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MINIMALITY AND COMPLEMENTARITY PROPERTIES ASSOCIATED WITH Z-FUNCTIONS AND M-FUNCTIONS

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ABSTRACT

A nonlinear generalization of square matrices with non-positive off-diagonal elements is presented, and an algorithm to solve the corresponding complementarity problem is suggested. It is shown that the existence of a feasible solution implies the existence of a least solution which is also a complementary solution. A potential application of this nonlinear setup in extending the well-known linear Leontief input-output systems is discussed.

MINIMALITY AND COMPLEMENTARITY PROPERTIES ASSOCIATED WITH Z-FUNCTIONS AND M-FUNCTIONS(1)

Introduction(2)

Given a mapping $f : R^n_+ \to R^n$ and a vector q in R^n, the complementarity problem is to find x in R^n such that

$$x \geq 0, \; f(x) + q \geq 0 \tag{1}$$

and

$$x'(f(x) + q) = 0 \tag{2}$$

If x satisfies (1) it is called a feasible solution, and if it also satisfies (2) it is a complementary solution.

In this paper we consider a nonlinear generalization of Z-matrices (i.e. square matrices with nonpositive off-diagonal elements), called Z-functions, as well as a generalization of M-matrices (i.e. Z matrices with positive principal minors) known as M-functions. We discuss properties of these classes of functions and then develop a scheme to solve the complementarity problem (1) - (2) defined by members of the classes. The scheme is a modification of an algorithm suggested by Chandrasekaran [1] for the solution of linear complementarity problems defined by Z-matrices.

It is shown that the modified algorithm produces a complementary solution to problem (1) - (2), provided one exists. In addition such a solution is the least element of the feasible set defined by (1); that is, the complementary solution x determined in the algorithm satisfies $0 \leq x \leq y$ for all $y \geq 0$ such that $f(y) + q \geq 0$. This result extends a theorem recently proved by Cottle and Veinott [2] for M-matrices.

In [8] the author shows that if additional assumptions are imposed on the Z-functions, yielding continuous surjective M-functions, then the algorithm can be viewed as a principal pivoting scheme. This approach leads to a natural extension of the Schur complement concept, defined with respect to square matrices. In [8] this extension is used to prove the nonlinear equivalent of the theorem which states that a Schur complement of an M-matrix is also an M-matrix. The iterative processes of Gauss-Seidel and Jacobi have a key role in the development and derivation of the complementarity algorithm.

Existing and potential applications of Z-functions and M-functions are included at the end of the paper.

While Z-matrices and M-matrices, also known as Minkowski matrices, have been studied extensively in the literature regarding both applied and theoretical aspects (see the work of Fiedler and Ptak [3], where most of the known results are included), it seems that very little attention has been given to nonlinear generalizations. One generalization that we focus on has been developed by Rheinboldt [7], whose motivation was to apply iterative schemes to nonlinear systems of equations. Rheinboldt's generalization is also studied by Moré [4].

We start by introducing the classes of Z-functions and M-functions.

Definitions and Preliminary Results

In this study we consider off-diagonally antitone functions, first introduced by Rheinboldt [7]. For our purposes a mapping $f(x)$ from R^n_+ into R^n with components $f_i(x)$, $i=1,\dots,n$, is <u>off-diagonally antitone</u> if for all x in R^n_+ and $i \neq j$, $i,j=1,\dots,n$ the scalar functions $F_{ij} : R^1_+ \to R^1$ defined by

$$F_{ij}(t) = f_i(x + te^j)$$

are nonincreasing. e^j is the j^{th} unit vector in R^n. $f(x)$ is said to be (strictly) diagonally isotone if for all x in R^n_+ the scalar functions

$$F_{ii}(t) = f_i(x + te^i), \quad i=1,\ldots,n$$

are (increasing) nondecreasing.

We define the classes of Z- and M-functions corresponding to Z- and M-matrices.

Definition (1). Let f be a mapping from R^n_+ into R^n.

(a) f is said to be a Z-function if it is off-diagonally antitone on R^n_+

(b) f is an M-function if it is a Z-function as well as inverse isotone on R^n_+ (i.e. for any x and y in R^n_+ $f(x) \leq f(y)$ implies that $x \leq y$).

Rheinboldt has studied M-functions and their application to nonlinear network flows. In this work we explore the Z- and M-functions in the context of complementarity theory and develop an algorithm to solve complementarity problems associated with these classes of functions.

In the algorithm which is later derived, we use the following nonlinear generalization of a principal submatrix due to Rheinboldt [7].

Definition (2). Let f be a mapping from R^n_+ to R^n and consider a permutation $(m(1),\ldots,m(n))$ of $(1,\ldots,n)$. Given an integer number p, $1 \leq p < n$, and real numbers $c_{p+1},\ldots,c_n$, we define the principal sub-function of dimension p, mapping R^p_+ to R^p;

$$g_i(x_1,\ldots,x_p) = f_{m(i)}\left(\sum_{j=1}^{p} x_j e^{m(j)} + \sum_{j=p+1}^{n} c_j e^{m(j)}\right), \quad i=1,\ldots,p \qquad (3)$$

For example, if $(m(1),\ldots,m(n)) = (1,\ldots,n)$, then for any $1 \le p \le n$ we get the leading principal function of dimension p

$$g_i(x_1,\ldots,x_p) = f_i(x_1,\ldots,x_p,c_{p+1},\ldots,c_n), \quad i=1,\ldots,p$$

Note that unlike the linear case, the dependence of a principal function on the constant terms $(c_{p+1},\ldots,c_n)$ cannot in general be represented as a separable term. Thus every principal function depends parametrically on the set of constants associated with it. A result concerning this dependence is given in [8].

For convenience of presentation the following notation is used to denote principal functions. If $(m(1),\ldots,m(n))$ is a permutation of $(1,\ldots,n)$ and $c_{p+1},\ldots,c_n$ are given constants, then the corresponding principal function will be denoted by $f_I(x_I,c_J)$ where $I = \{m(1),\ldots,m(p)\}$ and $J = \{m(p+1),\ldots,m(n)\}$. In most cases $c_{p+1} = \ldots = c_n = 0$, and we shall say that the corresponding principal function $f_I(x_I,0)$ is defined by the set of indices I.

The following result is an obvious consequence of the definition of a Z-function.

Lemma (1). Any principal function of a Z-function is in itself a Z-function.

The next lemma shows that principal functions preserve also the inverse isotonicity property.

Lemma (2). Any principal function of an M-function is an M-function.

This lemma was first proved by More and Rheinboldt [6], and independently by Tamir [8], who used a different approach.

It is interesting to note that in fact the inverse isotonicity principal functions of dimension 1,2 and n induce the same property on principal functions of any dimension p, $p=1,\ldots,n$.

Lemma (3). Let f be a continuous function from R_+^n to R^n. If f is strictly diagonally isotone and every principal function of dimension 2 is inverse isotone, then f is a Z-function.

Proof: Assume on the contrary that there exists indices $i < j$, vector x in R_+^n, and scalars s and t such that

$$f_i(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n) > f_i(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n)$$

$$\text{and} \quad s \geq t.$$

Clearly $s > t$. Using the strictly diagonal isotonicity property we have

$$f_j(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n) > f_j(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n).$$

The continuity assumption assures that the two strict inequalities are maintained if the i^{th} coordinate of $(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n)$ x_i, is increased somewhat to get $y_i > x_i$. Using the inverse isotoniticy property of the principal function defined by $\{i,j\}$, and the set of constants $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$, we get the contradiction

$$(x_i,s) \geq (y_i,t)$$

Note that Lemma 3 extends Proposition 1 of [2], that deals with M-matrices.

As a consequence of the lemma we have the following

Theorem (1). Let f be a continuous strictly diagonally isotone function from R_+^n to R^n. If f is inverse isotone and every principal function of dimension 2 is inverse isotone, then principal functions of any dimension are inverse isotone and f is an M-function.

The next lemma, dealing with principal functions of surjective (onto) M-functions, is proved by Rheinboldt in [7]. It should be observed, however, that this result is applicable only to functions which are defined and satisfy the M-property on the entire space R^n. The proof is based on the application of iterative solution procedures.

Lemma (4). Let $f : R^n \to R^n$ be continuous, surjective, off-diagonally antitone and inverse isotone on R^n. Then every principal function of f is continuous, surjective, off-diagonally antitone and inverse isotone on the corresponding subspace.

Notice that continuity is not assumed in Lemma 1 and Lemma 2.

Before turning to complementarity aspects related to the Z-functions and M-functions, we present two well known iterative processes used for the solution of systems of equations.

Consider the following n-dimensional system of equations, in the variables $x_1,\ldots,x_n$

$$\begin{array}{ccc} f_1(x_1,\ldots,x_n) & = & a_1 \\ \vdots & & \vdots \\ f_n(x_1,\ldots,x_n) & = & a_n \end{array} \qquad (4)$$

The (underrelaxed) Gauss-Seidel iteration for the solution of (4) is defined as follows

$$\begin{array}{ll} \text{Solve} & f_i(x_1^{k+1},\ldots,x_{i-1}^{k+1},x_i,x_{i+1}^k,\ldots,x_n^k) = a_i \quad \text{for } x_i. \\ \text{Set} & x_i^{k+1} = (1-w_k)x_i^k + w_k x_i, \quad i=1,\ldots,n, \qquad k = 0,1,2,\ldots. \end{array} \qquad (5)$$

The corresponding (underrelaxed) Jacobi iteration is:

$$\begin{aligned} &\text{Solve} \quad f_i(x_1^k,\ldots,x_{i-1}^k,x_i,x_{i+1}^k,\ldots,x_n^k) = a_i \quad \text{for } x_i. \\ &\text{Set} \quad x_i^{k+1} = (1-w_k)x_i^k + w_k x_i, \; i=1,\ldots,n \qquad k = 0,1,2,\ldots \end{aligned} \tag{6}$$

In both processes $\{w_k\}$ is a given sequence of relaxation factors, where $\epsilon \leq w_k \leq 1, k = 0,1,2,\ldots$ for some $\epsilon > 0$.

Rheinboldt [7], provides sufficient conditions for the applicability of Gauss-Seidel and Jacobi processes to systems of equations defined by strictly diagonally isotone and continuous Z-functions. The strict diagonal isotonicity property is necessary to guarantee the uniqueness of the iterates $\{x^k\}$ and $\{y^k\}$ defined by the Gauss-Seidel and Jacobi schemes. To obtain a result which applies to Z-functions as well, we define modified versions of these two iterative procedures.

Given the system of equations $f_i(x_1,\ldots,x_n) = a_i$, $i=1,\ldots,n$, and x^0 in R^n, the forward (unrelaxed) Jacobi iterates, $\{x^k\}$, are given by

$$\begin{aligned} &\text{Find} \quad x_i^* = \text{minimum } x_i \\ &\quad \text{subject to} \quad x_i \geq x_i^k \\ &\quad\quad \text{and} \quad f_i(x_1^k,\ldots,x_{i-1}^k,x_i,x_{i+1}^k,\ldots,x_n^k) = a_i. \\ &\quad \text{Set} \quad x_i^{k+1} = x_i^*, \quad i=1,\ldots,n, \qquad k=0,1,2,\ldots \end{aligned} \tag{7}$$

For a given y^0 in R^n the backward (unrelaxed) Jacobi iterates $\{y^k\}$ are defined by

$$\text{Find } y_i^* = \text{maximum } y_i$$

$$\text{subject to } y_i \leq y_i^k$$

$$\text{and } f_i(y_1^k,\ldots,y_{i-1}^k,y_i,y_{i+1}^k,\ldots,y_n^k) = a_i \qquad (8)$$

$$\text{set } y_i^{k+1} = y_i^*, \qquad i=1,\ldots,n, \qquad k=0,1,2,\ldots$$

The analogous definitions of the modified Gauss-Seidel iteration as well as those corresponding to the underrelaxed cases are clear, and we omit their formulation. It should be noted that all subsequent results, proved for the modified (unrelaxed) Jacobi process, are valid for the Gauss-Seidel iteration as well as for underrelaxed cases.

The modified process allows us to omit the strictly diagonal isotonicity property, required by Rheinboldt [7], and to establish the following theorem, which is applicable to continuous off-diagonally antitone functions.

<u>Theorem (2)</u>. Let $f : R_+^n \to R^n$ be a continuous, off-diagonally antitone function. Suppose that for some z in R^n there exist vectors x^0 and y^0 in R_+^n such that $x^0 \leq y^0$ and $f(x^0) \leq z \leq f(y^0)$. Then the corresponding (unrelaxed) Jacobi iterates $\{y^k\}$ and $\{x^k\}$, given by (7) and (8) and starting from y^0 and x^0, respectively, are uniquely defined and satisfy

$$x^0 \leq x^k \leq x^{k+1} \leq y^{k+1} \leq y^k \leq y^0, \; f(x^k) \leq z \leq f(y^k), \quad k = 0,1,2,\ldots$$

as well as

$$\lim_{k\to\infty} x^k = x^* \leq y^* = \lim_{k\to\infty} y^k$$

$$f(x^*) = f(y^*) = z \; .$$

The proof of this theorem is achieved by introducing slight modifications into the proof given by Rheinboldt to the case where the Z-function is also strictly diagonally isotone. Therefore we omit the proof.

We also point out that the result of Theorem (2) holds for the modified underrelaxed Jacobi process as well as for the corresponding modification of the Gauss-Seidel procedure.

Theorem (2) is the key result used to prove the validity of the following algorithm, which is applicable to complementarity problems corresponding to Z-functions.

The Complementarity Algorithm and the Main Results

Assume that $f : R^n_+ \to R^n$ is a continuous Z-function and that q is any vector in R^n.

Algorithm

Step 0. Let $I(1) = \{i \mid q_i + f_i(0) < 0\}$ in the initial form.

Step 1. If $I(1)$ is empty, stop; $x = 0$ is a complementary solution. Otherwise, set $w^0_{I(1)} = 0$ and go to Step 2.

Step 2. Consider the principal function, $f_I(x_I,0)$, defined by the current set of indices $I = I(t)$ and the set of constants $\{c_i = 0 \mid i \notin I\}$. Let the corresponding system of equations be

$$f_{i(j)} \left(\sum_{j=1}^{k} x_j \, e^{i(j)} \right) = - q_{i(j)}, \quad j = 1,\dots,k \tag{9}$$

where $I = I(t) = \{i(1),\dots,i(k)\}$, $i(j) < i(j+1)$, $j = 1,\dots,k-1$.

Apply the forward Jacobi process, (7), starting at

w_I^0 to the system (9) defined by $f_I(x_I,0)$. If the system of equations (9) has no solution, stop; the complementarity problem has no solution. Otherwise, let x_I^0 be the (positive) solution and go to Step 3.

Step 3. Let $I_1(t) = \{i \mid i \notin I(t),\ f_i(x_I^0,0) + q_i < 0\}$.

If $I_1(t)$ is empty, stop; $x = \sum_{j=1}^{k} x_j^0 e^{i(j)}$ is a complementary solution. Otherwise, define $I(t+1) = I(t) \cup I_1(t) = \{i(1),\ldots,i(k),\ldots,i(m)\}$. Define $w_{I(t+1)}^0$ by the following

$$w_j^0 = x_j^0 \quad \text{if } 1 \leq j \leq k \text{ and } w_j^0 = 0 \quad k < j \leq m.$$

Set $t+1 \to t$ and go to Step 2.

The next lemma leads to the proof of the validity of Algorithm for continuous Z-functions.

Lemma (5). Let y be any feasible solution to the complementarity problem defined by the continuous Z-function f and a vector q (i.e. $f(y) + q \geq 0$, $y \geq 0$), and t be the cycle index. If $x_{I(t)}^0$ denotes the solution generated by the modified Jacobi process in Step 2 at the t^{th} cycle, then

$$x_j^0 > 0, \quad j = 1,\ldots,k$$

and

$$y \geq x = \sum_{j=1}^{k} x_j^0 e^{i(j)}$$

where $I(t) = \{i(1),\ldots,i(k)\}$.

Proof:

Supposing that the algorithm terminates in r cycles and setting $I_1(0) = I(1)$, $I(0) = \emptyset$, $I_1(r) = \emptyset$ and $I(r+1) = I(r)$, we define $w^0_{I(r+1)} = x^0_{I(r)}$. Note that for $1 \leq t \leq r$, $w^0_{I(t)}$ is defined in Step 3 of the algorithm. Following this definition, the proof of the theorem will be complete if it is shown that for every t, $1 \leq t \leq r + 1$, $w^0_{I(t)}$ satisfies

$$\left\{\begin{array}{lll} y_{i(j)} \geq w^0_j & & 1 \leq j \leq k \\ w^0_j > 0 \ ; \ f_{i(j)} \ (w^0_{I(t)},0) + q_{i(j)} = 0 & & 1 \leq j \leq \bar{k} \\ w^0_j = 0 \ ; \ f_{i(j)} \ (w^0_{I(t)},0) + q_{i(j)} < 0 & & \bar{k} < j \leq k \end{array}\right\} \quad (10)$$

where $I(t) = I(t-1) \cup I_1(t-1) = \{i(1),\ldots,i(\bar{k})\} \cup \{i(\bar{k}+1),\ldots,i(k)\}$. From Step 1 of the algorithm it is clear that (10) holds for t=1. Suppose that (10) holds for some $t \geq 1$, where $I(t) = I(t-1) \cup I_1(t-1)$ $= \{i(1),\ldots,i(\bar{k})\} \cup \{i(\bar{k}+1),\ldots,i(k)\}$.

Let $I(t+1) = I(t) \cup I_1(t) = \{i(1),\ldots,i(k)\} \cup \{i(k+1),\ldots,i(m)\}$ be the set of indices generated in Step 3. We note that the principal function $f_{I(t)}(\cdot,0)$ and the k-vectors $\bar{x} = w^0_{I(t)}$, $\bar{y} = y_{I(t)}$, $z = -q_{I(t)}$ satisfy the assumptions of Theorem (2). The inequalities $\bar{x} \leq \bar{y}$ and $f_{I(t)}(\bar{x},0) \leq -q_{I(t)}$ are implied by the induction hypothesis, while $f_{I(t)}(\bar{y},0) \geq -q_{I(t)}$ follows from the feasibility of y and the off-diagonal antitonicity

$$f_{I(t)}(\bar{y},0) \geq f_{I(t)}(y) \geq -q_{I(t)} \ .$$

Thus we can apply Theorem (2), yielding a solution $x^0_{I(t)}$ that satisfies $f_{I(t)}(x^0_{I(t)},0) = -q_{I(t)}$ and $0 \leq w^0_{I(t)} \leq x^0_{I(t)} \leq y_{I(t)}$. $w^0_{I(t+1)}$ is defined by

$$w_j^0 = x_j^0 \text{ if } 1 \leq j \leq k \text{ and } w_j^0 = 0 \text{ for } k < j \leq m.$$

Therefore $y_{i(j)} \geq w_j^0$, $1 \leq j \leq m$ and $f_i(w^0_{I(t+1)},0) + q_i = 0$ for $i \in I(t)$. Furthermore, from the definition of $I_1(t)$ we have $f_i(w^0_{I(t+1)},0) + q_i < 0$ for $i \in I_1(t)$. To show that the j^{th} component, $1 \leq j \leq k$, of $w^0_{I(t+1)}$ is positive, we prove that $x_j^0 > 0$, $1 \leq j \leq k$. The induction hypothesis and $x^0_{I(t)} \geq w^0_{I(t)}$ yield the positivity of x_j^0 for $1 \leq j \leq \bar{k}$. Assume that $x_j^0 = 0$ and $\bar{k} < j \leq k$, then

$$0 = f_{i(j)}\ (x^0_{I(t)},0) + q_{i(j)} \leq f_{i(j)}\ (w^0_{I(t)},0) + q_{i(j)}$$

follows from the off-diagonal antitonicity and $x^0_{I(t)} \geq w^0_{I(t)}$. But the nonnegativity of the right hand side contradicts the induction hypothesis for $\bar{k} < j \leq k$. Thus the theorem follows.

The validity of Algorithm is a straight forward consequence of the last lemma.

<u>Theorem (3)</u>. Let $f : R^n_+ \to R^n$ be a continuous Z-function, and let q be an arbitrary vector in R^n. Then Algorithm when applied to the corresponding complementarity problem, finds a complementary solution or indicates that no feasible solution exists in at most n cycles. Infeasibility of (1) is indicated either by an unbounded sequence of iterates $\{x^k\}$ or by infeasibility of (7) for some iteration k, k=1,2,.... and component index i, i=1,...,n.

<u>Proof</u>: Assume first that the complementarity problem is feasible and let y be a nonnegative vector which satisfies $f(y) + q \geq 0$. Lemma (5) assures that the systems of equations defined in Step 2 have positive solutions which are obtained by applying the modified Jacobi scheme (7). The set of indices, corresponding to positive components of an arbitrary complementary solution, is increased every time Step 3 is visited; hence the process terminates in at most n cycles.

Lemma (5) implies also that a failure of the modified Jacobi process to converge to a positive solution indicates that the complementarity problem is not feasible. Specifically, the sequence $\{x^k\}$ generated by the forward Jacobi process is monotone increasing. Therefore, a failure to converge implies that for some iteration k, $k=1,2,\ldots,$ and component index i, $i=1,\ldots,n$, (7) is infeasible, or that $\{x^k\}$, $k=1,2,\ldots,$ is unbounded. Moreover, such a failure must occur in at most n cycles, provided the complementarity problem is not feasible.

We note that indeed each of the two indications of failure to converge may occur. The scalar linear function $f(x) = -x$ with $q = -1$ and starting point $x^0 = 0$ is an example of the first possibility, while the linear function $f(x_1,x_2) = (x_1-x_2, -x_1+x_2)$ with $q = (-1,-1)$ and starting point $x^0 = 0$ demonstrates the second.

Another important and immediate consequence of Lemma (5) is the following minimality property satisfied by the complementary solution produced by Algorithm.

<u>Corollary (1)</u>. Let $f : R^n_+ \to R^n$ be a continuous Z-function and let $q \in R^n$. Denote the feasible set defined by f and q by

$$X^+_q = \{x \mid f(x) + q \geq 0, \quad x \geq 0\} .$$

If X^+_q is not empty and x is the complementary solution produced by Algorithm, then $x \leq y$ for all y in X^+_q.

As demonstrated by the next two results a certain surjectivity property guarantees the existence of complementary solutions to (1) - (2) for all q in R^n, while inverse isotonocity assures the existence of at most one complementary solution.

Theorem (4) Let f be a continuous Z-function. (1) - (2) has a solution for any q in R^n if and only if $\{x | x \geq f(0)\} \subseteq f(R^n_+)$.

Proof: Sufficiency. From Corollary (1) it is sufficient to show that for any q in R^n there exists x in R^n_+ such that $f(x) + q \geq 0$. In fact it is sufficient to consider vectors q, satisfying $q \leq - f(0)$ only. But if $q \leq - f(0)$ then the surjectivity property yields the existence of $x \geq 0$ such that $f(x) + q = 0$.

Necessity. Consider u in R^n such that $u \geq f(0)$. Then there exists y in R^n_+ such that $f(y) - u \geq 0$ and $y(f(y)-u) = 0$. We prove that $f(y) = u$. Suppose first that $y_i > 0$ then $y(f(y)-u) = 0$ implies $f_i(y) = u_i$. If $y_i = 0$, we use the off-diagonal antitonicity property to obtain $0 \geq f_i(0) - u_i \geq f_i(y) - u_i \geq 0$; hence $f_i(y) = u_i$.

We note that a sufficient condition, (which is not necessary), for the existence of a complementary solution to (1) - (2) for all q in R^n, is presented by More [5]. He assumes that the continuous Z-function is order coercive, i.e. for each unbounded increasing sequence $\{x^k\}$ in R^n_+

$$\lim_{k\to\infty} f_i(x^k) = +\infty \quad \text{for some index } i.$$

The last theorem shows that order coercivity of continuous Z-functions implies $\{x | x \geq f(0)\} \subseteq f(R^n_+)$. To see that this condition is not necessary (i.e. that $\{x | x \geq f(0)\} \subseteq f(R^n_+)$ is indeed weaker than order coercivity for continuous Z-functions), we consider the scalar function $f(x) = x \sin x$. This function satisfies $R^1_+ \subseteq f(R^1_+)$ since $f((2n + \frac{1}{2})\Pi) \to \infty$, but it is not order coercive ($f(2n\ \Pi) = 0$ for all n).

Corollary (2): If f is a continuous M-function then for any q in R^n (1) - (2) has at most one complementary solution.

Proof: Let $y \in R^n$ be a solution to (1) - (2) corresponding to a vector q in R^n, and define $I = \{i | y_i > 0\}$. Following the preceding corollary, let x be the minimal element in X_q^+. Then $x_i = 0$, $i \notin I$. Considering the principal function defined by the set of indices I and zero constants, we obtain

$$q_I + f_I(y_I,0) = 0 \leq f_I(x_I,0) + q_I$$

where the equality sign follows from the complementarity condition $y'(f(y) + q) = 0$. Inverse isotonicity of the principal function yields $y \leq x$; hence $x = y$.

Several comments are in order. First, note that when f is an M-function the surjectivity property $\{x | x \geq f(0)\} \subseteq f(R_+^n)$ (which reduces to $R_+^n \subseteq f(R_+^n)$ if $f(0) = 0$) is equivalent to order coercivity. This is established by combining Theorem (4) and Corollary (2) of this study with Theorem 4.8 of [5]. When the off-diagonal antitonicity property is relaxed, order coercivity does not necessarily imply the surjectivity condition. As an example consider the function $f : R_+^2 \to R^2$ defined by $f(x_1,x_2) = \frac{1}{2}(x_1+x_2, x_1+x_2)$.

Finally, as pointed out by a referee, a result stronger than Corollary (2) is contained (implicitly) in [4] and [6]. Theorem 4.4 of [6] and Theorem 2.3 of [4] imply that Corollary (2) is true even when continuity is not assumed.

We mention several simplifications of the algorithm when applied to continuous M-functions. Note first that the forward and backward Gauss-Seidel and Jacobi schemes coincide with the original processes since f is strictly diagonally isotone. As any principal function of an M-function is injective (one-to-one) the system of equations defined in Step 2 has at most one solution in R^n_+. Hence any valid procedure, rather than the Gauss-Seidel and Jacobi iterative procedures, can be utilized to obtain the unique solution in R^n_+, provided one exists. When the Z-function is linear a finite procedure is applied to solve the linear equations (see [1]).

Suppose that f is continuous, off-diagonally antitone and inverse isotone on R^n (rather than R^n_+). We observe that if f is surjective then the equations system defined by (9), has a unique positive solution. This is implied by the surjectivity of the principal functions of f (Lemma (4)).

While studying polyhedral sets having a least element, Cottle and Veinott [2] proved the following theorem characterizing M-matrices in terms of complementary minimum solutions.

<u>Theorem (5)</u>. If A is an $n \times n$ matrix, the following are equivalent:

(1) A is an M-matrix.

(2) For each q in R^n the polyhedral set $X^+_q = \{x \mid Ax + q \geq 0,\ x \geq 0\}$ has a least element x_0 (i.e. $x_0 \in X^+_q$ and $x_0 \leq x$ for all $x \in X^+_q$) and x_0 is the only vector of X^+_q satisfying $x'(Ax+q) = 0$.

Corollary (1) provides a nonlinear generalization of the implication (1) $\Rightarrow$ (2), when Ax is replaced by any continuous M-function and (2) is replaced by (2') to assure the nonemptiness of X^+_q.

(2') For each $q \in R^n$ such that the feasible set $X^+_q = \{x \mid f(x) + q \geq 0,\ x \geq 0\}$ is nonempty, there exists a least element in X^+_q which is the only vector of X^+_q satisfying $x'(f(x) + q) = 0$.

Corollary (1) can be used to establish a characterization of Z-matrices in the spirit of Theorem (5).

<u>Theorem (6).</u> If A is an $n \times n$ matrix the following are equivalent

(1) A is a Z-matrix.

(2) For each q in R^n for which the polyhedral set $X_q^+ = \{x | Ax + q \geq 0, x \geq 0\}$ is not empty, there exists a least vector x in X_q^+ satisfying $x'(Ax + q) = 0$.

<u>Proof</u>: The implication (1) $\Rightarrow$ (2) follows from Corollary (1). To prove the converse statement we show that the off-diagonal elements of A are nonpositive. Assume, on the contrary, that for some $i \neq j$ the (i,j) entry, a_{ij}, is positive and consider the vector q defined to be the negative of the j^{th} column of A, i.e. $q = -(a_{1j},\ldots,a_{nj})$. The vector e^j, the j^{th} unit vector, belongs to X_q^+. In fact it belongs to X_p^+ for any vector $p \geq q$. Consider the vector $p = q + e^j$ and let x be the least element of X_p^+. Thus $0 \leq x \leq e^j$, which yields $x_k = 0$ for any $k \neq j$. This, in turn, implies that $x_j = 1$, since $(Ax)_i - a_{ij} \geq 0$. x is a complementary solution and thus requires

$$x_j((Ax)_j - a_{jj} + 1) = x_j(x_j a_{jj} - a_{jj} + 1) = 0$$

which contradicts $x_j = 1$.

We note that the implication (2) $\Rightarrow$ (1), proved in Theorem (6) for Z-matrices, does not necessarily hold for nonlinear continuous Z-functions. This is illustrated by the following

Example Let $f : R^2 \to R^2$ be defined as

$$f_1(x_1,x_2) = -x_1$$
$$f_2(x_1,x_2) = g(x_1) - x_2$$

where

$$g(x_1) = \begin{cases} -x_1 & \text{for } 0 \leq x_1 \leq 1 \\ -2 + x_1 & \text{for } 1 \leq x_1 \leq 2 \\ 0 & \text{otherwise .} \end{cases}$$

The set $X_q^+ = \{x | f(x) + q \geq 0, x \geq 0\}$ is nonempty if and only if $q \geq 0$. For $q \geq 0$, $x = 0$ is the least element in X_q^+ and it satisfies $x'(f(x) + q) = 0$. To see that f is not off-diagonally antitone, notice that $f_2(1,0) < f_2(2,0)$.

It is of interest to observe that Algorithm when applied to linear functions, reduces to the algorithm suggested by Chandrasekaran [1]. It should be noted, however, that the modified Jacobi process used in Step 2 is replaced there by a linear system of equations which has a unique nonnegative solution or none at all. In fact the matrix associated with this linear system is a (surjective) M-matrix provided a nonnegative solution exists. Hence, it follows that the Jacobi iteration will converge to the unique nonnegative solution of the relevant system provided one exists.

Corollary (1) implies that Chandrasekaran's algorithm finds the least solution to the linear complementarity problem defined by a Z-matrix.

Applications of Z-functions

We conclude this paper by discussing a potential application of Z-functions in extending well known linear Leontief input-output systems. We describe the simple Leontief Interindustry Model as follows. Consider n industries, each with one type of output (type i for industry i) during a given time period for production. Let a_{ij}, $1 \leq i, j \leq n$ be the number of units of type i required per unit of type j ($a_{ij} \geq 0$) and let b_i, $i = 1,\ldots,n$, be the number of units of type i required exogenously (e.g. a demand vector). A negative b_i is interpreted as availability of b_i units. If x_i denotes the number of units of type i to be produced, then the feasible production set is given by the polyhedral set

$$x_i \geq 0, \; x_i \geq b_i + \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1,\ldots,n. \qquad (11)$$

Setting $A = (a_{ij})$, $b = (b_1,\ldots,b_n)$ and $x = (x_1,\ldots,x_n)$, (11) becomes in matrix form

$$(I-A)x \geq b, \; x \geq 0.$$

The i^{th} row of $(I-A)x$ characterizes the net output of type i produced by the n industries, when x_j, $j = 1,\ldots,n$ units of type j are produced.

Motivated by the linear model we consider an interindustry system that produces n items. Suppose that $f_i(x_1,\ldots,x_n)$, $i = 1,\ldots,n$, is the net output of type i, $i = 1,\ldots,n$, produced by the system when the gross production is given by $x = (x_1,\ldots,x_n)$. For a given demand vector the feasible production set is given by the solutions to

$$x \geq 0, \quad f_i(x) \geq b_i, \quad i = 1,\ldots,n. \qquad (12)$$

Assume as in the linear case, that the mapping $f : R^n_+ \to R^n$ defined by the components $f_i(x)$, $i = 1,\ldots,n$ is off-diagonally antitone, continuous and maps the zero vector into itself. (The latter assumption is simply the fact that the net production is zero whenever there is no gross production.) We note that the off-diagonal antitonicity property and $f(0) = 0$ assure that if there is a positive demand of item i, b_i, then the system has to produce a positive gross production, x_i, of item i.

Given a demand vector b we can then apply Algorithm to yield a feasible production if one exists. Furthermore, the solution provided by Algorithm satisfies interesting minimality and complementarity properties (Corollary (1)). If we denote by x^0 the solution obtained by Algorithm, then $x^0 \leq x$ for any feasible production x satisfying (12).

We also observe that the least solution minimizes any isotone objective function $g : R^n_+ \to R^1$, (i.e. $x \leq y$ implies $g(x) \leq g(y)$), defined on the set of feasible productions.

It is our belief that the proposed nonlinear generalization of Leontief input-output model will be more applicable to real life situations where linearity assumptions have been found to be invalid. The author is currently engaged in a study which extends the above model to situations where several industries may produce the same type of product and thus face some competitive problems.

A different application of M-functions has been presented by Rheinboldt [7] who discussed the connection between nonlinear network flows and the class of M-functions.

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FOOTNOTES

1. Presented at the 8th International Symposium on Mathematical Programming, Stanford University, 1973

2. After this research was completed, Professor J. J. Moré informed the author that he had obtained a few of the results independently, but by different means. (See [4] and [5]).