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PROOF THAT THE EXISTENCE OF PURE INTEREST FIXES THE ADMISSIBLE FUNCTIONAL FORMS OF THE CARDINAL UTILITY FOR MONETARY INCOME

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0. INTRODUCTION

In stochastic theory of individuals' choice among alternatives whose (monetary) outcomes have differences in timing, it is customary in the literature to proceed by postulating the following concerning an individual's preferences: (1) for outcomes at any fixed time, the preferences are represented by a von Neumann-Morgenstern cardinal utility (which may or may not be assumed to be the same for each time slice of the choice space); and (2) for outcomes across time, the preferences accord with some rate of pure interest (having whatever time structure). It is taken for granted that this much postulation places no restriction on the functional form of the individual's NM-cardinal utility for monetary income, so that, in this regard, one is free to make whatever additional assumptions one wishes.

In this paper, I propose to show that, in the last mentioned respect, the literature is without theoretical foundations. More specifically, I propose to show that, if one makes the usual "consistency" assumptions concerning an individual's preferences in the realm of risk (which implies that the preferences may be represented by an NM-cardinal utility), and if it is assumed that the NM-cardinal utility functions are one and the same for every time slice, then, by assuming further that the individual's preferences across time accord with a nontrivial rate of pure interest (with whatever time structure), one necessarily fixes the functional forms which can serve as NM-cardinal utility for monetary income.

If my argument here is accepted, then it must also be accepted that either the literature dealing with individuals' choice when both risk and time preference prevail, e.g., capital budgeting theory, portfolio selection theory, etc., is in error in assuming that an individual's NM-cardinal utility for monetary income can have arbitrarily any functional form or, else, the foundations of the theory need questioning for not agreeing with an individual's having whatever NM-cardinal utility (or risk preference) for monetary income he may wish to have.

Not too roughly, my argument runs as follows. Given any time structure of pure interest, first I show, invoking "consistency," that, for every lottery promising monetary outcomes at some arbitrarily fixed time, one can construct a "time adjusted lottery" promising monetary outcomes at any specific time whatever, such that the individual is indifferent between owning one lottery or the other. Then, by the transitivity of the indifference relation, it follows that, for every lottery, the operation of taking its certainty equivalent must commute with that of adjusting for time. This translates into a functional equation which has determinate solutions when the pure interest rate is nonzero. So are fixed the functional forms which can serve as NM-cardinal utility for monetary income when both risk and a nontrivial rate of pure interest prevail.

Toward this, Section 1 below sets up the axiom system for the argument; Section 2 gives the construction of time adjusted lotteries and derives the functional equation which an individual's NM-cardinal utility for monetary income must satisfy in the presence of pure interest; Section 3 obtains the solutions of the functional equation; and Section 4 closes the study with a brief discussion of the results.

1. THE AXIOM SYSTEM

Since a simultaneous axiomatization of utility and probability is not an issue here, I will assume without discussion that the primitive notion of numerical probability is well defined and applicable to our individual. Also, to simplify exposition, I will work with discrete probability distributions instead of the generalized prbability distributions for which Stieltjes integrals and notation other than that below may be convenient. In regard to the notion of "consistency" of preference behavior, I will accept the following as basic and needing no further justification or foundation.

1.0 Principle of "Consistency": An individual's preferences will be said to be "consistent" only when he cannot, so to speak, make book against himself and end up winning -- or shall I say losing? -- in the process!

1.1 Definitions:

(1) Sure Outcome: A consequence which is well specified in the sense that it can always be determined with certainty whether the consequence obtains will be called a "sure outcome." A sure outcome may need specification in more details than one and, so, may not be represented as a one dimensional scalar, but rather as a many component

vector. Nonetheless, every sure outcome can be denoted by a single letter, and will here be so denoted.

(2) Simple Lottery: Let $\{c_1, \ldots, c_n\}$ be an arbitrary set of sure outcomes. The prospect ℓ of getting exactly one of $c_1, \ldots, c_n$ with probabilities $p_1, \ldots, p_n$, respectively, will be called a "simple lottery" with consequences in $\{c_1, \ldots, c_n\}$ and will be denoted by $\ell = \{(c_1 : p_1), \ldots, (c_n : p_n)\}$.

(3) Compound Lottery: Let $\{\ell_1, \ldots, \ell_m\}$ be an arbitrary set of simple lotteries with consequences in a set $\mathcal{O}$ of sure outcomes. Then, a prospect $L = \{(\ell_1 : q_1), \ldots, (\ell_m : q_m)\}$ of getting exactly one of $\ell_1, \ldots, \ell_m$ with probabilities $q_1, \ldots, q_m$, respectively, is called a "compound lottery" with consequences in $\mathcal{O}$. The simple lottery ℓ_L obtained from L by the classical laws of composition of probabilities will be called the "simple lottery associated with L." Compound lotteries of higher orders may similarly be defined and associated with simple lotteries.

(4) Constant or Sure Lottery: A simple lottery $\ell = \{(c : 1)\}$ sure to result in an outcome c is called a "constant" or a "sure lottery." Clearly, viewing it as a sure lottery is another way of viewing a sure outcome. Thus, for any sure outcome c, c will be identified with

$\{(c : 1)\}$, and the two notations will be used interchangeably to emphasize one or the other viewpoint, as convenient.

1.1.1 Remarks: (1) Given a simple lottery $\ell = \{(c_1 : p_1), \ldots, (c_n : p_n)\}$, another way of viewing it is to think of it as the compound lottery $\ell = \{(\ell_1 : p_1), \ldots, (\ell_n : p_n)\}$, where $\ell_i = \{(c_i : 1)\}$ $(i = 1, \ldots, n)$.

(2) Given a set $\mathcal{O}$ of sure outcomes, let $\mathcal{L}$ be the set of all lotteries, simple or compound, with consequences in $\mathcal{O}$. Then, for each $c \in \mathcal{O}$, identifying c with $\{(c : 1)\}$ embeds $\mathcal{O}$ in $\mathcal{L}$.

1.2 Standing Notation:

$\mathbb{R}$ = The set of real numbers;

$T = \{r \geq 0 \mid r \in \mathbb{R}\}$ denotes times, with 0 being the time at which the individual makes the choice;

$M = \mathbb{R}$ denotes the range of the individual's monetary income (or of the change in the monetary value of his asset position as at time 0). Although practical considerations put both an upper and a lower bound to the range, it is extended here, without loss of generality, to the entire real line for analytical convenience.

$M_t = M \times \{t\}$ denotes the set of all possible monetary incomes at time t $(t \in T)$;

$\mathcal{M} = \cup\{M_t \mid t \in T\}$ denotes the set of all possible monetary incomes at whatever time;

$\mathcal{L}$ = The set of all lotteries, simple or compound, having consequences in a suitably specified set of sure outcomes.

In the realm of risky alternatives which can be characterized as lotteries with consequences in some set $\mathcal{O}$ of sure outcomes, an individual's preference behavior is consistent in the sense of 1.1 above only if it satisfies the following two axioms.

1.3 <u>AXIOM I (of Complete Ordering)</u>: The individual has over $\mathcal{L}$ (and, hence, also over $\mathcal{O}$) a complete preference ordering $\preccurlyeq$ (to be read, "is not preferred to") which, for compound lotteries, is stated in terms of their associated simple lotteries, and which is continuous in the probabilities.

1.3.1 <u>N.B.</u> '$\sim$' will denote <u>indifference</u>, i.e., for any two lotteries $\ell', \ell'' \in \mathcal{L}$, $\ell' \sim \ell''$ iff $(\ell' \preccurlyeq \ell'')$ <u>&</u> $(\ell'' \preccurlyeq \ell')$.

1.4 <u>AXIOM II (of Substitutability or "Strong Independence")</u>: For any integer $n>0$, let $\ell_i', \ell_i'' \in \mathcal{L}$ be such that $\ell_i' \sim \ell_i''$ for all i , $i = 1, \ldots, n$, and let $q_i \geq 0$ $(i = 1, \ldots, n)$, with $\sum_{i=1}^{n} q_i = 1$, be any probabilities which are generated independently of the events generating the probabilities for ℓ_i' and ℓ_i'' $(i = 1, \ldots, n)$. Then, $L' = \{(\ell_1' : q_1), \ldots, (\ell_n'' : q_n)\} \sim \{(\ell_1'' : q_1), \ldots, (\ell_n'' : q_n)\} = L''$.

This much postulation yields the following result which, being standard in the literature, is stated here without proof.

1.5 Theorem (on the Existence of NM-cardinal Utility): Axioms I and II are sufficient (and necessary) for an individual's preferences over $\mathcal{L}$ to be represented by a "von Neumann-Morgenstern cardinal utility" $u: \mathcal{L} \to \mathbb{R}$, determined upto an affine transformation, and having the property that, for any simple lottery $\ell = \{(c_1 : p_1), \ldots, (c_n : p_n)\} \in \mathcal{L}$, $u(\ell) = \sum_{i=1}^{n} p_i u(c_i)$.

1.5.1 Remark: To specify u , it suffices to specify its restriction over the set $\mathcal{O} \subset \mathcal{L}$ of sure outcomes. No notational distinction will be made here between u and the restriction of u on $\mathcal{O}$ -- thus, allowing the latter to be written simply as $u: \mathcal{O} \to \mathbb{R}$.

Turn now to the case when $\mathcal{O}$ is the set $\mathcal{M}$ of all sure monetary incomes at whatever time and $\mathcal{L}$ is the set of all lotteries with consequences in $\mathcal{M}$, and assume that the individual's preferences over $\mathcal{L}$ satisfy the consistency Axioms I and II. Then, it is straightforward that his preferences satisfy the consistency conditions both in respect of each "time slice" of the choice space and in respect of the choices across time. In regard to the former, for any $t \in T$, let $\mathcal{L}_t \subset \mathcal{L}$ be the set of all lotteries having consequences only in $M_t \subset \mathcal{M}$. (<u>Remark</u>: Notice that $\cup\{\mathcal{L}_t \mid t \in T\} \subsetneq \mathcal{L}$!) Then, by 1.5, the individual's preferences on each time slice $\mathcal{L}_t$ can be represented by an NM-cardinal utility function $u_t: M_t \to \mathbb{R}$ $(t \in T)$. For any $t \in T$, u_t represents the individual's preferences <u>at time 0</u> for sure monetary incomes $m \in M$ at time t and, since u_t already carries the appropriate time subscript, the notation will be simplified by taking M as u_t's domain and writing u_t as $u_t: M \to \mathbb{R}$. In regard to the individual's preferences across time, it is usual to specify it in terms of his preferences among sure monetary outcomes $(m, t) \in \mathcal{M}$ as follows.

1.6 <u>AXIOM III (of Discounting at Pure Interest Rate)</u>: There exists a function $\rho: T \to [0, 1]$ such that, given any sure monetary outcome $(m, t) \in \mathcal{M}$, the individual is indifferent at time 0 between (m, t) and $(m', t + dt) \in \mathcal{M}$ iff $m' = m(1 + \rho(t)\,dt)$. The function ρ will be called the "<u>time structure of the individual's own pure interest rate</u>."

1.6.1 <u>Remarks</u>: (1) Using the sure lottery notation of 1.1.(4), the indifference postulated in the above axiom may be written as $\{((m, t) : 1)\} \sim \{((m', t+dt) : 1)\}$ iff $m' = m(1 + \rho(t)\,dt)$.

(2) Fix $t, t' \in T$ arbitrarily. Then, $(m, t) \sim (m', t')$ iff $m = m' e^{-\rho^* \Delta t}$, where $m, m' \in M$, $\Delta t = (t' - t)$ and $\rho^* = \frac{1}{\Delta t}\int_t^{t'} \rho(t)\,dt$. (Remark: Notice that $\rho^* \to \rho(t)$ as $\Delta t \to 0$.) Setting $\theta = e^{-\rho^* \Delta t}$, θ will be called the "certainty discount factor" corresponding to the ordered pair (t, t').

Comparison across time among arbitrary lotteries is then carried out using the following

1.7 Definition (of the Certainty Equivalent): For any $t \in T$, let $\ell_t = \{((m_1, t) : p_1), \ldots, ((m_n, t) : p_n)\} \in \mathcal{L}_t$ be arbitrary. Then, a sure monetary outcome $(m_\ell, t) \in M_t$ such that $(m_\ell, t) \sim \ell_t$ or, equivalently, such that $u_t(m_\ell) = \sum_{i=1}^{n} p_i\, u_t(m_i) = u_t(\ell_t)$, will be called a "certainty equivalent" of the lottery ℓ_t for the individual whose NM-cardinal utility on M_t is $u_t : M \to \mathbb{R}$.

1.7.1 Remark: With everything being as in 1.7, using the sure lottery notation of 1.1.(4), we may write $\{((m_\ell, t) : 1)\} \sim \ell_t$. This says (using Axiom II) that (m_ℓ, t) may be substituted for ℓ_t in all preference relations, and vice versa. In other words, (m_ℓ, t) is an asking price of ℓ_t, i.e., at time 0, the individual is indifferent between ℓ_t , assuming that he already owns it, and receiving for sure an income m_ℓ at time t .

The following assumption (which should not be hard to concede for all but a very mystical few!) guarantees a unique certainty equivalent for any lottery $\ell_t \in \mathcal{L}_t$ $(t \in T)$.

1.8 ASSUMPTION I (of Monotonicity of the Utility Functions): Each u_t $(t \in T)$ is a strictly increasing monotonic function of $m \in M$.

1.8.1 Remark: This assumption allows the definition of a preference order preserving map $CE_t\colon \mathcal{L}_t \to M_t$ $(t \in T)$, where $CE_t(\ell_t)$ is the unique certainty equivalent for any $\ell_t \in \mathcal{L}_t$.

This completes the preliminaries toward attending, in the next section, to the main issue, namely, what restrictions, if any, does Axiom III imply in respect of the individual's NM-cardinal utility functions u_t on time slices $M_t \subset \mathcal{M}$ $(t \in T)$.

2. TIME ADJUSTMENT OF LOTTERIES

From here on, assume that the individual's preferences conform to Axioms I - III.

In the nonstochastic realm, i.e., when no risk exists in fact, and none is even to be contemplated by our individual, it is straightforward that the individual's preferences on each time slice M_t $(t \in T)$ must accord with the following

2.1 Proposition (on the Consistency Condition for the Nonstochastic Case): For any $t' \in T$, and any $m_1, m_2 \in M$, $(m_1, t') \sim (m_2, t')$ iff $(\theta m_1, t) \sim (\theta m_2, t)$ for every $t \in T$, where θ is the certainty discount factor corresponding to (t, t').

If, as is usual, it is further assumed that each $u_t: M \to \mathbb{R}$ $(t \in T)$ is strictly increasing monotonic (see 1.8), then the above condition is no restriction for it holds anyway, and, so, for the nonstochastic case, the matter ends here.

But, contrary to what the literature takes for granted, the above does not close the matter for the stochastic case, as may be seen from the following informal argument: Suppose you own a lottery ticket $\ell_{t'} \in \mathcal{L}_{t'}$ which will give you, at time t', either an income m_1 with probability p or else an income m_2 with probability $(1 - p)$; and, fixing $t \in T$ arbitrarily, suppose that your certainty discount factor corresponding to (t, t') is θ.

Then, if the lottery were to draw the outcome (m_1, t'), you would be indifferent between m_1 for sure at t' and θm_1 for sure at t ; alternatively, if (m_2, t') were to be drawn, you would then be indifferent between m_2 for sure at t' and θm_2 for sure at t . Notice that the argument necessarily uses the discount factor as for <u>certainty</u> and <u>NOT</u> as "adjusted for risk" in some manner! Thus, comparing outcome for outcome, you are indifferent between $\ell_{t'}$ and the "<u>time adjusted lottery</u>" denoted as $\ell_t = \theta(\ell_{t'})_t \in \mathcal{L}_t$, which gives you, at time t instead of t' , incomes, each of which is θ times that given by $\ell_{t'}$ but has the same probability as in $\ell_{t'}$. In the next theorem, the same argument is made formally using no more postulation than that of Axims I - III.

2.2 <u>THEOREM I (on Time Adjusting of Lotteries)</u>: Fixing $t, t' \in T$ arbitrarily, let θ be the individual's certainty discount factor corresponding to (t, t'). Then, for any lottery $\ell_{t'} = \{((m_1, t') : p_1), \ldots, ((m_n, t') : p_n)\} \in \mathcal{L}_{t'}$, $\ell_{t'} \sim \ell_t = \{((\theta m_1, t) : p_1), \ldots, ((\theta m_n, t) : p_n)\} \in \mathcal{L}_t$.

<u>Proof</u>: Using the sure lottery notation of 1.1.(4), by Axiom III, $\{((m_i, t') : 1)\} \sim \{((\theta m_i, t) : 1)\}$ $(i = 1, \ldots, n)$. Express $\ell_{t'}$ as a compound lottery having for its outcomes the sure lotteries $\{((m_i, t') : 1)\}$ and, using Axiom II, substitute for these the corresponding sure lotteries $\{((\theta m_i, t) : 1)\}$ $(i = 1, \ldots, n)$. Then, ℓ_t is simply the associated simple lottery, so that, by Axiom I, $\ell_{t'} \sim \ell_t$.

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It is now possible to state the consistency condition which is the counterpart of 2.1 for the stochastic case.

2.3 <u>THEOREM II (on the Fundamental Consistency Condition)</u>: Let everything be as in the hypothesis of Theorem I above. Then, (m_ℓ, t') is a certainty equivalent of $\ell_{t'}$ iff $(\theta m_\ell, t)$ is a certainty equivalent of $\theta(\ell_{t'})_t$.

<u>Proof</u>: Assume that (m_ℓ, t') is a certainty equivalent of $\ell_{t'}$. Denote $d_\theta\colon \mathcal{L}_{t'} \to \mathcal{L}_t$ as the time discounting map defined by $d_\theta(\ell_{t'}) = \theta(\ell_{t'})_t$ $(\ell_{t'} \in \mathcal{L}_{t'})$. Then, by the transitivity of indifference, the following diagram commutes:

$$\begin{array}{ccc} \ell_t & \xleftarrow{\quad d_\theta \quad} & \ell_{t'} \\ \updownarrow & & \updownarrow \\ (m^*, t) & \xleftarrow{\quad d_\theta \quad} & (m_\ell, t') \end{array}$$

Thus, $(m^*, t) = (\theta m_\ell, t)$. Now, assume, instead, that $(\theta m_\ell, t)$ is the certainty equivalent of ℓ_t and repeat the argument noting that the certainty discount factor corresponding to (t', t) is $\frac{1}{\theta}$.

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The following is stated without proof for it is a straightforward corollary of the above theorem; it is given the status of a theorem here only to underline its importance.

2.4 <u>THEOREM III (on the Fundamental Functional Equation of Consistency)</u>: Fixing $t, t' \in T$ arbitrarily, let $u_t: M \to \mathbf{R}$ and $u_{t'}: M \to \mathbf{R}$ be the individual's NM-cardinal utility functions, and let θ be his certainty discount factor corresponding to (t, t'). For any positive integer n, let $m_1, \ldots, m_n \in M$ be arbitrary. Then, for any probabilities $p_1, \ldots, p_n$, with $p_i \geq 0 \quad (i = 1, \ldots, n)$ and $\sum_{i=1}^{n} p_i = 1$, and any $x \in M$,

$$u_{t'}(x) = \sum_{i=1}^{n} p_i u_{t'}(m_i) \Leftrightarrow u_t(\theta x) = \sum_{i=1}^{n} p_i u_t(\theta m_i)$$

The last three theorems are a direct consequence of Axioms I - III alone, i.e., of the requirements of "consistency" of behavior and the existence of pure interest rate for an individual. In the next section, a complete solution of the above functional equation is obtained with the assumption that each u_t $(t \in T)$ is strictly increasing monotonic (see 1.8) and also accords with the following

2.5 <u>ASSUMPTION II (of Time State Independence of Utility)</u>: Each $u_t: M \to \mathbf{R}$ $(t \in T)$ is one and the same function upto an affine transformation and is, thus, unambiguously denoted (without the time subscript) as $u: M \to \mathbf{R}$.

The alternative case when Assumption II does not necessarily hold (i.e., the case of "time state <u>dependence</u>" of utility) is investigated in a second paper.

3. THE ADMISSIBLE FORMS OF UTILITY FUNCTIONS

Let everything be as in 2.4 and assume that 1.8 and 2.5 hold. Then, $u\colon M \to \mathbb{R}$ satisfies the following functional equation

$$u(x) = \sum_{i=1}^{n} p_i\, u(m_i) \Leftrightarrow u(\theta x) = \sum_{i=1}^{n} p_i\, u(\theta m_i) \qquad \ldots\ldots\ldots\ldots (1)$$

for an arbitrary positive integer n, arbitrary $m_1, \ldots, m_n \in M$, arbitrary probabilities $p_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} p_i = 1$, and for every $\theta \in \Theta$, where

$$\Theta = \{e^{-\int_t^{t'} \rho(t)\,dt} \mid (t, t') \in T \times T\}. \qquad \ldots\ldots\ldots\ldots (2)$$

Note that $\theta \in \Theta$ implies that the interval $[\frac{1}{\theta}, \theta] \subset \Theta$.

If $\Theta = \{1\}$, i.e., if the time structure of the individual's own pure interest rate $\rho(t) \equiv 0$, then (1) holds trivially and any strictly increasing monotonic function $f\colon M \to \mathbb{R}$ will serve as an NM-cardinal utility function.

Now, assume that, for some $\theta^* \in \Theta$, $\theta^* \neq 1$. This implies that, if $u(m)$ is unbounded at any point $m^* > 0$ [resp. $m^* < 0$], then it is necessarily unbounded over every closed interval $[m, m'] \subset M$ with both $m, m' > 0$ [resp. both $m, m' < 0$]. To see this, choose $n = 2$ and, calling $m_1 = \delta$, $m_2 = y$, $p_1 = (1 - p)$ and $p_2 = p$, rewrite (1) as

$$u(x) = (1 - p)\,u(\delta) + p\,u(y) \Leftrightarrow u(\theta x) = (1 - p)\,u(\theta\delta) + p\,u(\theta y) \quad \ldots (3)$$

For some $y > \delta > 0$, assume that u is bounded over the interval

$[\delta, y]$. Pick $p > 0$ and $\theta \in [\frac{1}{\theta^*}, \theta^*]$, with $\theta > 1$, such that $\theta x < y$. Then, both $u(\theta\delta)$ and $u(\theta x)$ are finite; hence, $u(\theta y)$ is also finite. Thus, u is bounded on the interval $[\delta, \theta y]$. By repeating the argument j times, it follows that u is bounded on the interval $[\delta, \theta^j y]$ for $j = 1, 2, \ldots$. This shows that, for any $m \in M$, with $m \geq y$, $u(m)$ is finite for one can find some positive integer j_m such that $m \in [\delta, (\theta)^{j_m} y]$. A similar argument with $p > 0$, but with $\theta < 1$ such that $\theta x > \delta$ shows that $u(m)$ is finite for $0 < m \leq \delta$. The proof for the case when $m < 0$ is similar and is omitted.

There is nothing interesting about the case when $u(x)$ is unbounded everywhere except possibly at 0. From here on assume, therefore, that u is bounded on some (and, hence, all) positive closed interval(s) or on all negative closed intervals.

Take the case when u is bounded on all positive closed intervals $[\delta, y]$, with $0 < \delta < y$, and fix $\theta \neq 1$. In equation (3), let $y = 1$; then, for $p \in (0, 1]$, i.e., for $x \in (\delta, 1]$,

$$\frac{u(x) - u(\delta)}{u(1) - u(\delta)} = \frac{u(\theta x) - u(\theta\delta)}{u(\theta) - u(\theta\delta)} = p \qquad \ldots\ldots\ldots\ldots (4)$$

Now, let y vary and, taking $\delta < 1$, fix $x = 1$. As y ranges over the open interval $(1, \infty)$, p takes values ranging from near 1 to near 0. So, we may write, using (3),

$$\frac{u(y) - u(\delta)}{u(1) - u(\delta)} = \frac{u(\theta y) - u(\theta\delta)}{u(\theta) - u(\theta\delta)} = \frac{1}{p} \qquad \ldots\ldots\ldots\ldots (5)$$

By 1.8, $u(1) > u(\delta)$ and $u(\theta) > u(\theta\delta)$. Hence, (4) and (5) can be put together and rearranged into the following functional equation which holds for all $x > 0$:

$$u(\theta x) = \frac{1}{u(1) - u(\delta)} \left[u(1).u(\theta\delta) - \{u(\delta).u(\theta) + u(\theta\delta).u(x)\} + u(\theta).u(x) \right] \qquad \text{.......... (6)}$$

For the case when u is bounded on all negative intervals $[y, \delta]$, with $y < \delta < 0$, a similar argument as above gives the following functional equation for all $x < 0$:

$$u(\theta x) = \frac{1}{u(-1) - u(\delta)} \left[u(-1).u(\theta\delta) - \{u(\delta).u(-\theta) + u(\theta\delta).u(x)\} + u(-\theta).u(x) \right] \qquad \text{.......... (7)}$$

Further solution of the functional equations (6) and (7) now falls into two cases:

CASE I : u(x) remains bounded as x approaches 0 from the right or from the left.

Consider the case when $u(x)$ is bounded on a neighborhood of 0 and, hence, on every closed interval, whereby both equations (6) and (7) hold. Choose $\delta = 0$ and, with no loss in generality, set $u(0) = 0$ and $u(1) = 1$. Then, (6) and (7) reduce, respectively, to

$$u(\theta x) = u(\theta).u(x) \quad \text{for all} \quad x \geq 0 \qquad \text{.......... (8)}$$

$$u(\theta x) = \frac{u(-\theta)}{u(-1)} . u(x) \quad \text{for all} \quad x \leq 0 \qquad \text{.......... (9)}$$

In (3), substitute $\delta = -1$, $y = 1$ (so that $u(y) = 1$), and choose $0 < p < 1$ such that $x = 0$. Then,

$$\frac{u(-\theta)}{u(-1)} = u(\theta) \qquad \ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots \quad (10)$$

Thus, (8) holds for all real x. In case $u(x)$ is bounded only as it approaches 0 from the right, only (6) would hold. In this case, if $u(0)$ exists, with no loss of generality, set $u(0) = 0$, otherwise set $\underset{\delta_+ \to 0}{\mathrm{Lt}}\, u(\delta) = 0$, and set $u(1) = 1$. Then, (8) holds. In case $u(x)$ is bounded only as it approaches 0 from the left, only (7) would hold. Then, there is no loss in generality in setting $u(-1) = -1$, and $u(0) = 0$ if $u(0)$ exists, otherwise $\underset{\delta_- \to 0}{\mathrm{Lt}}\, u(\delta) = 0$, and in defining $u(\theta) = -u(-\theta)$ for all $\theta > 0$. Then, (7) reduces to (8) holding for all $x \le 0$. Hence, we have

3.1 <u>THEOREM IV (on the Functions Bounded near 0 and Admissible as Utility Functions)</u>: Accept Axioms I - III and Assumptions I and II. Assume further that the individual's own pure interest rate is nontrivial and denote θ to be the generic certainty discount factor corresponding to the ordered pair (t, t') $(t, t' \in T)$. Then, if $u(x)$ is bounded on a closed interval $[0, \delta]$ for some $\delta > 0$ or some $\delta < 0$, precisely one of the following obtains:

(1) u is bounded on all closed intervals and satisfies equation (11) below for all real x;

(2) u is bounded only on all non-negative closed intervals and satisfies (11) for all $x \ge 0$;

(3) u is bounded only on all non-positive closed intervals

and, defining $u(\theta) = -u(-\theta)$ for all $\theta > 0$, satisfies (11) for all $x \leq 0$

$$u(\theta x) = u(\theta) . u(x) ; \quad u(0) = 0 ; \quad u(1) = 1 \quad .... \quad (11)$$

<u>CASE II</u> : <u>$u(x)$ becomes unbounded both as $x \to 0$ from the left and from the right.</u>

If u is bounded on all positive closed intervals $[\delta, y]$ with $0 < \delta < y$, i.e., if (6) holds, then, by 1.8, as $\delta_+ \to 0$, $u(\delta) \to -\infty$, so that u is unbounded for all $x \leq 0$ and (7) cannot hold. Similarly, if (7) holds, then, as $\delta_- \to 0$, $u(\delta) \to +\infty$, so that u is unbounded for all $x \geq 0$ and (6) cannot hold.

Taking limits as $\delta \to 0$ and setting $\underset{\delta \to 0}{\text{Lt}}\ u(\delta)/u(\theta\delta) = 1/g(\theta)$, (6) and (7) reduce, respectively, to the following:

$$u(\theta x) = -u(1). g(\theta) + u(\theta) + u(x) . g(\theta) \text{ for all } x > 0 ..... \quad (12)$$

$$u(\theta x) = -u(-1).g(\theta) + u(-\theta) + u(x). g(\theta) \text{ for all } x < 0 ... \quad (13)$$

Since both (12) and (13) do not hold simultaneously, there is no loss in generality in defining for the case of (13)

$$u(x) = -u(-x) \quad \text{for all} \quad x > 0 . \quad .... \quad (14)$$

This transforms (13) into (12) which, then, holds for all $x \gtrless 0$.

Continuing with (12), without loss of generality, set $u(1) = 0$:

$$u(\theta x) = u(\theta) + u(x) . g(\theta) ; \quad u(1) = 0 \quad .... \quad (15)$$

Recall from (2) that $\theta \in \Theta$ implies $[\frac{1}{\theta}, \theta] \subset \Theta$. Clearly, for any $x \in [\frac{1}{\theta}, \theta]$,

$$u(\theta x) = u(\theta) + u(x).g(\theta) = u(x) + u(\theta).g(x) = u(x\theta) \quad \ldots\ldots \quad (16)$$

$$\frac{g(\theta) - 1}{u(\theta)} \equiv \frac{g(x) - 1}{u(x)} \equiv k \quad \text{for some finite } k \text{ and } \theta \neq 1 \neq x \quad \ldots\ldots \quad (17)$$

Therefore, either $g(x) \equiv 1$, in which case (15) reduces to

$$u(\theta x) = u(\theta) + u(x) ; \quad u(1) = 0 \quad \ldots\ldots\ldots\ldots \quad (18)$$

Alternatively,

$$u(x) = \frac{g(x) - 1}{k} ; \quad k \neq 0 \quad \ldots\ldots\ldots\ldots \quad (19)$$

Substituting (19) in (15), then, yields:

$$g(\theta x) = g(\theta) . g(x) ; \quad g(1) = 1 \quad \ldots\ldots\ldots\ldots \quad (20)$$

Note from (19) that, in this case, g and u are equivalent upto an affine transformation. Thus, (20) is exactly the same as (11), but it allows $u(x)$ to become unbounded in the neighborhood of 0 both from the left and from the right. Thus, we have

3.2 THEOREM V (on the Functions Unbounded at 0 and Admissible as Utility Functions): Let everything be as in Theorem IV. Then, if $u(x)$ is unbounded at 0, setting $u(-x) = -u(x)$, u satisfies equation (18) or else equation (20), and, in either case, for all $x > 0$ or else for all $x < 0$.

Equations (11), (18) and (20) are the classical Cauchy's functional equations, respectively having among their solutions only the following strictly monotonic functions

Eq.(11): $u(x) = |x|^{\beta} \cdot \text{Sign } x \,; \quad \beta > 0 \,; \quad x \geq 0 \text{ or } x \leq 0$ ..... (21)

Eq.(18): $u(x) = \beta \log |x| \cdot \text{Sign } x \,; \quad \beta > 0 \,; \quad x > 0$

or else $x < 0$ (not both) ... (22)

Eq.(20): $g(x) = |x|^{\beta} \cdot \text{Sign } x \,; \quad \beta < 0; \quad x > 0$

or else $x < 0$ (not both) ... (23)

This, finally, proves the following

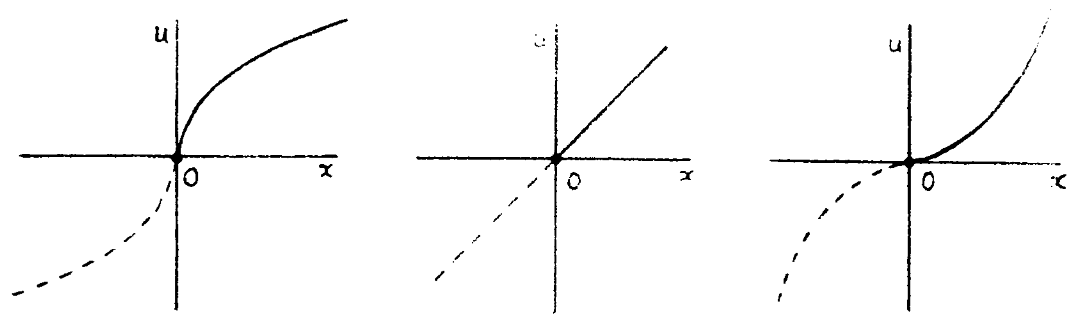
3.3 THEOREM VI (on All Admissible Utility Functions): The following are the only strictly increasing monotonic functions (upto an affine transformation) which satisfy Axioms I - III and the assumptions 1.8 and 2.5 together with the assumption that the individual has a nontrivial own rate of pure interest:

$u(x) = \gamma + \alpha |x|^{\beta} \cdot \text{Sign } x \,; \; \alpha > 0, \; \beta > 0; \; x \geq 0 \text{ or } x \leq 0$ .... (24)

$u(x) = \gamma + \beta \log |x| \cdot \text{Sign } x; \; \beta > 0; \; x < 0 \text{ or else } x > 0$ .... (25)

$u(x) = \gamma + \alpha |x|^{\beta} \cdot \text{Sign } x \,; \; \alpha < 0, \; \beta < 0; \; x < 0 \text{ or else } x > 0$. (26)

These "admissible" NM-cardinal utility functions are sketched in Figures 1 and 2 below.



Risk Loving	Risk Averse	Risk Neutral	Risk Averse	Risk Loving
$0 < \beta < 1$		$\beta = 1$	$\beta > 1$	
(1)		(2)	(3)	

$$u(x) = |x|^{\beta} . \text{Sign } x \; ; \quad \beta > 0$$

FIG. 1 : UTILITY FUNCTIONS WHICH REMAIN BOUNDED AS THEY APPROACH 0

Risk Loving	Risk Averse	Risk Loving	Risk Averse
$u(x) = \log\|x\| . \text{Sign } x$		$u(x) = - \|x\|^{-\beta} . \text{Sign } x$	
(1)		(2)	

Remark: u exists either for $x > 0$ or else for $x < 0$ (not both)!

FIG. 2 : UTILITY FUNCTIONS WHICH ARE UNBOUNDED AT 0

4. DISCUSSION AND SUMMARY

Several things should be noticed about the admissible NM-cardinal utility functions.

(1) For (26), the domain of definition of u may be the entire real line, or the non-positive or non-negative half line. If the domain is only a half line, then on the remaining part of the real line u is necessarily unbounded. If the domain is the entire real line, then the exponent β which defines $u(x)$ over all $x \geq 0$ is necessarily the same as that defining $u(x)$ over all $x \leq 0$.

(2) For (27) and (28), the domain of definition is either all $x > 0$ or else all $x < 0$ (but not both). Over the remainder of the real line u is necessarily unbounded.

(3) Any algebraic expression combining two or more admissible functions yields an admissible function only if the expression is reducible to one of the three admissible forms by some transformation.

(4) Each of the admissible functional forms represents risk averse preferences on the positive reals iff it represents risk loving preferences on the negative reals, and vice-versa.

(5) An NM-cardinal utility function is not admissible if it has a point of inflection at some point $x \neq 0$. This means that, for example, the doubly inflected Friedman-Savage utility function for money, or the concave-convex Markowitz utility

function for increments of (monetary) wealth is NOT admissible!

(6) An admissible NM-cardinal utility function is unbounded at any point $x \neq 0$ only if it is unbounded over all λx for all $\lambda > 0$.

(7) Except for the case $u(x) = x$, the functions defined by (24), (25) and (26) do not remain admissible for translation transformations (i.e., for $y = x + c$) -- a property which is needed to allow the asking price of a lottery to equal its bid price. Hence, if this additional requirement is to be met, then $u(x) = x$ is the ONLY admissible NM-cardinal utility function (upto an affine transformation).

(8) For each $t \in T$, the certainty equivalent map $CE_t: \mathcal{L}_t \to M_t$ is homogenous of degree one in respect of the scale of the risky outcomes; i.e., keeping all else the same, if you multiply all the outcomes of an arbitrary lottery $\ell \in \mathcal{L}_t$ by some constant $c > 0$, then the certainty equivalent of the new lottery is c times the certainty equivalent $CE_t(\ell)$.

Of course, all the above are the consequences of making Assumption II in conjunction with the Axiom I-III and the assumption (I) that the NM-cardinal utility functions are strictly increasing monotonic. However, regardless of the particular assumptions, the primary notions of the consistency of preference behavior, of mutually exclusive outcomes, of numerical probabilities and conditional probabilities, and of time adjustment (or time discounting) are all related together via the fundamental (consistency) condition of Theorem II, which must always hold: For every lottery, its time adjusted certainty equivalent must equal the certainty equivalent of the corresponding time adjusted lottery. This requirement allows but one degree of freedom: You may either choose arbitrarily the utility functions on each time slice M_t $(t \in T)$, or else may choose the structure of the calculus for time adjustment, but not both!