}{2(f(a) + E[\xi_0])^2} \quad \text{as } x^0 \to \infty.$$

From (9) and (11) it is clear that $0 < V(\alpha^*) - V(\hat{\alpha}) < L$ if μ_0 is suitably large. This gives some measure of the expected loss incurred by following the 'certainty equivalent' policy, α^*, rather than the true optimal policy $\hat{\alpha}$.

It can be seen from (6) that a^* is the optimal constant action if the contribution of the final term in the objective of (1) is neglected. Furthermore, $a^s \to a^*$ as $\mu_0 \to \infty$ since the last terms in the expression for $V(\alpha)$ have finite limits. In order to compare α^* with the optimal policy, $\hat{\alpha}$, it will be necessary to introduce some more terminology. For simplicity it will be assumed that the random variables, ξ_t, $t \geq 0$, may have any non-negative value and that $A = \{a \in R^m \mid a \geq a_{min} \geq 0\}$. Let $A(x) = \{a | a_{min} \leq a;\ f(a) \geq x\}$. $A(x_t)$ is the set of feasible actions which will guarantee completion of the task before time t+1. From the assumption about

A, A(x) is non-empty for all $x \geq 0$. If $a \in A^c(x)$, then under the above assumptions the activity may or may not be completed before t + 1. Let $v(x_t)$ be the expected cost of completing the project given that the state is x_t at time t. Then:

(12) $$v(x_t) = \min \{ g_1(x_t), g_2(x_t) \}$$

where:

(13) $$g_1(x_t) = \min_{a_t \in A^c(x_t)} \{ c(a_t + E[\gamma_0] + E[v(x_t - f(a_t) - \xi_t)] F(x_t - f(a_t)) + x_t E\left[\frac{c(a_t) + \gamma_t}{f(a_t) + \xi_t}\right] (1 - F(x_t - f(a_t))\}$$

(14) $$g_2(x_t) = x_t \min_{a_t \in A(x_t)} \{ E\left[\frac{c(a_t) + \gamma_t}{f(a_t) + \xi_t}\right] \}$$

and F is the probability distribution function for $\xi_0, \xi_1, \ldots$

From the previous analysis the optimal action, a_t^*, for large values of x_t will approximate the action, a^*, which minimizes $\frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]}$. However, if $x_t > 0$ is small enough and the decision is made to complete the activity during the next time period then, from (14) and the independence assumption, the optimal action $\hat{a}_t = a'$, where a' minimizes $(c(a) + E[\gamma_0]) E\left[\frac{1}{f(a) + \xi_t}\right]$. (It is assumed that the minimizing action is finite and non-negative in both of these cases). Evidently, there exists $x' > 0$ such that if $x_t > x'$ then $a^* \leq \hat{a}_t \leq a'$ and if $x_t \leq x'$ then $\hat{a}_t = a'$.

The formulation (12) to (14) provides insight both for the activity problem considered here and for renewal reward processes in general. For large x_t, $v(x_t) = g_1(x_t) \geq c^* x_t$. Also, the first two terms in (13) will predominate so that:

$$g_1(x_t) = \min_{a_t \in A^c(x_t)} \{c(a_t) + E[\gamma_0] + c^*(x_t - f(a_t) - E[\xi_0])\}$$

This functional equation is obviously solved by a^* as defined in (7). This confirms that the optimizing action for large x_t approximately minimizes the ratio of the expected cost to the expected output in each period (rather than the expectation of the ratio of the cost to output in each period - which seems, at first sight, to be an equally intuitive result).

The results for the discrete stages time-invariant random duration control problem can be summarized as follows:

Theorem 1:

The constant policy, $\alpha_t^* \equiv a^*$, $t \geq 0$ defined by (7), the optimal constant policy, α^s, and the optimal policy, $\hat{\alpha}$, satisfy:

$$V(\alpha^*) > V(\alpha^s) > V(\hat{\alpha}) > c^*\mu_0$$

If μ_0 is large enough, the opportunity cost involved in using a "certainty equivalent" policy, α^*, rather than the true optimal policy, $\hat{\alpha}$, satisfies:

$$0 < V(\alpha^*) - V(\hat{\alpha}) < \frac{\mathrm{var}[\xi_0](c(a^*) + E[\gamma_0])}{2(f(a) + E[\xi_0])^2}$$

As $x_t \to \infty$ the optimal action $\hat{a}_t \to a^*$. Furthermore, there exists $x' > 0$ such that if $x_t > x'$ then $a^* \leq \hat{a}_t \leq a'$ and if $x_t \leq x'$ then $\hat{a}_t = a'$ where a' minimizes the right-hand side of (14).

4. Continuous-Time Random Duration Model

In this section it will be assumed that the level of resources applied to the task can be adjusted continuously. In other words, the set of possible times at which a decision can be made is the positive real line $R^1_+ = [0,\infty)$. Let $x_t \in R^1_+$ be the amount of work left at time t. A continuous, time-invariant version of the dynamic equation, (1b), is $dx_t = -f(a_t)\,dt - d\xi_t$, where $f: R^m \to R^1_+$ is the production function and ξ_t is a continuous martingale with constant mean, g. Let $\xi_t = g + u_t$, where u_t is a continuous martingale with a zero mean. The system equation becomes:

$$(15) \qquad dx_t = -f(a_t)\,dt - g\,dt - du_t .$$

Similarly, a continuous, time-invariant version of the cost equation in (1) is

$$dc_t = c(a_t)\,dt + d\gamma_t ,$$

where $c: R^m \to R^1_+$ is the cost function and γ_t is a continuous martingale with constant mean h. Let $\gamma_t = h + w_t$, where w_t is a continuous martingale with a zero mean. The instantaneous cost is therefore:

$$(16) \qquad dc_t = c(a_t)\,dt + h\,dt + dw_t .$$

The initial condition, x_0, will be a random variable with mean μ_0. The random variables, x_0, u_t, w_t, $t \geq 0$ will be assumed to be independent of one another. For $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $\{x_0, u_s, w_s,\ s \leq t\}$. Let $(\Omega_t, \mathcal{F}_t, p_t)$ be the probability space of the dynamic process defined by (15) at time t. Let $A \subseteq R^m_+$ be a compact set of feasible actions. The admissible decision functions $\alpha_t: \Omega_t \to A$ at time t will be measurable with respect to $\mathcal{F}_t$. The objective of the system will be to finish the task (drive x_t to zero) at minimum cost. Let $T = \inf\{t | x_t = 0\}$. Then the objective is to find the admissible policy, α^*, which solves

(17) $$V(\alpha^*) = \min_{\alpha} \{E[\int_0^T (c(a_t) + h)\, dt + \int_0^T dw_t]\}$$
$$= \min_{\alpha} \{E[\int_0^T (c(a_t) + h)\, dt]\}$$

since w_t is continuous with zero mean.

Now consider a constant policy $\alpha_t \equiv a$, $t \geq 0$ and let x_0 be given. Then,

(18) $$x_T = 0 = x_0 - \int_0^T (f(a) + g)\, dt - u_T.$$

From the independence assumption, and since u_t, $t \geq 0$ is a zero-mean martingale, $E[u_T | x_0] = E[u_T] = 0$. So, from (18), $E[T|x_0] = \frac{x_0}{f(a) + g}$.
From (17) the cost of the constant policy is therefore

$$V(\alpha, x_0) = (c(a) + h)E[T|x_0]$$
$$= \frac{(c(a) + h)}{f(a) + g} x_0.$$

Let a^* be a solution of $\min_{a \in A} \left\{\frac{c(a) + h}{f(a) + g}\right\}$; $\alpha_t^* \equiv a^*$, $t \geq 0$ is an optimal constant strategy. Let $c^* = \frac{c(a^*) + h}{f(a^*) + g}$ and define a function $v: R^1 \to R^1$ by

(19) $$v(x) = \frac{(c(a^*) + h)}{(f(a^*) + g)} x = c^* x, \qquad x \geq 0.$$

<u>Theorem 2</u>:

Given x_0, the optimal strategy for the continuous-time problem is the constant policy $\alpha_t^* \equiv a^*$, $t \geq 0$.

Proof. Let β be any other admissible strategy, b_t the action taken at time t, T_β the random-activity completion time using policy β, and x_t^β, $t \geq 0$, the corresponding trajectory.

By the Ito differential rule, [9],

$$dv(x_t^\beta) = v_x dx^\beta + \tfrac{1}{2} v_{xx} dR_t$$

$$= c^*[-f(b_t) - g]\, dt - c^* du_t,$$

where dR_t is the incremental covariance of ξ_t and the second equality follows from (15) and (19), since $v_{xx} = 0$. Taking the stochastic integral of the last equation and then taking expectations gives

$$\text{(20)} \qquad -v(x_0) + E[v(x_{T_\beta}^\beta)] = -E[c^* \int_0^{T_\beta} (f(b_t) + g)\, dt],$$

where, again, use has been made of the fact that u_t is a zero-mean martingale so that $E[\int_0^{T_\beta} u_t dt] = 0$. Now, $v(x_{T_\beta}^\beta) = 0$ by definition of T_β since $x_{T_\beta} = 0$ a.e. and by definition of c^*, $c(b_t) + h \geq c^*(f(b_t) + g)$, $t \geq 0$. So, using these facts in (20):

$$\text{(21)} \qquad v(x_0) = c^* x_0$$

$$\leq E[\int_0^{T_\beta} (c(b_t) + h)\, dt] = V(\beta, x_0).$$

On the other hand, if $b_t \equiv a^*$, $t \geq 0$, equality is obtained in (21), since $v(x_0) = V(\alpha^*, x_0)$.

Corollary. The optimal policy for the continuous-time problem, defined by (15) to (17) is the constant action: $\alpha_t^* \equiv a^*$, $t \geq 0$, where a^* is a solution of $\min_{a \in A} \dfrac{c(a) + h}{f(a) + g}$.

The minimum expected cost is $V^*(\alpha^*) = \frac{(c(a^*) + h)}{f(a^*) + g}\mu_0 = c^*\mu_0$.

The expected completion time using the optimal policy is given by

$$E[T_{\alpha^*}] = \frac{\mu_0}{f(a^*) + g} .$$

Proof. The proof of the corollary follows immediately from the theorem after taking expectations with respect to x_0.

The optimal solution of the continuous-time problem depends on the distributions of ξ_t and γ_t only through the means of h and g. Hence, theorem 2 is an example of a "certainty equivalent" result. Furthermore, the expected cost due to the uncertainty in x^0 and ξ_t, $t \geq 0$ is zero. Since the optimal policy is a constant independent of x_t, $t \geq 0$ there is no advantage to be gained from making observations of the system state.

5. Optimal Solutions for Some Particular Technologies

The optimal (or nearly optimal) action, a^*, for the time-invariant free end time problems discussed in the previous sections is the solution to a problem of the form, $\min_{a \in A} \frac{c(a) + d}{f(a) + g}$, where d and g are constants representing the means of the additive disturbance terms in the cost and production functions. Since it has been assumed that c and f are continuous and that A is compact this problem always has a solution. Some simple examples are now stated, however the computational task involved in solving this problem is not always trivial.

Let the cost and production functions be given by:

$$c(a_t) + d = c_0 + c_1 a_t + c_2 a_t^2 \qquad , \; t \geq 0$$

$$f(a_t) + g = e a_t \qquad , \; t \geq 0$$

where $c_0, c_1, c_2 \in R'$ and $c_0, c_2, e > 0$. Then $a^* = \sqrt{c_0/c_2}$ if $a_{min} \leq \sqrt{c_0/c_2} \leq a_{max}$ and the optimal solution does not depend on any parameters of the production equation. However, this is a very special case. If a non-zero constant term is present in the production function, an optimal solution to this problem is the solution to a quadratic equation involving parameters from both the cost and production functions. If $c_2 = 0$ in the preceding example the solution would be unbounded except for the constraint on the actions. The optimal action is then:

$$a^* = \begin{cases} a_{min} & \text{if } c_1 e_0 \geq c_0 e_1 \\ a_{max} & \text{if } c_1 e_0 \leq c_0 e_1 \end{cases}$$

As another example, let the cost function be linear and the production function be of the Cobb-Douglas type:

$$c(a_t) + d = c_0 + c_1' a_t \qquad , \; t \geq 0$$

$$f(a_t) + g = b_0 \prod_{i=1}^{m} a_i^{b_i} \qquad , \; t \geq 0$$

where $c_0, c_1 > 0$, $c_0 \in R^1$, $c_1 \in R^m$, $b_i > 0$, $b_i \in R^1$, $0 \leq i \leq m$ and $\Sigma\, b_i < 1$. Also let $A = \{a \in R^m | a \geq 0\}$. Then the optimal action is given by:

$$a_i^* = \frac{c_0 b_i}{(1 - \Sigma b_i) c_i} \qquad ; \qquad 1 \leq i \leq m.$$

If $\Sigma b_i = 1$ the Cobb-Douglas production function gives constant returns to scale and if $\Sigma b_i > 1$ it gives increasing returns to scale. In both of these cases the solution would be unbounded if the action were not constrained.

For more general cases it will be necessary to use numerical approximation or specially devised algorithms in order to solve this problem. For the case of multidimensional linear cost and production functions for example, the algorithm given in [3] might be adopted.

6. The Design of Control Systems for Economic Activities of Random Duration

The significance of the results obtained for the time-invariant, free-end time activity models analyzed in the previous sections will now be discussed. It has been shown that a constant policy, a^*, is optimal for the continuous random duration and deterministic versions of this problem and that a constant policy is approximately optimal for the discrete random duration version. Intuitively, the action, a^*, minimizes the cost per unit output in each time period. This is also the action which minimizes the long-run average cost per unit of time in the corresponding infinite-horizon problem [7].

The fact that the optimal action is approximately constant, independent of the state, x_t, of the system, is quite surprising. This means that the choice of the optimal action is never affected by chance events. In a construction context for example, the action taken after two weeks of heavy rain and poor production should be the same as the action taken after two weeks of fine weather and good production. Of course, this result depends on the time invariance assumption. It is no longer true, for example, when additional penalties are incurred, if the activity is not finished before some given target date. The solutions for free-end time and fixed duration activity models can be very different. For the fixed-duration problem, where the production functions are linear and the cost functions are quadratic, the optimal action is linear and expected cost is quadratic in the state of the system, [1]. In the time invariant free-end time problem, however, the optimal (or near optimal) action is a constant and the expected cost is a linear function of the state of the system.

Perhaps the most interesting property of this activity model is that information systems which report the state of the system (amount of work remaining) have little or no value. The additional freedom of a "free-end time" makes the optimal actions less dependent on the state of the system and more a function of the particular technology. The benefits to be derived from the traditional information system which produces periodic 'progress reports' may not, therefore, be very significant. Again, this conclusion is dependent on the time invariance assumption. However, it does at least indicate the need for a careful economic analysis of the value and cost of this type of information in real problems.