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ON THE APPROXIMATION OF UPPER SEMI-CONTINUOUS CORRESPONDENCES AND THE EQUILIBRIUMS OF GENERALIZED GAMES

by

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Contents

In Section 1 we explain some of the definitions and terminology that we use.

In Section 2 we prove several theorems concerning the approximation of upper semi-continuous correspondences having for range a locally convex space. Theorems 1 and 2 (and Corollary 1) generalize certain approximation theorems by G. Haddad [16, pp. 1352–1354], G. Haddad and J. M. Lasry [17, pp. 299–300] and J. P. Aubin and A. Cellina [2, pp. 86–89] (see also F. S. De Blasi [10] and J. M. Lasry and R. Robert [21]). Although some of the details of the proofs of Theorems 1 and 2 are new, the basic ideas are taken from the above mentioned papers. Theorem 3 shows that the correspondences in the approximating families can be chosen so that they are regular (see Section 1). This theorem (see the remark following its proof) contains a classical result of M. Hukuhara [18, pp. 56–57]. Theorem 3 is proved using Theorem 2 and Propositions 1 and 4. Variants of Propositions 2, 3, and 4 were given in [19].

In Section 3 we give, among others, Theorem 5, which concerns the existence of equilibriums of generalized games (= abstract economies). The main purpose of this result is to replace in the W. Shafer-H. Sonnenschein equilibrium theorem for generalized games the continuity hypothesis by an upper semi-continuity one. The proof of Theorem 5 is based on the results on the approximation of upper semi-continuous correspondences obtained in Section 2. Theorem 5 is used in Section 4.

In Section 4 we establish Theorems 6 and 7. These theorems show that certain statements concerning the equilibrium of generalized games are equivalent to certain statements concerning minimax inequalities of Ky Fan type.

Theorems 1, 2, 3, 5, 6 and 7 are the main results of this paper.

1. – Notations and Terminology

Let X and Y be two sets and C a correspondence between X and Y. For every $x \in X$ and $y \in Y$ we write

$$C(x) = \{y \mid (x,y) \in C\} \quad \text{and} \quad \overset{-1}{C}(y) = \{x \mid (x,y) \in C\}.$$

Let X and Y be two topological spaces (in this paper we assume, although this is not always necessary, that all the topological spaces we consider are separated). The filter of neighborhoods of a point t in a topological space is denoted by $\mathcal{V}(t)$.

A correspondence C, between X and Y, has *open lower sections* if $\overset{-1}{C}(y)$ is open for every $y \in Y$. A correspondence having open graph has open lower sections.

A correspondence C, between X and Y, is *compact*[1] if for every $t \in X$ there is $V_t \in \mathcal{V}(t)$ such that $C(V_t)$ is relatively compact. If Y is compact any correspondence between X and Y is compact. A correspondence which is compact and has closed graph is upper semi-continuous. A correspondence which is compact, upper semi-continuous and has closed upper sections has closed graph.

For every correspondence C, between X and Y, we denote by $\overline{C}$ the correspondence which has for graph the adherence of the graph of C. The correspondence C is compact if and only if $\overline{C}$ is compact.

A correspondence C, between X and Y, is *quasi-regular* if:

(i) it has open lower sections;

(ii) $C(x)$ is non-void and convex for every $x \in X$;

(iii) $\overline{C}(x) = \overline{C(x)}$ for every $x \in X$.

If the correspondence C is compact (iii) is equivalent with: The correspondence C^1 between X and Y, defined by $C^1(x) = \overline{C(x)}$ for $x \in X$, is upper semi-continuous.

The correspondence C is regular if it is quasi-regular and has open graph.

It is easy to see that $\overline{C}$ is continuous if the correspondence C is quasi-regular and compact. Hence, $\overline{C}$ is continuous if C is quasi-regular and Y is compact.

For every subset A of a vector space we denote by $\gamma(A)$ the smallest convex set containing A.

In this paper, unless we say explicitly the contrary, we denote by E a *locally convex space.*

For other notations and terminology used here see the monographs by N. Bourbaki and the monograph by C. Castaing and M. Valadier, listed among the references at the end of this paper.

2. – Various approximation theorems.

Let X be a non-void set, Y a non-void subset of E and f a correspondence between X and Y. A family $(f_j)_{j\in J}$ of correspondences between X and Y, indexed by a non-void filtering[2] set J (we denote by $\leq$ the order relation in J) is an *upper approximating family for* f if:

A_I) $f \subset f_j$ for every $j \in J$;

A_{II}) for every $j \in J$ there is $j_* \in J$ such that $f_h \subset f_j$ if $h \in J$ and $j^* \leq h$;

A_{III}) for every $t \in X$ and $V \in \mathcal{V}_E(0)$ there is $j_{t,V} \in J$ such that

$$f_h(t) \subset f(t) + V$$

if $h \in J$ and $j_{t,v} \leq h$.

Remarks. – 1) We deduce from $A_I) - A_{III})$ that:

A_{IV}) for every $t \in X$ and $k \in J$

$$f(t) \subset \bigcap_{j\in J} f_j(t) = \bigcap_{k\leq j} f_j(t) \subset \overline{f(t)} \subset \bar{f}(t).$$

It follows that if $f(t)$ is closed for $t \in X$ then

$$f(t) = \bigcap_{j\in J} f_j(t)$$

for every $t \in X$.

2) Let f be a correspondence between X and Y and define the correspondence f' by[3] $f'(t) = \overline{f(t)}$ for $t \in X$. An upper approximating family for f' is also an upper approximating family for f.

3) Let X be a non-void set and Y a non-void subset of E. Let f be a correspondence between X and Y and let $(f_j)_{j\in J}$ be an upper approximating family for f. Let f' and

$(f_j')_{j\in J}$ be defined by $f'(t) = \overline{f(t)}$ and $f_j'(t) = \overline{f_j(t)}$ for every $t \in X$ and $j \in J$. Then $(f_j')_{j\in J}$ is an upper approximating family for both f and f'.

If X is a topological space we denote by $\mathcal{S}(X,Y;E)$ the set of all correspondences A between X and Y such that for every $t \in X$

$$A(t) = \sum_{j\in I} \varphi_j(t) A_j \tag{1}$$

where $(\varphi_j)_{j\in I}$ is a partition of unity of X and $(A_j)_{j\in I}$ is a family of *non-void closed convex* parts of Y. We denote by $\mathcal{S}^{(c)}(X,Y;E)$ the set of all $A \in \mathcal{S}(X,Y;E)$ which can be defined by (1) with a family $(A_i)_{i\in I}$ of *non-void compact convex* parts of Y. Observe that

$$\mathcal{S}^{(c)}(X,Y;E) = \mathcal{S}(X,Y;E)$$

if Y is compact.

A correspondence f belonging to $\mathcal{S}(X,Y;E)$ is lower semi-continuous. To prove this assertion it is enough to observe that for every $t \in X$ and $y \in f(t)$ there is a continuous selection of f which takes the value y at t. A correspondence belonging to $\mathcal{S}(X,Y;E)$ is not necessarily upper semi-continuous. As we shall see later, a correspondence belonging to $\mathcal{S}^{(c)}(X,Y;E)$ is both lower and upper semi-continuous (whence continuous).

THEOREM 1. – *Let X be a non-void paracompact space and Y a non-void closed convex subset of E. Let f be a correspondence between X and Y such that:*

1.1) f is upper semi-continuous on X;

1.2) $f(t)$ is non-void and convex for every $t \in X$.

Then there is an upper approximating family for f, consisting of correspondences belonging to $\mathcal{S}(X,Y;E)$.

Remarks. – 1) The proof of Lemma 1 below shows that $\overline{f(t)} \subset f_U(t)$ for every $t \in X$ and $U \in \mathcal{D}$. It follows that the family $(f_j)_{j\in J}$ constructed in the proof of Theorem 1 is also an upper approximating family for f', where $f'(t) = \overline{f(t)}$ for $t \in X$.

2) Let X be a topological space, Y a uniform space and $\mathcal{W}$ the uniform structure of Y. A correspondence f, between X and Y, is *Hausdorff upper semi-continuous* if for every $t \in X$ and $U \in \mathcal{W}$ there is $V \in \mathcal{V}(t)$ such that

$$f(x) \subset U(f(t))$$

for every $x \in X$. The statement and proof of Theorem 1 remain valid if the hypothesis 1.1) is replaced by: 1.1') *f is Hausdorff upper semi-continuous.*

The proof of Theorem 1 is based on three lemmas which we prove first.

Since X is paracompact, there is a uniform structure on X compatible with the topology of X. Let $\mathcal{D}$ be a basis of this uniform structure, consisting of *symmetric* sets.

For every $U \in \mathcal{D}$ we denote:

α) by U_1 an element of $\mathcal{D}$ such that

$$U_1 \circ U_1 \subset U;$$

β) by $(U_j)_{j \in I(U)}$ an open locally finite covering of X, which is a refinement of the covering $(U_1(x))_{x \in X}$; (we assume that $I(U)$ does not contain 1);

γ) by $(\varphi_{U,j})_{j \in I(U)}$ a partition of unity of X subordinated to the covering $(U_j)_{j \in I(U)}$.

For every $U \in \mathcal{D}$ and $j \in I(U)$ we denote by $t_{U,j}$ an element of X such that

$$U_j \subset U_1(t_{U,j}).$$

For every $U \in \mathcal{D}$ and $x \in X$ we denote by $I(U,x)$ the set

$$\{j \in I(U) | U_j \ni x\}$$

(observe that $I(U,x)$ is *finite* and that $\varphi_{U,j}(x) = 0$ if $j \notin I(U,x)$).

For every $U \in \mathcal{D}$ we denote by f_U the correspondence between X and Y defined by

$$f_U(x) = \sum_{j \in I(U)} \varphi_{U,j} C_{U,j} \tag{2}$$

for $x \in X$, where

$$C_{U,j} = \overline{\gamma(f(U(t_{U,j})))} \tag{3}$$

for $j \in I(U)$.

LEMMA 1. – *For every $U \in \mathcal{D}$ we have $f \subset f_U$.*

PROOF. – Let $x \in X$. If $j \in I(U,x)$ then $U_j \ni x$, whence

$$f(x) \subset f(U_j) \subset f(U(t_{U,j})) \subset C_{U,j}$$

If $y \in f(x)$ then

$$y = \sum_{j \in I(U,x)} \varphi_{U,j}(x) y \in \sum_{j \in I(U,x)} \varphi_{U,j}(x) C_{U,j},$$

whence $y \in f_U(x)$. Since $x \in X$ and $y \in f(x)$ were arbitrary, the lemma is proved.

LEMMA 2. – *If $V \in \mathcal{D}$, $U \in \mathcal{D}$ and $V \circ V \subset U_1$, then $f_V \subset f_U$.*

PROOF. – We observe first that if $i \in I(V,x)$ and $j \in I(U,x)$ then

$$C_{V,i} \subset C_{U,j} \tag{*}$$

Indeed, let $s \in V(t_{V,i})$; then $(t_{V,i}, s) \in V$. Since $i \in I(V,x)$ and $j \in I(U,x)$

$$x \in V_i \subset V_1(t_{V,i}) \quad \text{and} \quad x \in U_j \subset U_1(t_{U,j});$$

hence $(x, t_{V,i}) \in V_1$ and $(t_{U,j}, x) \in U_1$. We deduce

$$(t_{U,j}, s) \in V \circ V_1 \circ U_1 \subset U_1 \circ U_1 \subset U,$$

whence $s \in U(t_{U,j})$. Hence

$$V(t_{V,i}) \subset U(t_{U,j}). \tag{**}$$

From (**) and (3) we deduce immediately (*).

If $x \in X$ and $y \in f_V(x)$, then

$$y \in \sum_{i \in I(V,x)} \varphi_{V,i}(x) C_{V,i} \subset C_{U,j}$$

for every $j \in I(U,x)$. We deduce

$$y \in \sum_{j \in I(U,x)} \varphi_{U,j}(x) C_{U,j},$$

that is, $y \in f_U(x)$. Since $x \in X$ and $y \in f_V(x)$ were arbitrary, the lemma is proved.

LEMMA 3. – *Let $t \in X$, $W \in \mathcal{D}$ and let $M \subset E$ be convex and closed. If $f(x) \subset M$ for $x \in W(t)$ then $f_U(t) \subset M$ if $U \in \mathcal{D}$ and $U \circ U \subset W$.*

PROOF. – Let $j \in I(U,t)$. Then $t \in U_j \subset U_1(t_{U,j})$. If $z \in U(t_{U,j})$ we deduce $(t,z) \in U \circ U_1 \subset W$ so that $z \in W(t)$. Hence

$$f(U(t_{U,j})) \subset f(W(t)) \subset M$$

and hence $C_{U,j} \subset M$ for $j \in I(U,t)$. We conclude that

$$f_U(t) = \sum_{j \in I(U,t)} \varphi_{U,j}(t) C_{U,j} \subset M$$

Therefore, the lemma is proved.

We shall now prove Theorem 1: Let $J = \mathcal{D}$ (we write $U \leq V$ if and only if $U \supset V$). We shall show that $(f_U)_{U \in J}$ is an upper approximating family for f.

By Lemma 1, A_I) is satisfied.

Now let $U \in J$ and let $U_* \in J$ such that $U_* \circ U_* \subset U_1$. If $H \in J$ and $U_* \leq H$ then

$$H \circ H \subset U_* \circ U_* \subset U_1$$

and hence, by Lemma 2, $f_H \subset f_U$. Hence A_{II}) is satisfied.

Let $t \in X$ and $V \in \mathcal{V}_E(0)$. Let V_1 be a convex neighborhood of $0 \in E$ such that $V_1 + V_1 \subset V$. Since f is upper semi-continuous at t, there is $W \in \mathcal{D}$ such that

$$f(x) \subset f(t) + V_1 \subset \overline{f(t) + V_1} \subset f(t) + V$$

if $x \in W(t)$. Let $U_{t,V} \in J$ such that

$$U_{t,V} \circ U_{t,V} \subset W$$

If $H \in J$ and $U_{t,V} \leq H$ then

$$H \circ H \subset U_{t,V} \circ U_{t,V} \subset W$$

and hence, by Lemma 3,

$$f_H(t) \subset \overline{f(t) + V_1} \subset f(t) + V$$

Hence $A_{III})$ is also satisfied.

We conclude that $(f_U)_{U \in J}$ is an upper approximating family for f. Since $f_U \in \mathcal{S}(X, Y; E)$ for every $U \in J$ the theorem is proved.

Remarks. – 1) *It follows from (3) that* $C_{U,j} \subset \overline{\gamma(f(X))}$ *for every* U *and hence that*

$$f_U(X) \subset \overline{\gamma(f(X))}$$

for every U. *Hence, if* $\gamma(f(X))$ *is relatively compact the correspondences* $f_j (j \in J)$ *constructed in the proof of Theorem 1 belong to* $\mathcal{S}^{(c)}(X, Y, ; E)$ *and* $\gamma(f_j(X))$ *is relatively compact for every* $j \in J$.

2) The index set of the upper approximating family constructed in Theorem 1 is $\mathcal{D}$. Hence *the index set of the family depends on* X *but not on* f. A similar remark is valid for j^* (see $A_{II})$). For every $j = U$ in $\mathcal{D}$ we may choose $j^* = U_*$ where U_* is an element of $\mathcal{D}$ such that $U_* \circ U_* \subset U_1$. These remarks will be used in the proof of Theorem 2.

3) Every $A \in \mathcal{S}(X,Y;E)$ obviously has a continuous selection. It follows from Brouwer's fixed point theorem (for finite dimensional spaces) that if $X = Y$ and Y is compact and convex every $A \in \mathcal{S}(X,Y;E)$ has a fixed point. From this remark and from Theorem 1 one can easily deduce the Fan-Glicksberg generalization of Kakutani's fixed point theorem (see also G. Haddad [16]).

A closed convex part Y of E has the property (K) if for every compact part B of Y the set $\gamma(B)$ is relatively compact. Obviously Y has the property (K) if it is compact. If B is compact, then $\overline{\gamma(B)}$ is compact if and only if it is complete for $\tau(E,E')$ (M. G. Krein's theorem).

To simplify, we assume in the definition of *totally bounded* correspondences and in Corollary 1 below that X is a metric space (we denote by d the distance on X.)

A correspondence C, between X and Y, is *totally bounded* if $C(B)$ is relatively compact for every ball B (see G. Haddad and J. M. Lasry [17, p. 300]). It is obvious that a totally bounded correspondence is compact according to the definition adopted in this paper. The converse, however, is not true (for example, there are X, E, Y and $f \in \mathcal{S}^{(c)}(X,Y;E)$ such that f is not totally bounded). Nevertheless:

COROLLARY 1. – *Assume that the hypotheses of Theorem 1 are satisfied, that E has the property (K) and that:*

1.1″) *f is totally bounded.*

Then there is an upper approximating family for f, consisting of correspondences which belong to $\mathcal{S}^{(c)}(X,Y;E)$ and are totally bounded.

PROOF. – This result follows from Theorem 1 once we show that, under the hypotheses of the corollary, we can choose $\mathcal{D}$ so that, for every $U \in \mathcal{D}$, the correspondence f_U (defined by (2)) belongs to $\mathcal{S}^{(c)}(X,Y;E)$ and is totally bounded.

For this purpose, let

$$\mathcal{D} = \{W^{(\varepsilon)} \mid \varepsilon > 0\}$$

where

$$W^{(\varepsilon)} = \{x, y) \mid d(x, y) \leq \varepsilon\}$$

for every $s > 0$ (we may, instead, take $\mathcal{D} = \{W^{(1/n)} \mid n \in N\}$). If $U = W^{(\varepsilon)}$ we choose $U_1 = W^{(\varepsilon/2)}$.

Let $U = W^{(\varepsilon)}$ be an element of $\mathcal{D}$. Since the correspondence f is totally bounded and $U(t_{U,j})$ is a ball, $f(U(t_{U,j}))$ is relatively compact; since E has the property (K) we deduce that

$$C_{U,j} = \overline{\gamma(f(U(t_{U,j})))}$$

is compact for every $j \in J$. *Hence the correspondence* f_U *belongs to* $\mathcal{S}^{(c)}(X, Y; E)$.

Now let $W_r(x_0)$ be the ball of center x_0 and radius r and let x be an element of this ball. If $j \in I(U, x)$ then $x \in U_j \subset U_1(t_{U,j})$; it follows that if $z \in U(t_{U,j})$ then $d(x_0, z) \leq \lambda$ if $\lambda = r + \varepsilon/2 + \varepsilon$. Hence $U(t_{U,j}) \subset W_\lambda(x_0)$ and hence

$$C_{U,j} = \overline{\gamma(f(U(t_{U,j})))} \subset \overline{\gamma(f(W_\lambda(x_0)))}$$

for every $j \in I(U, x)$. Since $\overline{\gamma(f(W_\lambda(x_0)))}$ is convex we obtain

$$f_U(x) = \sum_{j \in I(U,x)} \varphi_{U,j}(x) C_{U,j} \subset \overline{\gamma(f(W_\lambda(x_0)))}.$$

Since $x \in W_r(x_0)$ was arbitrary we deduce $f_U(W_r(x_0)) \subset \overline{\gamma(f(W_\lambda(x_0)))}$. Since x_0 and r were arbitrary, we conclude that f_U is *totally bounded.*

Let X be a non-void topological space Y a non-void closed subset of E and $A \in \mathcal{S}^{(c)}(X, Y; E)$. It is easy to see that (we recall that the supports of the functions in a partition of unity form a locally finite family):

(a) *$A(t)$ is compact for every $t \in X$ and A is a compact correspondence.*

For completeness we prove that the correspondence A is compact: Indeed, let $t \in X$ and let $V_t \in \mathcal{V}(t)$ such that

$$I_t = \{j | V_t \cap \text{ Support } \varphi_j \neq \emptyset\}$$

is finite. Let

$$Y_t = \left(\sum_{j \in I_t} [0,1] A_j \right) \cap Y$$

Since Y_t is obviously compact and since $A(x) \subset Y_t$ for every $x \in V_t$ our assertion is proved.

(b) *A has closed graph and is continuous.*

Let $((x_t, y_t))_{t \in T}$ be an arbitrary family of elements belonging to the graph of A, indexed by a filtering set T and converging to (x, y). By using an ultrafilter $\mathcal{U}$ on T, finer than the filter of sections of T, we show that y is of the form $\sum_{j \in I} \varphi_j(x) a_j$ with $a_j \in A_j$ for every $j \in J$. We deduce that (x, y) belongs to the graph of A; since $((x_t, y_t))_{t \in T}$ was arbitrary we conclude that the graph of A is closed. Since by (a) the correspondence A is compact it follows that it is upper semi-continuous. Since A is lower semi-continuous we conclude that (b) is proved.

(c) *A has compact graph if X is compact.*

Assume that Y is *convex*. Let $(A^t)_{t \in T}$ be a family of correspondences belonging to $\mathcal{S}^{(c)}(X, Y; E)$ and $(\alpha_t)_{t \in T}$ a partition of unity of X. Then:

(d) *The correspondence A, between X and Y, defined by*

$$A(x) = \sum_{t \in T} \alpha_t(x) A^t(x)$$

for $x \in X$, belongs to $\mathcal{S}^{(c)}(X, Y; E)$.

Since the set $\{t \mid \alpha_t(x) \neq 0\}$ is finite, for every $x \in X$, A is well defined. Since Y is convex

$$A(x) = \sum_{t \in T} \alpha_t(x) A^t(x) \subset \sum_{t \in T} \alpha_t(x) Y \subset Y$$

for every $x \in X$. Hence A is a correspondence between X and Y.

By hypothesis for every $t \in T$ there is a partition of unity of X $(\varphi_j^t)_{j \in I(t)}$ and a family $(A_j^t)_{j \in I(t)}$ of non-void compact convex parts of Y such that

$$A_t(x) = \sum_{j \in I(t)} \varphi_j^t(x) A_j^t$$

for $x \in X$. Whence

$$\begin{aligned} A(x) &= \sum_{t \in T} \alpha_t(x) A^t(x) = \sum_{t \in T} \alpha_t(x) \left(\sum_{j \in I(t)} \varphi_j^t(x) A_j^t \right) \\ &= \sum_{t \in T} \sum_{j \in I(t)} (\alpha_t \varphi_j^t)(x) A_j^t = \sum_{(t,j) \in T^*} (\alpha_t \varphi_j^t)(x) A_j^t \end{aligned}$$

for $x \in X$ if $T^* = \bigcup_{t \in T} (\{t\} \times I(t))$. Since $(\alpha_t \varphi_j^t)_{(t,j) \in T^*}$ is a partition of unity of X it follows that A belongs to $\mathcal{S}^{(c)}(X, Y; E)$.

We now give the following variant of Theorem 1:

THEOREM 2. – *Let X be a non-void paracompact space and Y a non-void closed equilibrated convex part of E which has the property (K). Let f be a correspondence between X and Y such that:*

2.1) f is compact and upper semi-continuous;

2.2) $f(t)$ is non-void compact and convex for every $t \in X$.

Then there is an upper approximating family for f consisting of correspondences belonging to $\mathcal{S}^{(c)}(X, Y; E)$.

PROOF. – Let $(W_h)_{h \in H}$ be a locally finite open covering of X such that $f(W_h)$ is relatively compact for every $h \in H$. Let $(U_h)_{h \in H}$ be a locally finite open covering of X such that

$\overline{U}_h \subset W_h$ for every $h \in H$. For every $h \in H$ let u_h be a continuous mapping of X into $[0,1]$ such that $u_h(x) = 1$ for $x \in U_h$ and $u_h(x) = 0$ if $x \notin W_h$ (the space X is paracompact, whence normal). Let $(\varphi_h)_{h\in H}$ be a partition of unity of X subordinated to the covering $(U_h)_{h\in H}$.

For every $h \in H$ $u_h f$ is a correspondence, between X and Y, such that $(u_h f)(X)$ is relatively compact; moreover, $u_h f$ is upper semi-continuous on X (f is compact-valued) and $(u_h f)(x)$ is non-void and convex for every $x \in X$. Hence, for every $h \in H$, $u_h f$ satisfies the hypotheses of Theorem 1 and hence there is an upper approximating family $(f_j^{(h)})_{j\in J}$ for $u_h f$, consisting of correspondences belonging to $S^{(c)}(X,Y;E)$ (see Remarks 1) and 2) following the proof of Theorem 1 and observe that since E has the property (K) the set $\overline{\gamma((u_h f)(X))}$ is compact).

For every $j \in J$ let f_j be the correspondence, between X and Y, defined by

$$f_j(x) = \sum_{h\in H} \varphi_h(x) f_j^{(h)}(x)$$

for $x \in X$; by (d), above, f_j belongs to $S^{(c)}(X,Y;E)$.

Observe also that

$$f(x) = \sum_{h\in H} \varphi_h(x) f(x) = \sum_{h\in H} \varphi_h(x)(u_h f)(x),$$

for $x \in X$ (by 2.2)the set $f(x)$ is convex).

Since $u_h f \subset f_j^{(h)}$ for every $h \in H$ and $j \in J$, we deduce that $f \subset f_j$ for every $j \in J$. *Hence* $(f_j)_{j\in J}$ *satisfies* A_I).

If $j \in J$, $i \in J$ and $j^* \leq i$, then $f_i^{(h)} \subset f_j^{(h)}$ for every $h \in H$ (see Remark 2)following the proof of Theorem 1); we deduce $f_i \subset f_j$. *Hence* $(f_j)_{j\in J}$ *satisfies* A_{II}).

Now let $t \in X$, let $H(t) = \{h \mid U_h \ni t\}$ ($H(t)$ is finite) and let V be a convex neighborhood of $0 \in E$.

For every $h \in H(t)$ there is $j(t,h) \in J$ such that

$$f^{(h)}_{j(t,h)}(t) \subset (u_h f)(t) + V = f(t) + V.$$

Let $j(t)$ be an element of J superior to $j(t,h)^*$ for every $h \in H(t)$. Then

$$f^{(h)}_{j(t)}(t) \subset f(t) + V$$

for every $h \in H(t)$. Hence

$$f_{j(t)}(t) = \sum_{h \in H(t)} \varphi_h(t) f^{(h)}_{j(t)}(t) \subset \sum_{h \in H(t)} \varphi_h(t)(f(t) + V).$$

Since $f(t) + V$ is convex we obtain

$$f_{j(t)}(t) \subset f(t) + V.$$

We deduce

$$f_i(t) \subset f(t) + V$$

for every $i \in J$ such that $j_{t,V} \le i$ if $j_{t,V} = j(t)^*$.

Since $t \in X$ and the convex neighborhood V were arbitrary, we deduce that $(f_j)_{j \in J}$ *satisfies* $A_{III})$.

Hence the theorem is proved.

Remark. – Let $\mathcal{P}$ be a subset of the set of all partitions of unity of X. Denote by $\mathcal{S}_{\mathcal{P}}(X, Y; E)$ the set of all correspondences in $\mathcal{S}(X, Y; E)$ defined using partitions of unity belonging to $\mathcal{P}$; denote by $\mathcal{S}^{(c)}_{\mathcal{P}}(X, Y; E)$ the set of all correspondences in $\mathcal{S}^{(c)}(X, Y; E)$ defined using partitions of unity to $\mathcal{P}$. Assume that for every open locally finite covering $\mathcal{C}$ of X there exists a partition of unity belonging to $\mathcal{P}$ subordinated to $\mathcal{C}$. Then Theorem 1 and Corollary 1 remain valid if in their statements $\mathcal{S}(X, Y; E)$ and $\mathcal{S}^{(c)}(X, Y; E)$ are replaced by $\mathcal{S}_{\mathcal{P}}(X, Y; E)$ and $\mathcal{S}^{(c)}_{\mathcal{P}}(X, Y; E)$ (the same is true for Remark 1) following the proof of Theorem 1). Assume in addition that if $(\alpha_t)_{t \in T}$ and $(\varphi^t_j)_{j \in I(t)} (t \in T)$ belong to $\mathcal{P}$ then $(\alpha_t \varphi^t_j)_{(t,j) \in T^*}$, where $T^* = \bigcup_{t \in T}(\{t\} \times I(t))$, belongs to $\mathcal{P}$. Then Theorem 2 remains valid if in its statement we replace $\mathcal{S}^{(c)}(X,Y;E)$ be $\mathcal{S}_{\mathcal{P}}^{(c)}(X,Y;E)$. When X is metric the set $\mathcal{P}$ of all locally Lipschitz partitions of unity satisfies the above conditions.

Let X be a non-void set and Y a non-void subset of E. For every correspondence A between X and Y and set $U \subset E$ we denote by $A^{(U)}$ the correspondence between X and Y defined by

$$A^{(U)}(x) = (A(x) + U) \cap Y$$

for every $x \in X$; obviously $A \subset A^{(U)}$ if $U \ni 0$. If $A \in \mathcal{S}(X,Y;E)$ is given by (1) and if U is convex, then

$$A^{(U)}(t) = \left(\sum_{j \in J} \varphi_j(t)(A_j + U) \right) \cap Y$$

for every $t \in X$.

Denote by $\mathcal{B}$ a fundamental system of $0 \in E$. Then:

PROPOSITION 1. – *Let f be a correspondence between X and Y and let $(f_j)_{j \in J}$ be an upper approximating family for f. Then $(f_j^{(U)})_{(j,U) \in J \times \mathcal{B}}$ is an upper approximating family for f.*

For every $(j,U) \in J \times \mathcal{B}$

$$f_j^{(U)} = (f_j)^U.$$

The set $J \times \mathcal{B}$ is endowed with the usual order relation $\leq$: $(j',U') \leq (j'',U'')$ means $j' \leq j''$ and $U' \supset U''$.

PROOF. – The conditions $A_I)$ and $A_{II})$ are obviously satisfied. To verify $A_{III})$ choose $H \in \mathcal{B}$ such that $H + H \subset V$ and observe that if $(h,U) \in J \times \mathcal{B}$ and $(j_{t,H}, H) \leq (h,U)$ then

$$\begin{aligned} f_h^{(U)}(t) &\subset f_h(t) + U \subset f_h(t) + H \\ &\subset f(t) + H + H \subset f(t) + V. \end{aligned}$$

In the *next two propositions and the corollary* we assume that:

X is a non-void topological space;

Y is a non-void closed convex subset of E ;

U is a convex open neighborhood of $0 \in E$.

We do observe that for every compact set $L \subset Y$

$$\overline{(L+U)\cap Y} = (L+\overline{U})\cap Y.$$

PROPOSITION 2. – *Let A be a correspondence, between X and Y, compact and with closed graph. Then:*

2.1) $A^{(\overline{U})}$ is closed;

2.2) $A^{(\overline{U})}(x) = \overline{A^{(U)}(x)}$ for every $x \in X$;

2.3) $A^{(\overline{U})} = \overline{A^{(U)}}$.

PROOF. – Let $((x_i, y_i))_{i\in T}$ be a family of elements of $A^{(\overline{U})}$ indexed by a filtering set T which converges to (x,y). Then $(x_i)_{i\in T}$ converges to x and $(y_i)_{i\in T}$ converges to y. Since A is compact there is $V_x \in \mathcal{V}(x)$ such that $A(V_x)$ is relatively compact. We may and will assume that $x_i \in V_x$ for $i \in T$.

For every $i \in T$

$$y_i = a_i + u_i$$

with $a_i \in A(x_i)$ and $u_i \in \overline{U}$.

Let $\mathcal{U}$ be an ultrafilter on T finer than the filter of sections of T. Since $A(V_x)$ is relatively compact $\lim\limits_{(i,\mathcal{U})} a_i = a$ exists and, since the graph of A is closed, $(x,a) \in A$. Since $(y_i)_{i\in I}$ converges we deduce that $\lim\limits_{(i,\mathcal{U})} u_i$ exists and belongs, obviously, to $\overline{U}$. Hence

$$y = \lim_{(i,\mathcal{U})}(a_i + u_i) = a + u$$

and hence

$$y \in (A(x) + \overline{U}) \cap Y = A^{(\overline{U})}(x)$$

We deduce that $(x,y) \in A^{(\overline{U})}$ and hence that $A^{(\overline{U})}$ is closed.

To prove 2.2) of Proposition 2 we observe that

$$\overline{A^{(U)}}(x) = \overline{(A(x) + U) \cap Y} = (A(x) + \overline{U}) \cap Y = A^{(\overline{U})}(x)$$

(notice that $L = A(x)$ is a compact part of Y) for every $x \in X$.

To prove 2.3) of Proposition 2 let $(x,y) \in A^{(\overline{U})}$. By 2.2) we have $y \in \overline{A^{(U)}(x)}$. Hence there is a family $(b_j)_{j \in J}$ of elements of $A^{(U)}(x)$, indexed by a filtering set J, which converges to y. Since $(x, b_j) \in A^{(U)}$ for every $j \in J$ it follows that $(x,y) \in \overline{A^{(U)}}$. Hence $A^{(\overline{U})} \subset \overline{A^{(U)}}$. To prove the converse inclusion we observe that $A^{(U)} \subset A^{(\overline{U})}$ and that $A^{(\overline{U})}$ is closed.

PROPOSITION 3. – *Let A be a correspondence, between X and Y, compact, with closed graph, continuous and such that $A(x)$ is non- void and convex for every $x \in X$. Then $A^{(U)}$ is regular. If Y is compact $\overline{A^{(U)}}$ is continuous.*

PROOF. – That the graph of $A^{(U)}$ is open is proved, for instance, [19, p. 7][4]. By hypothesis $A(x)$ is non-void and convex for every $x \in X$, whence the same holds for $A^{(U)}(x)$ for every $x \in X$. By 2.2) and 2.3) of Proposition 2

$$\overline{A^{(U)}}(x) = A^{(\overline{U})}(x) = \overline{A^{(U)}(x)}$$

for every $x \in X$. Hence $A^{(U)}$ is regular.

The correspondence $A^{(U)}$ has open graph and hence it is lower semi-continuous; hence $A^{(\overline{U})}$ is lower semi-continuous. If Y is compact the correspondence $\overline{A^{(U)}}$ is compact; since it has closed graph it is upper semi-continuous. We conclude that $\overline{A^{(U)}}$ is continuous.

Remark. – If E is finite dimensional and if, in addition, U *is relatively compact*, then $A^{(\overline{U})}$ *is compact and continuous.*

Indeed, let $t \in X$ and $V_t \in \mathcal{V}(t)$ such that $A(V_t)$ is relatively compact; then

$$A^{(\overline{U})}(x) = (A(x) + \overline{U}) \cap Y \subset (A(V_t) + \overline{U}) \cap Y$$

if $x \in V_t$. Since $t \in X$ was arbitrary, $A^{(\overline{U})}$ *is compact.*

Since $A^{(\overline{U})}$ has closed graph, $A^{(\overline{U})}$ is upper semi-continuous. Since $A^{(\overline{U})} = \overline{A^{(U)}}$, $A^{(\overline{U})}$ is lower semi-continuous. Whence $A^{(\overline{U})}$ *is continuous.*

PROPOSITION 4. – *Let $A \in \mathcal{S}^{(c)}(X, Y; E)$. Then $A^{(U)}$ is regular. If Y is compact, $\overline{A^{(U)}}$ is continuous.*

Variants of Propositions 2, 3 and 4 were given in [19, p. 8].

THEOREM 3. – *Let X be a paracompact space and Y a non-void closed equilibrated convex part of E which has the property (K). Let f be a correspondence between X and Y such that:*

3.1) f is compact and upper semi-continuous;

3.2) $f(t)$ is non-void compact and convex for every $t \in X$.

Then there is a family $(f_j)_{j\in J}$ of correspondences between X and Y, indexed by a filtering set J, such that:

3.3) for every $j \in J$ the correspondence f_j is regular;

3.4) $(f_j)_{j\in J}$ and $(\overline{f_j})_{j\in J}$ are upper approximating families for f;

3.5) for every $j \in J$ the correspondence $\overline{f_j}$ is continuous if Y is compact.

PROOF. – Since the hypotheses of Theorem 2 are satisfied there is an upper approximating family $(g_k)_{k\in T}$ for f consisting of correspondences belonging to $\mathcal{S}^{(c)}(X, Y; E)$. Let $\mathcal{B}$ be a fundamental system of $0 \in E$ consisting of open convex neighborhoods of $0 \in E$ and let $J = T \times \mathcal{B}$. For every $(k, U) \in J$ let

$$f_{(k,U)} = (g_k)^{(U)}.$$

By Proposition 1 the family $(f_j)_{j\in J}$ is upper approximating for f. By Proposition 4, for every $j \in J$, f_j is regular.

Since f_j is regular we have

$$\overline{f_j}(x) = \overline{f_j(x)}$$

for every $x \in X$; hence $(\overline{f_j})_{j \in J}$ is also an upper approximating family for f (see Remark 3 following the definition of upper approximating families).

Hence 3.3) and 3.4) of Theorem 3 are proved. By Proposition 4 the correspondence f_j is continuous for every $j \in J$, if Y is compact. Hence Theorem 3 is proved.

Remark. – If E is finite dimensional and if $\mathcal{B}$ is a fundamental system of $0 \in E$ consisting of relatively compact, open convex neighborhoods of $0 \in E$, we deduce that $\overline{f}_j$ *is compact and continuous* for every $j \in J$ (see the remark following the proof of Proposition 3).

Let $(X_i)_{i \in I}$ be a family of sets and let $(Y_i)_{i \in I}$ be a family of subsets of E. For every $i \in I$ let f_i be a correspondence between X_i and Y_i, $\mathcal{C}(i)$ a set of correspondences between X_i and Y_i and $(f_j^{(i)})_{j \in J(i)}$ an upper approximating family for f_i consisting of correspondences belonging to $\mathcal{C}(i)$. It is useful to know that we may assume that the approximating families have the same set of indices. For completeness we show below how this can be done: Let J be the set of all pairs $(A, (j_i)_{i \in A})$ where A is a non-void finite part of I and $j_i \in J(i)$ for every $i \in A$. Introduce in J the order relation $\leq$ defined as follows:

$$(A, (j_i)_{i \in A}) \leq (B, (k_i)_{i \in B})$$

if $A \subset B$ and $j_i \leq k_i$ for $i \in A$. For every $i \in I$ and $j = (A, (j_s)_{s \in A})$ in J define

$$\begin{aligned} g_j^{(i)} &= f_{j_i}^{(i)} \quad \text{if } A \ni i \\ &= f_{i_0}^{(i)} \quad \text{if } A \not\ni i \end{aligned}$$

(we may assume that $J(i)$ has a smallest element i_0.) Then for every $i \in I$, $(g_j^{(i)})_{j \in J}$ is an upper approximating family for f_i consisting of correspondences belonging to $\mathcal{C}(i)$.

3. – On the equilibriums of generalized games.

In this section we assume that E is locally convex only in the statement and proof of Theorem 5.

A *generalized game* (= abstract economy), as defined in [22][5], is a triple

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

where $(X_i)_{i\in I}$ is a non-void family of non-void topological spaces and, for every $i \in I$, A_i and Q_i are correspondences between[6] X^I and X_i.

An *equilibrium* of $\mathcal{E}$ is an outcome $x^* \in X^I$ satisfying, for every $i \in I$:

E_I) $(x^*)_i \in \overline{A_i(x^*)}$;

E_{II}) $A_i \cap Q_i(x^*) = \emptyset$.

Assume that, for every $i \in I$, X_i is a subset of the topological vector space E. For every $i \in I$ denote by $\mathcal{C}_i$ (see [5] and [19, Section 2]) the set of all correspondences ψ between X^I and X_i such that:

h) $\psi(x)$ is convex for every $x \in X$;

hh) ψ has open lower sections;

hhh) $x_i \notin \psi(x)$ for every $x \in X^I$.

A correspondence φ between X^I and X_i is $\mathcal{C}_i$-*majorized* if for every $t \in X$ for which $\varphi(t) \neq \emptyset$ there are $\psi_t \in \mathcal{C}_i$ and $V \in \mathcal{V}(t)$ such that

$$\varphi(x) \subset \psi_t(x)$$

for every $x \in V$.

The following was proved in [19][7]:

THEOREM 4. – *The game $\mathcal{E}$ has an equilibrium if, for every $i \in I$:*

4.1) X_i is a convex compact subset of E;

4.2) A_i is quasi-regular;

4.3) $x_i \notin \gamma(Q(x_i)$ for every $x \in X^I$;

4.4) $A_i \cap Q_i$ is $\mathcal{C}_i$-majorized;

4.5) the set $\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$ is open.

Now consider the following properties:

4.2′) *A_i is regular.*

4.4′) *Q_i has open lower sections.*

4.4″) *Q_i is lower semi-continuous and $\mathcal{C}_i$–majorized.*

COROLLARY 2. – *The game $\mathcal{E}$ has an equilibrium if it has, for every $i \in I$, the properties 4.1), 4.2), 4.3) and* 4.4′).

PROOF. – Since Q_i has open lower sections, the correspondence $\gamma(Q_i)$, defined by $\gamma(Q_i)(x) = \gamma(Q_i(x))$ for every $x \in X^I$, has open lower sections (see [25, Lemma 5.1] or [24, Remark 2.3(b)]). Hence $A_i \cap \gamma(Q_i)$ has open lower sections. Since, for every $x \in X^I$, $A_i \cap \gamma(Q_i)(x)$ is convex and $x_i \notin A_i \cap \gamma(Q_i)(x)$, it follows that the correspondence $A_i \cap \gamma(Q_i)$ belongs to $\mathcal{C}_i$. Since

$$A_i \cap Q_i \subset A_i \cap \gamma(Q_i)$$

we deduce that $A_i \cap Q_i$ is $\mathcal{C}_i$-majorized. Since $A_i \cap \gamma(Q_i)$ has open lower sections,

$$\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$$

is open.

Hence $\mathcal{E}$ has the properties 4.1)–4.5) for every $i \in I$ and hence Corollary 2 is proved.

Corollary 2 is Theorem 2.5 of S. Toussaint ([24]). In an earlier paper N.C. Yannelis and N.D. Prabhakar ([25]) proved this result under certain additional hypotheses (for example, if E is locally convex and separated, if I is countable and if X_i is metrizable for every $i \in I$). We observe that the method of proof of these authors combined with an approximation procedure gives the result in Corollary 2 in locally convex spaces.

COROLLARY 3. – *The game $\mathcal{E}$ has an equilibrium if it has, for every $i \in I$, the properties 4.1), 4.2′) and 4.4″).*

PROOF. – Since A_i is open and Q_i is lower semi-continuous, $A_i \cap Q_i$ is lower semi-continuous and hence

$$\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$$

is open. Since Q_i is $\mathcal{C}_i$-majorized, 4.3) is satisfied and $A_i \cap Q_i$ is majorized.

Hence $\mathcal{E}$ has the properties 4.1)–4.5) for every $i \in I$ and hence Corollary 3 is proved.

That Corollary 3 is valid was stated by S. Toussaint [24, Remark 2.6(b)] (this remark suggested the formulation of Theorem 4).

We have now arrived at the main result of this section. Using one of the *approximation theorems* obtained in Section 2, we shall prove the:

THEOREM 5. – *The game $\mathcal{E}$ has an equilibrium if, for every $i \in I$:*

5.1) X_i is a convex compact subset of E;

5.2) $A_i(x)$ is non-void closed and convex for every $x \in E$;

5.3) A_i is upper semi-continuous;

5.4) Q_i is lower semi-continuous and $\mathcal{C}_i$-majorized;

5.5) The set $\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$ is open.

Remarks. – 1) The hypotheses 5.4) and 5.5) are satisfied if, for every $i \in I$:

1° *A_i is continuous*;

2° *Q_i is open*;

3° *$x_i \notin \gamma(Q_i(x))$ for every $x \in X^I$.*

Under these hypotheses the above result is the W. Shafer-H. Sonnenschein theorem on the equilibriums of generalized games (see [22] and [19, Section 6]).

2) The main purpose of Theorem 5 is to replace, in the W. Shafer-H. Sonnenschein theorem, the assumption that the correspondences A_i are continuous by the assumption that they are *upper semi-continuous*. An application of Theorem 5 is given in the next section.

PROOF. – By the approximation Theorem 3 of Section 2, for every $i \in I$, there is a family $(A_{ij})_{j\in J}$ indexed by a filtering set J, consisting of regular correspondences between X^I and X_i, such that both $(A_{ij})_{j\in J}$ and $(\overline{A}_{ij})_{j\in J}$ are upper approximating families for A_i.

The game

$$\mathcal{E}_j = ((X_i)_{i\in I}, (A_{ij})_{i\in I}, (Q_i)_{i\in I})$$

satisfies the hypotheses of Corollary 3 (or of Theorem 4) for every $j \in J$. Hence $\mathcal{E}_j$ has an equilibrium x_j^* for every $j \in J$.

Let $\mathcal{U}$ be an ultrafilter on J finer than the filter of sections of J and let

u) $$x^* = \lim_{(j,\mathcal{U})} x_j^*;$$

then, for every $i \in I$,

uu) $$(x^*)_i = \lim_{(j,\mathcal{U})} (x_j^*)_i.$$

Since $A_{ij} \cap Q_i(x_j^*) = \emptyset$ and $A_{ij} \supset A_i$ we deduce

$$A_i \cap Q_i(x_j^*) = \emptyset$$

for every $i \in I$ and $j \in J$. Using 5.5) we obtain

$$A_i \cap Q_i(x^*) = \emptyset$$

for every $i \in I$.

Since x_j^* is an equilibrium of $\mathcal{E}_j$ and since A_{ij} is regular we have

$$(x_j^*)_i \in \overline{A_{ij}(x_j^*)} = \overline{A_{ij}}(x_j^*)$$

whence

$$(x_j^*, (x_j^*)_i) \in \overline{A_{ij}}$$

for every $i \in I$ and $j \in J$. From u), uu) and property A_{II}) of upper approximating families we deduce

$$(x^*, (x^*)_i) \in \overline{A_{ij}}$$

for every $i \in I$ and $j \in J$. Since $(\overline{A_{ij}})_{j \in J}$ is also an upper approximating family of A_i we conclude

$$(x^*, (x^*)_i) \in A_i$$

for every $i \in I$.

Hence x^* is an equilibrium of $\mathcal{E}$ and hence the theorem is proved.

4. – Generalized games and inequalities of Ky Fan type

In this section we assume that E is locally convex only in S_{II}, S_{IV}, Theorem 7 and in the remarks concerning these statements.

We begin by introducing a few definitions so that we can shorten some of the statements below.

A *generalized game in functional form* (or simply, a game in functional form) is a triple

$$\mathcal{F} = ((X_i)_{i \in I}, (A_i)_{i \in I}, (\varphi_i)_{i \in I})$$

where $(X_i)_{i \in I}$ is a non-void family of non-void topological spaces and, for every $i \in I$, A_i is a correspondence between X^I and X_i and φ_i is a mapping of $X^I \times X_i$ into $\overline{R}$.

An *equilibrium* of $\mathcal{F}$ is an outcome $x^* \in X^I$ satisfying, for every $i \in I$:

E_I) $(x^*)_i \in \overline{A_i(x^*)}$;

E_{II}) $\varphi_i(x^*, y_i) \leq 0$ for every $y_i \in A_i(x^*)$.

If

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

is a generalized game and if, for every $i \in I$, φ_i is a mapping of $X^I \times X_i$ into $\overline{R}$ such that $Q_i = \{(x, y_i) \mid \varphi_i(x, y_i) > 0\}$ then the equilibrium outcomes of $\mathcal{E}$ and

$$\mathcal{F} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\varphi_i)_{i\in I})$$

are the same.

Let $(X_i)_{i\in I}$ be a non-void family of non-void convex subsets of E and φ_i a mapping of $X^I \times X_i$ into $\overline{R}$. We consider below the following properties:

(e') $x \mapsto \varphi_i(x, y_i)$ is lower semi-continuous on X^I for every $y_i \in X_i$.

(e'') $x_i \notin \gamma(\{y_i \mid \varphi_i(x, y_i) > 0\})$ for every $x \in X^I$.

Let $(X_i)_{i\in I}$ be a non-void family of non-void topological spaces and, for every $i \in I$, let A_i be a correspondence between X^I and X_i and φ_i a mapping of $X^I \times X_i$ into $\overline{R}$. For every $i \in I$ let α_i be the mapping of X^I into $\overline{R}$ defined by

$$\alpha_i(x) = \sup_{y_i \in A_i(x)} \varphi_i(x, y_i).$$

The families $(A_i)_{i\in I}$ *and* $(\varphi_i)_{i\in I}$ *are coherent if* $\{x \mid \alpha_i(x) > 0\}$ *is open for every* $i \in I$.

Observe that

$$\{x \mid \alpha_i(x) > 0\} = \{x \mid A_i \cap Q_i(x) \neq \emptyset\}$$

if

$$Q_i = \{(x, y_i) \mid \varphi_i(x, y_i) > 0\}.$$

Consider now the following four statements:

S_I. – A generalized game $\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$ has an equilibrium if the hypothesis 4.1), 4.2), 4.3) and 4.4′) *of Corollary* 2, *Section* 3, *are satisfied for every* $i \in I$.

For the next statement we need the hypothesis:

5.4′) Q_i *has open lower sections and* $x_i \notin \gamma(Q_i(x))$ *for every* $x \in X^I$.

S_{II}. – *A generalized game* $\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$ *has an equilibrium if the hypotheses* 5.1), 5.2), 5.3), 5.5) *of Theorem* 5, *Section* 3, *and* 5.4′) *above are satisfied for every* $i \in I$.

Since the hypothesis 5.4′) is obviously stronger than 5.4) of Theorem 5, Section 3, it follows that S_{II} *is true.*

S_{III}. – *A game in functional form* $\mathcal{F} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\varphi_i)_{i\in I})$ *has an equilibrium if, for every* $i \in I$:

1) X_i *is a convex compact subset of* E;

2) A_i *is quasi-regular;*

3) φ_i *has the properties* (e') *and* (e'').

S_{IV}. – *A game in functional form* $\mathcal{F} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\varphi_i)_{i\in I})$ *has an equilibrium if the families* $(A_i)_{i\in I}$ *and* $(\varphi_i)_{i\in I}$ *are coherent and if, for every* $i \in I$:

1) X_i *is a convex compact subset of* E;

2) $A_i(x)$ *is non-void closed and convex for every* $x \in E$;

3) A_i *is upper semi-continuous;*

4) φ_i *has the properties* (e') *and* (e'').

Then:[8]

THEOREM 6. – *The statements* S_I *and* S_{III} *are equivalent.*

THEOREM 7. – *The statements* S_{II} *and* S_{IV} *are equivalent.*

The proofs of the above theorems are very similar; this is why we shall prove here only one of them. We shall prove Theorem 7 since its proof requires some additional details.

PROOF. – Assume first that S_{II} is true. Let

$$\mathcal{F} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\varphi_i)_{i\in I})$$

be a game in functional form satisfying the hypotheses of S_{IV}. For every $i \in I$ let $Q_i = \{(x, y_i) \mid \varphi_i(x, y_i) > 0\}$. Obviously the generalized game

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

satisfies the hypotheses 5.1), 5.2) and 5.3). Since φ_i has the property (e'), Q_i has open lower sections; since φ_i has the property (e''), $x_i \notin \gamma(Q_i(x))$ for every $x \in X^I$. Hence $\mathcal{E}$ satisfies the hypotheses 5.4'). Since $(A_i)_{i\in I}$ and $(Q_i)_{i\in I}$ are coherent, $\mathcal{E}$ satisfies the hypothesis 5.5) also. Since we assumed that S_{II} is true, $\mathcal{E}$ has an equilibrium. Hence $\mathcal{F}$ has an equilibrium and hence S_{IV} is true.

Assume now that S_{IV} is true. Let

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

be a generalized game satisfying the hypotheses of S_{II}. For every $i \in I$ let φ_i be the characteristic function of Q_i; from 5.4') we deduce that φ_i has the properties (e') and (e''). Hence

$$\mathcal{F} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\varphi_i)_{i\in I})$$

has the property 4) of S_{IV}. Since $\mathcal{E}$ satisfies the hypothesis 5.5), the families $(A_i)_{i\in I}$ and $(\varphi_i)_{i\in I}$ are coherent. Obviously $\mathcal{F}$ has the properties 1), 2), 3) of S_{IV} also. Since we assumed that S_{IV} is true, $\mathcal{F}$ has an equilibrium. Hence $\mathcal{E}$ has an equilibrium and hence S_{II} is true.

Since S_I and S_{II} are true, we deduce from the above theorems that:

The statements S_{III} and S_{IV} are true.

Remarks. – 1) It is easy to see that a mapping φ_i of $X^I \times X_i$ into $\overline{R}$ has the property (e'') if and only if: *For every $x \in X^I$ there is a quasi-concave mapping g_x of X_i into $\overline{R}$*

such that $\varphi_i(x, y_i) \leq g_x(y_i)$ *for every* $y_i \in X_i$ *and* $g_x(x_i) \leq 0$. It follows that the results in Theorems S_{III} and S_{IV} (for I containing only one element) give various forms of the well-known Ky Fan[12,13] minimax inequality. It also follows that Theorem S_{III} (for I containing only one element) gives *the non-linear alternative* of H. Ben-El-Mechaiekh, P. Deguire and A. Granas ([3, Theorem 2, pp. 257-258]); when E is locally convex Theorem S_{IV} also gives this non-linear alternative. Theorem S_{IV} contains the Ky Fan minimax inequality with *constraints* as given, for example, in J. P. Aubin [1, pp. 279-282].

2) Let $\mathcal{F} = ((X_i)_{i \in I}, (A_i)_{i \in I}, (\varphi_i)_{i \in I})$ be a generalized game in functional form. Under certain supplementary hypotheses on $\mathcal{F}$ (stronger than the hypotheses of S_{III} and S_{IV}) we can deduce "directly" the existence of an equilibrium of $\mathcal{F}$ from the case when I contains only one element.

References

1. J. P. Aubin, Mathematical methods of game and economic theory, North-Holland, Amsterdam (1982).
2. J. P. Aubin and A. Cellina, Differential inclusions, Springer-Verlag, Berlin (1984).
3. H. Ben-El-Mechaiekh, P. Deguire, A. Granas, Une alternative non linéaire en analyse convexe et applications, Comptes Rendus de l'Académie des Sciences, Paris 295 (1982), 257–259.
4. H. K. Border, Fixed point theorems with applications to economics and game theory, Cambridge University Press, Cambridge (1985).
5. A. Borglin and H. Keiding, Existence of equilibrium actions and of equilibrium, Journal of Mathematical Economics 3 (1976), 313–316.
6. N. Bourbaki, Théorie des ensembles, Chs. 1–2, Hermann, Paris (1954).
7. N. Bourbaki, Topologie générale, Chs. 1–2 and 9, Hermann, Paris (1965 and 1960).
8. N. Bourbaki, Espaces vectoriels topologiques, Chs. 1–2, Hermann, Paris (1966).
9. C. Castaing and M. Valadier, Convex analysis and measurable multifunctions, Springer-Verlag, Berlin (1977).
10. F. S. De Blasi, Characterizations of certain classes of semi-continuous multifunctions by continuous approximations, Journal of Mathematical Analysis and Applications 106 (1985), 1–18.
11. G. Debreu, A social equilibrium existence theorem, Proceedings of the National Academy of Sciences, USA 38 (1952), 886–893.
12. Ky Fan, A minimax inequality and applications, in Inequalities III, O. Shisha ed., Academic Press, New York (1972), 103–113.
13. Ky Fan, Some properties of convex sets related to fixed point theorems, Mathematische Annalen 266 (1984), 519–537.
14. S. D. Flåm, Equilibria in non-cooperative games and competitive economies, Tamkang Journal of Mathematics 12(1) (1981), 47–57.
15. D. Gale and A. Mas-Colell, On the role of complete preferences in equilibrium theory, in Equilibrium and disequilibrium in economic theory, G. Schwödiauer ed., Reidel, Dordrecht (1977), 7–14.
16. G. Haddad, Topological properties of the sets of solutions for functional differential inclusions, Nonlinear Analysis, Theory, Methods & Applications 5 (12) (1981), 1349–1366.
17. G. Haddad and J. M. Lasry, Periodic solutions of functional differential inclusions and fixed points of σ-selectionable correspondences, Journal of Mathematical Analysis and Applications 96 (1983), 295–312.
18. M. Hukuhara, Sur l'application semi-continue dont la valeur est un compact convexe, Funkcialaj Ekvacioj 10 (1967), 43–66.
19. C. Ionescu Tulcea, On the equilibriums of generalized games, The Center for Mathematical studies in Economics and Management Science (August 1986), Paper No. 696.

20. M. A. Khan, On extensions of the Cournot-Nash theorem, in Advances in equilibrium theory, C.D. Aliprantis and others eds., Springer-Verlag, Berlin (1985).

21. J. M. Lasry and R. Robert, Analyse non linéaire multivoque, Cahiers de Mathématiques de la Décision, No. 7611, Université de Paris-Danphiné (1976).

22. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2 (1975), 345–348.

23. W. Shafer, Equilibrium in economies without ordered preferences or free disposal, Journal of Mathematical Economics 3 (1976), 135–137.

24. S. Toussaint, On the existence of equilibria in economies with infinitely many commodities and without ordered preferences, Journal of Economic Theory 33 (1984), 98–115.

25. N. C. Yannelis and N. D. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12 (1983), 233–245.

Footnotes

[1] The reader should observe that the compactness of a correspondence does not imply the relative compactness of its graph. A correspondence is compact if its graph is relatively compact; the converse, however, is not true. There are several other definitions of compactness of correspondences in the literature, and these definitions are not all equivalent.

[2] Filtering = directed.

[3] Here and in 3) below take the adherences in Y so that the new correspondences are again correspondences between X and Y.

[4] In the quoted paper, p. 7, line 14 from above: instead of "let $b \in A(x)$," read "let $b \in A(x) \cap (a + W)$."

[5] See also [4, 5, 11, 14, 15, 19, 20, 22, 23, 24, 25]. The equilibrium of a generalized game is defined as in [5].

[6] If $(X_i)_{i \in I}$ is a family of sets we denote by X^I the cartesian product $\prod_{i \in I} X_I$. If $x \in X^I$ we denote by x_i the coordinate of index i of x.

[7] In the paper quoted here we do not always assume that E is locally convex or separated. For example, Theorem 4 and Corollaries 2 and 3 remain valid if E is an arbitrary topological vector space and X_i is quasi-compact for every $i \in I$ (the adherences are taken in $X^I \times X_i$ and X_i, respectively). That some of the equilibrium theorems remain valid without assuming that E is separated was first observed by S. Toussaint ([24]).

[8] See the *geometric formulation of the minimax inequality* given in Ky Fan [13, Section 2].