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MONOPOLISTIC COMPETITION AND PREFERENCE DIVERSITY

by

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1. Introduction

In this paper we present a model of the demand for differentiated products which has as special cases two popular models used to analyze welfare and competition in monopolistically competitive markets. In one model aggregate demand for a group of products is generated by consumers distributed uniformly on a circle; the different products are available at stores located in various places on the circle; except for location all products are identical. The second model does not have such an appealing physical description; aggregate demand for a group of commodities is simply a function from the nonnegative orthant of $\mathbb{R}^N$ into itself. The physical description of the first model permits a straightforward analysis of competitive equilibrium given specifications of utility functions and costs of producing transported products. Salop (1979) provides an exemplary modern analysis. This model is attractive both because demand and costs have a definite (and easy to visualize) physical foundation and because it is relatively tractable. The amount of product variety is easily identified with the number of different stores located around the circle. One of the most compelling aspects of the model is that it permits strong conclusions about the relationship between the optimal amount of product variety and the amount which will result from the operation of a (monopolistically) competitive market. In many plausible cases competition will produce too much variety. Salop (1979) proves this for the case of linear transportation costs. Such a definite conclusion raises questions of robustness. Almost simultaneously Spence (1976) and Dixit-Stiglitz (1977) developed models of monopolistic competition which showed that the "too much variety" conclusion did not necessarily hold in general.

Standard procedure for analyzing market demand is to specify the demand functions of the individuals in the market and then to aggregate to obtain the market demand function. This is the way in which Salop (and his predecessors) analyzed competition on the circle. Spence and Dixit-Stiglitz did not do this. Instead they simply wrote down an aggregate benefit function which showed how social welfare depended on the amounts of the various products produced. It is easy to obtain market demand functions from this aggregate benefit function. Spence assumed a constant marginal utility of income so that demand functions are derivatives of the aggregate benefit function. Dixit-Stiglitz allowed for a diminishing marginal utility of income but the analysis is the same in spirit. Both Spence and Dixit-Stiglitz are silent on the origin of the aggregate benefit function. Spence says nothing and Dixit-Stiglitz (p. 298) state only that their utility function "can be regarded as representing Samuelsonian social indifference curves, or (assuming the appropriate aggregation conditions to be fulfilled) as a multiple of a representative consumer's utility."

While mute on the economic foundations of the aggregate benefit function, both Spence and Dixit-Stiglitz analyzed the economic consequences of the mathematical properties of the aggregate benefit function. Because symmetry encourages analysis, Spence and Dixit-Stiglitz paid particular attention to the case where the benefit function (and the resulting demand function) was symmetric. Note that demand functions and welfare functions are not symmetric in the model of competition on a circle. If, for example, only two of N possible stores are operating, it matters whether the two stores are next-door neighbors or whether they are as far apart from each other as possible. This violates symmetry, which means roughly that all that matters for the determination of prices is the frequency distribution of amounts which each store sells and not how these amounts are allocated to the different stores.

Spence and Dixit-Stiglitz were able to show that the conclusions reached in the model of competition on a circle were not robust, that in equilibrium there could be, depending on the form of the demand function, either too many or too few varieties produced.

The idea of starting the analysis with a particular multiproduct demand function has proved a fruitful one. Variants of Spence and Dixit-Stiglitz have been used to analyze many economic problems, including trade and technology transfer (Krugman, 1979; Feenstra and Judd, 1982), optimal tariffs (Venables, 1979), patent policy (Judd, 1985), public finance (Atkinson and Stiglitz, 1980, p. 208-217), and the incentives for merger (Deneckere and Davidson, 1985). Special cases of the model, in particular the so-called CES case where the aggregate benefit function has the form,

$$D(x_1,\ldots,x_n) = (\sum_{i=1}^{n} x_i^{\rho})^{1/\rho}$$

are tractable and lead to strong conclusions. Since this model is so attractive, it is natural to ask (as Spence and Dixit-Stiglitz did not) what lies behind it. The model's developers did not specify what distribution of tastes would give rise to the general symmetric demand function D(x). This problem motivates the recent papers by Sattinger (1984) and Perloff and Salop (1985). Each of these papers describes probability models which give rise to symmetric demand and welfare functions. The idea in each of these papers is that tastes are random. In the Perloff-Salop model the utility which an individual gets from a particular product is a random variable; the utility the same person gets from another product is also a random variable with the same distribution. These two random variables (and the random variable which is the utility from any other actual or potential product) are independently distributed. By changing the distribution of this random variable one gets different specifications of the model. Using a similar model Sattinger shows that if utility has a Pareto distribution then one obtains demand functions which look very much like those Dixit and Stiglitz obtained from the CES benefit function—the special case which has figured so prominently in applications. Like Spence and Dixit-Stiglitz, Sattinger shows that his model can lead in equilibrium to the provision of too few or too many products; Perloff and Salop do not treat these welfare issues.

Although this work does explain concretely what could give rise to a symmetric model, it does not clarify the relationship of the symmetric model to the earlier model of competition on the circle. The aim of this paper is to do precisely that. That is, we will describe a single model which has as special cases the model of competition on a circle and the symmetric models of Spence and Dixit-Stiglitz. We think our model explains why the competition on a circle model produces too many brands in competitive equilibrium while the symmetric model does not. The reason is that the symmetric model is much more competitive than the competition on the circle model; other things equal, equilibrium prices and (short run) profits are lower in the first model than in the second. Since (short run) profits are needed to cover fixed or set-up costs there are fewer stores or brands in zero profit equilibrium in the symmetric model than in the other model. Welfare analysis in the two models is different also. In the model of competition on a circle, there is a sense in which customers are quite similar. A few stores can efficiently supply most customers; the social gains from additional variety thus fall off quickly. In the symmetric model, tastes are very different; regardless of how many different varieties are marketed, some group will always benefit substantially from the introduction of a new variety. Other things equal the benefits from additional choice fall off more slowly.

A short way of describing our purpose in this paper is to say that it is to make precise this intuitive argument by giving meaning to the phrase "other things equal" in the preceding paragraph. In the next section we set out our basic model. It is a model most easily described in discrete terms where there are a finite number of goods and types of customers. This discreteness allows us in section III to use linear programming to characterize the relationship between demand and welfare in our model. In sections II and IV, we show how special cases of our model give rise to the demand functions associated with competition on a circle and with the symmetric representative consumer. While discreteness is attractive for some purposes, it makes analysis of welfare and competition clumsy. In sections V and VI we show how a limiting version of the model may be derived in which the number of potentially different brands is infinite. This limit model gives rise to exactly the same demand and welfare functions as used in the analysis of competition on the circle and in the Perloff-Salop version of the representative consumer model. These sections are quite technical and are good candidates for casual reading. In section VII we use the continuum model to show both that (for some attractive cases) symmetric models are more competitive than models of competition on the circle and that the social value of additional variety is greater in the symmetric model than in the model of competition on the circle.

II. The Discrete Model

A. Generalities

Our model is one of a market for a single good. It is a partial equilibrium model in that we consider interaction with other commodities in only the most rudimentary way--we assume, as does Spence, a constant marginal utility of income; the generalization to a diminishing marginal utility of income à la Dixit-Stiglitz is straightforward. The good sold on this market is lumpy and can be consumed only in integer units. Each consumer can consume, at most, one unit of the goods in question.

Suppose that there are T possible types (or brands) of the good: types are indexed by t running from 1 to T. It may aid the intuition to think of the market for washing machines. Each consumer must decide whether or not to purchase a washing machine; if he purchases one he must decide what brand to buy. The consumer decides what to do by computing costs and benefits of each choice and maximizing. Our partial equilibrium assumptions give this problem a particularly simple structure. Since marginal utility of income is constant, a consumer's preferences are completely described by the dollar value of consuming a particular type of washing machine. Thus, for consumer i, b_{it} is the benefit which consumer i reaps from consuming type t. If p_t is the cost of type t, he will consume s if $s = \text{argmax}(b_{it} - p_t)$ and $b_{is} - p_s \geq 0$.

The list $b_i = (b_{i1}, b_{i2}, \ldots, b_{iT})$ is a complete specification of consumer i's preferences. It has two, separable, aspects; one aspect is ordinal, the other is cardinal. The ordinal aspect of preferences is the ordering of brands. Let $\sigma_i(1)$ be i's most preferred brand, $\sigma_i(2)$ his second most preferred brand, and so on up to $\sigma_i(T)$, his least preferred brand. The list of brands in order of preference, $\sigma_i = (\sigma_i(1), \ldots, \sigma_i(T))$, is a permutation of the first T integers. The permutation σ_i has an easily interpreted inverse. Let π_i satisfy $\sigma_i(\pi_i(t)) = t$, for $t \in \{1,2,\ldots,T\}$; then $\pi_i(t)$ is the rank which customer i gives to machine t. If $\pi_i(7) = 2$, then brand 7 is i's second most favorite brand. The permutations σ_i (or π_i) comprise an ordinal description of i's preferences.

The second, cardinal, aspect of preferences concerns how satisfaction changes as consumer i moves from his most to his least preferred brand. Let $V_i(k)$ be the utility which consumer i gets from consuming his k^{th} favorite brand. Clearly $V_i(k)$ is a decreasing function of k. We will choose units so that $1 > V_i(1)$. Furthermore, the $V_i(k)$ are nonnegative as we are measuring utility in dollars. We illustrate the relationship of the two aspects of preference by writing

$$(1) \qquad b_{it} = V_i(\pi_i(t)).$$

In our model we assume that all consumers have the same kind of cardinal preferences but that their ordinal preferences differ. Everyone agrees about the value of the best, the next best, and the worst washing machine. People disagree about the identity of the best, the next best, and so on. This means that we can remove the subscript i from V in (1) and write

$$(2) \qquad b_{it} = V(\pi_i(t)).$$

For an example of a model which gives rise to preferences of the sort, consider a circle of circumference 1. Store t is located at points $(2t - 1)/2T$, $t = 1,\ldots,T$. Consumers of type i live at points $[2(i - 1)/2T]$, $i = 1,2,\ldots,T$. Stores supply (on an f.o.b. basis) washing machines of the same kind. See Figure 1. A washing machine is worth 1 (monetary) unit to a consumer when installed. Consumers pay transportation costs which depend on the distance from the store to their home measured in the clockwise direction. If ψ is an increasing function which gives delivery costs as a function of distance, and if $T = 3$, then

$$b_{11} = 1 - \psi(1/6),\ b_{12} = 1 - \psi(3/6),\ b_{13} = 1 - \psi(5/6)$$
$$b_{21} = 1 - \psi(5/6),\ b_{22} = 1 - \psi(1/6),\ b_{13} = 1 - \psi(3/6)$$
$$b_{31} = 1 - \psi(3/6),\ b_{32} = 1 - \psi(5/6),\ b_{13} = 1 - \psi(1/6)$$

This fits the model with

$$V(1) = 1 - \psi(1/6),\ V(2) = 1 - \psi(3/6) \text{ and } V(3) = 1 - \psi(5/6)$$

as the utility which consumers get from consuming their first, second, and third choices.

This model is considerably less competitive than the standard model of competition on the circle. Suppose that all stores are charging the same price. Then customer 1 will buy from store 1, customer 2 from store 2 and customer 3 from store 3. Suppose store 1 attempts to gain more customers by lowering its price. No consumer is just on the margin; a small price decrease will attract no customers. To attract customer 3 store 1 will have to lower its price enough to compensate for an increase in transportation costs from $\psi(1/6)$ to $\psi(3/6)$. To lure customer 2 away from store 2 the price decrease will have to make up for an increase in transportation costs from $\psi(1/6)$ to $\psi(5/6)$. If deliveries could be made in either direction then the model would be more competitive. But when deliveries can be made in two directions then the function $V(\cdot)$ is no longer strictly monotonic; consumer 1 is then indifferent between purchasing at store 1 and store 3.

However, this does not seem to be a particular problem. A little reflection should convince the reader that the development of our model of consumer demand in no way depends on the monotonicity of the value function.

If $V(\cdot)$ is any function mapping T into [0,1] and π is a permutation of $T = \{1,\ldots,T\}$ then $V(\pi(t))$ is the value which a consumer of type π places on brand t. Since we are free to pick $V(\cdot)$ we will, in section VI, pick $V(\cdot)$ to have a form which leads to the standard model of competition on the circle.

For this is suffices to have V(t) be an increasing function of the distance of t from the midpoint of T, $m(T) = (T + 1)/2$. Assume for simplicity that T is even. Let $V(t) = 1 - \hat{\psi}(|t - m(T)|)$ where $\hat{\psi}$ is an increasing function from $\{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{T-1}{2}\}$ to [0,1]. $\hat{\psi}$ represents delivery or transportation costs which are either real or psychological. We can further refine our model by defining transport costs in a way which does not depend on the value of T. Let

(3) $$V(t) = 1 - \psi(|\frac{t}{T} - \frac{1}{2}|)$$

where ψ is an increasing function from [0, 1/2] to 1. Again ψ is a transport cost function. In the sequel we will use this specification frequently. If $\psi(\cdot)$ is linear we say transport costs are linear; if $\psi(\cdot)$ is convex we say transport costs are convex.

As we indicated, a customer type is identified with a permutation of $T = \{1,2,\ldots,T\}$. As there are T! distinct permutations there are T! distinct types of customers.[1]

The demand side of the market is determined completely by a listing of the numbers of customers of each type. Let μ_i be the fraction of consumers in the market with preferences of type i. Then $\mu = (\mu_1,\ldots,\mu_{T!})$ specifies the

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[1]This is somewhat of an oversimplification. When $V(\cdot)$ has the form of equation (3) some permutations give rise to identical demand functions. This point is of no importance in the sequel and we shall ignore it.

structure of demand. More abstractly, demand is represented by a measure on the set of permutations of T.

B. Examples

1. The Symmetric Case

The perfectly symmetric model of Spence and Dixit-Stiglitz results when all preferences are equally likely, that is when

$$\mu_i = T!^{-1}, \quad i = 1,\dots,T! \tag{4}$$

In section IV below we show formally that (4) gives rise to demand functions which are symmetric.

2. The Case of Competition on the Circle

Now we examine demand structures, specifications of μ, which lead to the model of competition on the circle. It is best to start with an example. Suppose that T = 6 and let $\rho = (1,2,3,4,5,6)$ be the identity permutation. According to (2) this permutation represents the preferences of someone who lives halfway between stores 6 and 1 when stores are arranged on the circle as in Figure 2. Let ζ denote the shift operator so that $\zeta(\rho) = (6,1,2,3,4,5)$. Then $\zeta(\rho)$ represents the preferences of someone who lives between stores 5 and 6 on the same circle. Continuing $\zeta^2(\rho) = (5,6,1,2,3,4)$ is the preferences of the person living between 4 and 5; $\zeta^3(\rho)$ lives betwen 3 and 4, $\zeta^4(\rho)$ between 2 and 3, and $\zeta^5(\rho)$ between 1 and 2. With $\zeta^6(\rho) = \rho$ we have literally come full circle and are back home again between 6 and 1. The permutations $\zeta^i(\rho)$, $i = 1,2,\dots,6$ constitute a rotation group. If all elements of the rotation group contribute equally to demand so that

(5) $$\mu[\zeta^{i}(\rho)] = T^{-1}, \ i = 1,\ldots,T$$

the demand is as in the standard model of competition on the circle. Notice that only some permutations get positive weight in this scheme. The preference of someone with permutation (6,2,5,3,1,4) cannot be represented in terms of a location on the circle of Figure 2. In the sequel we define competition on the circle as preferences which can be represented in the form of equation (5) for some permutation ρ. We will also speak of the set of permutations $\zeta^{i}(\rho)$ as the rotation group. For most purposes the identity of the permutation ρ which generates the rotation group is immaterial and we will take ρ to be the identity.

This completes the development of the demand side of the model. Supply is very simple; it is a list of the amount of each type of brand available. Thus supply is represented by a vector $x \in \mathbb{R}^T$.

III. Equilibrium and Welfare

With this specification of the market and the parameters which determine supply and demand, we can ask (i) how supplies should be distributed to maximize welfare; (ii) how a competitive process would distribute supplies; and (iii) what prices are determined in the competitive equilibrium. As is often the case, considering a particular maximization problem (and its dual) provides answers to all three questions and shows once again that competition will allocate resources efficiently.

The problem of allocating supplies to customers so as to maximize consumer welfare is a straightforward linear programming problem. With the assumptions we have made about utility aggregate welfare is the sum of individual utilities. Thus, to allocate supplies to maximize welfare, it is only necessary to solve the following linear programming problem: find α_{it} to maximize:

$$(W) \qquad \sum_{i=1}^{T!} \sum_{t=1}^{T} \alpha_{it} V(\pi_i(t))$$

subject to

$$\alpha_{it} \geq 0, \quad i = 1,\dots,T!; \; t = 1,\dots,T.$$

$$\sum_{t=1}^{T} \alpha_{it} \leq \mu_i, \quad i = 1,\dots,T!$$

$$\sum_{i=1}^{T!} \alpha_{it} \leq x_t, \quad t = 1,\dots,T.$$

In this problem, α_{it} is the amount of good t allocated to people of type i; α_{it} is necessarily nonnegative. The constraint $\sum_{t=1}^{T} \alpha_{it} \leq x_t$ guarantees that no more of a type of good is allocated than is available. An allocation which satisfies the constraints of (W) is a feasible allocation.

Competition will solve (W). To see this, consider, as usual, the dual to (W): find y_t, p_t to minimize

$$(\hat{M}) \qquad \sum_{i=1}^{T!} y_i \mu_i + \sum_{t=1}^{T} p_t x_t$$

subject to

$$y_i + p_t \geq V(\pi_i(t)), \quad i = 1,\dots,T!; \quad t = 1,\dots,T.$$

$$y_i \geq 0, \quad i = 1,\dots,T!$$

$$p_t \geq 0, \quad t = 1,\ldots,T$$

The dual has the usual interpretation as the problem of finding nonnegative prices for resources (consumers and goods) which minimizes the total value of resources subject to the constraint that the revenue from sale of resources exceeds the returns obtained by using the resources to produce utility (which is measured in monetary units). A more interesting interpretation of the dual is available if we introduce the functions

$$f_i(p) = \max\,(0, \max_t\,(V(\pi_i(t)) - p_t)).$$

Thus, $f_i(p)$ measures the surplus consumer i gets if goods are sold at prices $p = (p_1,\ldots,p_T)$. At these prices consumer i will buy good t only if $f_i(p) \geq 0$ and if $V(\pi_i(t)) - p_t = f_i(p)$.

Consider now the problem: find $p_t \geq 0$ to minimize

$$\text{(M)} \qquad \sum_{i=1}^{T!} f_i(p)\mu_i + \sum_{t=1}^{T} x_t p_t$$

We claim that (M) and $(\hat{M})$ are equivalent.

Proposition 1: The solutions to $(\hat{M})$ and (M) have the same value. If (p^*,y^*) solves $(\hat{M})$, then p^* solves (M). Furthermore, if $\mu_i^* > 0$, then $y_i^* = f_i(p^*)$. Conversely, if p^* solves (M), then $(p^*,f(p^*))$ solves $\hat{M}$.

Proof: Since $(\hat{M})$ is feasible and bounded, it has a solution (p^*,y^*). Since y_i^* is feasible, $y_i^* \geq \max_t\,(V(\pi_i(t)) - p_t^*) = f_i(p^*) \geq 0$. If $\mu_i > 0$, then the solution to $(\hat{M})$ makes y_i as small as possible subject to this constraint.

Thus, $\sum_i \mu_i(f_i(p^*) - y_i^*) = 0$ and

$$\sum_{i=1}^{T!} f_i(p^*)\mu_i + \sum_{t=1}^{T} p_t^* x_t = \sum_{i=1}^{T!} y_i^*\mu_i + \sum_{t=1}^{T} p_t^* x_t.$$

If p^* is not a solution to (M) then there exists $\tilde{p} \geq 0$ such that

$$\sum_{i=1}^{T!} f_i(\tilde{p})\mu_i + \sum_{t=1}^{T} \tilde{p}_t x_t < \sum_{i=1}^{T!} y_i^*\mu_i + \sum_{t=1}^{T} p_t^* x_t.$$

Since $(\tilde{p}, f(\tilde{p}))$ is feasible for $(\hat{M})$, this contradicts the optimality of (p^*, y^*). The converse is easily proven, and is left as an exercise to the reader. []

A competitive equilibrium for the market with preferences distributed according to μ and stocks x is a price vector $p^* \geq 0$ and a feasible allocation α_{ij}^* such that

(1) $$\{t \mid \alpha_{it}^* > 0\} \subset \{t \mid V(\pi_i(t)) - p_t^* = f_i(p^*)\};$$

(2) $$p_t^* > 0 \Rightarrow \sum_i \alpha_{it}^* = x_t$$

and

(3) $$f_i(p^*) > 0 \Rightarrow \sum_t \alpha_{it}^* = \mu_i.$$

This definition is the natural one. Condition (1) states that if consumer i buys good t, good t must maximize his utility at prices p^*. The next two constraints are complementary slackness constraints. (2) states that if good t has a positive price, then it must be entirely allocated, while (3) says that if at prices p^* consumers of type i get positive consumer surplus, then all consumers of type i must consume one unit of some good. Together with (1), this constraint implies that the allocation is consistent with consumer preference maximization at prices p^*. Straightforward linear programming arguments establish that an equilibrium exists.

Proposition 2: (α^*, p^*) is a competitive equilibrium if and only if α^* is a solution to (W) and p^* a solution to (M).

Proof: Suppose (α^*) is a solution to (W) and (p^*) a solution to (M). Then $\alpha^*_{it} > 0$ implies $\mu_i > 0$ and $f_i(p^*) = y^*$; complementary slackness implies $V(\pi_i(t)) - p^*_t = y^*_i = f_i(p^*)$. Also complementary slackness implies that (2) and (3) hold.

Conversely, suppose (α^*, p^*) is a competitive equilibrium. Let $y^*_i = f_i(p^*)$, then α^* is feasible for (W) and p^* is feasible for (M). It will suffice to show that

$$\sum_i \mu_i f_i(p^*) + \sum_t x_t p^*_t = \sum_i \sum_t \alpha^*_{it} V(\pi_i(t)).$$

Let $I = \{i: f_i(p^*) > 0\}$ and $J = \{t: p_t > 0\}$. Then (1), (2) and (3) imply

$$\sum_i \mu_i f_i(p^*) + \sum_t x_t p^*_t = \sum_{i \in I} \mu_i f_i(p^*) + \sum_{j \in J} x_t p^*_t$$

$$= \sum_{i \in I} \sum_{t=1}^{T} \alpha^*_{it}(V(\pi_i(t)) - p^*_t) + \sum_{t \in J} \sum_{i=1}^{T!} \alpha^*_{it} p^*_t$$

$$= \sum_{i=1}^{T!} \sum_{t=1}^{T} \alpha^*_{it}(V(\pi_i(t)) - p^*_t) + \sum_{i=1}^{T!} \sum_{t=1}^{T} \alpha^*_{it} p^*_t = \sum_{i=1}^{T!} \sum_{t=1}^{T} \alpha^*_{it} V(\pi_i(t)). \qquad []$$

This result establishes that in our model social welfare is a function of aggregate supplies of goods, x, and that market clearing (inverse) demands exist and are "derivatives" of the social welfare function. To see this, suppose that the distribution of preferences, μ, is fixed and define W(x) to be the value of the solution to the problem (W) when supplies of goods are given by $x \in \mathbb{R}^T$. Cleary W(x) measures social welfare as a function of supplies x. If $p_t(x)$ is the dual variable corresponding to the constraint x_t in (W), then $p(x) = (p_1(x),\ldots,p_T(x))$ satisfies $p(x) = \partial W(x)$, where ∂W is the subgradient of $W(\cdot)$. Proposition 2 establishes that $p_t(x)$ are the equilibrium inverse demands. (M) shows that W(x) can be decomposed as the sum of consumer surplus and producer surplus.

IV. Symmetry

For a given transport function $\psi(\cdot)$ different distributions of taste, i.e., different measures μ, give rise to different welfare and demand functions. In section II.A above we stated that the symmetric model arose when all tastes were equally likely. In this section we justify this assertion.

The essence of the symmetric model is that welfare and demand depend only on the distribution of supplies. It does not matter which firm produces which amount. For example, to determine aggregate welfare it is enough to know that one firm produces two units, another five, and all others nothing. With this in mind we formally define symmetric functions as follows: let $y = (y_1,y_2,\ldots,y_N) \in \mathbb{R}^N$ and let σ be a permutation of the integers $1,2,\ldots,N$. Then let

$$y_\sigma = (y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(N)})$$

We will say a function $u\colon A \subset \mathbb{R}^N \to \mathbb{R}$ is symmetric if $u(x) = u(x_\sigma)$ for all x

$\in$ A and all permutations σ. A correspondence $D: A \subset \mathbb{R}^N \to B(\mathbb{R})^N$ is symmetric if $D(x_\sigma) = (D(x))_\sigma$, where $B(\mathbb{R}^N)$ are the Borel subsets of $\mathbb{R}^N$. Note that if $u: A \subset \mathbb{R}^N \to \mathbb{R}$ is symmetric and if $D: A \subset \mathbb{R}^N \to \mathbb{R}^N$ is the gradient of $u(\cdot)$, then $D(\cdot)$ is symmetric. We say the measure μ is symmetric if $\mu_i = 1/T!$, $i = 1,\ldots,T!$. Under symmetric measures all tastes are equally likely.

Proposition 3: If μ is symmetric, then surplus $W(x)$ is a symmetric function and (inverse) demand $p(x)$ is a symmetric correspondence.

Proof: For σ any permutation of $T = \{1,\ldots,T\}$ and x any point in $\mathbb{R}^T$ we have to prove that

(9) $$W(x_\sigma) = W(x) \text{ and } p_\sigma(x_\sigma) = p(x).$$

Let $p^* = p(x)$. We have shown that p^*, $f(p^*)$ are solutions to the dual problem $(\hat{M}_0)$: find $p_t, y_i \geqslant 0$ to minimize

$$(\hat{M}_0) \quad \sum_{i=1}^{T!} (T!)^{-1} y_i + \sum_{t=1}^{T} x_t p_t$$

subject to

$$p_t + y_i \geqslant V(\pi_i(t))$$

$$p_t \geqslant 0, \; y_i \geqslant 0$$

If supplies are changed to x_σ the corresponding dual changes to: find y_i and $p_t \geqslant 0$ to minimize

$(\hat{M}_\sigma)$ $$\sum_{i=1}^{T!} (T!)^{-1} y_i + \sum_{t=1}^{T} x_{\sigma(t)} p_t$$

subject to

$$p_t + y_i \geq V(\pi_i(t))$$

$$p_t \geq 0, \ y_i \geq 0$$

The constraints of $(\hat{M}_0)$ and $(\hat{M}_\sigma)$ are the same. The problems differ only in their objective function. Note that from the definition of $f(p^*)$, $p_\sigma^*, f(p_\sigma^*)$ is feasible for $(\hat{M}_\sigma)$. Furthermore, there is a one-to-one correspondence between the elements of $f(p^*)$ and $f(p_\sigma^*)$. Each consumer i corresponds to a permutation of T. Let ρ be the implied permutation. Then $f_\rho(p^*)$ is the consumer surplus derived by an individual when his first choice costs $p_{\rho(1)}^*$, his second choice $p_{\rho(2)}^*$, etc. Let τ be another permutation of T. Then $f_\tau(p_\rho^*)$ is the consumer surplus derived by an individual when his first choice costs $p_{\tau(\rho(1))}^*$, his second choice $p_{\tau(\rho(2))}^*$, etc. These amounts will be the same when $\sigma(t) = \tau(\rho(t))$, an equation which determines τ uniquely as

(1) $$\tau = \sigma\rho^{-1}$$

Since (1) defines a one-to-one correspondence between permutations, there is also a one-to-one correspondence between the elements of the two T! vectors $f(p^*)$ and $f(p_\rho^*)$. In particular

(2) $$\sum_{i=1}^{T!} (T!)^{-1} f_i(p^*) = \sum_{i=1}^{T!} (T!)^{-1} f_i(p_\rho^*).$$

The facts (1) and (2) are all we need to prove the theorem. $p_\sigma^*, f(p_\sigma^*)$ are feasible for $(\hat{M}_\sigma)$. Since $\sum p^*_{\sigma(t)} x_{\sigma(t)} = \sum p^*_t x_t$, it follows that $p^*_\sigma, f(p^*_\sigma)$ gives the same value to the objective function of (M_σ) as $p^*, f(p^*)$ does to the objective function of $(\hat{M}_0)$. If $p^*_\sigma, f(p^*_\sigma)$ do not solve $(\hat{M}_\sigma)$, then there is a vector q such that q,f(q) is feasible for $(\hat{M}_\sigma)$ and gives a lower value to the objective function. But then the arguments just made show that if $\pi = \rho^{-1}$, then $q_\pi, f(q_\pi)$ are feasible for $(\hat{M}_0)$ and give a lower value to the objective function than $p^*, f(p^*)$. This contradiction establishes that $p^*_\sigma = p(x_\sigma)$ and $W(x_\sigma) = W(x)$. []

V. Limiting Results

The linear programs (W) and (M) do not yield inverse demands $P_t(x,\mu)$ or aggregate benefit functions $W(x,\mu)$ which lend themselves easily to computational analyses. In particular, $P_t(x,\mu)$ is a piecewise constant convex valued correspondence. Correspondingly, the benefit function $W(x,\mu)$ is concave and differentiable almost everywhere. In this section, we show that when the number of produceable brands is limited to a (possibly large but) finite number, and when the number of conceivable brands is very large (in some precise fashion), demand functions for both the symmetric model and the model of competition on the circle become tractable. The set of conceivable brands should be interpreted here as the largest set of products any individual consumer could possibly imagine (such as the spectrum of colors cars may come in). Due to the presence of fixed costs, however, only a finite number of these brands may ever be produced. We refer to the latter set as the set of produceable brands. As the construction below shows, the limiting demand functions are not sensitive to the initial specification of the set of produceable brands. Thus, if P and P$'$ are two specifications of produceable sets such that $P' \supset P$, the limiting demand functions corresponding to P', when restricted to the set P, will be identical to the limiting demand functions obtained from P. In other words, there is an internally consistent way of letting the set of produceable brands increase due to, e.g., a decrease in the fixed cost of setting up a plant.

Let there by N produceable brands, positioned equidistantly in the half open unit interval I:

$$P = \{j/N,\ j = 1,\ldots N\} \subset I = (0,1]$$

At each stage k ($k = 0,1,2,\ldots$), the number of conceivable brands will be doubled by placing the new brands midway between the pre-existing conceivable brands. Thus:

$$T_k = \{\text{conceivable brands at stage } k\} = \{j/T_k;\ j = 1,\ldots,T_k\}$$

where $T_k = 2^kN$. Both the set of produceable brands P, and the set of conceivable brands at stage k, T_k, are embedded in the half-open interval (0,1] so that we can accommodate the model of spatial competition on the circle. This is accomplished by identifying the point 0 with the point 1, as if by bending the unit interval around itself and glueing the endpoints together.

Since at each stage k the number of conceivable brands is doubled, the number of possible preference patterns increases from $(2^{k-1}N)!$ at stage $(k - 1)$ to $(2^kN)!$ at stage k. One way to visualize what is going on is to imagine that each preference pattern at stage $(k - 1)$ is replaced, at stage k, by $I_k = (2^kN)!/(2^{k-1}N)!$ preference patterns, each of which is equally likely. In other words, the consumer at stage (k - 1) can be thought of as not distinguishing between the conceivable brands at stage (k - 1) and the neighboring brands that are introduced at stage k. He is then split up into I_k consumers who do distinguish between all brands at stage k, and whose preference patterns are, in some sense, close to his. Formally, let

$$\Pi_k = \{\text{permutations of } T_k\}$$

The cardinality of Π_k is thus $T_k!$ Corresponding to Π_k, we also have Ψ_k, the set of preferences at stage k:

$$\Psi_k = \{V(\pi);\ \pi \in \Pi_k\}$$

where $V: I \to [0,1]$ is the valuation function of the "representative" consumer.

We can extend the permutation $\pi \in \Pi_k$ to the whole interval as follows:

$$\pi^*(\alpha) = \pi(j_k(\alpha)) + (\alpha - j_k(\alpha)), \qquad \alpha \in (j_k(\alpha) - \frac{1}{T_k}, j_k(\alpha)]$$

where

$$j_k(\alpha) = \underset{\{j \in T_k:\ j > \alpha\}}{\operatorname{argmin}} |j - \alpha|.$$

π^* is a member of Π, the set of measure preserving bijections of I. Π can be thought of as the set of permutations of I. Figure 3 illustrates this extension process for the case k = 0, N = 4. Let $\Pi_k^* = \{\pi^*: \pi \in \Pi_k\}$ and $\Psi_k^* = \{V(\pi^*);\ \pi^* \in \Pi_k^*\}$. The purpose of this construction is to let Π_k^* and Ψ_k^* live in the infinite space of permutations and preferences that will be constructed in section VI. Observe that $\Pi_k^* \subset \Pi_{k+1}^*$ and $\Psi_k^* \subset \Psi_{k+1}^*$ for all $k \geq 0$.

Corresponding to Π_k^* and Ψ_k^*, we can define stage-k "symmetric" measures μ_k and "distance" measures ν_k as follows:

$$\mu_k = \frac{1}{T_k!} \sum_{\pi^* \in \Pi_k^*} p_{\pi^*}$$

where p_{π^*} is the point measure at π^* (i.e., $p_{\pi^*}(E) = 0$ if $\pi^* \notin E$ and $p_{\pi^*}(E) = 1$ otherwise). If we define $R_k^* \subset \Pi_k^*$ as the stage k rotation group, we can express ν_k as

$$\nu_k = \frac{1}{T_k} \sum_{\pi^* \in R_k^*} p_{\pi^*}$$

Thus, μ_k gives equal weight to all preference patterns in Ψ_k^*, whereas ν_k only gives equal weight to those preference patterns that correspond to rotations of the representative preference pattern V.

The plan for the remainder of this section is to try and describe the limiting joint distribution of the valuations for any subsets of goods in P. This will allow us to describe the limiting direct demand functions explicitly. We also show that the demand correspondences of the finite models converge uniformly to the limiting demand functions. This implies that the solutions to the programs (W) and (M) can, for k large enough, be approximated arbitrarily closely by their limiting counterparts.

A. <u>The Symmetric Case</u>

For each k, the probability space (Π_k^*, μ_k) defines two stochastic processes on I:

$$\{\pi_k^*(\alpha);\ \alpha \in I\} \text{ and } \{V(\pi_k^*(\alpha));\ \alpha \in I\}$$

The random variable $\tilde{\alpha}_k: \pi_k^* \to \pi_k^*(\alpha)$ describes the distribution of values that α is permuted to at stage k. Similarly, the random variable $V(\tilde{\alpha}_k)$ describes the distribution of valuations for good α. In appendix I (Lemma 1), we prove that $\tilde{\alpha}_k \Rightarrow \tilde{\alpha}$ (the arrow "=>" denotes weak convergence), where $\tilde{\alpha}$ is uniform on I. Furthermore, for any finite collection $\{\alpha, \beta, \gamma, \ldots\}$ contained in P (or, for that matter, I), the joint distribution of $\{\tilde{\alpha}_k, \tilde{\beta}_k, \tilde{\gamma}_k, \ldots\}$ converges weakly to the joint distribution of i.i.d. uniformly distributed random variables.

Hence, the distribution of valuations $\{V(\pi_k^*(\alpha));\ \alpha \in P\}$ for any finite subset P of I converges weakly to a collection of i.i.d. random variables, each with measure λV^{-1}. The measure λV^{-1} is defined, for every Borel set B, as $\lambda V^{-1}(B) = \lambda(V^{-1}(B))$, where λ is the Lebesgue measure on I. Thus, when V is strictly decreasing, the distribution of this random variable can be expressed as: $\text{Prob}[\tilde{V} \leq v] = 1 - V^{-1}(v)$. When V is smooth with first derivative bounded away from zero, $\tilde{V}$ has a continuous density given by $1/[V'(V^{-1}(v))]$.

B. Competition on the Circle

Again, the probability spaces (Π_k^*, ν_k) define two stochastic processes on I: $\{\pi_k^*(\alpha);\ \alpha \in I\}$ and $\{V(\pi_k^*(\alpha));\ \alpha \in I\}$. Lemmas 1 and 2 of Appendix I apply, and for all $\alpha \in I$: $\tilde{\alpha}_k \Rightarrow \tilde{\alpha}$, where $\tilde{\alpha}$ is uniform on I. However, for any finite subcollection P of I, $\{\tilde{\alpha},\ \alpha \in P\}$ are no longer independent for any k, nor in the limit. We describe the limiting distribution below (proof omitted).

For any $\alpha \in I$, define $\phi_\alpha: I \to I$ by $\phi_\alpha(x) = (x - \alpha) \bmod 1$. Let $\{B_\alpha;\ \alpha \in P\}$ be sets of the form $[a,b)$, where $a \leq b \in I$. The system of all sets of the form $\{x \in \mathbb{R}^N:\ x_1 \in B_{\alpha_1}, \ldots, x_N \in B_{\alpha_N}\}$ (called cylinder sets), form a semi-ring on I^N, where N is the cardinality of P. In order to describe the limiting distribution, it is sufficient to describe it on this semi-ring (see, e.g., Aliprantis and Burkinshaw, Chapter 3). Letting $P = \{\alpha_1,\ldots,\alpha_N\}$, we have:

$$\mathrm{Prob}[\tilde{\alpha}_1 \in B_{\alpha_1},\ldots,\tilde{\alpha}_N \in B_{\alpha_N}] = \lambda(\bigcap_{i=1}^{N} \phi_{\alpha_i}(B_{\alpha_i}))$$

where λ is the Lebesgue measure on I. Figure 4 illustates this for the case $N = 2$. The limiting joint distribution of valuations can now be described. Define V_N: $I^N \to I^N$ by $V_N(x_1,\ldots,x_N) = (V(x_1),\ldots,V(x_N))$. Then if we let ν be the measure constructed above, the distribution of $(\tilde{V}_{\alpha_1},\ldots,\tilde{V}_{\alpha_N})$ is νV_N^{-1}, i.e., for all $H \subset I^N$: $\nu V_N^{-1}(H) = \nu(V_N^{-1}(H))$.

Let us denote the limiting distribution of valuations for the set of produced goods ω^s in the symmetric case and ω^d in the model of competition on the circle. We can now easily describe limiting demands. Let $A_i(p_1,\ldots,p_N)$ be the following subset of $\mathbb{R}^N$:

$$A_i(p_1,\ldots,p_N) = \{(V_1,\ldots,V_N): V_i \geq p_i, V_j \leq V_i + p_j - p_i, \forall j \neq i\}$$

Limiting demand at prices $(p_1,\ldots,p_N)$ is then, for the symmetric case,

$$\omega^s(A_i(p_1,\ldots,p_N)) = \int_{p_i}^{1} \prod_{j\neq i} G(v + p_j - p_i)dG(v)$$

with $G(v) = \mathrm{Prob}[\tilde{V} \leq v]$. In other words, the limiting demand function is continuous, and coincides with the demand function advocated by Perloff and Salop (1985). In order that limiting demand be well-behaved (i.e., continuous) for the model of competition on the circle, it is well-known that the transportation cost function must be strictly convex (see, e.g., D'Aspremont et al., 1979). We will make this assumption, which amounts to strict concavity of V, henceforth. For most of what follows in this section, however, the assumption is not necessary. It is only made to facilitate the statement of our results. Limiting demand for the model of competition on the circle is then:

$$d_i(p_1,\ldots,p_N) = \omega^d(A_i(p_1,\ldots,p_N))$$

Observe that the nature of this demand function depends on the particular set of produced goods P chosen, as ω^d does.

At this stage, we would like to establish a link between the finite demands and the finite distributions on the one hand, and between the finite and limiting demands on the other hand. Because the finite demands are correspondences rather than functions, the statement of this relationship is somewhat cumbersome. First, we show that the stage k demand functions are continuous almost everywhere. Then we argue that, excluding a set of Lebesgue measure zero, finite demands converge pointwise to limiting demands. Finally, we sharpen the result and prove that the finite-stage demand correspondences converge uniformly to the limiting demand functions. This means that for large enough k the limiting demand functions are uniformly close to the stage-k demand correspondences.

Define $G = V(\bigcup_{k=0}^{\infty} T_k) = \{V(x);\ x \in \bigcup_{t=0}^{\infty} T_k\}$ and $H = G - G = \{w\colon w = v_1 - v_2;\ v_1,v_2 \in G\}$. Clearly, both G and H are countable. Define $\Lambda^c \subset I^N$ as:

$$\Lambda^c = \{p \in I^N\colon p_i \in G \text{ for some } i, \text{ or } p_i - p_j \in H \text{ for some } i \neq j;\ i,j = 1,\ldots,N\}$$

Lemma 4 of Appendix I shows that Λ^c has Lebesgue measure zero. We then have the following.

Proposition 4: The stage-k demand correspondence $d^k(\cdot)$ is a continuous function on Λ for all k (both for the symmetric model and the model of competition on the circle).

Proof: On Λ, every consumer type $V \in \Psi_k^*$, where $\Psi_k^* = V(\pi_k^*)$, has strict preference of one good over all others, or buys nothing. In other words, each consumer's demand $d_V(p_1,\ldots,p_N)$ is a singleton for $(p_1,\ldots,p_N) \in \Lambda$. Thus aggregate demand, $d^k(p_1,\ldots,p_N) = \sum_{V \in \Psi_k^*} \omega_k(V) d_V(p_1,\ldots,p_N)$, where ω_k is the stage k distribution of valuations, is a singleton for $(p_1,\ldots p_N) \in \Lambda$. Since $d^k(\cdot)$ is upper-hemicontinuous (it is a nonempty, closed convex-valued correspondence), and $d^k(\cdot)$ is a singleton on Λ, it is a continuous function on Λ. []

In fact, on Λ, we can describe $d^k(\cdot)$ explicitly:

$$d_i^k(p_1,\ldots,p_N) = \omega_k(A_i(p_1,\ldots,p_N))$$

where ω_k is either ω_k^s or ω_k^d. Since $\omega_k \Rightarrow \omega$, and since $A_i(p_1,\ldots,p_N)$ is an ω-continuity set for each $(p_1,\ldots,p_N)$, we have:

$$d_i^k(p_1,\ldots,p_N) \to d^k(p_1,\ldots,p_N)$$

for every $(p_1,\ldots p_N) \in \Lambda$.

Theorem 4 of the Appendix proves, for the symmetric model, the following stronger result: on Λ, $d^k(\cdot) \to d(\cdot)$ uniformly. The same result is valid for the model of competition on the circle, but the proof is omitted for the sake of brevity. We can now prove the main result of this section.

Proposition 5: $d^k(\cdot) \to d(\cdot)$ uniformly on I^N.

Proof: We already know the statement is true on $\Lambda \subset I_N$. In other words, $\forall \varepsilon > 0 \;\; \exists k(\varepsilon): \forall k \geq k(\varepsilon): |d^k(z) - d(z)| < \varepsilon$ for all $z \in \Lambda$. The theorem will be proved when we can substitute I^N for Λ in the above statement. Thus, let $k \geq k(\varepsilon)$ and $x \in d^k(z)$ for some $z \in \Lambda^c$. Because $d^k(\cdot)$ is a closed, convex valued correspondence, $\exists\, z_m, z_m' \in \Lambda$: $z_m, z_m' \to z$ and $d^k(z_m) \leq x \leq d^k(z_m')$. We can now make the following estimate:

$$|x - d(z)| \leq \max\{|d^k(z_m) - d(z)|,\; |d^k(z_m') - d(z)|\}$$

for all m. But, $|d^k(z_m) - d(z)| \leq |d^k(z_m) - d(z_m)| + |d(z_m) - d(z)|$, and similarly for z_m'. The first term on the right side of the inequality is less than $\varepsilon/2$, by uniform convergence of $d^k(\cdot)$ to $d(\cdot)$ on Λ; the second term goes to zero as $m \to \infty$ by continuity of $d(\cdot)$. Thus, $|x - d(z)| \leq \varepsilon$. []

One might wonder whether the limiting operations we carried out above might be extended to other measures besides the rotation group measure and the symmetric measure. One simple extension is this. For any stage k, the rotation group measure is an extreme point of the unit simplex of measures in $\mathbb{R}^{n(k)}$, with $n(k) = (2^k N)!$. It is also easily verified that the symmetric measure is the center of gravity of this set of measures. One might then want to think about convex combinations of the rotation group measure and the symmetric measure, i.e., imagine a world which is populated by individuals a fraction λ of which looks like the individuals that make up the model of competition on the circle, and a fraction $(1 - \lambda)$ of which looks like the individuals making up the symmetric model. Denote this measure $(1 - \lambda)\mu_k + \lambda\nu_k$ by τ_k. If P is the set of produced goods, the projection of

τ_k onto P will just be $(1 - \lambda)\omega_k^s + \lambda\omega_k^d$, which converges weakly to $(1 - \lambda)\omega^s + \lambda\omega^d$. In other words, the limiting demand function will just be a convex combination of the limiting demand functions in the symmetric model and the model of competition on the circle.

VI. A Continuum Version of the Model

Imagine a world with a continuum of conceivable goods: I. Consumers have preferences which are arbitrarily scrambled up versions of a representative preference pattern $V(\cdot)$ on I. In other words, the set of allowable preferences is $\Psi' = \{V(f);\ f \in X\}$, where $X = I^I = \{f\colon I \to I\}$. We will now show that it is possible to define a measurable structure Σ on X, and measures μ and ν on (X,Σ) such that $\mu_k \Rightarrow \mu$ and $\nu_k \Rightarrow \nu$. This will imply that for any finite subset $P \subset I$, $\mu_{[P]} = \omega^S_{(P)}V_N$ and $\nu_{[P]} = \omega^d_{(P)}V_N$, where $\mu_{[P]}$ denotes the projection of μ onto P. We will thus have constructed a limiting world which, for any finite subset P of I, has the same distribution of valuations over P as the limiting distributions derived in section V.

First, we endow X with the product topology Ξ. Observe that this topology is not metrizable (Munkres, p. 131). Let $C(X) = \{\phi\colon X \to \mathbb{R},\ \phi$ continuous$\}$ and $\Delta = \{U = \phi^{-1}(0),\ 0$ open in $\mathbb{R},\ \phi \in C(X)\}$. Elements of Δ are called open Baire sets. Finally, let Σ be the σ-algebra generated by Δ. Members of Σ are called Baire sets. The reason for introducing the Baire σ-algebra is that this is the minimal σ-algebra with respect to which weak convergence can be defined.

Let S be the system of all cylinder sets on X, of the form:

$$A = \{f \in X\colon f(t_1) \in B_1, \ldots, f(t_n) \in B_n\}$$

where $t_1, \ldots, t_n$ is a finite subset of I, and each B_i is of the form $[a,b)$ with

$a \leqslant b$ in I. S is a semiring (Prokhorov and Rozanov, p. 79). Furthermore, as the σ-algebra generated by S coincides with Σ (Ash, pp. 190-195), it suffices to define μ and ν on S only. Let A be the cylinder set described above. Then we take

$$\nu(A) = \lambda\left(\bigcap_{i=1}^{n} \phi_{t_i}(B_i)\right)$$

and

$$\mu(A) = \prod_{i=1}^{n} \lambda(B_i)$$

It is easily verified that μ and ν are indeed measures on S. We can extend μ (and ν) from (X,S) to (X,Σ) in the standard fashion:

$$\mu(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\} \text{ is a sequence in S with } A \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

where A is an arbitrary element of Σ. The measure μ constructed above is known as the product measure. We are now ready to state the following:

Definition: Let μ_n, μ be measures on (X,Σ). Then μ_n converges weakly to μ, denoted by $\mu_n \Rightarrow \mu$ if and only if $\int_X f d\mu_n \rightarrow \int_X d f\mu$ for all $f \in C(X)$.

Alternative characterizations of weak convergence of measures on arbitrary topological spaces can be found in Varadarajan (1965, in particular Theorem 2, p. 182). With this definition we can state:

Proposition 6: Let μ_k, μ and ν_k, ν be defined as above. Then $\mu_k \Rightarrow \mu$ and $\nu_k \Rightarrow \nu$.

Proof: This follows directly from Theorem 8 of Varadarajan (1965, p. 186), and the fact that we proved earlier that for any finite subset P of I,

$\mu_{k[P]} \Rightarrow \mu_{[P]}$ and $\nu_{k[P]} \Rightarrow \nu_{[P]}$. □

Let us recapitulate for a moment. We just constructed a world in which the distribution of valuations coincides, for any finite subset P of I, with the limiting distribution of valuations over P derived in section V. In particular, this implies that the demand functions for "limiting world" coincide with the limiting demand functions calculated in section V. While this justifies the use of the name "limiting world," there is a problem with the approach developed so far: the space of preferences Ψ' is too large. All functions V(f) are allowed: f may be measurable, nonmeasurable, etc. Our limiting framework is thus much different from that of the finite models, in which all preferences were permutations of a representative preference pattern. We will now show that this "deficiency" can be remedied. In other words, we show that our two models could just as well have been defined directly on a continuum of (conceivable) goods.

Let $\Pi \subset X$ be the set of (Lebesgue) measure preserving bijections on I. The members of Π can be thought of as the "permutations" of I. Let Ξ_π be the subspace topology that Π inherits from (X,Ξ). Furthermore, let $C(\Pi) = \{f: \Pi \to \mathbb{R},\ f \text{ is continuous}\}$ and $\Delta_\pi = \{f^{-1}(O),\ f \in C(\Pi),\ O \text{ open in } \mathbb{R}\}$. We refer to the members of Δ_π as the open Baire sets of Π. Finally, let $\sigma(\Delta_\pi)$ be the σ-algebra generated by Δ_π. Our objective is now to prove, for the symmetric model, the following proposition:

<u>Proposition 7</u>: There exists a pair (Λ_π,μ_π), where Λ_π is a σ-algebra of subsets of Π, and μ_π a probability measure on Λ_π, such that:

(1) for each $A \in \Lambda_\pi$ and $T \in \Pi$, $\mu_\pi(TA) = \mu_\pi(A)$

(2) $\sigma(\Delta_\pi) \subset \Lambda_\pi$

(3) $\mu_k \Rightarrow \mu$

A few words of explanation seem in order here. Conditions (1) is a left invariance condition. It says that if a Λ_π-measurable set of permutations is "permuted," then it remains Λ_π-measurable, and its μ_π-measure is unaffected. Thus, (1) just requires that μ_π is the analogue on Π of the finite measures μ_k on Π_k^*. For the model of competition on the circle, condition (1) would read:

(1)' for each $A \in \Lambda_\pi$ and each T_α, $\nu_\pi(T_\alpha A) = \nu_\pi(A)$

where T_α: $x(t) \to x((t + \alpha) \bmod 1)$. In other words, (1)' requires that if a Λ_π-measurable set of permutations is shifted to the right by a distance of α, then it remains Λ_π-measurable, and its ν_π measure is unaffected. Again, (1)' just expresses the requirement that ν_π be the analogue on Π of the (finite) measures ν_k on Π_k^*. Condition (2) requires that Λ_π contain the Baire sigma-algebra $\sigma(\Delta_\pi)$, so that condition (3) makes sense. The last condition ensures that for any finite set of produced goods P, the finite demand functions converge to the limiting demand functions derived from $(X, \sigma(\Delta_\pi), \mu_\pi)$.

Proof of the Proposition:

(1) and (2) follow directly from the theorem in Deneckere (1985).

(3) follows from Theorem 6 in Appendix I. []

The proof of Proposition 7 for the model of competition on the circle is somewhat simpler than for the symmetric model, because there clearly exists such a limiting world, namely the one described, e.g., by Salop. The details of the proof are relegated to Appendix II, for the first part of the theorem. The second part follows directly from Theorem 6 in Appendix I.

VII. Comparing Optimal and Competitive Outcomes in the Two Models

A. Intuition

Here we use the machinery developed in the last two sections to compare the symmetric and distance models. We study the relationship between the equilibrium number of stores (or varieties) which would result from (monopolistic) competition under free entry and the optimal (or welfare maximizing) number of stores. Our notion of competition is, we think, the natural one. We assume stores are price setters and look for symmetric Nash equilibria with zero profits in each model. We assume stores have the same technology. They produce output with fixed costs F and constant variable costs c. Thus in equilibrium profits over and above variable costs just cover fixed costs.

Our standard of optimality is also the natural one. As discussed in section III, the sum of producer and consumer surplus is a correct measure of welfare in our model; in particular it makes sense to sum the consumer surplus which different consumers receive into one measure of aggregate consumer surplus. The welfare maximizing number of stores or varieties is the number of stores which, if all stores sold output at marginal cost would maximize the sum of consumer surplus and profits at stores.

We show (in Proposition 8) under fairly general conditions that in the distance model price competition and free entry results in too many stores, too much variety. Under slightly less general conditions, the opposite result holds in the symmetric model (this is Proposition 9). In the symmetric model competition leads to too few stores. These results are not entirely new; they restate under slightly different and possibly more general conditions what is already known.

What is new is that our model explains (at least in part) _why_ these results should hold. We compare the symmetric model and the distance model with the same technology. The same $\psi(s)$ gives the cost to a customer of going to a store located s units away from him. As discussed in Section II, different distributions of customer's preferences lead to the symmetric model and the distance model. We use the convenient demand functions of the limiting forms of these models, derived in Sections V and VI to make a ceteris paribus comparison.

We find that the symmetric model is more competitive than the distance model. Prices and profits per store are lower in monopolistic competition equilibrium. With price competition and free entry there are fewer stores in equilibrium in the symmetric model than in the distance model. The intuitive explanation for this result is that in the symmetric model if a store raises its prices it will lose customers to all other stores; in the distance model a store which raises prices will lose customers only to the two stores which are its nearest neighbors. Consequently for any fixed number of stores prices are lower (in equilibrium) in the symmetric case than in the model of competition on the circle. Since prices are lower, profits per store are lower. With free entry the equilibrium number of stores is determined by the condition that profits over variable costs must cover fixed costs. Thus if stores operating in the two models have the same fixed and variable costs, a zero profit (free entry) equilibrium will support fewer stores in the symmetric model than in the distance model. This intuitive argument is made precise in Proposition 11.

While competition supports fewer stores in the symmetric model than in the distance model, welfare would be optimized with more stores in the symmetric model than in the distance model. We show, in Proposition 12, that the optimal number of stores is greater in the symmetric model than in the distance model. In either model the benefits (as measured by total consumer surplus) of adding additional stores decrease as the number of stores increases. Benefits fall faster in the distance model than in the symmetric model. To see why, observe that in the distance model if there are N stores no customer is consuming a brand more than $(2N^{-1})$ away from his most preferred brand. The increase in consumer surplus which accrues to any customer when an additional store is added is at most $\psi(2N^{-1}) - \psi(0)$. As N increases this gets quite small. In the distance model tastes of consumers are relatively similar; a few stores or varieties will satisfy everyone. In the symmetric model tastes are very dissimilar. No matter how many varieties are produced some consumers will find them all equally repugnant. If a new brand is introduced, it will be the first choice of some of these dissatisfied consumers. No matter how many stores there are, the introduction of a new store will increase some consumer's welfare by $\psi(2^{-1}) - \psi(0)$. The gains from adding additional stores do not decrease so quickly in this model.

While we believe the intuition is quite persuasive, our results are not as strong as might be hoped for. Many of them are asymptotic or large number results. Specifically the proofs we give of Propositions 9, 11, and 12 are only valid as fixed costs approach zero (and thus the optimal and competitive number of stores both become large).

B. Notation

We now introduce concepts and notation which allow us to state and prove the propositions alluded to above. We define many of the same concepts for the symmetric model and the model of competition on the circle and use superscripts to distinguish between them. Thus, $D^s(p,N)$ is the demand at a store in the symmetric model when all N stores which have entered the market are charging p; $D^d(p,N)$ is the demand at a store in the distance model when all N stores which have entered the market are charging p. Also, $d^i(q,p,N)$ is the demand at a store (in model i; i = s or i = d) which charges q when the N − 1 other stores in the model are charging price p. Also, p_N^i is the Nash equilibrium price (in model i) when N stores are in the market, N_m^i is the number of stores in equilibrium with free entry in model i and N_o^i is the optimal number of stores.

The underlying technology of production is that each potential store has fixed costs of F (which it must pay to enter the market) and produces output at constant marginal cost c.

The transport cost function is ψ; $\psi(z)$ is the cost to a consumer of traveling a distance z to a store. As discussed above, ψ is most easily interpreted as representing some sort of psychological cost. In each model consumers must travel no more than a distance of 1/2. This is because we chose units so that competition in the distance model takes place on a circle with a circumference of unity. Since consumers can travel in either direction, no consumer ever has to travel more than a distance of 1/2 to get to any store in the market (on the circle). In the corresponding symmetric model the maximum distance between any customer and any brand is also 1/2. We make the following assumptions about the transportation cost function:

(i) $\psi(0) = 0$;

(ii) $\psi'(z) > 0$; $\psi''(z) \geq 0$ for $z \geq 0$

and

(iii) $\psi(1/2) < 1 - c$.

The first two assumptions are not very restrictive: it costs nothing to move nowhere; transport costs are increasing and convex. Convexity is necessary for some of our results. It also seems to be important for the existence of equilibrium in the distance model. The third assumption is somewhat more substantive. The good is attractive enough that even the worst brand--that a distance of 1/2 away--is worth buying if it is sold at marginal cost or slightly more. This assumption simplifies our proofs considerably; we do not know how many of our results go through if we abandon it.

Welfare in our model is the sum of producer and consumer surplus. In the distance model all consumers buy one unit of the product so welfare is given by:

1 - (Total transportation costs + variable costs of production
+ fixed costs of production)

The optimal number of stores is the number which minimizes the sum of total transportation costs (which decrease as the number of stores increases) and fixed costs of production (which increases by F each time a store is added). If there are N stores selling their products at marginal cost total transportation costs are given by

$$T^d(N) = 2N \int_0^{(2N^{-1})} \psi(x)\, dx.$$

since assumption (iii) implies that everybody will be served at that price. Thus, for large N the increase in consumer surplus from adding another store is

$$(1) \qquad G_N^d = - \frac{dT^d(N)}{dN}$$

For the symmetric case we calculate G_N^s, the welfare gain (net of fixed cost) from adding an additional brand, slightly differently. Suppose that in the symmetric model N stores offer goods for sale at marginal cost c and that these store produce to meet demand. Exclusive of fixed costs the stores make zero profits. Consumer surplus is

(2) $$\int_c^1 z dH^N(z)$$

where

(3) $$H(z) = 1 - 2\psi^{-1}(1 - z).$$

This is because, as explained in Section VI, the consumer can be thought of as picking the brand which gives him most utility from a sample of brands each of which generates a random utility according to the distribution function $H(\cdot)$. Since (by assumption (iii)) all brands give utility greater than c, all consumers, even those picking from a menu of their least favorite brands, will get some consumer surplus. The increase in consumer surplus from adding another brand is

$$G_N^s = \int_c^1 z dH^{N+1}(z) - \int_c^1 z dH^N(z).$$

Note that

$$G_N^s = \int_0^1 z dH^{N+1}(z) - \int_0^1 z dH^N(z)$$

(4) $$= \int_0^1 H^N(z)(1 - H(z))\, dz.$$

The first step follows from assumption (iii); the second is integration by parts.

In both the distance and the symmetric model N_o^i satisfies

$$G_N^i = F$$

as both G_N^i are decreasing functions of N.

Let Π_N^i be profits per store (exclusive of fixed costs) in model i when there are N stores in the market. A consequence of free entry is that N_m^i satisfies

(5) $$\Pi_N^i = F, \quad i = s,d.$$

C. Results

We now state and prove our main results.

Proposition 8: If (i) and (ii) hold then $N_m^d > N_o^d$.

Proof: We prove this by calculating that $\Pi_N^d > G_N^d$. Since G_N^d is decreasing in N, this suffices. The standard analysis of equilibrium in this model shows that

(6) $$\Pi_N^d = \frac{\phi'((2N^{-1}))}{N^2}$$

(To derive (6) note that $d^d(q,p,N) = 2(q - c)x$ where x, the greatest distance anyone travels to shop at the store charging q when all other stores are charging p, satisfies

$$q + \psi(x) = p + \psi(N^{-1} - x).$$

Implicitly differentiating and using the fact that in equilibrium $x = (2N^{-1})$ we see that $\frac{\partial x}{\partial q} = -\frac{1}{2\psi'(2N^{-1})}$. Thus, since the equilibrium condition which determines p_N^d is $x + (q - c)(\partial x/\partial q) = 0$, $p_N^d = \psi'(2N^{-1})/N$, and (6) follows.

Now it follows from (1) that

$$G_N^d = N^{-1}\psi(2N^{-1}) - 2\int_0^{(2N^{-1})}\psi(x)\, dx \tag{7}$$

$$= \int_0^{(2N^{-1})}\psi'(x)2x\, dx \leq \psi'(2N^{-1}) \int_0^{(2N^{-1})}2x\, dx$$

$$= \frac{\psi'(2N^{-1})}{4N^2} < \frac{\psi'((2N^{-1}))}{N^2} = \Pi_N^d.$$

The first inequality follows from the convexity of ψ. This completes the proof. []

Proposition 9: If F is sufficiently small $N_o^s \geq N_m^s - 1$.

Proof: We prove this by calculating $\Pi_N^s < G_N^s$. This suffices since G_N^s is decreasing in N.

$$G_N^s = \int_0^1 H^N(z)(1 - H(z))dz.$$

Since ψ is convex, (3) implies that H is concave. However, $H(0) = 0$ and $H(1) = 1$ so that $H(z) \geq z$ for $0 \leq z \leq 1$. Thus,

$$G_N^s = \int_0^1 H^N(z)(1 - H(z)\, dz \geq \int_0^1 z^N(1 - H(z))\, dz$$

$$= (N + 1)^{-1} z^{N+1}(1 - H(z))\Big|_{z=0}^{z=1} - \int_0^1 (N + 1)^{-1} d(1 - H(z))$$

$$= (N + 1)^{-1} \int_0^1 z^{N+1} h(z)\, dz$$

where h(z) is the density corresponding to H. Now if h is continuous:

$$\lim_{N\to\infty} \int_0^1 N\, z^N h(z)\, dz \to h(1).$$

Thus

(8) $$G_N^s \geq (N+1)^{-1} \int_0^1 z^{N+1} h(z)\, dz \approx (N + 1)^{-2} h(1)$$

Note that if ψ is strictly convex then H is strictly concave so that the inequality in (8) is, in this case, necessarily a strong inequality.

Now consider $\Pi_N^s = (p_N^s - c)D^s(p_N^s, N)$. Substituting the first order conditions which p_N^s must satisfy we see that

(9) $$\Pi_N^s = - [D^s(p_N^s, N)]^2 / d_1^s(p_N^s, p_N^s, N).$$

where $d_1(q,p,N)$ is the derivative of (q,p,N) with respect to the first argument. We show in Lemma 10 below that

(10) $$\lim_{N\to\infty} p_N^s = c$$

and that

$$\lim_{N\to\infty} d_1^s(p_N^s, p_N^s, N) = -h(1).$$

Together assumptions (iii) and (10) imply that

(12) $$D^s(p^s_N, N) \approx N^{-1}$$

for large N. Combining (8), (9), (11) and (12) we see that

$$\lim_{N\to\infty} \frac{G^s_N}{\Pi^s_N} \geq h(1)^2 \cdot \left(\frac{N}{N+1}\right)^2$$

Thus it will suffice to show that $h(1) > 1$. Since $h(z) = H'(z)$ it is easy to calculate that $h(1) = 2/\psi'(0)$. Note that assumptions (i) and (ii) imply that if $\psi'(0) \geq 2$, then $\psi(1/2) \geq 1$ which contradicts assumption (iii). We conclude that for large N, $G^s_N \geq \Pi^s_N[(N/N+1)]^2$ which establishes Proposition 9.

Lemma 10: $\lim_{N\to\infty} p^s_N = c$ and $\lim_{N\to\infty} d^s_N(p,p,N) = h(1)$.

Proof: p^s_N satisfies $(p - c)d^s_1(p,p,N) + D^s(p,N) = 0$. Since $D^s(p,N) = N^{-1}(1 - H^N(p)) \leq N^{-1}$, $\lim_{N\to\infty} D^s(p,N) = 0$. Thus $\lim_{N\to\infty} p^s_N = c$ if $\lim_{N\to\infty} |d^s_1(p,p,N)| > 0$. Since

$$d^s(q,p,N) = \int_q^1 H(z - q + p)^{N-1} h(z)dz,$$

$$d^s_1(p,p,N) = -h(p)H^{N-1}(p) - \int_p^1 h(z)\, dH^{N-1}(z).$$

But the measure $H^{N-1}(p)$ converges weakly to the measure with a point mass at one. Since $h(\cdot)$ is continuous, $\lim_{N\to\infty} |d^s_1(p,p,N)| = h(1) = 2/\psi'(0) > 0.$ □

Proposition 11: If F is sufficiently small, $N^d_m > N^s_m$.

Proof: We first show that if fixed costs are sufficiently small then profits per store are greater in the distance model than in the symmetric model when there are the same number of firms in existence:

(13) If F is sufficiently small $\Pi_N^d > \Pi_N^s$.

The first order conditions for profit maximization state that p_N^i must satisfy

$$p - c = \frac{D^i(p,N)}{d_1^i(p,p,N)}.$$

Thus,

$$(14) \qquad \frac{\Pi_N^d}{\Pi_N^s} = \frac{d_1^s(p,p,N)}{d_1^d(p,p,N)} \frac{D^d(p,N)^2}{D^s(p,N)^2}$$

Since for large N, $D^d(p,N) \approx N^{-1}$, while $D^s(p,N) \leqslant N^{-1}$, the second fraction in (14) has a limit not exceeding 1. Now

$$\lim_{N\to\infty} d_1^d(p,p,N) = \lim_{N\to\infty} \psi'((2N)^{-1})^{-1} = \psi'(0)^{-1}.$$

We saw above that $\lim_{N\to\infty} d_1^s(p,p,N) = 2/\psi'(0)$. Thus,

$$\lim_{N\to\infty} \frac{\Pi_N^d}{\Pi_N^s} \geqslant 2,$$

which establishes (13).

We know from (6) and (iii) that Π_N^d is decreasing in N; in the distance model profits per store decrease as the number of stores decrease. Thus, since N_c^s satisfies $\Pi_N^s = F$, (13) implies that $N_c^d > N_c^s$. $\square$

Proposition 12: If F is sufficiently small $N_o^s > N_o^d$.

Proof: Clearly it will suffice to show

$$\lim_{N\to\infty} \frac{G_N^s}{G_N^d} > 1.$$

For large N we recall from (7) that $G_N^d \leq \psi'((2N)^{-1})/4N^2 \approx \psi'(0)/4N^2$ while from (8) we have $G_N^s \geq h(1)/(N + 1)^2 = 2/\psi'(0)(N + 1)^2$ Recalling that $\psi'(0) < 2$ we have

$$\lim_{N\to\infty} \frac{G_N^s}{G_N^d} > \frac{8}{\psi'(0)^2} > 2.$$

This completes the proof. []

Propositions 8, 9 and 11 only make sense if a (symmetric) monopolistically competitive equilibrium exists both in the symmetric model and in the model of competition on the circle. It can be shown that if conditions (i)-(iii) are further strengthened to $|\phi'(0)| > 0$ and $|\psi''| < \infty$, then, for sufficiently large N, a unique symmetric equilibrium exists in both models.

D. Example

Consider the case of linear transportation costs: $\psi(z) = 2z$, and zero marginal cost of production. Then $H(z) = 1 - 2\psi^{-1}(1 - z) = z$. Hence welfare in the symmetric model and in the distance model are easily calculated as:

$$W^S(N) = \int_0^1 x \, dH^N(x) - NF = \frac{N}{N+1} - NF$$

$$W^H(N) = 1 - NF - 2N \int_0^{(2N)^{-1}} 2x dx = 1 - NF - 1/(2N)$$

Differentiating these expressions and setting the derivative equal to zero yields:

$$N_o^s = \frac{1}{\sqrt{F}} - 1, \quad N_o^d = \frac{1}{\sqrt{2F}}$$

From (6) we see that equilibrium profits in the distance model satisfy

$$\Pi_N^d = \frac{2}{N^2}$$

and thus $N_m^d = \sqrt{2/F}$.

For the symmetric model, $d^s(q,p,N) = \int_p^1 H^{N-1}(v + p - q)dH(v)$, and hence $d_1^s(q,q,N) = -1$, and $D^s(q,N) = N^{-1}(1 - q^N)$. Then (9) yields

$$\Pi_N^s = \frac{(1 - p_N^s)^2}{N^2}$$

so that, for large N, $N_m^s \approx 1/\sqrt{F}$.

For this example, there is a negligible tendency towards overentry in the symmetric model ($N_m^s = N_o^s + 1$), but a very pronounced overentry in the distance model ($N_m^d = 2N_o^d$). For fixed N, monopolistic competition equilibrium profits in the distance model are twice as high as in the symmetric model, yielding $N_m^d \approx \sqrt{2}\, N_m^s$. Finally, the optimal number of firms in the symmetric model exceeds that in the distance model, the ratio being equal to $\sqrt{2}$.

Appendix I

Let $E_\alpha^k = \{\pi(\alpha): \pi \in \Pi_k^*\} = \{\alpha + x - j_k(\alpha): x \in T_k\}$ for $\alpha \in I$. E_α^k is the set of values that α is permuted to at stage k. Correspondingly, let $V_\alpha^k = V(E_\alpha^k) = \{V(\pi(\alpha)); \pi \in \Pi_k^*\}$ be the set of valuations that α is permuted to at stage k.

Lemma 1: Let $\alpha \in A = \bigcap_{k=1}^{\infty} T_k$. Then $\tilde{\alpha}_k \Rightarrow \tilde{\alpha}$, where $\tilde{\alpha}$ is distributed uniformly on (0,1], and the random variable $\tilde{\alpha}_k$ is defined as $\tilde{\alpha}_k: \pi_k^* \to \pi_k^*(\alpha)$.

Proof: Since $\alpha \in A$, $\exists\, r: \forall\, s \geq r\ E_\alpha^s = T_s$. Let $x \in E_\alpha^s$ for some $s > r$. Then $\mathrm{Prob}[\tilde{\alpha}_k \leq x] = x$ for all $k \geq s$. Thus $F_\alpha^k(x) = \mathrm{Prob}[\tilde{\alpha}_k \leq x] \to x \equiv F(x)$ for $x \in \bigcup_{s \geq r} E_\alpha^s = A$, where $F(\cdot)$ is the distribution function for $\tilde{\alpha}$. Let $x \notin A$. Since A is dense in I, $\exists\, x_n, y_n \in A$ such that $x_n < x < y_n$, $x_n \uparrow x$ and $y_n \downarrow x$. But

$$F_\alpha^k(x^n) \leq F_\alpha^k(x) \leq F_\alpha^k(y^n), \quad \forall\, k,n$$

Hence, upon letting $k \to \infty$:

$$F(x^n) \leq \underline{\lim}\, F_\alpha^k(x) \leq \overline{\lim}\, F_\alpha^k(x) \leq F(y^n)$$

Upon taking limits as $n \to \infty$, we have:

$$x = F(x) = \lim_{k\to\infty} F_\alpha^k(x) \qquad []$$

Lemma 2: Let $\alpha \in A^c$, the complement of A. Then $\tilde{\alpha}_k \Rightarrow \tilde{\alpha}$, where $\tilde{\alpha}$ is distributed uniformly on I.

Proof: Let $E_\alpha = \bigcup_{k=0}^{\infty} E_\alpha^k$. Then E_α is dense in I. Let $z \in E_\alpha$. Then $z = \alpha + x_k - j_k(\alpha)$ where $x_k \in T_k$, for each k. We have: $\text{Prob}[\tilde{\alpha}_k \leq z] = x_k \to z = F(z)$ since $\alpha - j_k(\alpha) \to 0$ as $k \to \infty$. Since E_α is dense in I, the rest of the proof then proceeds as in Lemma 1. []

Theorem 3: For every $k \geq 0$, let (Π_k^*, μ_k) be the set of permutations at stage k, and the symmetric measure at stage k. For any finite subset $P \subset I$, the joint distribution of $\{\tilde{\alpha}_k;\ \alpha \in P\}$ converges weakly to a collection of i.i.d. uniformly distributed random variables.

Proof: In view of Lemmas 1 and 2, we need only prove independence. Below, we treat the case where the cardinality of P is 2, and α, β are both in A. The rest of the proof is left as an exercise to the reader.

Let $r_\alpha = \min\{k: E_\alpha^k = T_k\}$ and $r = \max(r_\alpha, r_\beta)$. For $k \geq r$:

$$\text{Prob}[\tilde{\alpha}_k = x_1,\ \tilde{\beta}_k = x_2] = \begin{cases} 0 & \text{, if } x_1 = x_2 \text{ or } x_i \notin T_k \\ \dfrac{1}{T_k(T_k - 1)}, & \text{otherwise} \end{cases}$$

Thus, if $x_1 = j_1/T_k < x_2 = j_2/T_k$ are both in T_k we have for $k \geq r$:

$$\text{Prob}[\tilde{\alpha}_k \leq x_1,\ \tilde{\beta}_k \leq x_2] = \frac{j_1(j_2 - 1)}{T_k(T_k - 1)} \to x_1 x_2 = F(x_1, x_2) \text{ as } k \to \infty,$$

where $F(x_1,x_2)$ is the joint distribution function of a pair of i.i.d. uniformly distributed random variables. We have proved the validity of the theorem for $(x_1,x_2) \in A \times A$, a dense subset of I^2. As observed in Lemma 1, this suffices. []

Lemma 4: The Lebesgue measure of Λ^c, $\lambda(\Lambda^c)$, is zero.

Proof:

$\Lambda^c = \{z \in I^N | \exists\, i = 1,\ldots,N: z_i \in G\} \cup \{z \in I^N | z_i - z_j \in H,\ i \neq j;\ i,j = 1,\ldots,N\}$.

But $\{z \in I^N | \exists\, i = 1,\ldots,N: z_i \in G\} \subset \bigcup_{i=1}^{N} \Delta(i,\varepsilon_n)$ where

$\Delta(i,\varepsilon_n) = \bigcup_{z_i \in G} [B(\varepsilon_n 2^{-m}, z_i) \times \prod_{j \neq i} I]$, m indexes G, and $B(\alpha,x)$ is a ball of radius α around x. Hence,

$$\lambda\{z \in I^N | \exists\, i = 1,\ldots,N: z_i \in G\} \leq \sum_{i=1}^{N} \lambda(\Delta(i,\varepsilon_n)) = 2N\varepsilon_n.$$

Letting $\varepsilon_n \to 0$, we see that the left side of this inequality is zero. Fix $i \neq j$, i and j in $\{1,\ldots,N\}$. The set $\{z \in I^N: z_i - z_j = c\}$ with $c \in H$ is a hyperplane of dimension $(N - 1)$ and thus has Lebesgue measure zero. Since H is countable, $\lambda\{z \in I_N: z_i - z_j \in H\} = 0$ for fixed $i \neq j$. Because there are only finitely many choices of i and j to be made, this establishes the result. []

Theorem 5: On Λ, $d^k(\cdot) \to d(\cdot)$ uniformly, where $d^k(\cdot)$ and $d(\cdot)$ are the stage k and limiting demand functions for the symmetric model.

Proof: We will only describe the main steps of the proof here; details are avilable from the authors upon request. On Λ, $d_j^k(z)$ is a singleton:

$$d_j^k(z) = \frac{1}{T_k!} \# \{\pi \in \Pi_k^*: V(\pi(\alpha_j)) - z_j = f_\pi(z)\},$$

where $f_\pi(z) = \max\{0, \max_{j=1,\ldots,N} (V(\pi(\alpha_j)) - z_j)\}$ is consumer surplus derived by a consumer with preference pattern $V(\pi)$, when the set of produced goods is $P = \{\alpha_1,\ldots,\alpha_N\}$, and goods are sold at prices $(z_1,\ldots,z_N)$. Without loss of generality, let $j = N$. Then

$$d_N^k(z) = \sum_{\substack{t \in V(T_k) \\ t > z_N}} \text{Prob}[V(\pi(\alpha_N)) = t]\ \text{Prob}[V(\pi(\alpha_i)) < t - z_i + z_N,\ \forall\, i \neq j \,|\, V(\pi(\alpha_N)) = t]$$

Now,

$$\text{Prob}[V(\pi(\alpha_N)) = t] = \frac{\#\ \{\pi \in \Pi_k^*:\ V(\pi(\alpha_N)) = t\}}{T_k!} = \frac{1}{T_k},$$

and

$$\text{Prob}[V(\pi(\alpha_i)) < t - z_i + z_j,\ \forall\, j \neq i \,|\, V(\pi(\alpha_N)) = t] =$$

$$\sum_{\substack{a_1 \in V(T_k) \\ a_1 \neq t \\ a_1 > t - z_N + z_1}} \text{Prob}[V(\pi(\alpha_1)) = a_1 \,|\, V(\pi(\alpha_N)) = t]\ \text{Prob}[V(\pi(\alpha_i)) < t + z_i - z_N,\ \forall\, i = 2,\ldots,N \,|\, V(\pi(\alpha_N)) = t \text{ and } V(\pi(\alpha_1)) = a_1]$$

The first term in this expression is equal to $1/(T_k - 1)$, and the last term can in turn be expanded. So:

$$d_j^k(z) = \frac{1}{T_k} \sum_{\substack{t \in V(T_k) \\ t > z_N}} \frac{1}{T_{k-1}} \sum_{\substack{a_1 \in V(T_k) \\ a_1 \neq t \\ t_1 < t - z_N + z_1}} \frac{1}{T_{k-2}} \cdots \frac{1}{T_k - (N-1)} \sum_{\substack{a_{N-1} \in V(T_k) \\ a_{N-1} \neq t, a_1, a_2, \ldots, a_{N-2} \\ a_{N-1} < t + z_{N-1} - z_n}}$$

In other words, the term after the first summation sign is calculated as sampling without replacement. We claim that this term may be approximated uniformly in (z,t) by a calculation involving sampling with replacement:

$$\frac{1}{T_k} \sum_{\substack{a_1 \in V(T_k) \\ a_1 < t - z_N + z_1}} \frac{1}{T_k} \sum_{\substack{a_2 \in V(T_k) \\ a_2 < t - z_N + z_1}} \cdots \frac{1}{T_k} \sum_{\substack{a_{N-1} \in V(T_k) \\ a_{N-1} \neq t, a_1, a_2, \ldots, a_{N-2} \\ a_{N-1} < t + z_{N-1} - Z_N}}$$

i.e., there exist $L_N < \infty$ and ε_k: $\varepsilon_k T_k \to 0$ and the difference between the two calculations is bounded by $L_N \varepsilon_k$. Finally, we claim that sampling with replacement can be approximated uniformly in (z,t) by a Riemann integral, i.e., that

$$\frac{1}{T_k} \sum_{\substack{t \in V(T_k) \\ t > z_N}} \frac{1}{T_k^{N-1}} \prod_{i=1}^{N-1} \Big(\sum_{\substack{a_i \in V(T_k) \\ a_i < t - z_N + z_i}} \Big)$$

is uniformly close to

$$\int_{z_N}^{1} \prod_{i \neq N} G(t - z_N + z_i) dG(t),$$

where $G(t) = \text{Prob}[\tilde{V} \leq t]$. []

<u>Theorem 6</u>: $\mu_k \Rightarrow \mu_\pi$ on (Π, Σ_π)

<u>Proof</u>: We already know that $\mu_k \Rightarrow \mu$ on (X, Σ). Thus, for every open set G:

$$\underline{\lim}\, \mu_k(G) \geq \mu(G)$$

However, for every $E \in \Delta_\pi$, $\exists\, G \in T$: $E = G \cap \Pi$, and so

$$\mu_k(E) = \mu_k(G), \quad \mu_\pi(E) = \mu_\pi(G \cap \Pi) = \mu(G)$$

Thus, $\underline{\lim}\, \mu_k(E) \geq \mu_\pi(E)$. Thus, by Theorem 2 of Varadarajan (p. 182), $\mu_k \Rightarrow \mu_\pi$. [].

Appendix II

The purpose of this Appendix is to prove the following:

Theorem: There exists a pair (Λ_π, ν_π), where Λ_π is a σ-algebra of subsets of Π, and ν_π a probability measure on Λ_π, such that:

(1) for each $A \in \Lambda_\pi$, and each $T_\alpha \in \Gamma$, $\nu_\pi(T_\alpha A) = \nu_\pi(A)$

(2) $\sigma(\Delta_\pi) \subset \Lambda_\pi$

where $\Gamma = \{T_\alpha: X \to X \mid T_\alpha: x(t) \to x((t + \alpha) \bmod 1),\ \alpha \in I\}$

A few words of explanation: (1) is a "shift" invariance condition. It says that if a Λ_π-measurable set of "permutations" of I is shifted α units to the right, then it remains Λ_π measurable, and its ν_π measure is unaffected. The theorem would remain true for "shifts to the left." Condition (2) requires that Λ_π contain the Baire σ-algebra on Π. This is useful when we consider weak convergence of measures to ν_π.

The outer measure ν^* is defined from the measure space (X,S,μ) as:

$$\nu^*(A) = \inf\{\sum_{n=1}^{\infty} \nu(A_n): \{A_n\} \text{ is a sequence in S with } \bigcup_{n=1}^{\infty} A_n \supset A\}$$

for every subset A of X. A subset E of X is called ν-measurable if it is measurable with respect to the outer measure generated by ν, i.e.,

$$\nu^*(E) + \nu^*(E^c) = 1.$$

The set of E in X for which this statement holds will be called Λ. Λ is a σ-algebra on X, and ν^* is a probability measure on (X,Λ).

Lemma 1: $\nu^*(T_\alpha A) = \nu^*(A)$, $\forall A \subset X$, $\forall T_\alpha \in T$.

Proof: There exist sequences $\{O_n^i\}_{n=1}^\infty \subset S$ such that, for each i, O_n^i are disjoint, $\bigcup_{n=1}^\infty O_n^i \supset A$ and $\sum_{n=1}^\infty \nu(O_n^i) \downarrow \nu^*(A)$ as $i \to \infty$. Let $P_n^i = T_\alpha O_n^i$. Then P_n^i are disjoint, $\bigcup_{n=1}^\infty P_n^i \supset T_\alpha A$ and $P_n^i \in S$. Thus:

$$\nu^*(T_\alpha A) \leqslant \sum_{n=1}^\infty \nu(P_n^i) = \sum_{n=1}^\infty \nu(O_n^i) \downarrow \nu^*(A)$$

The last equality follows from

$$\nu(P_n^i) = \lambda(\bigcap_{j=1}^m \phi_{t_j}(B_m)) = \lambda(\bigcap_{j=1}^m \phi_{(t_j+\alpha)\text{mod } 1}(B_m)) = \nu(O_n^i)$$

with $O_n^i = \{x\colon x(t_1) \in B_1,\ldots,x(t_m) \in B_m\}$. The inequality $\nu^*(A) \leqslant \nu^*(T_\alpha A)$ is proved in the same manner. []

We will now try to prove that Λ contains the open sets: $\Xi \subset \Lambda$. To this extent, we first note:

Lemma 2: Let $A \in \Xi$, i.e., $A = \bigcup_{\beta\in B} A_\beta$, with $A_\beta = \prod_{\alpha\in I} J_{\alpha\beta}$ are basic open sets (and thus, $J_{\alpha\beta} = I$ for all but finitely many α, $J_{\alpha\beta}$ open for all α,β). Then

$$\nu^*(A) = \lambda(\bigcup_{\beta\in B} \bigcap_{\alpha\in I} \phi_\alpha(J_{\alpha\beta}))$$

Proof: Let $Z_\beta = \bigcap_{\alpha\in I} \phi_\alpha(J_{\alpha\beta})$, and $E = \bigcup_{\beta\in B} Z_\beta$. Notice that Z_β is open in I. We claim now that there exists $B' \subset B$, B' countable, such that

$\lambda(\bigcup_{\beta\in B'} Z_\beta) = \lambda(E)$. Let $(\alpha_1,\alpha_2,\ldots)$ be an enumeration of the rationals in E. For each pair (α_i,α_j), let $G_{\alpha_i\alpha_j} = \{Z_\beta\colon \alpha_i \in Z_\beta, (\alpha_i - \alpha_j, \alpha_i + \alpha_j) \subset Z_\beta\}$. To each pair (α_i,α_j), we now associate H_{α_i,α_j}, where H_{α_i,α_j} is empty if G_{α_i,α_j} is empty, and H_{α_i,α_j} is an arbitrary element of G_{α_i,α_j} otherwise. Let $B' = \{B: Z_\beta = H_{\alpha_i\alpha_j}$ for some $\alpha_i,\alpha_j\}$. Then B' is at most countable, and $\bigcup_{\beta\in B'} Z_\beta = E$. The latter statement follows since if $x \in E$ is irrational, $\exists\ \alpha_i$ close to x, and α_j rational such that $x \in H_{\alpha_i,\alpha_j}$. []

Lemma 3: Suppose $A \in \Xi$, $G \subset A$ and $\nu^*(G) < \nu^*(A)$. Then there exists a basic open set O: $O \subset A\backslash G$.

Proof: In calculating the outer measure of G, there is no loss of generality in covering G with basic open sets, instead of members of S. Hence there exist $O_n \in \Xi$: $O_n \supset G$, $\nu^*(O_n) < A$ and $\nu^*(O_n) \downarrow \nu^*(G)$. Without loss of generality, we may assume $O_n \subset O_{n+1}$. Fix $\tilde{n}$, and pick $x \in A\backslash O_{\tilde{n}}$. Let O be a basic open set containing x, s.t. $O \subset A\backslash O_{\tilde{n}}$. Then $O \subset A\backslash O_n\ \forall\ n \geq \tilde{n}$, and hence $O \subset A\backslash G$. []

We are now ready to prove:

Lemma 4: Every open set is ν-measurable.

Proof: Let $A \in \Xi$, and $\{G_n^i\}_{n=1}^\infty$ a sequence in S such that, for each i, G_n^i are disjoint, $\bigcup_{n=1}^\infty G_n^i \supset A^c$, and $\sum_{n=1}^\infty \nu(G_n^i) \downarrow \nu^*(A^c)$. Without loss of generality, $G^i = \bigcup_{n=1}^\infty G_n^i$ may be assumed to be a decreasing sequence. Thus, $H^i = (G^i)^c \subset A$ satisfies $H^i \subset H^{i+1}$. Let $H = \bigcup_{i=1}^\infty H^i \subset A$. If we can prove that $\nu^*(H) = \nu^*(A)$, the theorem will be proven, since then

$$\nu^*(A) + \nu^*(A^c) = \nu^*(H) + \lim_{n\to\infty} \nu^*(G^i) = \lim_{n\to\infty}(\nu^*(H^i) + \nu^*(G^i)) = 1$$

Suppose, to the contrary, that $\nu^*(H) < \nu^*(A)$. By Lemma 3, there exists

$O \subset A$: $H \cap O = \phi$. Thus, $G = \bigcap_{n=1}^{\infty} G^i \supset O \cup A^c$, and

$$\nu^*(A^c) = \lim_{n\to\infty} \nu^*(G^i) \geq \nu^*(G) \geq \nu^*(O) + \nu^*(A^c) > \nu^*(A^c),$$

by the finite additivity of ν^*, a contradiction. []

Let us define $\Lambda_\pi = \Lambda \cap \Pi = \{A \cap \Pi: A \in \Lambda\}$, and ν_π on Λ_π as $\nu_\pi(A \cap \Pi) = \nu^*(A \cap \Pi)$. Then Λ_π is a σ-algebra on Π, and ν_π a probability measure on (Π, Σ_π), provided that $\nu^*(\Pi) = 1$ (Billingsley, 1979, p. 38). Furthermore, Λ_π is constructed so as to contain the Baire σ-algebra $\sigma(\Delta_\pi)$. Indeed, Λ_π contains all open sets of Π (in the subspace topology).

<u>Lemma 5</u>: $\nu^*(\Pi) = 1$

<u>Proof</u>: Suppose, to the contrary, that $\nu^*(\Pi) < 1$. As observed in the proof of Lemma 4, there exist basic open sets A_n such that $\bigcup_{n=1}^{\infty} A_n \supset \Pi$ and $\nu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \nu(A_n) < 1 - \varepsilon$ for some $\varepsilon > 0$. Let $A = \bigcup_{n=1}^{\infty} A_n \in \Xi$. With $\nu^*(A)$ as defined in Lemma 2, we see that $\bigcup_{\beta\in B} \bigcap_{\alpha\in I} \phi_\alpha(J_{\alpha\beta})$ does not cover I. Let $x \in I$ not be covered. Then $\pi(t) = (t + x) \bmod 1$ is not covered by A, a contradiction. []

Finally, we note that the shift-invariance ν^* on Λ carries over to ν_π on Λ_π.

<u>Lemma 6</u>: For each $A_\pi \in \Lambda_\pi$ and $T_\alpha \in \Gamma$, $\nu_\pi(T_\alpha A_\pi) = \nu_\pi(A_\pi)$

<u>Proof</u>: From Billingsley (1979, p. 38), it follows that if ν_π^* is the outer measure generated by ν_π, then $\nu_\pi^*(A \cap \Pi) = \nu^*(A \cap \Pi)$ for all $A \in \Lambda$. Thus, $\nu_\pi^*(T_\alpha A_\pi) = \nu^*(T_\alpha A_\pi) = \nu^*(A_\pi)$, i.e., $T_\alpha A_\pi$ is ν_π-measureable,and

$\nu_\pi(T_\alpha A_\pi) = \nu_\pi(A_\pi)$. $\square$

This concludes the proof of the theorem.

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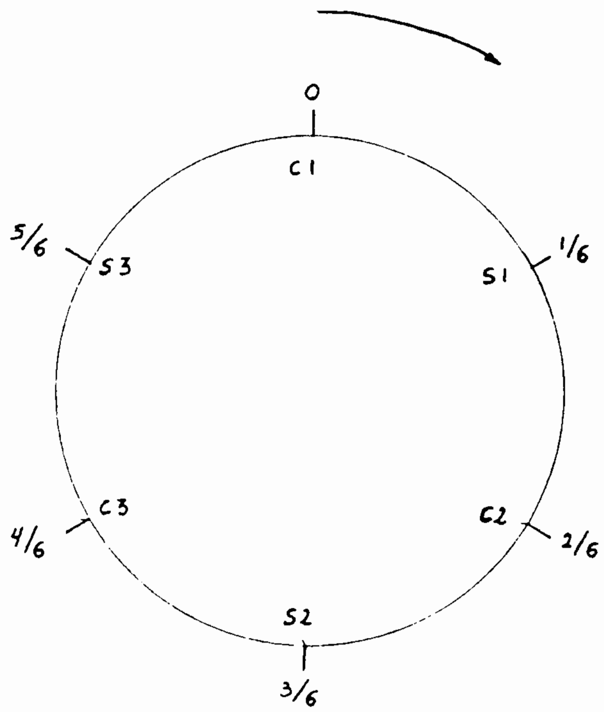

Figure 1: A model of one-sided competition on the circle with T = 3. (The symbol C refers to consumers, the symbol S to stores.)

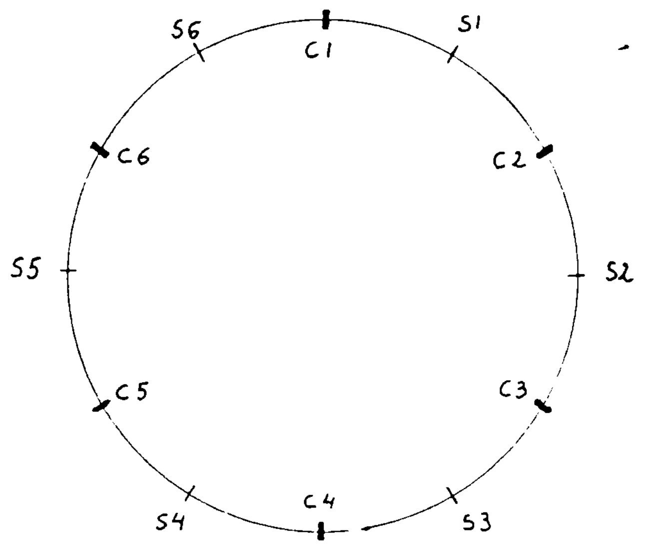

Figure 2: The model of competition on the circle with T = 6.

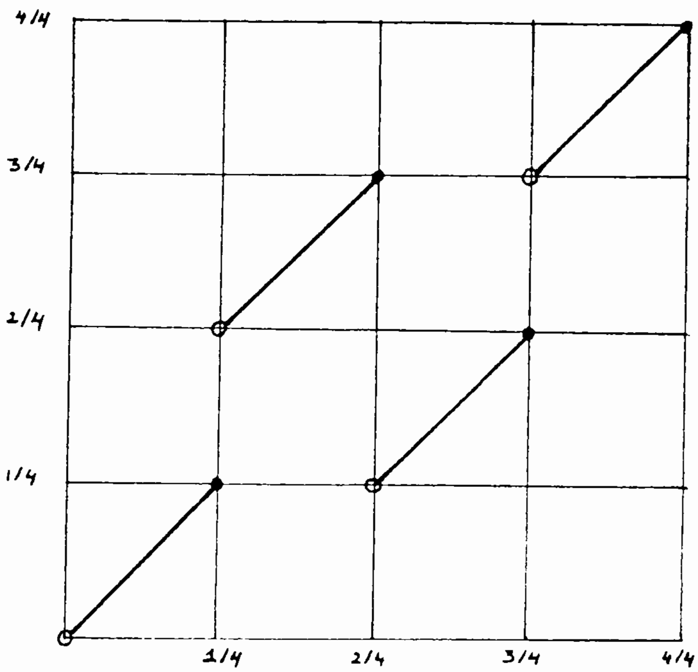

Figure 3: Extending a permutation in T_4 to a measure-preserving bijection on the unit interval [0,1).

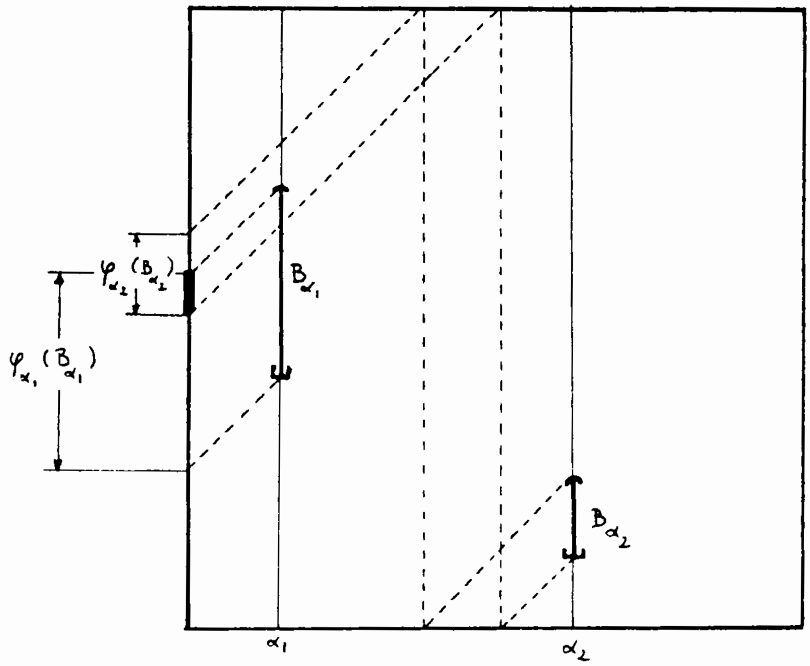

Figure 4: Computation of the limitation rotation group measure using the maps $\{\phi_\alpha\}$.