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INTERNAL PRICING AND COST ALLOCATION FOR EFFICIENT DECENTRALIZED CONTROL

by

Avraham Beja[1]
and
Israel Zang[2]

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[1]Tel Aviv University.

[2]Tel Aviv University. Currently visiting Northwestern University.
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Abstract

This paper argues, contrary to the prevelant academic opinion, that joint-cost allocation can be used efficiently for evaluating the profitability of planned activities in informationally decentralized organizations. The issue is studied in the framework of a very general class of Decentralized Control Rules. These rules are procedures that could be implemented in a multi-unit organization, in order to verify whether it would be profitable to eliminate one or more of its activities (but not to specify which of the activities should best be eliminated). Cost allocations are identified as a subclass of exceedingly simple procedures. All potential procedures are examined on three different dimensions: the accuracy of their conclusions, the amount of information that must be collected before a conclusion is reached, and the complexity of the internal communication involved in the process. The analysis shows that, under reasonable conditions, the simple cost-allocating procedures are unsurpassable in all three relevant dimensions.

Keywords: Cost-allocation, Decentralization, Mechanism design, Asymmetric Information.

I. Introduction and Summary

Large organizations almost invariably use some accounting method that allocates joint profits or costs to individual activities. There is also evidence that firms use joint-cost allocations not only for taxation and other regulatory purposes, but also as inputs to their (internal) decisions.[(1)] When asked about this, managers will typically say that joint costs exist and must not be ignored, and hence they want to verify that the overall profitability of each of the firm's activities is adequate.[(2)] To economists, this view is dubious at best. Allocating joint profits and costs to individual activities has been compared to "clapping one's hands, then trying to defend how much of the sound is attributable to each hand".[(3)] Textbooks forewarn that "attempts to use allocated joint-costs...to evaluate the profitability of finished goods are likely to yield misleading conclusions".[(4)] Some authorities give a more categorical statement of the predominant opinion: "there is broad mainstream academic agreement that joint-cost allocations are useless for planning and evaluation purposes, and theoretical concensus that they are wholly arbitrary and needless".[(5)]

This paper argues that joint-cost allocations can be used efficiently in planning the firm's operations and in evaluating the profitability of proposed activities. We address the issue in its operational setting: given a proposed plan, would it be profitable to eliminate some of the planned activities? The immediate concern is with a straight yes-or-no answer (the more difficult problem of how to revise the plan will have to be tackled only if such a revision is found to be necessary). When information is incomplete and asymmetric, organizational procedures designed to answer this yes-or-no question must be evaluated on three different dimensions: the accuracy of their conclusions, the amount of information that must be collected, and the complexity of the required coordination. The analysis shows that, under reasonable conditions, procedures based on cost allocation are unsurpassed in all three relevant dimensions.

The following overview will clarify our framework and highlight the main results. We posit a multi-divisional organization operating as an informationally decentralized "team". A decision is tentatively adopted on the basis of incomplete information about its profitability. There are "joint profits", which can be observed by central management, and "divisional profits" which the respective divisions can assess to any desired degree of precision, with commensurate effort. The organization wants to ascertain that under the tentative plan all divisions are "adequately profitable", i.e., that elimination of some of the planned activities cannot increase profits. To verify this, central management transmits messages to divisions, instructing them to collect information about their profits until either a reassuring response or an alarming response is warranted (according to the received instructions). A "decentralized control rule" is a specification of the messages, the responses, and the aggregation of these responses into an overall yes-or-no conclusion. In "internal-pricing procedures", the messages are accounting charges (or premiums) made to divisions, and each division gives a reassuring response if its profits, adjusted for the accounting charge, are surely non-negative, or an alarming response if its adjusted profits are surely negative. Cost-allocating procedures are internal pricing procedures where all active divisions get an accounting adjustment, the divisional adjustments sum up exactly to the overall joint cost, and the overall conclusion is "positive" if and only if all divisions have non-negative adjusted profits.

Control rules can terminate with errors of two kinds: concluding that no activity-eliminations can increase profits when in fact some can, and vice versa. A decentralized control rule is "conservative" if it never terminates with an error of the first kind. All control rules are partially ordered by their performance as follows: (i) every conservative rule dominates all non-conservative rules, and (ii) one conservative rule dominates another conservative rule if it generates fewer errors (of the second kind) or if it reaches the same conclusions using less information. A control rule is "efficient" if it is not strongly dominated by any other rule. The purpose of this study is to establish the efficiency of internal-pricing procedures in general and cost-allocating procedures in particular.

Broadly speaking, the conditions necessary to establish this efficiency are very mild. They fall into three categories, relating to (i) the divisions' specialization in collecting information, (ii) the range of values that the divisions' profits can take, and (iii) profit complementarities between divisions. The main results are:

1. If the indirect inferences about one division's profits which can be drawn from the information of other units in the organization are "limited" in some sense, then every control rule, however complicated, is (weakly) dominated by some efficient internal-pricing procedure (theorem 3).
2. If, in addition, there is for every division some potential realization of the division's profits under which it will be profitable to make the division inactive, then every conservative cost-allocating procedure is efficient (theorem 7).
3. If, in addition, the organization's profit structure exhibits a degree of complementarity which is similar to (but slightly stronger than) division-wise super-additivity, then every control rule is (weakly)

dominated by some efficient cost-allocating procedure (theorem 6).

Concisely summarized, the overall conclusion of this paper is that, under fairly weak conditions, a decentralized control rule (however complicated) is efficient if and only if it is equivalent in performance to a conservative cost-allocation procedure (theorem 8).

These results lend support to the use of joint-cost allocations as instruments for evaluation and control when complex organizations plan their activities. But our interpretation of cost allocations contrasts common managerial interpretations in at least two important respects. First, cost allocations typically cannot (and clearly should not be expected to) give meaningful answers as to <u>which</u> activities should best be eliminated. In the concluding section of this paper, we give an example of an <u>efficient</u> cost allocation where an activity which shows positive adjusted profits should be eliminated, while at the same time an activity which shows accounting losses should best be maintained. The second important difference in interpretation is that our system is specifically designed also to identify situations where it is <u>not</u> profitable to eliminate <u>any</u> one activity <u>alone</u>, and yet it is profitable to eliminate two or more activities <u>jointly</u>.

The paper is organized as follows. Section II contrasts our approach with alternative approaches in the literature on cost allocation. Section III describes the organizational setup and the difficulties in ascertaining "adequate profitability". Decentralized control rules are formally introduced in Section IV. The performance of these rules is analyzed in Section V, where the notions of conservatism and efficiency are formalized. Section VI introduces internal-pricing control rules: their properties are analyzed in some detail and their efficiency under the stated conditions is established. Section VII does the same for cost-allocation procedures. Section VIII gives some comments on the role of core allocations in our framework. Finally, Section IX compares our findings with some of the prevalent views on internal pricing and joint-cost allocation. To assist the flow of the presentation, all the proofs are presented in an appendix.

II. Alternative Approaches to Cost Allocation

Without attempting to survey the vast literature on cost allocation, we briefly compare our approach to some alternative approaches in the classical and the more recent literature. The classical view is perhaps best represented by Stigler, who writes: "any allocation of common costs to the product is irrational if it affects the amount of the product produced, for the firm should produce the product if its price is at least equal to its minimum marginal cost".(6) We concur with the last part of this statement. But our analysis shows that it does not imply the first part. Marginal analysis, correctly interpreted, must relate not only to infinitesimal margins, but also to "broader margins" such as the complete elimination of product-lines. In our decentralized setting, broad-margin-analysis induces, not contradicts, the use of cost allocations for decision-making.

In recent research, attempts to explain the prevalence of cost allocations tend to rely on the incentives problem in a principal-agent relationship.(7) These studies do not suppose or argue that adjusted divisional profits are surrogates for the division's "profitability". Rather, divisional accounts are set as arbitrary bases for the compensation of division managers, designed to make the decisions that they take in their own self-interests be consistent with the designer's goals. Basing compensation schemes on divisional profits sounds reasonable, but why the adjustments take the particular form of full joint-cost allocations remains questionable.[8] In fact, an important result in the theory of economic control mechanisms sheds serious doubt on the power of the incentives problem to explain cost allocation: in a fairly general set-up, full cost-allocation, incentive-compatibility and optimality of decisions are not mutually consistent.[9] Note that both this approach and ours rely on decentralization, but our model is in the pure "team" framework.[10] The adjusted divisional accounts in our model are more in spirit with the "naive" interpretation of "divisional profits".

Many studies single out particular cost allocation methods as their recommended procedures. These studies usually don't give an explicit statement of the decision problem that the suggested method is supposed to serve better than other methods. Shubik[11] argued that official accounting statements must serve diverse users facing a variety of decision problems, and suggested that if joint-costs are to be allocated the choice of method should be based on general "equity" principles to which all users can agree. His reinterpretation of the Shapley (1953) axioms singles out the allocations given by the Shapley value of the associated "divisional profit game". Other axioms lead to the "Auman-Shapley prices".[12] Quite a few cost allocation schemes have been examined in the accounting literature for their core properties, and many have been found to be in the core whenever it is non-empty.[13] In this respect it is of interest to note that under our condition of profit complementarity the divisional profit game is convex and thus its core is non-empty. In our framework, this complementarity gives rise to efficient cost allocations which are in the core. But our core allocations arise in a manner which is "dual" to the usual requirement that the profits allocated to any subset of divisions be sufficiently high:

"conservatism" requires that the <u>costs</u> allocated to every subset of divisons be sufficiently high (i.e. allocated profits must be sufficiently low). Yet, efficiency makes the allocations "tidy" (i.e. sum up to the overall joint cost), and this tidiness then drives the allocations to be in the core. In the absence of the required profit complementarity, the core may be empty but, as long as the organization has limited indirect inferences, (non-tidy) internal pricing procedures still dominate all decentralized control rules.

III. A Decision Problem in a Multi-Unit Organization

We study an organization composed of n operating divisions (alternatively "departments", or "product-lines"), and a central management. The organization operates as a "team", in that none of its units has incentive or power to deviate from organizational procedures designed to maximize total profits. Asymmetric information and interdependence of the divisions' actions require that decisions be coordinated.

Let potential decisions of the organization be denoted by $x = (x_1,\ldots,x_n)$, where x_i, an element of an arbitrary set A_i, is the action to be taken by division i under the joint decision x. $A \subset \prod_{i=1}^{n} A_i$ denotes the set of all joint decisions. Let T(x) denote the total profits associated with a joint decision x. Evaluating T(x) requires organizational effort in data collection, computation and communication. This effort is the focus of our analysis. The information structure that we posit is as follows:

1) The profit function T is decomposable as

$$T(x) = J(x) + \sum_{i=1}^{n} D_i(x_i) \qquad (1)$$

where $D_i(x_i)$ is a "divisional profit/cost component" which depends only on the action x_i of division i, and where J(x) is a "joint" profit/cost component which captures all aspects that depend on more than just the isolated action of any one division.

2) The organization's prior information is summarized by a probability measure on $\{J(x), D_1(x_1), \ldots, D_n(x_n);\ x \in A\}$. This measure is common to all of the organization's units.

3) Central management can identify the joint profit structure J, which implies the values of $\{J(x),\ x \in A\}$. Evaluation of J(x) for particular decisions x requires computational effort (e.g., running a very complex computer algorithm).

4) Each division can collect information about its own divisional profits. Information about $D_i(x_i)$, collected by division i, narrows down the support of the marginal distribution of $D_i(x_i)$. With sufficient data collection effort, the division can accurately identify $D_i(x_i)$. Of course, some inferences about a division's profits can also be drawn from the information collected by other units (by applying Bayes law to the prior).

Under the obstacles imposed by this information structure, it may be practically impossible for the organization to evaluate all plausible alternatives in order to make an optimal choice among them. Suppose that on the basis of incomplete information the organization reaches some tentative joint decision. The profits associated with this decision and with many (perhaps all) of its alternatives are not precisely known. Yet, the organization wishes to ascertain that the decision is not inferior to a selected number of simple alternatives. In particular, the organization wishes to avoid undertaking any "unprofitable activities", i.e., activities whose elimination will increase profits. This is formalized as follows.

Suppose that for i=1,...,n there is one particular divisional action $\theta_i \in A_i$, which is interpreted as "letting division i be inactive", such that

$D_i(\theta_i)$ is known (to all) with certainty. Without loss of generality, we assume the profit functions to be normalized, and let $J(\theta_1,\ldots,\theta_n) = 0$ and $D_i(\theta_i) = 0$ for $i=1,\ldots,n$. Also, let $N = \{1,\ldots,n\}$ and for any $x \varepsilon A$ and $S \subset N$ let $x_S \varepsilon A$ be defined by

$$(x_S)_i = x_i \qquad \text{if } i \in S$$
$$= \theta_i \qquad \text{otherwise.}$$

That is, we assume that if x is feasible, so are the alternatives x_S, $S \subset N$ (note also that $x_N = x$).

Let x be a tentative joint decision. If all activities are "adequately profitable" then letting any subset S of divisions be inactive will not increase profits, and x must satisfy

$$T(x_N) \geq T(x_{N-S}) \text{ for all } S \subset N \tag{2}$$

where N-S denotes the complement of S in N. Using our decomposition and normalization, this can be rewritten as

$$J(x_N) + \sum_{i\varepsilon S} D_i(x_i) - J(x_{N-S}) \geq 0 \quad \text{for all } S \subset N \tag{3}$$

We shall refer to this set of inequalities as the Adequate Profitability Condition (abbreviated APC). APC is clearly a necessary, but not sufficient, condition for optimality. It simply tests the profitablity of adjusting the tentative decision x by letting any subset of divisions be inactive while maintaining all other activities at their level in x.

Due to the difficulty in assessing profits and costs for all the alternatives x_{N-S}, it may be far from trivial for the organization to ascertain whether a tentative decision x does or does not satisfy APC in the current environment. One possible procedure that can fully resolve the "adequate profitability issue" at considerable organizational effort would be for all divisions to assess their divisional profits with full accuracy and transmit these values to central management; management on its part will assess $J(x_S)$ for all S and test APC directly as in expression (3). Such a procedure can identify not only whether profitable activity-eliminations exist, but also which activities should best be eliminated. Yet, the procedure may be highly inefficient on two counts. First, it requires a precise evaluation of all $D_i(x_i)$, whereas frequently some bounds on these values may be sufficient to give a definite answer. Second, it requires central management to evaluate $J(x_S)$ for a large nubmer of alternatives x_S, and this number may be exorbitant if the organization has many active divisions. These difficulties indicate that it may be better for the organization to employ first some preliminary procedures designed only to verify whether or not APC holds: the more difficult problem of <u>which</u> activities to eliminate will have to be tackled only if activity eliminations are deemed to be potentially profitable. Such procedures should be evaluated both in terms of the accuracy of their conclusions and in terms of the organizational effort required to operate them. We shall show later that "internal pricing procedures" and "cost allocating procedures" have noteworthy merits in both of these respects.

IV. Decentralized Control Rules

Procedures that attempt to verify APC must eventually end in a yes-or-no statement, and in this sense they are what is sometimes called "control rules". Other examples include quality control rules, which specify whether or not to accept an ordered lot of items, or industrial control which determines whether or not to readjust the setting of a production process. In general, a *control rule* is a procedure which eventually terminates with either an 'OK' or an 'Alarm'.

In a multi-division setting, control rules may or may not be decentralized. In a decentralized control rule, divisions follow divisional control procedures, based on information and instructions received from central management, and on information about the current environment that they collect directly on the spot. Each of the sub-procedures eventually ends with a divisional OK or Alarm, and these in turn determine the overall conclusion.

We choose to start our analysis of procedures for verifying APC with a relatively large class of control rules. Given our structured multi-divisional setting, these rules are decentralized. Subsequent analysis will identify which of these rules are better than others in various respects. The class of procedures to be considered in this paper, simply termed "Decentralized Control Rules" or DCR's, can be generally described as follows.

1) Central management transmits messages to some (possibly all) divisions, based on the current joint profit/cost structure J and on the tentative decision x. Potentially, the messages may be as detailed and complicated as may be deemed useful. The rule's first parts specify which divisions are to receive messages, and what these messages will be.
2) Divisions that receive a message respond to it according to the information available to them about the divisional profit $D_i(x_i)$, as reflected by the conditional probability distribution of $\tilde{D}_i(x_i)$, given current information. The rule's divisional response subprocedure specifies, for every possible message and every probability distribution, whether the division should respond with a divisional OK, respond with a divisional Alarm, or collect additional information about $\tilde{D}_i(x_i)$. The collection of information continues until either an OK or an Alarm is the

prescribed response (which always holds if $D_i(x_i)$ is accurately evaluated, but can also hold for fuzzier assessments).

3) Finally, the rule's last part specifies how the divisions' responses are aggregated into an overall OK or an overall Alarm.

Formally, let E_J be a class of normalized joint profit/cost functions, viz.

$$E_J \subset \{J:A \rightarrow R,\ J(\theta_1,\ldots,\theta_n) = 0\} .$$

Let $E_m \subset E_J \times A$ be a set of pairs (J,x) under which central management may choose to initiate a control procedure in an attempt to verify APC. Also, let U denote a class of univariate real random variables u. U includes all degenerate random variables, and to simplify notation we shall not distinguish between the real number $\alpha \in R$ and the degenerate random variable u which takes the realization α with probability one. For any set S, the standard notation 2^S is used to denote the collection of all subsets of S, viz. $2^S = \{Q:Q \subset S\}$. Finally, note that divisions which are initially inactive under the tentative decision (and known with certainty to have zero profits) play no role whatsoever in the procedures described above. Therefore, we choose to simplify the exposition and avoid burdensome notation by assuming that all divisions are active, i.e., $(J,x) \in E_m$ implies $x_i \neq \theta_i$ for all $i \in N$. We then have

<u>Definition 1</u>: A <u>Decentralized Control Rule</u> for APC (abbreviated DCR), is a four-tuple F = (f,s,r,q) such that for every $(J,x) \in E_m$

(i) $s(J,x) \subset N$ is the set of divisions that receive a message from central management.

(ii) For all $i \in s(J,x)$, $f_i(J,x) \in M$ is the message sent to division i, where M is an arbitrary set (the "message space").

(iii) r: $M \times U \rightarrow$ {OK, Alarm, Continue} determines the "response rule" as follows: when division i happens to receive a message m it uses the random variable u_i^c, which represents its conditional current assessment of $D_i(x_i)$ given its current information; then if $r(m,u_i^c)$ = OK the division responds with a divisional ρ_i = OK, if $r(m,u_i^c)$ = Alarm it responds with a divisional ρ_i = Alarm, and if $r(m,u_i^c)$ = Continue the division collects additional information about $D_i(x_i)$, updates its current assessment u_i^c, and reiterates the process. To guarantee an eventual response, $r(m,\alpha) \in$ {OK,Alarm} for all $m \in M$ and all degenerate distributions $\alpha \in R$, and it is assumed that, as $r(m,u_i^c)$ = Continue persists, the collection of information converges to an exact evaluation of $D_i(x_i)$, where an OK/Alarm response is prescribed and the iterations must stop.

(iv) $q(J,x) \subset 2^{s(J,x)}$ determines the overall conclusion (OK or Alarm) on the basis of the responses ρ_i as follows: if there is some $Q \in q(J,X)$ such that ρ_i = OK for all $i \in Q$, the rule terminates with an overall OK, otherwise it terminates with an overall Alarm (i.e., $q(J,x)$ is a set of "coalitions" in $s(J,x)$ that can "force" an overall OK).

It is evident that the class of DCR's, as defined above, is quite general. In particular, the arbitrary message space may include conditional instructions of any conceivable kind, and the divisions' response rule can take into account the likelihood of all alternative potential realizations of the division's profits. The major restriction of DCR's is that they are decentralized, in that all iterations of the process take place only at the divisional level, and in that a division's eventual response must be either an unqualified OK or an Alarm. These restrictions are clearly motivated by obvious organizational considerations which are left out of the formal model. It should also be noted that, as defined, a DCR does not prescribe the divisions' paths of information collection, provided only that they converge, if necessary, to an exact evaluation of the divisional profit. However, we shall see later that this omission is not restrictive, but rather adds to the flexibility of good control procedures.

As an example for the use of one familiar accounting procedure as a DCR, suppose $A = R^n_+$, $E_J \subset \{J: R^n_+ \to R\}$, $E_m = E_J \times R^n_{++}$, and let $F = (f,s,r,q)$ be defined by

$$s(J,x) = N$$

$$f_i(J,x) = -x_i J(x)/\sum x_i \quad \text{(where M=R)}$$

$$r(m,u) = \text{Continue if } u \text{ is non degenerate}$$

$$r(m,\alpha) = \text{Ok if } m \leq \alpha$$

$$= \text{Alarm if } m > \alpha$$

$$q(J,x) = \{N\}$$

The reader is advised to interpret to himself the workings of this procedure, and to assess informally how effective or ineffective it is in verifying APC at a reasonable organizational effort.

V. The Performance of DCRs

To distinguish between more effective and less effective decentralized control rules, one must analyze the "performance" of DCRs over a range of potential situations. The domain of "environments" over which the performance of DCRs is to be evaluated is defined as follows. For every $(J,x) \epsilon E_m$ let $H(J,x) \subset R^n$ denote the support of the conditional distribution of $D_1(x_1),\dots,D_n(x_n)$, given $\{J(x), x \epsilon A\}$, but prior to the incorporation of divisional information (i.e., $y \epsilon H(J,x)$ is a potential realization of the divisional profits, given J and x). It will be assumed that all these sets are convex.(14) An <u>environment</u> is an element (J,x,y) such that $(J,x) \epsilon E_m$ and $y \epsilon H(J,x)$. Let E denote the set of all environments.

A DCR is a procedure which attempts to distinguish between 'good' environments which satisfy APC and 'bad' ones that don't. Let the set of realizations of $D_1(x_1),\ldots,D_n(x_n)$ that are 'good' for $(J,x) \in E_m$ be denoted by $G(J,x)$, viz.

$$G(J,x) = \{y \in H(J,x):\ J(x_N) + \sum_{i\in S} y_i \geq J(x_{N-S}) \quad \forall\, S \subset N\} .$$

To avoid trivialities, it is assumed that management initiates a control procedure only when APC is in doubt, i.e., for all $(J,X) \in E_m$ both $G(J,x)$ and its complement in $H(J,x)$ are non-empty.

When (and if) a DCR F terminates with an Alarm under an environment (J,x,y) such that $y \in G(J,x)$, or with an OK when y is in the complement of $G(J,x)$, F generates a "classification error", or "false signal". A false signal leads to misguided action. When the control rule used to verify APC terminates with an OK, the organization presumably proceeds on the premise that APC is satisfied in the current environment. If this premise is false, maintaining all the activities in x involves a (possibly substantial) opportunity loss. When the procedure terminates with an alarm, management presumably acts on the premise that APC is violated, and initiates a search for a subset of divisions that would better be made inactive. If this premise is false and APC in fact holds, the search will ultimately prove to be futile. We shall assume that the opportunity loss when a potential improvement is discarded due to an unwarranted OK is immeasurably more harmful than the wasted effort in a futile search due to a false alarm.

Beyond the obvious interest in minimizing false signals, it is also desirable that the information which responding divisions must collect before a response is prescribed would not be unnecessarily excessive. Of course, there is little merit in a DCR which requires very little information if it does so by terminating prematurely, thereby frequently generating false signals. Indeed, the consideration of economy in collecting information during the control process is secondary to that of avoiding false alarms: when the organization acts on the false premise that APC is violated, the (futile) search for a subset of activities whose elimination will show the best improvement over the proposed decision typically involves an exact evaluation of all $D_i(x_i)$, and this more than offsets any earlier savings in data collection.

These considerations suggest a clear hierarchy in the evaluation of decentralized control rules. First and foremost, a DCR should avoid unwarranted OK's. Secondly, it should minimize false alarms. Thirdly, it should economize in the collection of information. Each of these desiderata implies a (partial) ordering of DCRs, and the hierarchical structure suggests that the three criteria be applied lexicographically.

To make these ideas precise, we shall need some further notation. First, we must identify what the eventual outcomes of a given DCR can be. This is complicated by the fact that, in general, the outcome of a DCR may depend not only on the current environment but also on what the divisions happen to learn about the environment during their process of collecting information. To analyze what a DCR F will do, we must therefore consider all potential "status-points" of the process, where the status also involves a "profile" of the divisions' assessments about their divisional profits. Precisely, a status is an element (J,x,y,v), where $(J,x,y) \in E$ is an environment and where $v \in U^n$ is an n-tuple of random variables $(v_1,\dots,v_n)$, $v_i \in U$ being the i^{th} division's assessment of its divisional profits. Each of these assessments must be consistent with the environment in which it arises, and also with what is initially known about the division's profits. For the first consistency condition, we require that, for all i, $y_i \in$ support v_i.

For the second condition, let $\overline{H}_i(x)$ denote the support of the (unconditional) marginal prior distribution of $D_i(x_i)$. Given x, it is initially known that $D_i(x_i) \epsilon \overline{H}_i(x)$, and we therefore require that, for all i, support $v_i \subset \overline{H}_i(x)$. Elements $v \epsilon U^n$ which satisfy these two conditions will be termed "potential information profiles under (J,x,y)". A status-point, then, consists of an environment and a potential information profile under this environment. The performance of F can then be analyzed via a "path-function" W_F which specifies, for every possible status-point, what F will do next. W_F is defined by:

$$
\begin{aligned}
W_F(J,x,y,v) &= \text{OK} && \text{if there is some } Q \epsilon q(J,x) \text{ such that} \\
& && r(f_i(J,x),v_i) = \text{OK for all } i \epsilon Q. \\
&= \text{Alarm} && \text{if, for all } Q \epsilon q(J,x), \\
& && r(f_i(J,x),v_i) = \text{Alarm for some } i \epsilon Q \\
&= \text{Reiterate} && \text{otherwise.}
\end{aligned}
$$

Note that W_F does not explicitly depend on y (i.e., W_F is constant on the y-component of the status) but the adopted notation helps clarify the relationship between potential information profiles and their environments. The notation also clarifies the range of possible final outcomes of F in each environment. In particular, it identifies the environments in which F will definitely terminate with an OK, those in which it will definitely terminate with an Alarm, and those in which F's final outcome also depends on the divisions' paths of information collection. Looking at both W_F and G thus identifies all of F's potential "classification errors".

Before returning to our three performance criteria, we shall first use the path-function W to identify an important class of "conservative" procedures. These are DCRs which can terminate with an OK only if APC holds, thus completely eliminating the danger of a misclassification error of the first kind.

Definition 2: A DCR F is (non-trivially) conservative (abbreviated CDCR) if

(i) $W_F(J,x,y,v)=$ OK implies $y \in G(J,x)$.

(ii) For every $(J,x) \in E_m$ there is some status-point such that $W_F(J,x,y,v)=$ OK (non-triviality).

Clearly, conservative DCRs exist. Since avoiding unwarranted OKs is the first and foremost performance criterion, it is also clear that all non-conservative DCRs are inferior to the conservative ones. We can therefore restrict our performance analysis to CDCRs only. Noting that all CDCRs are of course equivalent in the first criterion, we proceed to compare CDCRs by the second one, i.e., by their propensities to generate false alarms.

In general, the propensities of two CDCRs to generate false alarms are not directly comparable, because these alarms may occur at different status-points. F_1 may generate false alarms in some environments where F_2 does not, while in other environments F_2 generates false alarms and F_1 doesn't. Also, there may be environments where F_1 generates false alarms under some potential information profiles while F_2 generates false alarms under other information profiles. However, there are some pairs of CDCRs where it can definitely be said that "F_2 generates at least as many false alarms as F_1". This is true if in every environment under which F_1 may generate a false alarm at some status-point, F_2 must terminate with a false alarm, no matter what sequence of status points is encountered during the process. When this applies, we shall write $F_1 \leq_A F_2$. Note that if in some environment a CDCR F_1 generates an alarm and a CDCR F_2 generates an OK, then the alarm generated by F_1 must be a false alarm (F_2 being conservative), and it can't be true that $F_1 \leq_A F_2$. The binary relation '$\leq_A$' is therefore defined on the class of CDCRs for the environments E as follows: $F_1 \leq_A F_2$ if for all environments $(J,x,y) \in E$ and any two potential information profiles v^1, v^2 under (J,x,y), $W_{F_2}(J,x,y,v^2) = OK$ implies $W_{F_1}(J,x,y,v^1) \in \{OK, Reiterate\}$. It is evident that '$\leq_A$' is a partial order, and we denote its strict component by '$<_A$' and its equivalence component by '$=_A$', in the usual way.(15) In short, then, $F_1 \leq_A F_2$ means that "F_2 generates at least as many false alarms as F_1", $F_1 <_A F_2$ means that "F_2 generates more false alarms than F_1" and $F_1 =_A F_2$ means that "F_1 and F_2 generate false alarms in precisely the same environments". In general, it may be that neither of the above holds.

In a similar fashion, we define another binary relation, denoted '$\leq_I$', to reflect the third criterion, i.e., economy in data-collection. Let the relation $F_1 \leq_I F_2$ apply if $W_{F_1}(J,x,y,v) = Reiterate$ implies $W_{F_2}(J,x,y,v) = Reiterate$. Again '$\leq_I$' is a partial order, with its associated components '$<_I$' and '$=_I$', and $F_1 \leq_I F_2$ if and only if F_2 always requires the collection of at least as much information as does F_1.

Applying the three performance criteria lexicographically induces a dominance relationship between DCRs. We shall say that a DCR F_1 (weakly) dominates F_2 if (i) F_1 is conserveative and F_2 is not, or (ii) both are conservative and $F_1 <_A F_2$, or (iii) both are conservtive, $F_1 =_A F_2$ and $F_1 \leq_I F_2$. F_1 strongly dominates F_2 if F_1 dominates F_2 and F_2 does not dominate F_1. As usual, F is said to be "efficient" if it is not strongly dominated. Since this notion plays a central part in our analysis, we put it down as a formal definition.

Definition 3: A DCR F is efficient if it is conservative and there is no CDCR F' such that $F' <_A F$ or $F' =_A F$ and $F' <_I F$.

Finally, we note that when two DCRs F_1 and F_2 weakly dominate each other then $F_1 =_A F_2$ and $F_1 =_I F_2$, which in turn imply $W_{F^1} = W_{F^2}$. Such rules are termed performance-equivalent, or in short equivalent: they always terminate at the same status-points and with the same conclusions, but they may use very different means to get there. The four-tuple (f,s,r,q) puts emphasis on the means: which messages are sent and to whom, how responding divisions behave, and how responses are aggregated to the procedure's conclusion. On the other hand, W summarizes the signalling quality and the information-collection effort associated with the procedure.

In the terminology of general allocation mechanisms,(16) equivalent DCRs are alternative implementations of their mutual path-function W. The "design" problem, then, is to choose an attractive implementation of a desired W. Efficiency is of course highly desirable, provided that it can be attained with a sufficiently simple implementation.

VI. The Efficiency of Internal-Pricing-DCRs

Much has been said and written in various contexts on the efficiency of price-mechanisms as concise means for the communication of complex information between agents. In the same spirit, we shall now show that, for verifying APC in certain classes of environments, control rules based on internal pricing can perform as well as any of the other DCRs which employ much more complex means of communication.

Internal-pricing-DCRs are exceedingly simple organizational procedures. Based on its information about the joint profit/cost structure J and on the tentative decision x, central management charges divisions with "internal prices" (when an internal price is negative, this is interpreted as a "premium" credited to the division). Each division so charged assesses its potential divisional profits net of the charged internal price (or augmented by the credited premium). The division is then classified as 'OK' if its adjusted profits are surely non-negative, or as an 'Alarm' if they are surely negative: the division is responsible for the collection of sufficient information about its potential profits and costs until either of these classifications is warranted. This is formalized by

<u>Definition 4</u>: An <u>internal-pricing DCR</u> (or in short a price-DCR) is a DCR $F = (f,s,r,q)$ such that range $f \subset R$ and $r = r^o$ where r^o is defined by:

$$
\begin{aligned}
r^o(m,u) &= \text{OK} && \text{if } \alpha \geq m \text{ for all } \alpha \in \text{support } u \\
&= \text{Alarm} && \text{if } \alpha < m \text{ for all } \alpha \in \text{support } u \\
&= \text{Continue} && \text{otherwise.}
\end{aligned}
$$

In the search among all DCRs for those which are efficient, price-DCRs are marked with two salient limitations: their real-valued message-space and their special response rule r^o. Since r^o makes straightforward use of the price messages, it would seem that the real valued message space of price DCRs is their most restrictive feature. Our first result rectifies this false impression. We first identify three distinctive properties of r^o, listed as P1-P3 below, which may or may not be also satisfied by other response rules to arbitrary messages. Theorem 1 then reveals what can and what cannot restrict the performance of price DCRs.

<u>P1</u> (monotonicity). For all $m \in$ range f and all $\alpha \in R$, $r(m,\alpha) =$ OK and $\alpha' > \alpha$ imply $r(m,\alpha') =$ OK.

<u>P2</u>. For all $m \in$ range f, $\min\{\alpha \in R: r(m,\alpha) = \text{OK}\}$ exists. Note that this combines three statements: the set is non-empty, it is bounded from below, and its infinum is attained.

P3. For all $m \in$ range f and all $u \in U$

$$r(m,u) = \begin{cases} OK & \text{if for all } \alpha \in \text{ support } u \ \ r(m,\alpha) = OK \\ Alarm & \text{if for all } \alpha \in \text{ support } u \ \ r(m,\alpha) = Alarm \\ Continue & \text{otherwise.} \end{cases}$$

Theorem 1.

a) For every CDCR F there is a CDCR F' satisfying P3 such that $F' \leq_A F$.

b) If a CDCR F satisfies P1 and P3, there is an internal-pricing CDCR F^0 such that $F^0 \leq_A F$.

c) If a DCR F satisfies P1-P3, there is an internal-pricing DCR F^0 such that $W_{F^0} = W_F$.

Proofs of this and of our subsequent results are presented in an appendix.

Theorem 1 gives a surprising perspective on the inherent limitations of price DCRs. It shows (part c) that P1-P3 fully characterize the performance of these procedures. Relative to P1-P3, the simple message space of price DCRs is not restrictive at all. Of the three properties, the highly stylized structure of P3 never restricts the signaling accuracy of price DCRs (part a) and P2 is similarly harmless in the presence of P1 and P3 (part b). The key factor, then, is the extremely intuitive monotonicity property P1, which simply asserts that "if lower profits are OK, then higher profits are certainly OK". But however intuitive P1 may be, it is still restrictive: there are CDCRs whose signalling accuracy cannot be immitated or surpassed by any price-DCR. One example of such a DCR which also satisfies P2 and P3 (but of course not P1) is presented in the appendix.

While in general the performance of price-DCRs may be limited, our next two theorems give conditions on the class E of potential environments under which the signalling accuracy and ultimately the overall performance of price-DCRs is at least as good as that of any DCR. These conditions, stated below as E1 and E2, relate to the indirect inferences that can be drawn about a division's profits from data which is observable by other units in the organization.

Condition E1: E has limited cross-inferences if, for every $(J,x) \in E_m$, $H(J,x) = H_1(J,x) \times \ldots \times H_n(J,x)$, where $H_i(J,x) = \{y_i : y \in H(J,x)\}$.

Condition E2: E has limited indirect-inferences if it has limited cross-inferences and, for every $(J,x) \in E_m$ and $i \in N$, $H_i(J,x) = \overline{H}_i(x)$

Recall that $\overline{H}_i(x)$ is the support of the marginal prior distribution of $D_i(x_i)$, and note that $H_i(J,x)$ is the support of the respective conditional distribution, given J. "Limited cross-inferences" means that any potential realization of one division's profits cannot be totally excluded on the basis of information about the realized profits of other divisions. "Limited indirect inferences" means that realizations which are initially believed to be possible cannot be totally excluded also on the basis of central management's information about the current joint-profit structure.

Theorem 2: If E has limited cross-inferences (E1) then for every CDCR F there is an internal-pricing CDCR F^o such that $F^o \leq_A F$.

Under limited cross-inferences, the signalling accuracy of price-DCRs is thus unsurpassed. But, in general, the information-collection effort required by price-DCRs can be improved upon if central management can use its own information to update at the outset the divisions' assessments of their profits. Under limited indirect inferences, this cannot narrow down the supports and hence will not shorten the information-collection process.

Theorem 3: If E has limited indirect inferences (E2) then every DCR is (weakly) dominated by some efficient internal-pricing DCR. In particular, for every efficient DCR F there is an internal-pricing DCR F^o such that $W_{F^o} = W_F$.

The communication simplicity of internal-pricing DCRs can hardly be surpassed if the class E of environments is sufficiently rich. The results of this section establish that, under suitable conditions on E, their signalling accuracy and their economy in collecting information also cannot be surpassed by any decentralized control rule.

VII. Cost-Allocating DCRs

Having established the efficiency of internal-pricing-DCRs, we now turn our attention to a subclass of price-DCRs. In these procedures charges (or premiums) are allocated to all active divisions, an overall OK depends on all divisions showing non-negative adjusted profits, and the net charge to all divisions equals the overall joint cost (or the net premium equals the overall joint profit). In fact, these rules' allocations coincide with the familiar (tidy) "cost allocation" procedures.

Definition 5: A cost-allocating DCR is an internal-pricing DCR $F = (f,s,r^o,q)$ satisfying

$$\left.\begin{array}{ll} \textbf{P4.} & s(J,x) = N. \\ \textbf{P5.} & q(J,x) = \{s(J,x)\} \\ \textbf{P6.} & \sum_{i\in N} f_i(J,x) = -J(x). \end{array}\right\} \quad \text{for all } (J,x) \in E_m$$

We wish to identify conditions on E under which all efficient DCRs can be imitated by internal-pricing-DCRs which are also cost-allocating. To clarify the different issues involved in this question, we shall concentrate separately on properties P4–P5, which are jointly equivalent to $q(J,x) = \{N\}$, and on property P6, which requires that the allocations be "tidy". Each of these two parts involves different conditions on E. For analyzing the first one, consider

Condition E3. All divisions are "vulnerable": division i is vulnerable if for all $(J,x) \in E_m$ there is some $y \in H(J,x)$ such that $J(x) + y_i < J(x_{N-\{i\}})$.

Recall that a DCR is initiated by central management only if there is some potential realization of the divisional profits under which it would be profitable to let a subset of the divisions be inactive. If a division is "vulnerable", it means that one can never exclude the possibility that letting this particular division alone be inactive would be profitable.

Theorem 4. If E has limited cross-inferences and all (active) divisions are vulnerable, then every CDCR must satisfy P4–P5.

Conditions E1 through E3 have two things in common. First, they all relate to the sets H(J,x) of potential realizations of the divisional profits. Second, they all require that these sets, and thus the class E for which performance of DCRs is to be evaluated, be sufficiently "rich" in some sense.[(17)] The last condition that we shall study differs from the first three in both of these respects. First, it relates to pairs $(J,x) \in E_m$, not to the potential realizations of divisional profits associated with each pair. Second, it requires that E_m be sufficiently "narrow" in some sense. In essence, our last condition on E focuses on the performance of control rules only in those environments where the profit structure shows a certain degree of "profit complementarity" between divisions.

We shall say that (J,x) exhibits "increasing returns to scope" if, for every triple S_1, S_2, S_3 of disjoint subsets of N, (J,x) satisfies

$$J(x_{S_1 \cup S_2}) - J(x_{S_2}) \leq J(x_{S_1 \cup S_2 \cup S_3}) - J(x_{S_2 \cup S_3}) \tag{7.1}$$

Equation (7.1) says that, in the decision x, letting the divisions in S_1 be active rather than inactive when initially the divisions in both S_2 and S_3 are active adds at least as much to profits as making S_1 active when initially only divisions in S_2 are active. Clearly, in this respect the implications for joint profit and for total profit are identical. Note also that by letting $S_2 = \emptyset$ (7.1) becomes

$$J(x_{S_1}) + J(x_{S_3}) \leq J(x_{S_1 \cup S_3}) \quad \forall \quad S_1, S_3 \subset N \tag{7.2}$$

Thus (7.1) implies division-wise super-additivity of profits. Indeed, it can be argued that division-wise super-additivity of profits, and increasing returns to scope, are the major reasons for the evolution of large multi-divisional organizations.(18) It is in these large organizations that the problems of decentralized control are most acute. Hence, it is of utmost interest to study the performance of DCRs especially in environments that exhibit increasing returns to scope.

Condition E4. E exhibits increasing returns to scope, written in short $E \subset E^*$, if, for all $(J,x) \in E_m$, (7.1) is satisfied for all disjoint $S_1, S_2, S_3 \subset N$.

Theorem 5. If $E \subset E^*$ then conservative cost-allocating DCRs exist. Furthermore, every internal-pricing-CDCR F^o which satisfies P4-P5 is (weakly) dominated by some cost allocating DCR F*.

Theorems 4 and 5 show that, under various conditions on the class E of environments in which a DCR is supposed to perform well, two of the basic features of cost-allocating DCRs arise from considerations of conservatism, and the third then arises from considerations of efficiency. Combining these results with the earlier Theorem 3, we get an overall statement on the efficiency of cost-allocating DCRs.

Theorem 6. If $E \subset E^*$ has limited indirect inferences and all (active) divisions are vulnerable, then every DCR is (weakly) dominated by some efficient cost-allocating DCR. In particular, for every efficient DCR F there is a cost-allocating DCR F* such that $W_{F^*} = W_F$.

Theorem 6 gives a strong case for the use of cost-allocation procedures to verify APC in a relevant class of environments. The set of efficient cost-allocating DCRs is not only non-empty, it is indeed performance-equivalent to the set of all efficient DCRs. But the theorem does not indicate which of the cost allocation procedures are efficient and which are not. The next theorem answers this question.

Theorem 7. If E has limited indirect inferences and all (active) divisions are vulnerable, then every cost-allocating CDCR is efficient.

In other words, Theorem 7 says that, under the stated conditions, if a cost allocation is conservative then it is also efficient. Recall that if $E \subset E^*$ then conservative cost-allocations exist, and note that the conservatism of a cost allocation can be fully ascertained by central management on the basis of its own information.

An immediate corollary to Theorems 6 and 7 shows the direct correspondence between cost-allocating-CDCRs and efficient CDCRs. Since this neatly summarizes the analysis, it is stated as our last theorem.

Theorem 8. When $E \subset E^*$ has limited indirect inferences and all (active) divisions are vulnerable, then a DCR F is efficient if and only if there is a cost-allocating CDCR F* such that $W_{F^*} = W_F$.

VIII. Some Comments on Core Allocations

In recent research on cost allocation, considerable attention has been given to allocations which are in the core of the associated "divisional profit game". It has been argued that if the organization is a loose alliance of autonomous divisions which can retract at will to form alternative alliances, then the portion of joint costs allocated to any subset of divisions must not be higher than the costs that they would incur as a sub-organization operating on its own. Such considerations are of course irrelevant in our pure "team" context. Yet, these core allocations are interestingly related to the allocations of conservative and efficient price-DCRs in our framework. This section is devoted to a number of brief comments intended to clarify this relationship. "Core-DCRs" are cost-allocating DCRs satisfying

P7 (core allocation). For all $(J,x) \in E_m$, $\sum_{j \in S} f_j(J,x) \leq -J(x_S) \quad \forall S \subset N$.

The following observations can be readily verified (see appendix):

1. Every core-DCR satisfies part (i) of definition 2. Hence, if a core-DCR is non-trivial (i.e. satisfies part (ii) of definition 2) then it is conservative.
2. Under the conditions of theorem 6, every cost-allocating CDCR is a core-DCR.
3. However, not all core-DCRs need be non-trivial (hence conservative) even if the conditions of theorems 6 or 8 hold. Therefore, core-DCRs cannot replace the cost-allocating CDCRs in theorems 7 and 8, as stated.

   To resolve this last issue, consider the following strengthening of condition E3.

Condition E3(a): All divisions are "pivotal": division i is pivotal if for every $(J,x) \in E_m$ there is $y \in H(J,x)$ such that

$$J(x_N) + y_j < J(x_{N-\{j\}}) \qquad \text{for all } j \neq i$$

and

$$J(x_N) + \sum_{j \in S} y_j > J(x_{N-S}) \qquad \text{if } S \subset N \text{ and } i \in S.$$

If a division is pivotal, there is a potential realization of the divisional profits such that letting the division be inactive is not profitable while letting any of the other divisions alone be inactive is profitable. Note that if one division is pivotal then all other divisions are vulnerable, hence when all divisions are pivotal all divisions are also vulnerable. Our last observation is:

4. If all divisions are pivotal then every core-DCR is non-trivial, hence conservative.

The impact of these observations is summarized by the following version of theorem 8:

Theorem 8(a): If $E \subset E^*$ has limited indirect inferences and all divisions are pivotal, then a DCR F is efficient if and only if there is a core-DCR F** such that $W_{F^{**}} = W_F$.

IX DCRs and Management Practice

The last section of this study is devoted to some comparisons between our framework and the corresponding concepts in accounting and managerial practice. Besides the apparent similarities, there are also differences - some of them substantial - which must be noted.

The first clarification relates to matters of terminology, and therefore is of relatively minor importance. In our cost-allocating DCRs, charges to some divisions and credits to other divisions may occur simultaneously. In accounting, such adjustments will usually be labelled (at least in part) as "transfer-prices", not joint profit/cost allocations. Conversely, some organizations occasionally use "partial" (as opposed to "tidy") cost-allocations, in which only part of the (central-management) overhead cost is allocated to divisions.(19) In our terminology these are internal-pricing, not cost-allocating adjustments - regardless of their common sign. More importantly, the partial cost-allocations observed in corporate practice cannot be "conservative": in internal-pricing CDCRs that satisfy P4-P5 the charges to all divisions must sum up to at least the overall joint cost.(20)

In prevalent managerial attitudes, joint-cost allocations are perceived as means to assess "overall profitability". Our framework appears to capture this basic concept: when all divisional accounts in a conservative price-DCR show non-negative adjusted profits, no elimination of activities can increase overall profits, and in this sense all divisions are indeed "adequately profitable". The interpretation of the adjusted divisional accounts when some divisions show "accounting losses" is a more subtle matter. First, recall that decentralized control rules cannot be guaranteed to have perfect performance, and conservative DCRs can generate "false alarms". Managers seem to appreciate this possibility when they refrain from eliminating divisions which show accounting losses - a not uncommon occurrence. But even when the 'Alarm' associated with the existence of divisions which show accounting losses is "genuine" (not "false"), divisional accounts must not be construed as evidence on the profitability of each division individually.

Decentralized control rules cannot, and should not be expected to, specify which divisions should best be eliminated.

This point is illustrated by the following example. Consider an organization consisting of three divisions with planned sales levels of 1, 4 and 5 (million dollars), respectively, and suppose that in this range the joint profit/cost (in thousands) is given by

$$J(x) = 10(2-x_1)(5-x_2)(12-x_3) - 1200$$

where x_i is the i^{th} division's sales level. The organization's sales plan involves an overall joint cost of 1130 (thousand). Suppose that this cost is allocated to divisions in proportion to their dollar sales levels, viz. 113, 452, and 565 (thousands), respectively. Note that the organization exhibits increasing returns to scope and that the allocations specified above are "conservative". Assuming that divisional profits can take any value in a sufficiently wide range, these allocations are also "efficient". Table 1 depicts six possible combinations of the divisions' profit levels. In all six cases, overall profits are positive, the adjusted account of division 1 is also positive, and the adjusted account of division 2 is always negative.

Insert Table 1 about here

In case #1, the signs of the divisions' adjusted profits give correct indications of their profitability: division 2 is the only division with negative adjusted profits, and also the only division which can profitably be made inactive. However, the value of the adjusted profits does not give an accurate indication of how "profitable" or "unprofitable" the divisions are.

For example, making division 2 inactive will save only 30, not 202 (thousand).

Cases #2 and #3 demonstrate "false alarms": divisions 2 and 3 have negative adjusted profits in both cases, but neither of these nor any other subset of divisions can be profitably made inactive.

The last three cases are even more interesting. In case #4, only division 2 shows negative adjusted profits, but it is more profitable to make both division 1 and division 2 inactive. In case #5, both division 2 and division 3 show negative adjusted profits, but it is only profitable to make division 2, not 3, inactive. Finally, in case #6 divisions 2 and 3 again show negative adjusted profits, but letting either one of these alone (or both of them jointly) be inactive will not be profitable. Yet, it is profitable to let divisions 1 and 2 be inactive, even though division 1 shows positive adjusted profits (and division 3 doesn't).

When managers identify such peculiarities, they tend to attribute them to the particular cost allocation which is being used (indeed, the reader may be inclined to react in the same way...). But our analysis shows that these phenomena are not idiosyncratic. The allocations in Table 1 are efficient, and hence they cannot be improved upon. Other allocations will give rise to similar phenomena for other potential realizations of the divisional profits. Yet, this does not detract from the power of price-DCRs in general, and cost allocations in particular, to identify potentially profitable eliminations of activities, as effectively as can possibly be done under the bounds imposed by decentralization. On the other hand, a choice among the infinitely many conservative cost allocations cannot be made on the basis of efficiency considerations alone.

Table 1

$x_1 = 1 \qquad x_2 = 4 \qquad x_3 = 5$

$$J(x) = 10(2-x_1)(5-x_2)(12-x_3) - 1200 = -1130$$

Case No.	Unadjusted Divisional Profits			Cost Allocations 113	452	565
				Adjusted Profits		
	1	2	3	1	2	3
1	400	250*	600	287	-202*	35
2**	400	300	450	287	-152	-115
3**	300	350	500	187	-102	-65
4	300*	250*	600	187*	-202*	35
5	400	250*	500	287	-202*	-65
6	300*	300*	550	187*	-152*	-15

*Making the division inactive is profitable in this case.

**False alarm.

Footnotes

1. See e.g. Thomas (1977) pp. 5-8.
2. See also Fremgen and Liao (1981, p. 65) for the responses of managers in a survey sponsored by the National Association of Accountants. When asked why they allocate common costs, the desire to assure "adequate profitability" was the most frequently cited motive.
3. Ijiri (1975) p. 184.
4. Kaplan (1982) p. 409.
5. Thomas (1977) p. 4. Kaplan (1977, p. 52) uses almost identical language.
6. Stigler (1966) p. 165.
7. See Biddle and Steinberg (1984, p. 6-7) for a partial survey.
8. In the absence of a more definite argument, Zimmerman (1979) argues that cost allocation can be an efficient means for motivating managers because "the firm is already calculating cost allocations for tax and external reporting purposes, (and) the additional bookkeeping costs of reporting these allocations internally are minimal" (p. 508).
9. See Groves (1985) and the references cited there.
10. In the sense of Marschak and Radner, e.g. The Ecomomic Theory of Teams (1972).
11. Shubik (1962). See also Shubik (1985).
12. E.g. Billera and Heath (1982), Mirman and Tauman (1982), and Mirman Samet and Tauman (1983).
13. For a survey of these studies, see Biddle and Steinberg (1984).
14. The assumed convexity is essential only in theorems 1(b), 7 and 8, but it is also used elsewhere for convenience.

15. Viz. $F_1 <_A F_2$ if $F_1 \leq_A F_2$ and not $F_2 \leq_A F_1$, and $F_1 =_A F_2$ if $F_1 \leq_A F_2$ and $F_2 \leq_A F_1$.

16. Cf. Hurwicz (1973).

17. The following example demonstrates the mildness of conditions E1–E3. Suppose (as is often done in various contexts) that $D_1(x_1)\ldots,D_n(x_n)$ has a joint-normal distribution, with means, variances and covariances that depend on J and x. Then E1–E3 hold automatically whenever $x_i \neq \theta_i$ implies $\sigma_i(J,x) > 0$, however small. E1–E3 may of course also hold with bounded supports.

18. Baumol et. al (1982) use the term "economies of scope" for super-additivity, which they discuss extensively. In our terminology, economies of scope mean that the associated "divisional profit game" is convex. for a discussion of this concept, see Shapley (1971).

19. See, for example, Mautz and Skausen's (1968) report on a survey by the Financial Executives Research Foundation.

20. Surprisingly, conservative DCRs share this property with the "Groves Mechanism" which is designed to elicit truthful revelation of the divisional profit <u>function</u> from division managers pursuing their own self-interests (see e.g. Groves, 1985, and Groves and Loeb 1979). Note also that partial <u>profit-allocation</u> (not usually exercised in corporate practice) may be consistent with conservative DCRs.

21. See Fiedler M. and V. Ptak "on Matrices with non-Positive off-Diagonal Elements and Positive Principal Minors", <u>Czech. Math. J.</u> Vol 12, No 87 (1962) pp. 382-401 (Theorem 4.3 - 2°, 11°).

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Appendix: Proofs of Theorems

Proof of theorem 1:

(a) Given a CDCR $F=(f,s,r,q)$ let $F'=(f,s,r'q)$ be defined by

$$r'(m,\alpha)=OK \text{ if there is a status } (J,x,y,v) \text{ and } i\varepsilon s(J,x)$$
$$\text{such that } m=f_i(J,x),\ \alpha=y_i,\ r(m,v_i)=OK \text{ and } W_F(J,x,y,v) = OK.$$
$$= \text{Alarm otherwise}$$

and $r'(m,u)$ is then defined by P3 for all $u\varepsilon U$.

Clearly F' is non-trivial. Suppose $W_{F'}(J,x,y,v) = OK$. Then there is $Q\varepsilon q(J,x)$ such that $\forall\ i\varepsilon Q\ r'(f_i(J,x),v_i) = OK$, and by P3 $r'(f_i(J,x),y_i) = OK$. Then, by the construction of r', $\forall\ i\varepsilon Q$ there is $u^i \varepsilon U$ such that $r(f_i(J,x),u^i) = OK$ and $y_i \varepsilon$ support $u^i \subset \overline{H}_i(x)$. Hence there is a status $(J,x,y,\overline{v})$ such that $\forall\ i\varepsilon Q\ \overline{v}_i = u^i$ and $r(f_i(J,x),\overline{v}_i) = OK$, implying $W_F(J,x,y,\overline{v}) = OK$ and (F being conservative) $y \varepsilon G(J,x)$. Thus F' is a CDCR.

Now suppose $W_F(J,x,y,v) = OK$. Then there is $Q\varepsilon q(J,x)$ such that $\forall\ i\varepsilon Q$ $r(f_i(J,x),v_i) = OK$, hence $r'(f_i(J,x),y_i) = OK$. Hence if (J,x,y,v') is a status then, by P3, $y_i\varepsilon$ support v_i' implies $r'(f_i(J,x),v_i') \neq$ Alarm $\forall\ i\varepsilon Q$. Hence $W_{F'}(J,x,y,v') \neq$ Alarm. Thus $F' \leq_A F$.

(b) Given a CDCR $F=(f,s,r,q)$ satisfying P1 and P3, define $F^0=(f^0,s^0,r^0,q^0)$ by

$$q^0(J,x) = \{Q\varepsilon q(J,x):\ \text{for some status } (J,x,y,v)\ \ r(f_i(J,x),v_i) = OK\ \forall\ i\varepsilon Q\} \tag{A.1}$$

$$s^0(J,x) = \{i \varepsilon s(J,x):\ i\varepsilon Q \text{ for some } Q\varepsilon q^0(J,x)\} \tag{A.2}$$

and for $i\varepsilon s^0(J,x)$

$$f_i^0(J,x) = \inf\{\alpha\varepsilon R:\ \text{there is a status } (J,x,y,v) \text{ such that, } \alpha=y_i,\ W_F(J,x,y,v) = OK \text{ and } r(f_i(J,x),v_i) = OK\} \tag{A.3}$$

Note that the conservatism of F and the construction of s^0 imply

$J(x_{N-\{i\}}) - J(x) \leq \overset{0}{f}_i(J,x) < \infty$.

Clearly F^0 is non-trivial. Suppose $W_{F^0}(J,x,y,v) = OK$. Then there is $Q \varepsilon q^0(J,x)$ such that $y_i \geq \overset{0}{f}_i(J,x)\ \forall\ i \varepsilon Q$. By (A.1), there is a status $(J,x,\overline{y},\overline{v})$ such that $\forall\ i \varepsilon Q$ $r(f_i(J,x),\overline{v}_i) = OK$, and then by (A.3) $\overline{y}_i \geq \overset{0}{f}_i(J,x)$. For t=1,2,... let $z^t = (\overline{y}+(t-1)y)/t$. By the convexity of H(J,x), $z^t \varepsilon H(J,x)$. Now, for $i \varepsilon Q$, if $z_i^t = \overline{y}_i$ then $r(f_i(J,x),z_i^t) = OK$ by P3 and if $z_i^t \neq \overline{y}_i$ then $z_i^t > \min(\overline{y}_i,y_i) \geq \overset{0}{f}_i(J,x)$ hence $r(f_i(J,x),z_i^t) = OK$ by (A.3), P3 and P1. Hence $W_F(J,x,z^t,z^t) = OK$, implying $z^t \varepsilon G(J,x)$ for all t. $z^t \to y$ then implies that y satisfies APC and $y \varepsilon G(J,x)$. Thus F^0 is a CDCR.

The proof that $F^0 \leq_A F$ is similar to that of part (a).

(c) Given a DCR F=(f,s,r,q) satisfying P1-P3, define $F^0=(f^0,s,r^0,q)$ by

$$\overset{0}{f}_i(J,x) = \min\{\alpha \varepsilon R:\ \ r(f_i(J,x),\alpha) = OK\}$$

By P2, these minima exist. By P1, $r^0(\overset{0}{f}_i(J,x),\alpha)=r(f_i(J,x),\alpha)\ \forall\ \alpha \varepsilon R$, $i \varepsilon s(J,x)$. Then by P3 $r^0(\overset{0}{f}_i(J,x),u) = r(f_i(J,x),u)\ \forall\ u \varepsilon U$, hence $W_{F^0} = W_F$.

An example of an efficient CDCR which is not "price-compatible"

Let n=3 and suppose that for all $(J,x) \varepsilon E_m$

$$J(x_S) = J(x) < 0\ \forall\ S \neq \emptyset$$

and

$$H(J,x) = \{y \varepsilon R^n:\ \ y_i \geq 0\ \forall\ i \varepsilon N,\ y_2 \leq y_1\}$$

Let F=(f,s,r,q) be defined by

$$s(J,x) = \{1,2,3\}$$

$$q(J,x) = \{\{1,2\},\ \{1,3\},\{2,3\}\}$$

$$f_i(J,x) = (i,-J(x))$$

$r((1,\beta),\alpha) = \text{OK}$ if $0.2\,\beta \le \alpha < 0.3\,\beta$ or $0.7\,\beta \le \alpha$

$r((2,\beta),\alpha) = \text{OK}$ if $0.3\,\beta \le \alpha$

$r((3,\beta),\alpha) = \text{OK}$ if $0.8\,\beta \le \alpha$

with $r(m,\alpha) = \text{Alarm}$ otherwise, and $r(m,u)$ then defined by P3 for all $u \in U$.

Proof of theorem 2. Given a CDCR $F=(f,s,r,q)$ let $F^0=(f^0,s^0,r^0,q^0)$ be defined by (A.1) – (A.3) as in the proof for part (b) of theorem 1. Non-triviality of F implies non-triviality of F^0. Suppose $W_{F^0}(J,x,y,v) = \text{OK}$. Then there is $Q \in q^0(J,x)$ such that $y_i \ge f_i^0(J,x)\ \forall\ i \in Q$. By (A.3), for every $i \in Q$ there is a sequence of status points $(J,x,z^i(t),v^i(t))$, $t=1,2,\ldots$ such that $r(f_i(J,x),v_i^i(t)) = \text{OK}$ for all t and $z_i^i(t) \to f_i^0(J,x)$. Then, by E1, for every t there is a status $(J,x,z^0(t),v^0(t))$ where

$$z_i^0(t) = \begin{cases} z_i^i & \text{for } i \in Q \\ y_i & \text{for } i \in N-Q \end{cases}$$

$$v_i^0(t) = \begin{cases} v_i^i(t) & \text{for } i \in Q \\ y_i & \text{for } i \in N-Q \end{cases}$$

and $z^0(t) \to \bar{z}$, where $\bar{z}_i = f_i^0(J,x)$ if $i \in Q$ and $\bar{z}_i = y_i$ if $i \in N-Q$. By construction, $W_F(J,x,z^0(t),v^0(t)) = \text{OK}$ for all t, hence $z^0(t) \in G(J,x)$. Let

$$G^*(J,x) = \{\xi \in R^n:\ \sum_{i \in S} \xi_i \ge J(x_{N-S}) - J(x)\ \forall\ S \subset N\} \tag{A.4}$$

and note that $G(J,x) = G^*(J,x) \cap H(J,x)$ and $G^*(J,x)$ is closed, hence $\bar{z} \in G^*(J,x)$. But $y \ge \bar{z}$ implies $y \in G^*(J,x)$, and $y \in H(J,x)$ then implies $y \in G(J,x)$. Thus we have established that $W_{F^0}(J,x,y,v) = \text{OK}$ implies $y \in G(J,x)$ and that F^0 is a CDCR.

$F^0 \le_A F$ is then proved as in theorem 1.

Proof of theorem 3.

Let Γ^0 denote the set of all internal-pricing CDCRs $F=(f,s,r^0,q)$ such that for all $(J,x)\varepsilon E_m$ and $i\varepsilon s(J,x)$, $f_i(J,x)\varepsilon \overline{H}_i(x)$. Note that for every internal-pricing CDCR F there is $F^0 \varepsilon \Gamma^0$ such that $W_{F^0} = W_F$. Also, define for every DCR F and every $(J,x)\varepsilon E_m$

$$V_F(J,x) = \{y\varepsilon H(J,x):\ W_F(J,x,y,y) = OK\} \tag{A.5}$$

Note that when F^1, F^2 are internal pricing CDCRs the following three statements are equivalent:

(1) For all $(J,x)\ \varepsilon\ E_m$ $V_{F^2}(J,x) \subset V_{F^1}(J,x)$

(2) $F^1 \leq_A F^2$

(3) F^1 (weakly) dominates F^2

(3) ==> (2) ==> (1) is immediate, and (1) ==> (3) follows from the special structure of r^0.

The proof of the theorem consists of two parts:

1. First, we establish that every DCR F is weakly dominated by some $F^0\varepsilon\Gamma^0$. This needs proof only when F is a CDCR. By theorem 2, and by our preliminary analysis above, there is $F^0\varepsilon\Gamma^0$ such that $F^0 \leq_A F$. We shall now show that if $F^0 =_A F$ then $F^0 \leq_I F$. Let

$$Z(v) = \{z\varepsilon R^n:\ z_i\varepsilon \text{ support } v_i\ \forall\ i\varepsilon N\}.$$

By E2, if (J,x,y,v) is a status then $\forall\ z\varepsilon Z(v)$ (J,x,z,v) is a status. Suppose $W_{F^0}(J,x,y,v)$ = Reiterate. Then there is a $Q\varepsilon q^0(J,x)$ such that $\forall\ i\varepsilon Q$ $r^0(f^0_i(J,x),v_i) \neq$ Alarm, hence there is $z^1\varepsilon Z(v)$ such that $W_{F^0}(J,x,z^1 z^1) = OK$. Also, $\forall\ Q\varepsilon q^0(J,x)$ there is $i\varepsilon Q$ such that $r(f^0_i(J,x),v_i) \neq OK$, hence there is $z^2\varepsilon Z(v)$ such that $W_{F^0}(J,x,z^2,z^2)$ = Alarm. Now if $W_F(J,x,y,v) = OK$ then $W_F(J,x,z^2,v) = OK$, contradicting $F^0 \leq_A F$, and if $W_F(J,x,y,v)$ = Alarm then $W_F(J,x,z^1,v)$ = Alarm, contradicting

$F \leq_A F^0$. Hence when $F^0 =_A F$ $W_{F^0}(J,x,y,v)$ = Reiterate implies $W_F(J,x,y,v)$ = Reiterate, and thus $F^0 \leq_I F$, so that F^0 dominates F. In particular, if F is efficient then $W_{F^0} = W_F$.

2. Finally, we show that every $F^0 \in \Gamma^0$ is weakly dominated by some efficient $F^*=(f^*,s^*,r^0,q^*)\in\Gamma^0$. Given $F^0=(f^0,s^0,r^0,q^0)$, s^*,f^* and q^* are defined as follows.

Let $(J,x)\in E_m$ be given.

For every pair $(\bar{s},\bar{q})$ such that $\bar{s} = \{i(1),\ldots,i(k)\} \subset N$ and $\bar{q} \subset 2^{\bar{s}}$, define

$$\Gamma(\bar{s},\bar{q}) = \{F'=(f',s',r',q')\in\Gamma^0: \; s'(J,x)=\bar{s},\; q'(J,x) = \bar{q},\; V_{F^0}(J,x) \subset V_{F'}(J,x)\}$$

$$B(\bar{s},\bar{q}) = \{\xi\in R^k: \text{ for some } F'\in\Gamma(\bar{s},\bar{q})\ \xi_j = f'_{i(j)}(J,x) \text{ for } j=1,\ldots,k\}$$

Clearly $B(s^0(J,x), q^0(J,x)) \neq \emptyset$. We shall now show that whenever $B(\bar{s},\bar{q}) \neq \emptyset$ there is a "minimal" element $b(\bar{s},\bar{q}) \in B(\bar{s},\bar{q})$, i.e., $\xi\in B(\bar{s},\bar{q})$ and $\xi \leq b(\bar{s},\bar{q})$ imply $\xi = b(\bar{s},\bar{q})$. Let $\{\xi(t),t=1,2,\ldots\}$ be any decreasing sequence in $B(\bar{s},\bar{q})$. $B(\bar{s},\bar{q})$ is bounded from below (by $J(x_{N-\{i(j)\}})-J(x)$ for $j=1,\ldots,k$), hence $\xi(t)$ has a limit point, say $\xi(t) \to \bar{\xi}$. Let $F^{(t)}$ denote the corresponding sequence in $\Gamma(\bar{s},\bar{q})$ and let $\underline{F} = \underline{F}(\bar{\xi}) = (\underline{f},\underline{s},r^0,\underline{q})$ be defined by $\underline{s} = s^{(1)}$, $\underline{q} = q^{(1)}$, and $\underline{f}_{i(j)}(J,x) = \bar{\xi}_j$ for $j=1,\ldots,k$ and $\underline{f}_i(J',x') = f_i^{(1)}(J',x')\ \forall\ i\in s^{(1)}(J',x')$ for all $(J',x') \neq (J,x)$. Suppose $W_{\underline{F}}(J,x,y,v) = OK$. Then there is $Q\in\bar{q}$ such that $y_{i(j)} \geq \bar{\xi}_j$ whenever $i(j)\in Q$. $F^{(t)}\in\Gamma(\bar{s},\bar{q})$ implies $F^{(t)}\in\Gamma^0$, hence for all $i\in\bar{s}$ $f_i^{(t)}(J,x)\in\bar{H}_i(x) = H_i(J,x)$. Hence, as in the proof of theorem 2, there is a sequence $z^0(t)$ such that for all $i\in N-Q$ $z_i^0(t) = y_i$ and for all $i\in Q$ $z_i^0(t) = f_i^{(t)}(J,x)$, and then $z^0(t)\in G(J,x)$, $z^0(t)\to\bar{z}$ $y \geq \bar{z}$ and $y\in H(J,x)$ imply $y\in G(J,x)$. Thus $\underline{F}$ is a CDCR and it is readily verified that $\underline{F} \epsilon \Gamma^0$. Also, since $\xi^{(t)}$ is a decreasing sequence, $V_{F(t)}(J,x) \subset V_{\underline{F}}(J,x)$ hence $V_{F^0}(J,x) \subset V_{\underline{F}}(J,x)$, and thus $\underline{F} \epsilon \Gamma(\overline{s},\overline{q})$ and $\overline{\xi} \epsilon B(\overline{s},\overline{q})$. This establishes the existence of (at least one) minimal element $b(\overline{s},\overline{q})$ of $B(\overline{s},\overline{q})$, and for the corresponding $\underline{F} = \underline{F}(b(\overline{s},\overline{q}))$ it is true that if $F' \epsilon \Gamma(\overline{s},\overline{q})$ and $V_{\underline{F}(b(\overline{s},\overline{q}))}(J,x) \subset V_{F'}(J,x)$ then

$V_{\underline{F}(b(\overline{s},\overline{q}))}(J,x) = V_{F'}(J,x)$.

After repeating this for all the (finitely many) pairs $(\overline{s},\overline{q})$ such that $\Gamma(\overline{s},\overline{q}) \neq \emptyset$, let $\underline{\underline{F}} = \underline{F}\ (b(\overline{\overline{s}},\overline{\overline{q}}))$ be such that

$$V_{\underline{\underline{F}}}(J,x) \subset V_{\underline{F}(b(\overline{s},\overline{q}))}(J,x) \text{ implies } V_{\underline{\underline{F}}}(J,x) = V_{\underline{F}(b(\overline{s},\overline{q}))}(J,x).$$

Then let $s^*(J,x) = \overline{\overline{s}}$, $q^*(J,x) = \overline{\overline{q}}$ and $f^*_{i(j)}(J,x) = b(\overline{\overline{s}},\overline{\overline{q}})_j$ for $j=1,\ldots,k$.

Applying this definition to all $(J,x) \epsilon E_m$ defines $F^* \epsilon \Gamma^0$ such that $\forall\ (J,x) \epsilon E_m\ \ V_{F^0}(J,x) \subset V_{F^*}(J,x)$, hence F^* weakly dominates F^0. To see that F^* is efficient, suppose to the contrary that F^* is strongly dominated by some DCR F'. Then by part (*f*) above there is some $F'' \epsilon \Gamma^0$ which dominates F' and thus also strongly dominates F^* and F^0. Hence there exists $(J,x) \epsilon E_m$ such that $F'' \epsilon \overline{\Gamma}(s''(J,x), q''(J,x))$, $V_{F^*}(J,x) \subset V_{F''}(J,x)$ and $V_{F^*}(J,x) \not\subset V_{F''}(J,x)$, which contradicts the construction of F^*. Thus F^* must be efficient and the proof is complete.

Proof of theorem 4. Let $F=(f,s,r,q)$ be a CDCR. We first show that if $W_F(J,x,y,v) = OK$ on Q, i.e., if (J,x,y,v) is a status, $Q \epsilon q(J,x)$, and $r(f_i(J,x),v_i) = OK\ \forall\ i \epsilon Q$, then $Q=N$. Suppose to the contrary that $i \epsilon N-Q$. By E3, there is $\overline{\overline{y}}\ \epsilon\ H(J,x)$ such that $J(x) + \overline{\overline{y}}_i < J(x_{N-\{i\}})$. Let $\overline{y} \epsilon R^n$ be defined by $\overline{y}_i = \overline{\overline{y}}_i$ and $\overline{y}_j = y_j$ for $j \neq i$, and by E1 $\overline{y} \epsilon H(J,x)$. Also, let $\overline{v} \epsilon U^n$ be defined by $\overline{v}_i = \overline{\overline{y}}_i$ and $\overline{v}_j = v_j$ for $j \neq i$, and then $W_F(J,x,\overline{y},\overline{v}) = OK$ on Q. But this is a contradiction, since F is a CDCR and by construction $\overline{y}$ is not in $G(J,x)$. Hence $Q=N$.

To conclude the proof, note that by non-triviality of all CDCRs there is for every $(J,x)\varepsilon E_m$ some status such that $W_F(J,x,y,v) = OK$, hence $W_F(J,x,y,v) = OK$ on N, and if $Q\varepsilon q(J,x)$ then necessarily $W_F(J,x,y,v) = OK$ on Q, implying Q = N. Thus $q(J,x) = \{N\}$.

Proof of Theorem 5.

To prove the theorem, we need some preliminaries, which we state as lemmas for convenience. For every $(J,x)\varepsilon E_m$, $y\varepsilon R^n$ and $A \subset N$, define

$$d_A(y;J,x) = -\sum_{i\varepsilon A} y_i + J(x_{N-A}) - J(x)$$

and note that if $y\varepsilon G^*(J,x)$, where $G^*(J,x)$ is defined by (A.4), then $d_A(y;J,x) \leq 0$.

Lemma 1. If $E \subset E^*$, then for every $(J,x)\varepsilon E_m$, $A,B \subset N$ and $y\varepsilon G^*(J,x)$

$$d_{A \cup B}(y;J,x) + d_{A \cap B}(y;J,x) \geq d_A(y;J,x) + d_B(y;J,x)$$

Proof. Let $S_1 = A-B$, $S_2 = N-(A \cup B)$, $S_3 = B-A$ and $S_4 = A \cap B$. Then $\{S_1,\ldots S_4\}$ is a partition of N, and

$$S_1 \cup S_2 = N - B \qquad S_1 \cup S_4 = A$$

$$S_2 \cup S_3 = N - A \qquad S_3 \cup S_4 = B$$

$$S_1 \cup S_2 \cup S_3 = N - (A \cap B) \qquad S_1 \cup S_3 \cup S_4 = A \cup B$$

Then

$$d_{A \cup B}(y;J,x) + d_{A \cap B}(y;J,x) = -\sum_{i\varepsilon S_1} y_i - \sum_{i\varepsilon S_3} y_i - 2\sum_{i\varepsilon S_4} y_i$$

$$+ J(x_{S_2}) + J(x_{S_1 \cup S_2 \cup S_3}) - 2J(x)$$

$$d_A(y;J,x) + d_B(y;J,x) = -\sum_{i\varepsilon S_1} y_i - \sum_{i\varepsilon S_3} y_i - 2\sum_{i\varepsilon S_4} y_i$$

$$+ J(x_{S_2 \cup S_3}) + J(x_{S_1 \cup S_3}) - 2J(x)$$

and the desired inequality follows directly from (7.1).

Lemma 2: If $(J,x)\epsilon E_m$ satisfies (7.1) and $y\epsilon G^*(J,x)$ satisfies

$d_A(y;J,x) = d_B(y;J,x) = 0$ then $d_{A\cup B}(y;J,x) = d_{A\cap B}(y;J,x) = 0$.

Proof: This follows from lemma 1, since for $y\epsilon G^*(J,x)$ $d_S(y;J,x) \leq 0$ for all $S \subset N$.

Lemma 3: Suppose $(J,x) \epsilon E_m$ satisfies (7.1). If $y\epsilon G^*(J,x)$ and $d_N(y;J,x,) < 0$, there exists $j\epsilon N$ such that

$d_S(y;J,x) < 0$ whenever $j\epsilon S$, $S \subset N$.

Proof: Suppose to the contrary that for every $i\epsilon N$ there is a subset $S_i \subset N$ containing $\{i\}$ such that $d_{S_i}(y;J,x) = 0$. Clearly, $\bigcup_{i\epsilon N} S_i = N$. Moreover, lemma 2 implies that $d_{S_1 \cup S_2}(y;J,x) = 0$ and applying lemma 2 recursively we obtain $d_N(y;J,x) = 0$, a contradiction.

The proof of theorem 5 now consists of two parts.

1. Given an internal pricing CDCR $F^0=(f^0,s^0,r^0,q^0)$ satisfying P4-P5, a cost allocating CDCR $F^*=(f^*,s^0,r^0,q^0)$ which dominates F^0 is found by the following algorithm, which defines $f^*(J,x)$ for every $(J,x)\epsilon E_m$.

For $i=1,\ldots n$ let $g_i^0(J,x) = \max\{f_i^0(J,x), \inf H_i(J,x)\}$.

For $i=1,\ldots n$, define recursively

$\lambda_i(J,x) = \max\{d_S(g^{i-1}(J,x);J,x): \; S \subset N, \; i\epsilon S\}$

$g_j^i(J,x) = g_j^{i-1}(J,x) + \lambda_i(J,x)\delta_{ij} \; \forall \; j\epsilon N$ (δ_{ij} is Kronecker's delta.)

Let $F^*(J,x) = g^n(J,x)$.

To see that F^* is satisfactory, note first that F^0 being a CDCR implies $g^0(J,x)\epsilon G^*(J,x)$. Assume inductively that (i) $d_S(g^{i-1}(J,x);J,x) \leq 0 \; \forall \; S \subset N$ and (ii) for all $j=1,\ldots,i-1$ there is $S \subset N$ such that $j\epsilon S$ and $d_S(g^{i-1}(J,x);J,x) = 0$. For $i=1$ (i) follows from $g^0(J,x)\epsilon G^*(J,x)$ and (ii) holds trivially. By construction, if (i) and (ii) hold for i they also hold for i+1. Hence (i) and (ii) hold for i=1,...n, and in fact also for i=n+1. Then from (i) it follows that $f^*(J,x) \varepsilon G^*(J,x)$ and F^* is a CDCR, and then it follows from (ii) and lemma 3 that $d_N(f^*(J,x);J,x) = 0$ and F^* is cost allocating. Also from (i) $g_j^{i+1}(J,x) \leq g_j^i(J,x) \ \forall \ i,j$, hence if $y \varepsilon H(J,x)$ and $y_i \geq f_i^0(J,x) \ \forall \ i \varepsilon N$ then $y_i \geq f_i^*(J,x) \ \forall \ i \varepsilon N$ and thus $F^* \leq_A F^0$. Since both F^0 and F^* are price DCRs, it follows that F^* weakly dominates F^0.

2. To conclude the proof of the theorem as stated, we note that conservative price DCRs satisfying P4–P5 exist. For example, for $(J,x) \varepsilon E_m$ take some $y \varepsilon G(J,x)$ and let $f_i^0(J,x) = y_i$ for all $i \varepsilon N$.

<u>Proof of theorem 6.</u> Let F be a CDCR. By theorem 3, there is an efficient internal-pricing CDCR F^0 which weakly dominates F. By theorem 4, F^0 must satisfy P4–P5, and by theorem 5 there is a cost allocating DCR F^* which dominates F^0. Hence F^* is an efficient CDCR which dominates F.

<u>Proof of theorem 7.</u>

1. To start, we show that, under the stated conditions, if F is a cost-allocating CDCR then, for every $(J,x) \varepsilon E_m$ and $i \varepsilon N$, $f_i(J,x) \varepsilon H_i(J,x)$.

First observe that if there is $k \varepsilon N$ such that $f_k(J,x) > y_k \ \forall \ y \varepsilon H(J,x)$ then non-triviality is violated. Second, suppose that there is $k \varepsilon N$ such that $f_k(J,x) < y_k \ \forall \ y \varepsilon H(J,x)$. By E3, there is $z^{(k)} \varepsilon H(J,x)$ such that $z_k^{(k)} < J(x_{N-\{k\}})-J(x)$, and then by E1 there is $\overline{y} \varepsilon H(\overline{J},x)$ such that $\overline{y}_i \geq f_i(J,x) \ \forall \ i \varepsilon N$ (thus $W_F(J,x,\overline{y},\overline{y})$ = OK) and $\overline{y}_k = z_k^{(k)}$ so that $\overline{y}$ is not in $G(J,x)$, contradicting conservatism. With the convexity of $H(J,x)$, these observations imply $f_i(J,x) \varepsilon H_i(J,x)$ for all $i \varepsilon N$.

If $f(J,x)$ is viewed as an element in R^n, we can write in short $f(J,x) \varepsilon H(J,x)$.

2. Now let $F=(f,s,r^0,q)$ be a cost-allocating CDCR, and suppose, contrary to the theorem, that there exists a DCR F' that strongly dominates F. By theorem 3, there is an internal pricing CDCR $F^0=(f^0,s^0,r^0,q^0)$ that dominates F', and by theorem 4 $s^0(J,x) = N$ and $q^0(J,x) = \{N\} \forall (J,x)\epsilon E_m$. By transitivity, F^0 strongly dominates F, and there is some $(J,x)\epsilon E_m$ such that $V_F(J,x) \subset V_{F^0}(J,x)$ and $V_F(J,x) \neq V_{F^0}(J,x)$, where V is defined by (A.5). Let $y\epsilon V_{F^0}(J,x) - V_F(J,x)$, i.e., $y_i \geq \overset{0}{f}_i(J,x) \forall i\epsilon N$ and $y_k < f_k(J,x)$ for some $k\epsilon N$. Then $f_k(J,x) > \overset{0}{f}_k(J,x)$. But by our analysis above $f^0(J,x)\epsilon H(J,x)$ and, F^0 being a CDCR, $f^0(J,x)$ must satisfy $\sum_N \overset{0}{f}_i(J,x) \geq - J(x)$. Together with $\sum_N f_i(J,x) = - J(x)$ and $f_k(J,x) > \overset{0}{f}_k(J,x)$, this implies $\overset{0}{f}_\ell(J,x) > f_\ell(J,x)$ for some $\ell\epsilon N$. Hence $f(J,x) \epsilon V_F(J,x) - V_{F^0}(J,x)$, a contradiction of $V_F(J,x) \subset V_{F^0}(J,x)$. Thus F must be efficient.

Proof of theorem 8.

Follows immediately from theorems 6 and 7.

Proofs of observations in section VIII

1) Let $F = (f,s,r^0 q)$ be a core-DCR. Then P6 and P7 imply

$$\sum_{i\epsilon S} f_i(J,x) \geq J(x_{N-S}) - J(x_N) \forall (J,x)\epsilon E_m,\ S \subset N.$$

Suppose that $W_F(J,x,y,v) = OK$. Then $y_i \geq f_i(J,x,) \forall i\epsilon N$ and consequently

$$\sum_{i\epsilon S} y_i \geq J(x_{N-S}) - J(x_N) \forall S \subset N,$$

that is $y\epsilon G(J,x)$ and part (i) of definition 2 holds.

2) Let $F^0=(f^0,s^0,r^0,q^0)$ be a cost-allocating CDCR. Vulnerability, E1, and the convexity of H(J,x) imply (as in part 1 of the proof for theorem 7) that $f^0(J,x)\epsilon H(J,x) \forall (J,x)\epsilon E_m$. $W_{F^0}(J,x,f^0(J,x),f^0(J,x)) = OK$ and the conservatism of F^0 imply

$$\sum_{i \in S} f_i^0(J,x) \geq J(x_{N-S}) - J(x) \;\forall\, S \subset N \tag{A.6}$$

and since F^0 is cost allocating it also satisfies

$$\sum_{i \in N} f_i^0(J,x) = - J(x) \tag{A.7}$$

Substracting (A.6) from (A.7) gives P7, hence F^0 is a core-DCR.

4) Let $F^0=(f^0,s^0,r^0,q^0)$ be a core-DCR. Let $(J,x)\varepsilon E_m$ and for every $j\varepsilon N$ let $y^j \varepsilon H(J,x)$ satisfy, as in E3(a),

$$y_i^j < J(x_{N-\{i\}}) - J(x) \;\forall\, i \neq j \tag{A.8}$$

$$\sum_{i \in S} y_i^j > J(x_{N-S}) - J(x) \;\forall\, S \subset N,\; j \varepsilon S \tag{A.9}$$

To show that F^0 is non-trivial, we will establish the existence of $\lambda \varepsilon R^n$ such that $\lambda \geq 0$ $\sum \lambda_i = 1$ and

$$\sum_{j \varepsilon N} \lambda_j y^i \geq f^0(J,x). \tag{A.10}$$

Let $T = (t_{ij})$ be the n x n matrix given by

$$t_{ij} = y_i^j - f_i^0(J,x) \;\forall\, i,j \;\varepsilon\; N$$

Since F^0 is a core-DCR, (A.6) and (A.7) hold, and by (A.6) and (A.8)

$$t_{ij} < 0 \quad \forall\, i,j \varepsilon N,\;\; i \neq j \tag{A.11}$$

Also, by (A.7) and (A.9) with S = N

$$\sum_{i \varepsilon N} t_{ij} = \sum_{i \varepsilon N} y_i^j - \sum_{i \varepsilon N} f_i^0(J,x) > 0 \tag{A.12}$$

Finally note that (A.10) holds if and only if $T\lambda \geq 0$. However, (A.11) and (A.12) imply[(21)] that T^{-1} exists and $T^{-1} \geq 0$. Let $\xi \varepsilon R^n$ satisfy $\xi \geq 0$ and $T^{-1}\xi \neq 0$. Then let

$$\lambda = T^{-1}\xi \;/\; \sum_{i \varepsilon N} (T^{-1}\xi)_i$$

λ satisfies $\lambda \geq 0$, $\sum \lambda_i = 1$, and $T\lambda \geq 0$, so that $z = \sum_{j \varepsilon N} \lambda_j y^j \varepsilon H(J,x)$ and $W_{F^0}(J,x,z,z) = OK$, hence F^0 is non-trivial. ‖