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ON THE EXISTENCE OF SUNSPOT EQUILIBRIA IN AN OVERLAPPING GENERATIONS MODEL

by

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[3,4]

I. Introduction and Summary

There are several aspects of dynamic economies that the Arrow-Debreu model does not satisfactorily capture. The open-ended time horizon introduces an infinite number of consumers and commodities, causing the failure of the first welfare theorem. Secondly, markets take place in time, so consumers are naturally restricted to trade only in those markets which operate during their lifetime. This restricted participation poses no problem under the assumption of perfect foresight; see Shell [1971]. However, combined with the uncertainty created by an unknown future, restricted market participation cannot be analyzed within the Arrow-Debreu framework.

Samuelson [1958] constructs a model in the general equilibrium tradition with overlapping generations of infinitely lived consumers. There are an infinite number of markets, but consumers only trade and receive utility when they are alive. The overlapping generations model has been extended and advanced by Gale [1973], Cass, Okuno, Zilcha [1979], and Balasko and Shell [1980, 1981]. These models capture the double infinity of consumers and commodities resulting from an infinite time horizon, along with the potential for a multiplicity of equilibria. However, the perfect foresight assumption requires consumers to know all prices, and the natural restrictions imposed by time are inessential without uncertainty.

Modelling the uncertainty created by an unknown future is a difficult task. The first attempt to attack this problem from a general equilibrium perspective was a joint research project by David Cass and Karl Shell, as reported in Shell [1977] and Cass and Shell [1983]. The Shell [1977] paper provides an overlapping generations economy in which preferences and endowments are stationary but consumption depends on a random variable unrelated to the rest of the economy. This extrinsic uncertainty affects the economy even though consumers have rational expectations. Cass and Shell [1983] analyze a static model with the only uncertainty being realizations of sunspots. They find that if some individuals are excluded from trading contingent securities based on sunspots, then there are rational expectations equilibria for which sunspots affect consumption levels. Azariadis [1981] studies the overlapping generations model from a macroeconomic perspective, in the tradition of Lucas [1972]. He shows that there can be self-fulfilling prophecies in the sense that cycles persist because people believe they will persist.

This paper provides the framework to study the effects of extrinsic uncertainty in a more systematic way than has previously been attempted in the overlapping generations literature. Prices and consumption levels in period $t = 1,2,3...$ are specified for each history of sunspot realizations through period t. For a given sunspot history, consumers will not in general know tomorrow's price, since it may depend on tomorrow's sunspot realization. However, if they know the probability of each realization and the associated prices, they can act so as to maximize expected utility. An equilibrium is a set of prices and consumption levels in which for all sequences of sunspot realizations markets clear and consumers maximize expected utility. If sunspots affect the allocation of resources for some consumer given some history of sunspot realizations, we have a sunspot equilibrium.

In Azariadis [1981] and Spear [1984], extrinsic uncertainty is required to have a stationary effect on the economy. As a result, they require preferences, endowments, and the money supply to be left stationary, and income effects must be strong enough to cause the offer curve to bend backwards. I am able to establish the existence of sunspot equilibria with weaker assumptions on preferences and endowments, assumptions which are quite standard in the overlapping generations literature. The crucial assumption is that there be a range of perfect foresight equilibria. The indeterminacy of perfect foresight equilibrium, commonly found in monetary models, allows me to establish the existence of sunspot equilibria.

My broader definition of a sunspot equilibrium ensures the existence of sunspot equilibria under fewer assumptions and focuses on the role of indeterminacy, but there is a cost. Stationary sunspot equilibria may be interesting in their own right, making the additional restrictions and assumptions worthwhile. For example, there is a relationship between stationary sunspot equilibria and deterministic cycles, as explored by Azariadis and Guesnerie [1984]. There is also the argument that a nonstationary sunspot equilibrium places impossible requirements on the information gathering abilities of consumers. As a result, stationary sunspot equilibria are more compelling than nonstationary ones. It could be argued that rational expectations only makes sense if the equilibrium is stationary.

I think that the above critique is not in conflict with the theme of this paper. In a temporary equilibrium framework, or some other framework that does not assume rational expectations, extrinsic uncertainty is bound to play a role. By requiring rational expectations, I have tied my hands and made the existence of sunspot equilibria as hard as possible.

Even under the rational expectations hypothesis, expectations cannot be endogenized. Far from being "flukes," sunspot equilibria arise from the indeterminacy inherent in overlapping generations models. Sunspots can affect the economy in much the same way animal spirits affect the economy in Keynes' _General Theory_.[1] In actual, complicated, nonstationary economies, there are likely to be no stationary equilibria, and we may not see the economy follow any of the nonstationary ones. But relaxing the restriction that consumers know the model should increase and not decrease the role for extrinsic uncertainty.

In section 2, I set up the model and define "sunspot equilibrium," and section 3 contains the existence proof. Sections 4 and 5 consider the case of a stationary economy. Under the maintained hypothesis of multiple perfect foresight equilibria, I show that any initial consumption level consistent with perfect foresight is consistent with a sunspot equilibrium. For any realization of sunspots in any period, consumption in the sunspot equilibrium must be within the range consistent with perfect foresight. In section 6, I consider the welfare implications of sunspot versus deterministic equilibria and provide two examples. Some concluding remarks are presented in section 7.

II. The Model

In each period, $t = 0,1,2...$, one consumer is born and lives for that period and the next one. The consumer in each generation is indexed by his birthdate. The economy starts in period 1, and continues into the infinite future. In each period, there is a single perishable consumption good and a completely durable fiat money. Let x_t^s be the consumption of consumer t in period s. This is a pure exchange economy with no production; the endowment of consumer t in period s is ω_t^s (for $s = t,\ t + 1$). For consumer t, we have $(x_t^t, x_t^{t+1}) \in \mathbb{R}_{++}^2$ and $(\omega_t^t,\ \omega_t^{t+1}) \in \mathbb{R}_{++}^2$. The consumption and endowment points for Mr. 0 are $x_0^1 \in \mathbb{R}_{++}$ and $\omega_0^1 \in \mathbb{R}_{++}$, respectively.

Each consumer has a utility function $u^t(x_t^t, x_t^{t+1})$ that is assumed to be strictly montonic, strictly concave, and continuously differentiable. Also, the closure of each indifference surface is assumed to lie in the strictly positive orthant. Each consumer pays a tax τ_t, given in units of fiat money. When τ_t is positive the consumer is taxed, and when it is negative the consumer is given a transfer. We interpret $(-\sum_{t=0}^{k} \tau_t)$ as the money supply in period k + 1.

Under perfect foresight, in which consumers know future prices with certainty, maximization problems are:

Mr. 0: $\max u^0(x_0^1)$

$$\text{s.t. } p^1 x_0^1 = p^1 \omega_0^1 - p^m \tau_0$$

$$x_0^1 > 0$$

Mr. t: $\max u^t(x_t^t, x_t^{t+1})$

$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^m \tau_t$$

$$x_t^t > 0$$

$$x_t^{t+1} > 0$$

Market clearing requires: $x_t^t + x_{t-1}^t = \omega_t^t + \omega_{t-1}^t$ for $t = 1,2,\ldots$

Here, p^t is the price of period t consumption and p^m is the price of money, with the normalization $p^1 = 1$, giving us a present price system. Balasko and Shell [1981] Prop. 3.1, show that the price of money must be constant in the certainty world, so it is appropriate to have just one price, p^m, instead of $p^{m,t}$. If p^m were not constant, some consumer would have the opportunity to make arbitrage profits by selling high-priced money and buying cheaper money. This arbitrage opportunity is obviously inconsistent with equilibrium.

Balasko and Shell [1980] establish the existence of an equilibrium price sequence in a more general model with many commodities per period. For the case of stationary money supply, preferences, and endowments, and a single commodity per period, Cass, Okuno, Zilcha [1979] show that either $p^m = 0$ is the only equilibrium money price or there is an interval of values consistent with equilibrium.

The only uncertainty which I consider takes the form of sunspots, with a new realization in each period starting with period one. The set of possible sunspot types in period t is

$$\Omega_t = \{1,2,3,\ldots,n\}.$$

The set of states of nature is

$$\Omega = \mathop{\times}_{t=1}^{\infty} \Omega_t .$$

Also, let $\Omega^t = \mathop{\times}_{i=t}^{\infty} \Omega_i$ and let B_t be the obvious finest σ-algebra on $\mathop{\times}_{i=1}^{t} \Omega_i$.

We are now ready to define the σ-algebra specifying the set of events taking place through period t, B^t.

$$(1) \qquad B^t = \{A \subset \Omega \mid A = a \times \Omega^{t+1}, \text{ where } a \in B_t\}$$

Let B be the smallest σ-algebra on Ω containing B^t for t = 1,2,3,... .

The stochastic process generating sunspots is described by a probability measure P on the measurable space (Ω,B). It follows that the conditional probabilities of sunspot realizations, given the history of previous realizations, are well defined. If $s^t \in \times_{i=1}^{t} \Omega_i$ is the history of sunspots through peirod t and $s_{t+1} \in \Omega_{t+1}$ is a particular sunspot type for period $t + 1$, then $\pi(s^t, s_{t+1})$ represents the probability of type s_{t+1} in period $t + 1$, conditional on the particular history s^t. To simplify notation, we will occasionally denote this conditional probability as $\pi_{s_{t+1}}$ when the history s^t is obvious from the context.

At the beginning of each period, the sunspot type is revealed to everyone, and then the goods market opens. The realization of sunspots in period one, s_1, is therefore known at the outset. Young consumers know all prices, quantities, and sunspot realizations up to and including their first period, but they do not know the future. The period t consumption decision of Mr. t must be based only on the information available at the time, so x_t^t is B^t-measurable. Consumers in their last period simply dump their remaining wealth on the market, so x_t^{t+1} is B^{t+1}-measurable. Consumers know the entire stochastic process generating sunspots. A Rational Expectations Equilibrium for this economy is one in which markets clear and consumers maximize expected utility.

Definition 2.1: A Rational Expectations Equilibrium (R.E.E.) for $(u,\omega,\tau,\Omega,B,P)$ is a set of prices $(p^m, 1, p^2, p^3, \ldots)$ and consumptions $(x_0^1, x_1^1, x_1^2, x_2^2, x_2^3, \ldots)$ that satisfy:

(2a) For all $t \geq 1$, x_t^t, x_{t-1}^t, and p^t are B^t-measurable functions of the state of nature. p^m is B^1-measurable.

(2b) $x_0^1 - \omega_0^1 = -p^m \tau_0$.

(2c) For all $t \geq 1$ and all $s^t \in \times_{i=1}^{t} \Omega_i$, x_t^t and x_t^{t+1} solve

$$\max \sum_{s_{t+1}=1}^{n} \pi(s^t, s_{t+1}) u^t(x_t^t, x_t^{t+1})$$

$$(*) \qquad \text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^m \tau_t$$

$$x_t^t > 0 \text{ is } B^t\text{-measurable}$$

$$x_t^{t+1} > 0$$

$$\text{(2d)} \qquad \text{(market clearing) } \omega_{t-1}^t + \omega_t^t = x_{t-1}^t + x_t^t$$

Definition 2.2: A Sunspot Equilibrium (S.E.) is a R.E.E. in which for some t and some $s^t \in \times_{i=1}^{t} \Omega_i$, there exist $\alpha, \beta \in \Omega_{t+1}$ such that $x_t^{t+1}(\alpha) \neq x_t^{t+1}(\beta)$ and $(\pi_{s_1})(\pi_{s_2}) \ldots (\pi_{s_t})(\pi_\alpha)(\pi_\beta) \neq 0$ hold. It is then said that sunspots matter for consumer t. A R.E.E. that is not a S.E. is a non-sunspot equilibrium.[2]

It will become important to distinguish equilibrium consumption levels for different sunspot realizations. We will use the notation $x_t^t(s^t)$ and $x_t^{t+1}(s^t, s_{t+1})$, or when the sunspot history is obvious from the context, x_t^t and $x_t^{t+1}(s_{t+1})$. Prices will be denoted as $p^t(s^t)$ and $p^{t+1}(s^t, s_{t+1})$, or simply p^t and $p^{t+1}(s_{t+1})$.

I interpret the market structure here to be a sequence of spot markets connected to each other by money. That is, goods are traded for money, a commodity that pays off in all events next period. There are no contingent commodity markets as in Debreu [1959], so we are free to normalize the price of money on each spot market. Implicit in Definition 2.1 is the normalization $p^{m,t}(s^t) = p^m(s^1)$ for all t and s^t.

This normalization has the attractive feature that all of the budget constraints a consumer faces on each spot market can be expressed as the single equation, (*). Whichever state occurs, lifetime expenditures must be financed by lifetime endowments and transfers. Another nice feature is the simple relationship between a non-sunspot equilibrium and the corresponding equilibrium for the certainty economy. If $(\hat{p}, \hat{p}^m)$ is an equilibrium for the certainty economy, then we have for the corresponding non-sunspot equilibrium,

$$p^m = \hat{p}^m$$

and

$$p^t(s^t) = \hat{p}^t$$

for all t and s^t.

Normalizing the price of money to be constant across all histories can be done, as long as money always has positive value. There is a class of sunspot equilibria excluded by this normalization (and excluded from Definition 2.1), the class in which money is worthless for some histories but not for others. Rather than complicate the definition of a sunspot equilibrium, we will retain Definition 2.1 throughout the text. However, Example 2 in Section VI illustrates one of these "excluded" equilibria.

When there are multiple perfect foresight equilibria, as there often are, it is easy to construct a S.E. Let all prices and quantities follow one perfect foresight path if $s_1 = 1$, another perfect foresight path if $s_1 = 2$. This somewhat trivial form of S.E., a randomization over perfect foresight equilibria,[3] is included in Definition 2.2 for the sake of mathematical symmetry. There is also an economic reason. Because of the multiplicity of perfect foresight equilibria, an initial condition determined outside the model fixes the price of money and commodity prices. However, since the initial condition is not determined by the fundamentals of the economy, it may be useful to think of it being caused by sunspots. When the sunspot realization at the beginning of period 1 affects the price of money, the interpretation is that sunspots are determining the initial condition. When we want to consider the initial condition fixed independently of sunspots, we can set $\pi_{s_1} = 1$ for $s_1 = 1$ without loss of generality.

The following story provides some intuition for the way sunspots affect prices. The initial condition sets the price of money and first period consumption. Loosely speaking, it also sets an expectation of what tomorrow's price is likely to be. Although there may be a unique perfect foresight path that fulfills this expectation, there are an infinite number of pairs of different prices (with associated probabilities) for which today's action is also rational. Thus, there could be some random process for which tomorrow's price is either the first or second of the pair according to tomorrow's realization. But why would tomorrow's young generation allow the sunspots to affect their demand and thereby affect the price? Because different realizations set different expectations of prices the day after tomorrow, which induces different behavior from expected utility maximizers.

III. Existence of Sunspot Equilibrium in a General Model

Before proceeding to the existence proof, we will need a few assumptions and definitions.

A) Utility functions are strictly monotonic, twice continuously differentiable, strictly concave, and the closure of each indifference surface lies in the strictly positive orthant.

B) $\omega_0 \in \mathbb{R}_{++}$ and $\omega_t \in \mathbb{R}^2_{++}$ for $t = 1,2,3,\ldots$.

C) $[0,\bar{p}^m] \subset p^m(\omega,\tau)$ for some $\bar{p}^m > 0$.[4]

D) For all t and all $s^t \in \overset{t}{\underset{i=1}{\times}} \Omega_i$, there are at least two realizations $\alpha,\beta \in \Omega_{t+1}$ such that $\pi_\alpha > 0$ and $\pi_\beta > 0$.

E) It is not the case that $\tau_0 + \tau_1 = 0$ and $\tau_t = 0$ for $t > 1$.

Assumption A is quite standard for this kind of model. Spear [1984] makes the same assumption; Balasko and Shell [1983] use strict quasi-concavity instead of strict concavity becuase they only deal with perfect foresight equilibria; Cass and Shell [1983] do not make the closure assumption, nor do Azariadis and Guesnerie [1984].

Assumption B, that endowments are strictly positive, is made by all of the above authors except Azariadis and Guesnerie [1984], who impose enough resource relatedness to allow zero endowments.

Assumption C says that there is an interval of money prices $[0,\bar{p}^m]$, each consistent with a perfect foresight equilibrium. This indeterminacy is a common feature of overlapping generations models. It is satisfied whenever the money supply becomes forever zero at some finite date (cf. Balasko and Shell [1983]), or in a stationary model whenever the marginal rate of substitution at ω is less than one.

Assumption D simply says that there are always at least two possible realizations of sunspots. Different realizations cannot yield different consumption levels when there is only one possible realization. Assumption E rules out the trivial case of a zero money supply for all time.

<u>Definition 3.1</u>: Let the function $G(\frac{p^{t+1}}{p^t}; x_t^t, \frac{p^m}{p^t})$ be defined by

$$G(\frac{p^{t+1}}{p^t}; x_t^t, \frac{p^m}{p^t}) = u_1(x_t^t, x_t^{t+1}) - u_2(x_t^t, x_t^{t+1}) \frac{p^t}{p^{t+1}}.$$

In Definition 3.1, x_t^{t+1} is determined by $\frac{p^{t+1}}{p^t}$, x_t^t, and $\frac{p^m}{p^t}$ through the budget constraint, (*). The consumer superscript on the utility function has been and will continue to be suppressed, and u_1 and u_2 are the first partials of u. The function G represents the marginal utility of consumption in youth, taking into account its effects on future consumption via the budget constraint. If $\tau_t = 0$, G is a function solely of $\frac{p^{t+1}}{p^t}$ and x_t^t, since then p^m/p^t has no effect on G.

<u>Fact 1</u>: Under Assumptions A, B, and C, and if $\tau = (0,0,0,\ldots)$, there can be no sunspot equilibrium.

<u>Proof</u>: Equilibrium requires:

$$x_0^1 - \omega_0^1 = - p^m\tau_0 = 0, \text{ so we have } x_0^1 = \omega_0^1 \text{ and } x_1^1 = \omega_1^1.$$

Given

$$x_t^t = \omega_t^t, \text{ Mr. t's budget constraint}$$

$$p^t(x_t^t - \omega_t^t) + p^{t+1}(x_t^{t+1} - \omega_t^{t+1}) = 0$$

implies

$$x_t^{t+1} = \omega_t^{t+1} \text{ and } x_{t+1}^{t+1} = \omega_{t+1}^{t+1} \text{ hold.}$$

By induction, all agents consume their endowments, and sunspots cannot matter.

<u>Fact 2</u>: Under Assumptions A, B, and C, and $\tau = (\tau_0,\tau_1,0,0,\ldots)$ with $\tau_0 + \tau_1 = 0$ and $\tau_0 \neq 0$ there are trivial sunspot equilibria but no non-trivial ones.

<u>Proof of Fact 2</u>: Suppose we have an S.E., with first period consumption given by the initial condition. Equilibrium requires that we have

$$x_0^1 - \omega_0^1 = -p^m\tau_0$$

$$(x_1^1 - \omega_1^1) + p^2(x_1^2 - \omega_1^2) = -p^m\tau_1.$$

By market clearing, we have $p^2(x_1^2 - \omega_1^2) = -p^m(\tau_0 + \tau_1) = 0$, which implies $x_1^2 = \omega_1^2$ holds for all realizations of sunspots in period 2.

Given $x_{k-1}^k = \omega_{k-1}^k$ holds for all realizations of sunspots in period $k > 1$, market clearing implies we have $x_k^k = \omega_k^k$. It follows from the budget constraint that $p^{k+1}(x_k^{k+1} - \omega_k^{k+1}) = 0$ holds. Therefore, $x_k^{k+1} = \omega_k^{k+1}$ holds for all realizations of sunspots in period $k + 1$. By induction, consumption does not depend on sunspot realizations in any period, which means we cannot have a S.E. other than trivial ones, where the initial condition is determined by first period sunspots.

<u>Theorem 1</u>: Under Assumptions A,B,C,D, and E, there exists a nontrivial S.E. Furthermore, sunspots can matter for an arbitrary finite number of consumers who carry a nonzero total money supply into their second period.

<u>Proof</u>: Properties of certainty equilibria, indexed by p^m, will be used to construct a sunspot equilibrium. Utility maximization for Mr. 1 to Mr. k and market clearing in periods 1 thorugh k + 1 yield the following system of equations in a certainty equilibrium.

(3.1) $$u_1(x_1^1,x_1^2) - \frac{1}{p^2}\, u_2(x_1^1,x_1^2) = 0$$

$\vdots$

(3.k) $$u_1(x_k^k,x_k^{k+1}) - \frac{p^k}{p^{k+1}}\, u_2(x_k^k,x_k^{k+1}) = 0$$

$$(4.1) \qquad p^1(\omega_1^1 - x_1^1) + p^m\tau_0 = 0$$

$$p^2(\omega_2^2 - x_2^2) + p^m(\tau_0 + \tau_1) = 0$$

$$\vdots$$

$$p^k(\omega_k^k - x_k^k) + p^m(\sum_{t=0}^{k-1} \tau_t) = 0$$

$$(4.k+1) \qquad p^{k+1}(x_k^{k+1} - \omega_k^{k+1}) + p^m(\sum_{t=0}^{k} \tau_t) = 0$$

In place of x_t^{t+1} (for $t = 1,2,3,\ldots,k-1$) we can substitute $\omega_{t+1}^{t+1} + \omega_t^{t+1} - x_{t+1}^{t+1}$, which follows from goods-market clearing. The system is of the form

$$F(x_1^1, x_2^2, \ldots, x_k^k, x_k^{k+1}, p^2, p^3, \ldots, p^{k+1}; p^m) = 0.$$

with $2k + 1$ unknowns and $2k + 1$ equations, parametrized by p^m.

The Jacobian of the above system is given by

$$J = \begin{matrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{matrix} \quad \text{where}$$

$J_{11} =$

$$\begin{matrix}
u_{11} - \frac{1}{p^2} u_{12} & -(u_{12} - \frac{1}{p^2} u_{22}) & 0 & \cdots & 0 & 0 \\
0 & u_{11} - \frac{p^2}{p^3} u_{12} & -(u_{12} - \frac{p^2}{p^3} u_{22}) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -(u_{12} - \frac{p^{k-1}}{p^k} u_{22}) & 0 \\
0 & 0 & 0 & \cdots & u_{11} - \frac{p^k}{p^{k+1}} u_{12} & -(u_{12} - \frac{p^k}{p^{k+1}} u_{22})
\end{matrix}$$

$J_{21} =$

$$\begin{matrix}
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -p^2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -p^k & 0 \\
0 & 0 & 0 & \cdots & 0 & p^{k+1}
\end{matrix}$$

$J_{12} =$

$$
\begin{matrix}
u_2/(p^2)^2 & 0 & 0 & \cdots & 0 & 0 \\
-\dfrac{u_2}{p^3} & \dfrac{p^2 u_2}{(p^3)^2} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \dfrac{p^{k-1} u_2}{(p^k)^2} & 0 \\
0 & 0 & 0 & \cdots & -\dfrac{u_2}{p^{k+1}} & \dfrac{p^k u_2}{(p^{k+1})^2}
\end{matrix}
$$

$J_{22} =$

$$
\begin{matrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\omega_2^2 - x_2^2 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \omega_k^k - x_k^k & 0 \\
0 & 0 & 0 & \cdots & 0 & x_k^{k+1} - \omega_k^{k+1}
\end{matrix}
$$

Whenever $|J| \neq 0$, we can apply the implicit function theorem to conclude that x_k^{k+1} is a continuously differentiable function of p^m. At $p^m = 0$, the economy is at autarky so J_{22} is composed of zeros. The determinant of J is expressed as follows:

$$(5) \qquad |J| = (-1)^{k(k+1)}(-1)(-p^2) \dots (-p^k)p^{k+1} \; |J_{12}|$$

Equation (5) can be simplified to:

$$(6) \qquad |J| = (-1)^k \prod_{t=1}^{k} u_1(x_t^t, x_t^{t+1}).$$

The expression in (6) cannot be zero because the first partials of each utility function are positive. By the implicit function theorem, there is a neighborhood of nonnegative money prices for which $(\partial x_k^{k+1})/(\partial p^m)$ is well defined and continuous.

At $p^m = 0$, Cramer's rule implies

$$\frac{\partial x_k^{k+1}}{\partial p^m} = \frac{-\sum_{t=0}^{k} \tau_t}{p^{k+1}}.$$

We can choose k so that Mr. k carries a nonzero money supply into period k + 1, so we have $\sum_{t=0}^{k} \tau_t \neq 0$ and therefore $(\partial x_k^{k+1})/(\partial p^m) \neq 0$. Thus, there is a neighborhood of nonnegative money prices for which $(\partial x_k^{k+1})/(\partial p^m) \neq 0$. Let the smaller of the above two neighborhoods be $[0,p^{m*}]$.

<u>Claim 1</u>: There is a neighborhood $[0,p^{m**}]$ for which $p^m \in [0,p^{m**}]$ implies p^k is bounded.

<u>Proof of Claim 1</u>: The implicit function theorem applied to the system of equations (3.1)-(4.k+1) implies p^k is a continuously differentiable function of p^m. When $p^m = 0$ occurs, p^k is at the unique autarky price. By continuity, there is an interval $[0,p^{m**}]$ for which p^k is within a neighborhood of the autarky price. Thus, p^k can be bounded.

<u>Claim 2</u>: For small enough p^m, the offer curve must strictly cross the

vertical line $x_k^k = \omega_k^k + \frac{p^m}{p^k} \sum_{t=0}^{k-1} \tau_t$ in x_k^k, x_k^{k+1} space.

<u>Proof of Claim 2</u>: When $\omega_k^k + \frac{p^m}{p^k} \sum_{t=0}^{k-1} \tau_t > \omega_k^k - \frac{p^m}{p^k} \tau_t$ holds, the vertical line is to the right of the translated endowment point,[5] so the result follows.

When

$$\omega_k^k + \frac{p^m}{p^k} \sum_{t=0}^{k-1} \tau_t = \omega_k^k - \frac{p^m}{p^k} \tau_k \text{ holds,}$$

we have $\sum_{t=0}^{k} \tau_t = 0$, which contradicts our choice of k.

The remaining case is when we have $\omega_k^k + \frac{p^m}{p^k} \sum_{t=0}^{k-1} \tau_t < \omega_k^k - \frac{p^m}{p^k} \tau_k$, so the vertical line is to the left of the translated endowment. Since the marginal rate of substitution is a continuous function of (x_k^k, x_k^{k+1}), we know that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| (x_k^k, x_k^{k+1}) - (\omega_k^k, \omega_k^{k+1}) \right\| < \sqrt{2}\, \delta \text{ implies} \tag{7}$$

$$\left\| \frac{u_1(x_k^k, x_k^{k+1})}{u_2(x_k^k, x_k^{k+1})} - \frac{u_1(\omega_k^k, \omega_k^{k+1})}{u_2(\omega_k^k, \omega_k^{k+1})} \right\| < \varepsilon. \tag{8}$$

Here, "$\|\cdot\|$" denotes the Euclidean norm.

Pick ε arbitrarily, which gives rise to δ. Because p^k and $[u_1(\omega_k^k, \omega_k^{k+1})/u_2(\omega_k^k, \omega_k^{k+1})]$ are bounded, we can choose p^m so that

$$\frac{p^m}{p^k} \left| \sum_{t=0}^{k-1} \tau_t \right| < \delta \tag{9}$$

(10) $$\frac{p^m}{p^k}|\tau_k| < \delta$$

(11) $$\frac{p^m}{p^k}\left|\sum_{t=0}^{k-1} \tau_t\right| < \frac{\delta}{\varepsilon + \frac{u_1(\omega_k^k, \omega_k^{k+1})}{u_2(\omega_k^k, \omega_k^{k+1})}} .$$

The first two inequalities (9) and (10) guarantee that (7) is satisfied by all points in the triangle formed by vertices

$$(\omega_k^k - \frac{p^m}{p^t}\tau_k, \omega_k^{k+1}); \ (\omega_k^k + \frac{p^m}{p^k}\sum_{t=0}^{k-1}\tau_t, \omega_k^{k+1}); \text{ and}$$

$$(\omega_k^k + \frac{p^m}{p^k}\sum_{t=0}^{k-1}\tau_t, \omega_k^{k+1} + \delta).$$

Mr. k's (continuous) offer curve goes through the first vertex and either crosses the vertical side of the triangle or crosses the hypotenuse. If it crosses the hypotenuse, the marginal rate of substitution at that point equals the slope of the budget line, which is greater than

$$\varepsilon + \frac{u_1(\omega_k^k, \omega_k^{k+1})}{u_2(\omega_k^k, \omega_k^{k+1})}.$$

This contradicts (8), so the offer curve crosses the vertical side and the claim is proved.

Fix p^m small enough so that (9), (10), (11); $p^m \in (0, \bar{p}^m)$, $p^m \in (0, p^{m*})$, and $p^m \in (0, p^{m**})$ all hold. Since the offer curve intersects the x_k^k line, there must be some prices at which Mr. k would want to consume more than x_k^k and some at which he would want to consume less. By quasi-concavity of the utility function, there exists p_α^{k+1} and p_β^{k+1} arbitrarily close to p^{k+1} such that

$$G\left(\frac{p_\alpha^{k+1}}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right) > 0 \text{ and } G\left(\frac{p_\beta^{k+1}}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right) < 0 \text{ hold.}$$

Define $\tilde{G}$ by

$$(12)\qquad \tilde{G} = \min\left[G\left(\frac{p_\alpha^{k+1}}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right),\ -G\left(\frac{p_\beta^{k+1}}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right)\right].$$

Since G is continuous, it must take on all values between $-\tilde{G}$ and $\tilde{G}$ at prices between p_β^{k+1} and p_α^{k+1}.

Consider a particular sunspot history $s^k \in \overset{k}{\underset{t=1}{\times}} \Omega_t$, and let the probabilities of realizations of period k + 1 sunspots, conditional on s^k, be denoted $\pi_1, \pi_2, \ldots, \pi_n$. We know that at least two of these probabilities are positive, and let the second largest value be π_j. Choose $p^{k+1}(s^k, j) \in [p_\beta^{k+1},\ p_\alpha^{k+1}]$ so that we have

$$G\left(\frac{p^{k+1}(s^k, j)}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right) = \tilde{G}.$$

For $s_{k+1} \neq j$, choose $p^{k+1}(s^k, s_{k+1})$ so that we have

$$G\left(\frac{p^{k+1}(s^k, s_{k+1})}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right) = -\frac{\pi_j}{1 - \pi_j}\,\tilde{G}.$$

Therefore, we have

$$(13)\qquad \sum_{s_{k+1}=1}^{n} \pi_{s_{k+1}}\, G\left(\frac{p^{k+1}(s^k, s_{k+1})}{p^k};\ x_k^k,\ \frac{p^m}{p^k}\right) = 0,$$

which is the first order condition for expected utility maximization. By repeating the argument for all $s^k \in \times_{t=1}^{k} \Omega_t$, we know that all consumers through Mr. k are maximizing expected utility and markets clear through period k.

The resulting values of $x_k^{k+1}(s^k,s_{k+1})$ will be contained in the neighborhood of x_k^{k+1}. Since x_k^{k+1} is a continuously differentiable function of p^m, each such value of $x_k^{k+1}(s^k,s_{k+1})$ is consistent with perfect foresight at prices

$$(\hat{p}^m(s^k,s_{k+1});\ 1;\ \hat{p}^2(s^k,s_{k+1});\ \hat{p}^3(s^k,s_{k+1});\ldots).^6$$

Claim 3: If we extend the sunspot tree by

$$(14)\qquad p^t(s^t) = \frac{\hat{p}^t(s^k,s_{k+1})p^{k+1}(s^k,s_{k+1})}{\hat{p}^{k+1}(s^k,s_{k+1})} \text{ for all } \begin{array}{l} s^k \in \times_{i=1}^{k} \Omega_i \\ s_{k+1} = 1,\ldots,n \\ t = k+2,\ k+3,\ \ldots \end{array}$$

then we have a S.E.[7]

Proof of Claim 3: Market clearing in period k + 1 implies we have for all (s^k,s_{k+1})

$$p^m\left(\sum_{t=0}^{k} \tau_t\right) = p^{k+1}(s^k,s_{k+1})[\omega_k^{k+1} - x_k^{k+1}(s^k,s_{k+1})]$$

and

$$\hat{p}^m(s^k,s_{k+1})\left(\sum_{t=0}^{k} \tau_t\right) = \hat{p}^{k+1}(s^k,s_{k+1})[\omega_k^{k+1} - x_k^{k+1}(s^k,s_{k+1})]$$

Therefore, we have

$$(15)\qquad p^m = \frac{p^{k+1}(s^k,s_{k+1})\hat{p}^m(s^k,s_{k+1})}{\hat{p}^{k+1}(s^k,s_{k+1})}.$$

Given the realization $s_{k+1} \in \Omega_{k+1}$ and the history s^k, Mr. t's budget constraint for $t > k$ is

$$(16) \qquad p^t(s^k,s_{k+1})[x_t^t - \omega_t^t] + p^{t+1}(s^k,s_{k+1})[x_t^{t+1} - \omega_t^{t+1}] = -p^m\tau_t$$

Substituting (14) and (15) into (16), we find that the budget constraint faced by Mr. t in the sunspot economy is equivalent to

$$(17) \qquad \hat{p}^t(s^k,s_{k+1})[x_t^t - \omega_t^t] + \hat{p}^{t+1}(s^k,s_{k+1})[x_t^{t+1} - \omega_t^{t+1}] = -\hat{p}^m(s^k,s_{k+1})\tau_t$$

This is the same constraint as that faced by Mr. t under the perfect foresight equilibrium indexed by (s^k,s_{k+1}). Therefore he demands the same amount here in the proposed S.E. as he does in the perfect foresight equilibrium, and markets continue to clear. By choosing prices according to (14) we have shown that the utility-maximizing consumption levels also clear markets, so $(p^m;\ 1;\ p^2;\ldots;p^k;\ p^{k+1}(s^k,s_{k+1});\ p^{k+2}(s^k,s_{k+1});\ldots)$ is a S.E. where we let (s^k,s_{k+1}) run over the entire set $\overset{k+1}{\underset{i=1}{\times}} \Omega_i$.

For $\kappa > k$, we can branch the tree again if we have $\sum_{t=0}^{\kappa} \tau_t \neq 0$ and p^m small enough.[8] For any sunspot history $s^\kappa \in \overset{\kappa}{\underset{i=1}{\times}} \Omega_i$, there exist $p_\alpha^{\kappa+1}$ and $p_\beta^{\kappa+1}$ where

$$G\left(\frac{p_\kappa^{\kappa+1}}{p^\kappa(s^\kappa)};\ x_\kappa^\kappa(s^\kappa),\ \frac{p^m}{p^\kappa(s^\kappa)}\right) > 0$$

and

$$G\left(\frac{p_\kappa^{\kappa+1}}{p^\kappa(s^\kappa)};\ x_\kappa^\kappa(s^\kappa),\ \frac{p^m}{p^\kappa(s^\kappa)}\right) < 0.$$

Let the probabilities of period $\kappa + 1$ sunspots, conditional on s^κ, be denoted

$\pi_1, \pi_2, \ldots, \pi_n$. by the argument used above, there are prices arbitrarily close to $p^{\kappa+1}(s^k, s_{k+1})$ defined in (14), where we have

$$\sum_{s_{\kappa+1}=1}^{n} \pi_{s_{\kappa+1}} G\left(\frac{p^{\kappa+1}(s^\kappa, s_{\kappa+1})}{p^\kappa(s^\kappa)}; x_\kappa^\kappa(s^\kappa), \frac{p^m}{p^\kappa(s^\kappa)}\right) = 0.$$

Thus, Mr. κ maximizes expected utility by choosing $x_\kappa^\kappa(s^\kappa)$, the quantity that clears the market.

By the implicit function theorem, $x_\kappa^{\kappa+1}(s^\kappa, s_{\kappa+1})$ is consistent with a perfect foresight equilibrium for each history s^κ and realization $s_{\kappa+1} = 1,2,\ldots,n$. Index each of these prefect foresight equilibria by $(s^\kappa, s_{\kappa+1})$ so that the perfect foresight price at which $x_\kappa^{\kappa+1}(s^\kappa, s_{\kappa+1})$ gets consumed is $(\hat{p}^m(s^\kappa, s_{\kappa+1}); 1; \hat{p}^2(s^\kappa, s_{\kappa+1}); \ldots)$.

For $t = 1,2,\ldots,\kappa$, define prices in the S.E. as in (14). For $t > \kappa + 1$, define prices according to

$$(18) \qquad p^t(s^t) = p^t(s^\kappa, s_{\kappa+1}) = \frac{\hat{p}^t(s^\kappa, s_{\kappa+1}) p^{\kappa+1}(s^\kappa, s_{\kappa+1})}{\hat{p}^{\kappa+1}(s^\kappa, s_{\kappa+1})}.$$

We have defined sunspot contingent prices where markets clear and consumers maximize expected utility. Furthermore, sunspots matter for Mr k and Mr. κ. For any consumer who carries a nonzero money supply into his second period, we can repeat the argument (a finite number of times) to create a S.E. where sunspots matter for him as well. []

Definition 3.2: The tax-transfer policy $\tau = (\tau_0, \tau_1, \tau_2, \ldots)$ is said to be strongly balanced if there is a finite k with the property $\sum_{i=0}^{t} \tau_i = 0$ for $t > k$.[9]

Corollary 1: Assume that A, B and D hold. Furthermore, assume that the tax transfer policy τ is strongly balanced and that there is a nonzero money supply at some date after period one (i.e., $\sum_{t=0}^{k} \tau_t \neq 0$ for some $k > 0$). Then there is a sunspot equilibrium.

Proof: Balasko and Shell [1984] show that for strongly balanced tax-transfer policies property C holds. We can now apply Theorem 1.

Corollary 2: For every economy satisfying A, B and D, there exists a τ for which a S.E. exists.

Proof: Choose τ to satisfy the hypothesis of Corollary 1.

IV. Existence of Sunspot Equilibrium in a Stationary Model

For the remainder of this paper, we will assume that utility functions and endowments are stationary, and that the money supply is constant at 1, i.e., $\tau = (-1,0,0,0,\ldots)$. It is assumed that the indifference curve through the endowment has a slope less than unity at ω. We require this assumption to guarantee a role for money in this economy; otherwise only autarky is consistent with equilibrium.

In Figure 1, $\omega_t^t - \hat{x}$ corresponds to the largest offer (excess supply) of first period consumption on the (reflected) offer curve of Mr. t below the 45° line. If first and second period consumption are gross substitutes as in Fig. 1b, then $\hat{x}$ also represents the golden rule first period consumption level.

Recapitulating the assumptions made so far, we have:

A') The stationary utility function u is continuously differentiable, strictly monotonic, and strictly concave.

B') Endowments are stationary at ω, and $\tau = (-1,0,0,\ldots)$.

F) The indifference curve through ω has a slope strictly between zero and one at the endowment point.

Lemma 4.1: Fix period t consumption at $\bar{x}_t^t$ with $\hat{x} < \bar{x}_t^t < \omega_t^t$ holding, and fix $p^t > 0$. Then, under A', B', and F there exists p_α^{t+1} and p_β^{t+1} such that

$$G(\frac{p_\alpha^{t+1}}{p^t};\ \bar{x}_t^t) > 0 \text{ and } G(\frac{p_\beta^{t+1}}{p^t};\ \bar{x}_t^t) < 0$$

both hold. Furthermore, there are perfect foresight price paths $(p_\alpha^{t+1}, p_\alpha^{t+2}, \ldots)$ and $(p_\beta^{t+1}, p_\beta^{t+2}, \ldots)$ consistent with consumption level

(19) $$x_{t+1}^{t+1}(s) = \omega_{t+1}^{t+1} + \frac{p^t}{p^{t+1}(s)}(\bar{x}_t^t - \omega_t^t), \text{ for } s = \alpha, \beta.$$

Proof: Demand[10] is continuous, so for all $\varepsilon > 0$ there exists $\delta > 0$ such that

(20) $$\left|\frac{p^t}{p_\alpha^{t+1}} - \frac{u_1(\omega)}{u_2(\omega)}\right| < \delta \text{ implies } \left|x_t^t(\frac{p^t}{p_\alpha^{t+1}}) - \omega_t^t\right| < \varepsilon.$$

Since the offer is upward sloping near ω, we have:

(21) $$0 < \frac{p^t}{p_\alpha^{t+1}} - \frac{u_1(\omega)}{u_2(\omega)} < \delta \text{ implies } \omega_t^t - x_t^t(\frac{p^t}{p_\alpha^{t+1}}) < \varepsilon.$$

Also, for all $\delta > 0$ there exists $p_\alpha^{t+1} < \infty$ such that

(22) $$0 < \frac{p^t}{p_\alpha^{t+1}} - \frac{u_1(\omega)}{u_2(\omega)} < \delta \text{ holds.}$$

Choose ε less than $(\omega_t^t - \bar{x}_t^t)$ and find a δ that satisfies (21); then find p_α^{t+1} from (22). If we have $p_\alpha^{t+1} \leq p^t$, pick a smaller δ from (21). For small enough δ, $[u_1(\omega)/u_2(\omega)] < 1$ implies $p_\alpha^{t+1} > p^t$.

If follows from (5) and our choice of ε that we have

$$\omega_t^t - x_t^t\left(\frac{p^t}{p_\alpha^{t+1}}\right) < \omega_t^t - \bar{x}_t^t$$

which implies

$$\bar{x}_t^t < x_t^t\left(\frac{p^t}{p_\alpha^{t+1}}\right).$$

That is, the actual amount of current consumption is less than the optimal amount at the price ratio p^t/p_α^{t+1}. More current consumption increases utility; strict quasiconcavity implies

$$G\left(\frac{p_\alpha^{t+1}}{p^t};\ \bar{x}_t^t\right) > 0.$$

Let $\bar{x}_t^t - \hat{x} = \varepsilon'$ hold. Since demand is continuous and takes the values $\hat{x}$ and ω_t^t, there must be some price at which it takes any given value in between. In particular, there exists p_β^{t+1} such that we have

$$0 < x_t^t\left(\frac{p^t}{p_\beta^{t+1}}\right) - \hat{x} < \varepsilon'.$$

Therefore, we also have

$$x_t^t\left(\frac{p^t}{p_\beta^{t+1}}\right) < \bar{x}_t^t.$$

The amount of current consumption is more than the optimal amount at price p^t/p_β^{t+1}, so strict concavity implies

$$G\left(\frac{p_\beta^{t+1}}{p^t};\ \bar{x}_t^t\right) < 0 \text{ occurs.}$$

It remains to show that there are perfect foresight equilibria with price paths $(p_\alpha^{t+1}, p_\alpha^{t+2}, \ldots)$ and $(p_\beta^{t+1}, p_\beta^{t+2}, \ldots)$ consistent with consumption level

$$x_{t+1}^{t+1}(s) = \omega_{t+1}^{t+1} + \frac{p^t}{p^{t+1}(s)} (\bar{x}_t^t - \omega_t^t)$$

for $s = \alpha, \beta$. It follows from equation (19) that whenever we have

$$\frac{p^t}{p^{t+1}(s)} < 1, \qquad \omega_{t+1}^{t+1} - x_{t+1}^{t+1}(s) < \omega_t^t - \bar{x}_t^t < \omega_t^t - \hat{x}$$

holds, so there is a perfect foresight path for initial consumption $x_{t+1}^{t+1}(s)$. We have chosen p^t/p_α^{t+1} less than one. The ratio p^t/p_β^{t+1} is close to and can be chosen less than the price at which $\hat{x}$ is demanded, and that price cannot exceed one. []

Theorem 2: Choose any initial condition $\bar{x}_1^1$ where we have $\hat{x} < \bar{x}_1^1 < \omega_1^1$. Then under A', B', D and F, there exists a S.E. in which sunspots matter for all consumers except Mr. 0.

Proof: Apply Lemma 4.1 for $t = 1$ to get p_α^2, p_β^2. Let

$$\tilde{G} = \min \left[G\left(\frac{p_\alpha^2}{1}; \bar{x}_1^1\right), -G\left(\frac{p_\beta^2}{1}; \bar{x}_1^1\right)\right].$$

Since G is continuous, it must take on all values between $-\tilde{G}$ and $\tilde{G}$ at prices between p_β^2 and p_α^2.

Let the probabilities of the various realizations be given by $\pi_1, \pi_2, \ldots, \pi_n$. For some j, $0 < \pi_j \leq 1/2$ holds, by condition D. Choose $p^2(1,j) \in [p_\beta^2, p_\alpha^2]$ so that $G(\frac{p^2(1,j)}{1}; \bar{x}_1^1) = \tilde{G}$ holds. For $i \neq j$, choose $p^2(1,i) \in [p_\beta^2, p_\alpha^2]$ so that we have $G(\frac{p^2(1,j)}{1}; \bar{x}_1^1) = -\frac{\pi_j}{1 - \pi_j} \tilde{G}$.

Therefore $\sum_{s=1}^{n} \pi_s G(\frac{p^2(1,s)}{1}; \bar{x}_1^1) = 0$, which is the first order condition for expected utility maximization.

For each history s^k, we can reapply Lemma 4.1 for $t = k$, and by the above argument we can find $p^{k+1}(s^k, s_{k+1})$ where the period k market clears and Mr. k maximizes expected utility. Furthermore, consumption is strictly between $\hat{x}$ and ω_k^k, so we can reapply the argument for k + 1. For finite k, there are a finite number of histories, so if markets clear and consumers maximize through period k, they will through period k + 1. By construction, sunspots matter for each consumer, and the proof by induction is complete.

V. Some Properties of Sunspot Equilibria

Theorem 3: In a S.E. for which we have $p^m \geq 0$ and A', B', and F, then for any sunspot history, $\hat{x} \leq x_t^t \leq \omega_t^t$ must hold.

Proof: If we have $x_t^t > \omega_t^t$ then money has a negative value in period t. Either $p^m < 0$ or $p^t < 0$ must occur. By hypothesis, $p^m \geq 0$ holds, and negative equilibrium commodity prices are inconsistent with monotonicity. Therefore, $x_t^t \leq \omega_t^t$ must hold.

Now we show that $x_t^t < \hat{x}$ leads to a contradiction. By the definition of $\hat{x}$, either the offer curve is always to the left of $\omega_t^t - x_t^t$, or it takes that value strictly above the 45° line.

In the former case, $G(\frac{p^{t+1}}{p^t}; x_t^t) > 0$ is true for all p^{t+1}/p^t, which contradicts expected utility maximization.

In the latter case, there exists $\varepsilon > 0$ such that:

1) The point $(\omega_t^t - x_t^t,\ \omega_t^t - x_t^t + \varepsilon)$ is on the offer curve;

2) $(\omega_t^t - x_t^t, z)$ is on the offer curve implies we have $z \geq \omega_t^t - x_t^t + \varepsilon$ (see Figure 2).

At any price ratio p^t/p^{t+1} where we have

$$\frac{p^t}{p^{t+1}} < \frac{\omega_t^t - x_t^t + \varepsilon}{\omega_t^t - x_t^t},$$

then

$$G(\frac{p^{t+1}}{p^t};\ x_t^t) > 0$$

holds. For a S.E. to exist there must be at least one path with strictly positive probability for which we have

$$\frac{p^t}{p^{t+1}} \geqslant \frac{\omega_t^t - x_t^t + \varepsilon}{\omega_t^t - x_t^t}.$$

It follows that $x_t^{t+1} - \omega_t^{t+1} \geqslant \omega_t^t - x_t^t + \varepsilon$ holds true along that path. By market clearing in period t + 1, $\omega_{t+1}^{t+1} - x_{t+1}^{t+1} \geqslant \omega_t^t - x_t^t + \varepsilon$ holds.

If the offer curve is always to the left of $\omega_t^t - x_t^t + \varepsilon$, we are done. Otherwise, there exists $\varepsilon' > 0$ such that:

1) the point $(\omega_{t+1}^{t+1} - x_{t+1}^{t+1},\ \omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon')$ is on the offer curve;

2) $(\omega_{t+1}^{t+1} - x_{t+1}^{t+1},\ z)$ is on the offer curve implies we have $z \geqslant \omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon'$.

Lines through the origin can intersect the offer curve only once, which implies:

$$\frac{\omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon'}{\omega_{t+1}^{t+1} - x_{t+1}^{t+1}} > \frac{\omega_t^t - x_t^t + \varepsilon}{\omega_t^t - x_t^t}.$$

The fact that $\omega_{t+1}^{t+1} - x_{t+1}^{t+1} > \omega_t^t - x_t^t$ implies $\varepsilon' > \varepsilon$.

At any price ratio p^{t+1}/p^{t+2} where we have:

$$\frac{p^{t+1}}{p^{t+2}} < \frac{\omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon'}{\omega_{t+1}^{t+1} - x_{t+1}^{t+1}},$$

then

$$G(\frac{p^{t+2}}{p^{t+1}};\ x_{t+1}^{t+1}) > 0$$

occurs. For a S.E. to exist there must be at least one path with strictly positive probability for which we have:

$$\frac{p^{t+1}}{p^{t+2}} \geq \frac{\omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon'}{\omega_{t+1}^{t+1} - x_{t+1}^{t+1}}.$$

Along that path, $x_{t+1}^{t+2} - \omega_{t+1}^{t+2} \geq \omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon'$ must hold. Therefore, we have $x_{t+1}^{t+2} - \omega_{t+1}^{t+2} \geq \omega_{t+1}^{t+1} - x_{t+1}^{t+1} + \varepsilon \geq \omega_t^t - x_t^t + 2\varepsilon$. Market clearing implies $\omega_{t+2}^{t+2} - x_{t+2}^{t+2} \geq \omega_t^t - x_t^t + 2\varepsilon$. Repeating the same argument k - 2 more times, there must be a path with each branch having strictly positive probability for which we have:

$$\omega_{t+k}^{t+k} - x_{t+k}^{t+k} \geq \omega_t^t - x_t^t + k\varepsilon.$$

By making k large enough, there must be a path of finite length that violates market clearing. The probability of the entire path, given the sunspot history up to period t, is the product of a finite number of strictly positive numbers, which must itself be positive. We have a contradiction, so $\hat{x} \leq x_t^t$ holds. □

This restriction on paths consistent with equilibrium is related to the literature on decentralized capital accumulation.[11] In those models, there are price paths which are period-by-period consistent with rational behavior but which eventually drive some price down to zero. This contradicts market clearing, so the path cannot be an equilibrium. Here we have paths in which consumers behave rationally, but where p^{t+1}/p^t is being driven down.

Eventually a consumer will be expected to supply more than his entire endowment, or else the offer curve bends backwards, in which case the path is inconsistent with equilibrium. The restriction that we have $\hat{x} \leq x_t^t \leq \omega_t^t$ applies to sunspot equilibria just as it applies to non-sunspot equilibria.

The asymptotic properties of perfect foresight equilibria are fairly well known. Convergence to autarky, convergence to the golden rule steady state, chaos, and cyclical behavior are all possible, depending on the offer curve. It is more difficult to describe the asymptotic properties of sunspot equilibria because there are potentially an infinite number of paths rather than just one.

It is clearly possible in a S.E. to reach autarky with probability one, or to reach the golden rule steady state with probability one. Azariadis [1981] has shown that when sunspots form a stationary first order Markov process, random cycles can result, in which the economy avoids autarky and the golden rule steady state. There are equilibria in which some paths reach autarky, others reach the steady state, while some paths cycle or exhibit chaos. Without placing restrictions on the offer curves, the only general statement about asymptotic behavior seems to be that consumption must stay within the bounds established by Theorem 3.

VI. Welfare Properties, With Two Examples

An appropriate measure of welfare for sunspot equilibria is ex ante expected utility. Thus, an allocation (possibly contingent on sunspots) is Pareto optimal if there is no other allocation yielding at least the same expected utility to all agents and strictly higher expected utility to at least one agent. All expectations are taken in period one.

Theorem 4: Sunspot equilibria are not Pareto optimal.[12]

Proof: Consumer t's ex ante expected utility in a S.E. is given by:

$$(23)\quad \sum_{s_2=1}^{n} \sum_{s_3=1}^{n} \cdots \sum_{s_t=1}^{n} \sum_{s_{t+1}=1}^{n} \pi_{s_2}\pi_{s_3} \cdots \pi_{s_t}\pi_{s_{t+1}} u[x_t^t(s^t),\ x_t^{t+1}(s^t,s_{t+1})]$$

Consider the allocation $\{(\bar{x}_t^t,\bar{x}_t^{t+1})\}_{t=1}^{\infty}$, which gives each agent his expected consumption, independent of sunspots. Mr. 0 gets the same utility that he gets in the S.E. For $t \geq 1$, $u(\bar{x}_t^t,\bar{x}_t^{t+1})$ equals

$$(24)\quad u[\sum_{s_2=1}^{n} \cdots \sum_{s_t=1}^{n} \sum_{s_{t+1}=1}^{n} \pi_{s_2} \cdots \pi_{s_t}\pi_{s_{t+1}} x_t^t(s^t);$$
$$\sum_{s_2=1}^{n} \cdots \sum_{s_t=1}^{n} \sum_{s_{t+1}=1}^{n} \pi_{s_2} \cdots \pi_{s_t}\pi_{s_{t+1}} x_t^{t+1}(s^t,s_{t+1})]$$

Because utility is a strictly concave function, the expression in equation (24) is strictly greater than the expression in equation (23). It remains to show that $\{(\bar{x}_t^t,\bar{x}_t^{t+1})\}_{t=1}^{\infty}$ is feasible.

$$\bar{x}_t^t + \bar{x}_{t-1}^t = \sum_{s_2} \cdots \sum_{s_t} \sum_{s_{t+1}} \pi_{s_2} \cdots \pi_{s_t}\pi_{s_{t+1}} x_t^t(s^t)$$
$$+ \sum_{s_2} \cdots \sum_{s_t} \pi_{s_2} \cdots \pi_{s_t} x_{t-1}^t(s^t)$$

$$\bar{x}_t^t + \bar{x}_{t-1}^t = \sum_{s_2} \cdots \sum_{s_t} [\pi_{s_2} \cdots \pi_{s_t} x^t(s^t)(\sum_{s_{t+1}} \pi_{s_{t+1}})]$$
$$+ \sum_{s_2} \cdots \sum_{s_t} [\pi_{s_2} \cdots \pi_{s_t} x_{t-1}^t(s^t)]$$

$$\bar{x}_t^t + \bar{x}_{t-1}^t = \sum_{s_2} \cdots \sum_{s_t} [\pi_{s_2} \cdots \pi_{s_t}][x_t^t(s^t) + x_{t-1}^t(s^t)]$$

$$= (\omega_t^t + \omega_{t-1}^t) \sum_{s_2} \cdots \sum_{s_t} (\pi_{s_2} \cdots \pi_{s_t}) = \omega_t^t + \omega_{t-1}^t$$

Therefore, $\{(\bar{x}_t^t, \bar{x}_t^{t+1})\}_{t=1}^{\infty}$ is feasible and dominates the sunspot equilibrium. []

It is a well-known fact that non-sunspot equilibria are often Pareto inefficient.[13] This fact calls for a comparison between the sunspot equilibria for a given initial condition and the non-sunspot equilibria for the same initial condition. It is not surprising that the non-sunspot equilibrium can dominate all sunspot equilibria. However, Example 1 shows that there are also economies for which a sunspot equilibrium dominates the unique non-sunspot equilibrium with the same initial condition. Thus, if a benevolent social planner is stuck with a bad initial condition, he may find it useful to introduce randomness to a previously deterministic economy.

Example 1:[14]

$u(x_t^t, x_t^{t+1}) = \log x_t^t + \log x_t^{t+1}$, $\omega = (2,1)$

initial condition: $p^m = 1/4$

market clearing implies:

(25) $$x_t^t = 2 - \frac{1}{4p^t};$$

(26) $$x_t^{t+1} = 1 + \frac{1}{4p^{t+1}};$$

$$x_1^1 = 7/4.$$

The non-sunspot equilibrium is found by maximizing utility subject to the budget constraint. We have

$$(27) \qquad \frac{1}{x_t^t} - \frac{p^t}{p^{t+1}} \frac{1}{x_t^{t+1}} = 0.$$

Solving equations (25), (26), and (27) yields the equilibrium price path

$$(28) \qquad p^{t+1} = \frac{4p^t - 1}{2}; \; p^1 = 1.$$

Consumption vectors for the first few consumers are:

$$x_1 = (\frac{7}{4}, \frac{7}{6}) \quad \text{and} \quad x_2 = (\frac{11}{6}, \frac{11}{10}).$$

Sunspot Equilibria

The sunspot tree is depicted in Figure 3.

In period one, there is probability π^2 of having the next generation consume x_2^2; otherwise the economy jumps to and stays at the steady state consumption level of 3/2. If the economy finds itself away from the steady state period t, then it jumps to the steady state and remains there with probability $1 - \pi^{t+1}$, and proceeds to x_{t+1}^{t+1} with probability π^{t+1}. This is a class of sunspot equilibria because the π's may be chosen arbitrarily to fix the x's. Maximizing expected utility yields:

$$(29) \qquad \pi^{t+1}[\frac{1}{x_t^t} - \frac{p^t}{p^{t+1}} \frac{1}{x_t^{t+1}}] + (1 - \pi^{t+1})[\frac{1}{x_t^t} - 2p^t 2/3] = 0,$$

where p^t and p^{t+1} are prices along the non golden rule path. The price level at the golden rule steady state is always p = 1/2. Equations (25), (26), and

(29) imply

$$(30)\qquad \pi^{t+1} = \left(\frac{2 - 4p^t}{8p^t - 1}\right)\left(\frac{4p^{t+1} + 1}{1 - 2p^{t+1}}\right).$$

Equation (30) shows that the probability of approaching autarky is bounded above zero. We have:

$$\underset{t=2}{\overset{T}{\text{Product}}}\ [\pi^t] \geq 2/7\ \frac{4p^T + 1}{2p^T - 1} \geq 4/7,$$

which implies

$$\text{Prob (autarky)} = \lim_{T\to\infty} \underset{t=2}{\overset{T}{\text{Product}}}\ [\pi^t] \geq 4/7.$$

Since only one path approaches autarky, its probability is easily defined.

We next turn to a comparison of ex ante expected utilities. Along the non-sunspot path, utilities are as follows:

$$u(x_1^1, x_1^2) = \log 7/4 + \log 7/6;$$

$$u(x_2^2, x_2^3) = \log 11/6 + \log 11/10;$$

$$u(x_t^t, x_t^{t+1}) < \log 11/6 + \log 11/10, \text{ for } t = 2,3,...$$

Now consider the S.E. in which $\pi^2 = 4/7$ holds. In this S.E., the economy either jumps to the golden rule steady state and stays there or jumps to the autarkic steady state and stays there. Expected utilities are as follows:

$$Eu(x_1^1, x_1^2) = \log 7/4 + 3/7 \log 3/2;$$

$$Eu(x_t^t, x_t^{t+1}) = 4/7 \log 2 + 6/7 \log 3/2, \text{ for } t = 2,3,\ldots$$

A quick calculation reveals that the expected utility of the S.E. is greater than the non-sunspot level for every consumer starting with Mr. 1. Mr. 0's utility level, determined by the initial condition, remains the same under both equilibria.

Sunspot-induced Cycles

One reason to study sunspot equilibria is for the insights they may provide to cyclical economic fluctuations. It seems likely that "animal spirits" is a psychological variable that influences capital investment and overall economic activity. This model does not include production, but a reinterpretation of the commodities allows the model to exhibit crude business cycles. In the Lucas-Azariadis style, we let ω_t^t represent the leisure endowment of Mr. t in his youth and ω_t^{t+1} his consumption endowment in old age. Young consumers get utility only from leisure and old consumers get utility only from the consumption good. Young consumers are able to transform their leisure directly into the consumption good, which they sell to the old consumers. By this reinterpretation, $\omega_t^t - x_t^t$ represents the labor supply (employment level) in period t.

Azariadis shows the existence of sunspot-induced cycles where the economy follows a stationary first order Markov process. To achieve stationarity, preferences must be restricted somewhat. However, the movements of the economy are not gradual; going from peak to trough only takes one period. More realistic looking cycles are possible in his framework if the transition matrix is N × N instead of 2 × 2.

Example 2 illustrates a procedure for constructing sunspot equilibria that cycle, and the offer curve is not required to bend backwards. The economy performs something of a random walk, making small movements period to period with the overall movement looking like random cycles or random cycles off a trend.

<u>Example 2</u>

$$u(x_t^t, x_t^{t+1}) = \log x_t^t + \log x_t^{t+1}, \ \omega = (2,1)$$

The features of the equilibrium will be described before fixing an initial condition. There are N consumption levels $(x^k)_{k=1}^N$ ranging from the steady state level $x^1 = 3/2$ to the autarky level $x^N = 2$. The economy follows a first order Markov process with stationary transition matrix:

$$
\begin{array}{c|ccccccccccc|}
 & 1 & & & & & & & & & & N \\
1 & 1 & 0 & 0 & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & 1/2 & 0 & 1/2 & & 0 & 0 & & 0 & 0 & 0 & 0 \\
 & 0 & 1/2 & 0 & & 1/2 & 0 & & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & & & & \\
 & 0 & 0 & 0 & & 0 & 0 & \cdots & 1/2 & 0 & 1/2 & 0 \\
 & 0 & 0 & 0 & & 0 & 0 & & 0 & \pi & 0 & 1-\pi \\
N & 0 & 0 & 0 & & 0 & 0 & & 0 & 0 & 0 & 1
\end{array}
$$

If the economy reaches the golden rule or autarkic steady state, it stays there. If x^{N-1} is reached, the economy proceeds to x^{N-2} with probability π and it proceeds to autarky with probability $1 - \pi$. Otherwise, consumption levels (for the young) move up or down one notch with probability 1/2.

The exact equilibrium is found by solving for the values of $x^2, x^3, \ldots, x^{N-1}$ and specifying π and the starting point. Assume the current consumption level is x^k. The first order conditions are given by

$$\frac{2}{x^k} = \frac{2 - x^{k+1}}{2 - x^k}\left(\frac{1}{3 - x^{k+1}}\right) + \frac{2 - x^{k-1}}{3 - x^{k-1}}\left(\frac{1}{2 - x^k}\right).$$

This expression can be simplified to:

$$\frac{4 - 2x^k}{x^k} = \frac{2 - x^{k+1}}{3 - x^{k+1}} + \frac{2 - x^{k-1}}{3 - x^{k-1}} \tag{31}$$

We can arbitrarily specify x^2, and equation (31) generates each successive consumption level from x^3 to x^{N-1}. The problem is that this difference equation eventually produces values above 2, which are inconsistent with equilibrium. We resolve this problem by choosing N low enough (or x^2 close enough to 1.5) so that we have $x^{N-1} < 2$. Then choose π so that x^{N-1} solves the first order conditions for expected utility maximization. For instance, the following is an equilibrium:

$x^1 = 1.5;$

$x^2 = 1.6;$

$x^3 = 1.8;$

$x^4 = 2;$

$\pi = 7/18;$

initial condition: $x_1^1 = 1.6$.

It should be noted that there is one major qualitative difference between this example exhibiting gross substitutability and those exhibiting gross complementarity. As time unfolds here, the economy eventually reaches one of the steady states; the probability of reaching the golden rule depends on where the economy starts. Thus, the impact of sunspots eventually disappears.[15] In the gross complement examples, the consumption levels can be picked so as not to include either steady state, so sunspots will always affect the economy without dying out.

VII. Concluding Remarks

By allowing arbitrary stochastic processes to cause sunspot equilibria, I am able to incorporate the equilibria of Shell [1977] and Azariadis [1981] into the same framework. This more general view of sunspots allows me to prove the existence of equilibria under less restrictive assumptions than those used previously. However, sunspot equilibria will no longer necessarily be stationary. Even when they are constructed to be stationary as in Example 2, the impact of sunspots may die out unless preferences are restricted somewhat. I also show that sunspot equilibria are not Pareto optimal in terms of ex ante expected utility, although they may dominate the corresponding non-sunspot equilibrium.

This model can be extended in several interesting directions. With more than one commodity per period, sunspots can affect the general price level and relative prices. It seems that sunspot equlibria will be even more prevalent than they are in the one commodity model.

One theme of this and other work on sunspots is that an open-ended future creates its own uncertainty. Not even rational expectations and completely foreseen market fundamentals can destroy the impact of psychological variables. The vast multiplicity of sunspot equilibria creates the problem of not knowing which equilibrium will arise. A promising line of future research is to analyze mechanisms which coordinate expectations, whether by consumer interactions or focal points.

When intrinsic uncertainty is added to the model, interesting complications arise. Small changes in realizations of the fundamentals could have a large impact on the economy if people allow them to affect their expectations. Budget deficits, for example, could influence the economy directly through fundamentals and indirectly through expectations.

Notes

[1]He was explaining investment behavior and there is no capital in this model, but it seems likely that "animal spirits" can be largely understood as extrinsic uncertainty.

[2]The appropriate definition of a S.E. is in terms of consumption rather than prices. If the money supply is always zero, we are in autarky. However, we can find prices $p^{t+1}(\alpha)$ and $p^{t+1}(\beta)$, unequal and consistent with the autarky equilibrium. The definition of a sunspot equilibrium should exclude this possibility.

[3]This kind of S.E. is discussed in Cass-Shell [1983].

[4]$P^m(\omega,\tau)$ is the set of perfect foresight equilibrium money prices for the economy with endowments ω and tax-transfer policy τ. See Balasko and Shell [1981], especially section 7, for a discussion.

[5]When the tax is nozero, an individual's endowment of goods and money, graphed in commodity space, will depend on the value of the tax. Mr. k's translated endowment is $(\omega_k^k - (p^m/p^k)\tau_k,\ \omega_k^{k+1})$. One of the difficulties of this proof is that the offer curves also shift as p^m/p^k changes.

[6]Here $\hat{p}^{k+1}(s^k,s_{k+1})$ is a perfect foresight equilibrium price, indexed by (s^k,s_{k+1}). This should be distinguished from the (proposed) sunspot equilibrium value of p^{k+1} when the sunspot history is (s^k,s_{k+1}), which is denoted $p^{t+1}(s^k,s_{k+1})$.

[7]This S.E. is being constructed so that realizations of sunspots after period k + 1 have no effect on prices. Thus, if $s^t = (s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_t)$ holds then we have $p^t(s^t) = p^t(s^k, s_{k+1})$.

[8]If the initial choice of p^m was too high, we can choose a lower p^m and redo the proof up until now.

[9]See Balasko and Shell [1983], section 4.

[10]Mr. t's ordinary demand function for current consumption is denoted $x_t^t(\frac{p^t}{p^{t+1}})$.

[11]See Shell, Sidrauski, Stiglitz [1969], especially section 3.

[12]This theorem was first proved for a static model by Cass and Shell [1983]. It was also stated in a more general overlapping generations context by Balasko [1983]. He was not assuming that consumers have Von-Neumann Morgenstern expected utility functions, and the proof was somewhat intricate. I am presenting the proof in this simpler framework to highlight its essential features.

[13]This point was made in Samuelson [1958]. Balasko and Shell [1980, especially section 4] show that all equilibria are weakly Pareto optimal in the sense that they cannot be dominated by allocations differing from the equilibrium allocation by a finite number of components.

[14]This type of S.E. was put forth by Azariadis and Guesnerie [1984].

[15]By eventually, I mean that the probability of the economy reaching one of the steady states by period t approaches 1 as t approaches infinity. At the steady states, sunspots can have no effect.

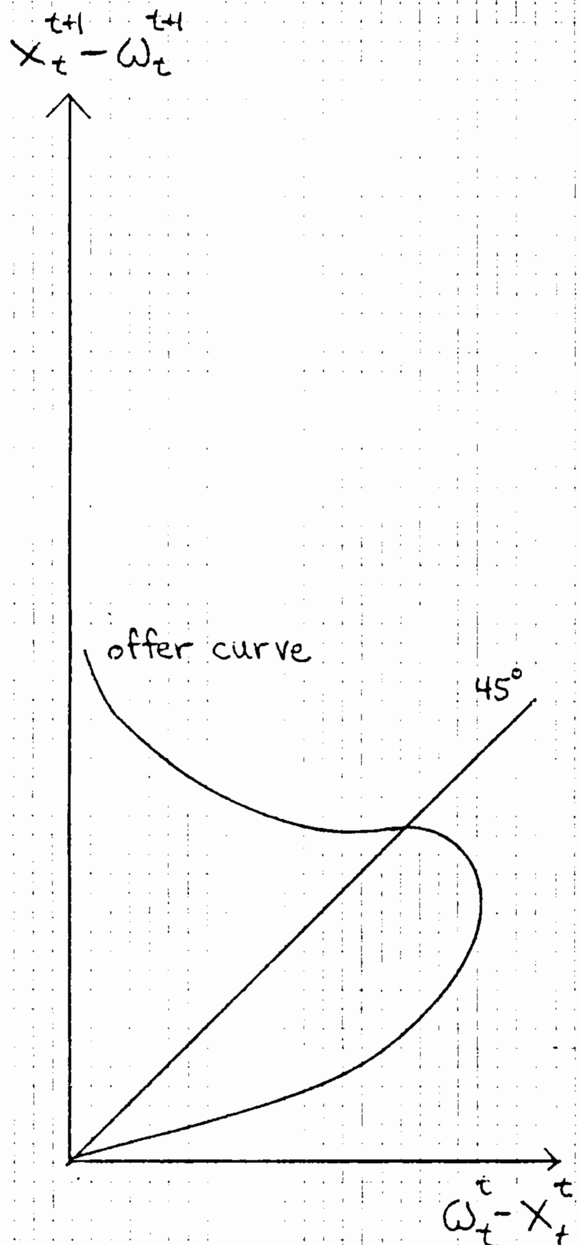


Figure 1a

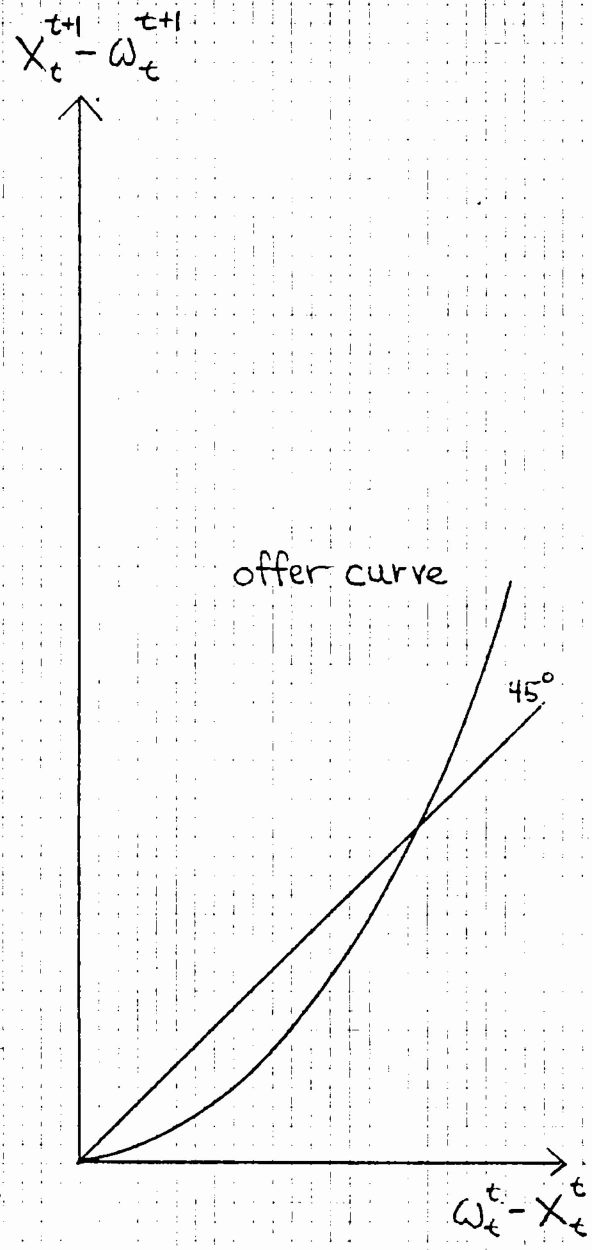


Figure 1b

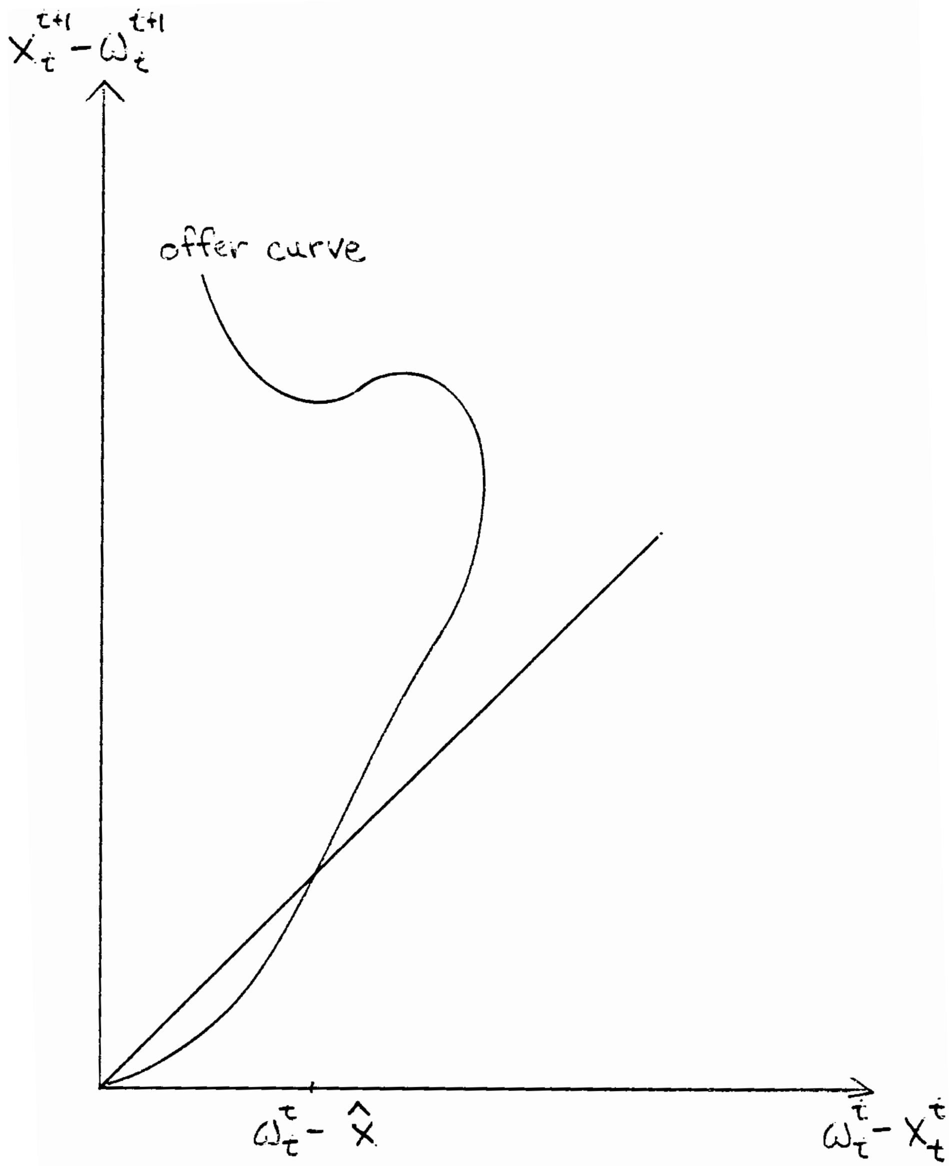
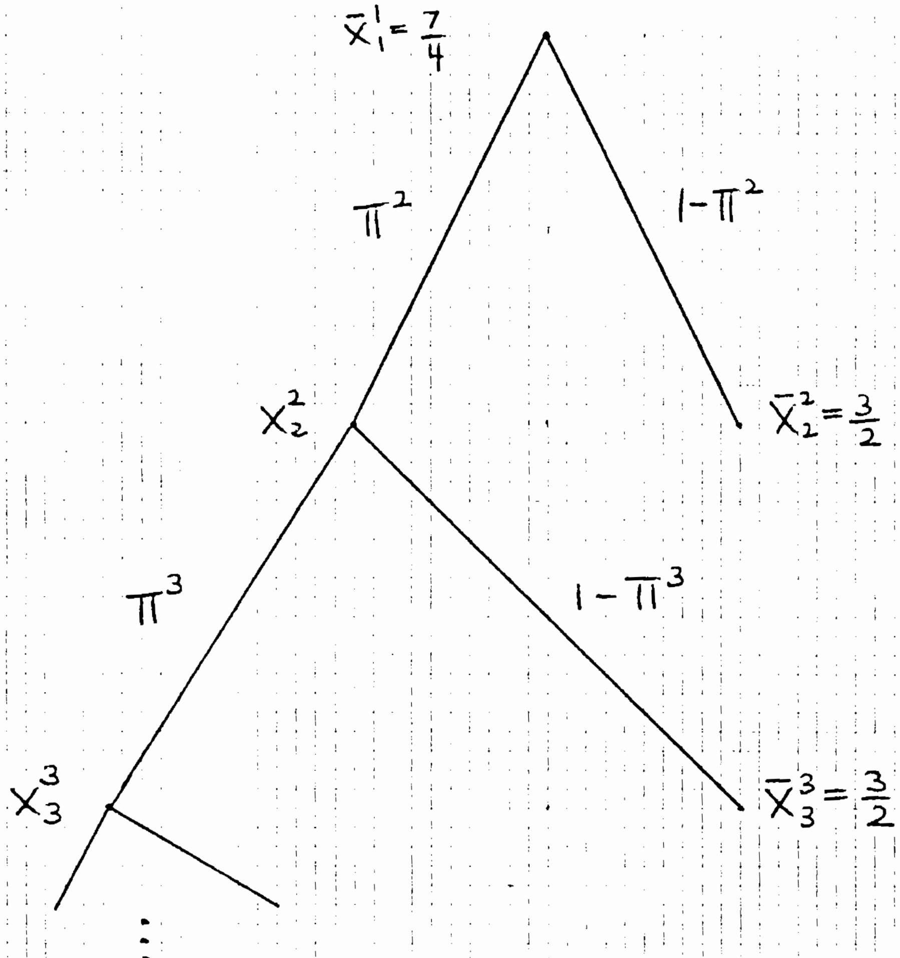


Figure 2



$\bar{x}_1^1 = \frac{7}{4}$
π^2
$1-\pi^2$
x_2^2
$\bar{x}_2^2 = \frac{3}{2}$
π^3
$1-\pi^3$
x_3^3
$\bar{x}_3^3 = \frac{3}{2}$

Figure 3

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