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ONE IS ALMOST ENOUGH FOR MONOPOLY

by

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ONE IS ALMOST ENOUGH FOR MONOPOLY

It has been argued that two factors — product durability and (potential) entry — may force a monopolist to price at marginal cost. This paper shows that when these two forces coexist, the tendency towards competition may be negated. First, we prove that durable goods <u>oligopolists</u> without commitment powers may attain joint profits arbitrarily close to those of a monopolist with perfect commitment power. Second, we demonstrate that the presence of a potential entrant may enable a durable goods <u>monopolist</u> to act as if she had commitment power. Thus, potential as well as actual entry may restore monopoly power.

1. Introduction

Classic economic analysis suggests that a monopolist has the ability to exercise market power: to charge greater than the competitive price and, often, to earn supranormal profits. Adam Smith was certainly not the first to make this observation when, in The Wealth of Nations, he wrote, "The monopolists, by keeping the market constantly under-stocked, by never fully supplying the effectual demand, sell their commodities much above the natural price, and raise their emoluments, whether they consist in wages or profits, greatly above their natural rate." Nor was he the last; most economics texts today faithfully recite a similar story.

Recently, two strands of research have called into question the conventional wisdom about monopoly. The first, introduced by Coase [1972], considers the problem of the durable goods monopolist. Coase observed the following intuition: Suppose that a monopolist were to offer a durable good for sale at the static monopoly price, and suppose that all consumers who valued the good at greater than the static monopoly price were to purchase the good. Then, if the static monopoly price exceeded the marginal cost of the good, the monopolist would have every incentive to cut the price of the good in order to generate additional sales; moreover, the process would continue until price equaled marginal cost. Consumers, anticipating this price-slashing behavior, would choose to postpone purchasing the good when faced with the static monopoly price. Rational consumer behavior forces the maximizing monopolist to introduce the good at close to marginal cost.

Bulow [1982] analyzed Coase's reasoning in a finite-horizon model. In the final period, a monopolist who lacks commitment power charges the static monopoly rental price for the residual demand curve. By backward induction, Bulow calculates the monopolist's best action in each earlier period, and shows that it is always to charge unambiguously less than the static monopoly price.

Stokey [1979, 1981] formalized and proved Coase's intuition in a series of two papers. In the first, she demonstrated that a monopolist who can make binding commitments about her future sales does best by introducing the good at the static monopoly price and never cutting the price afterwards. In other words, the best intertemporal price discrimination is no intertemporal price discrimination. In the second paper, Stokey considered the problem of the durable goods monopolist who lacks commitment powers. She constructed an equilibrium of the infinite-horizon model which is the limit of the unique equilibria of finite-horizon versions of the same model. The price path associated with that equilibrium (including the initial price) converges to marginal cost as the length of each period goes to zero; this proposition is commonly known as the "Coase Conjecture."[1]

Gul, Sonnenschein and Wilson [1986] modeled this situation as an infinite-horizon game between a single firm and a continuum of consumers. The notion of "no commitment" is captured by the requirement of subgame perfection of the equilibrium. These authors discovered a continuum of additional subgame perfect equilibria in this game, but proved that an interesting subclass (weak-Markov equilibria) behave like Stokey's backward induction equilibrium--they also satisfy the Coase Conjecture.[2]

The second strand of literature challenging the common wisdom about monopoly was introduced by Baumol, Bailey, Panzar, and Willig (see, for example, Baumol et al. [1982]). These researchers focus on the role of potential, costlessly-reversible entry in determining the monopoly outcome. In essence, they seize upon the result of Bertrand that "two is enough for competition"--under constant marginal (and average) cost, the unique duopoly Nash equilibrium in price strategies is for both firms to engage in marginal-cost-pricing. The Contestable Markets literature goes further by arguing that under certain conditions of monopoly and entry (most notably, zero sunk cost), "one is almost enough for competition".

In this paper we attempt to combine and extend the reasoning of the Coase Conjecture and Contestable Markets. We prove that for durable goods oligopoly, as the time interval between successive offers approaches zero, all joint payoffs between zero and static monopoly profits are attainable. Moreover, we show that actual entry into the market is unnecessary to obtain this result: a durable goods monopoly with a potential entrant (who, in equilibrium, never enters) may also earn static monopoly profits. We further find the full set of equilibrium joint profits associated with arbitrary discount factors. Our results indicate that even when firms discount the future significantly, effective collusion can still occur.

Closely related to the present paper is Faruk Gul's "Foundations of Dynamic Oligopoly," also appearing in this issue of the Rand Journal of Economics. Many of the results in these two papers are quite similar. However, Gul emphasizes the limiting result for more general demand curves and treats the case of asymmetric market shares. We provide a precise description of optimally collusive equilibria for a parameterized family of demand curves, for all discount factors, and stress the model of potential entry. Thus, the two papers are complementary.

In Section 2 of the paper, we describe the model and establish two theorems on subgame perfect equilibria (SPE's). Section 3 and Appendix A characterize optimally collusive and minimally collusive SPE's. In section 4, we find the entire set of joint profits supported by SPE's, for all discount factors δ, and we indicate its limiting behavior as $\delta \to 1$. We establish analogous results for monopoly with potential entry in section 5. Some details are reconciled to Appendix B. In the conclusion, we compare the present results to those of a sequel paper (Ausubel and Deneckere, 1986), which literally studies durable goods monopoly.

2. A Durable Goods Oligopoly

It is reasonable to suppose that an oligopoly behaves more competitively than a monopoly. Hence, if one believed that profits in a durable goods monopoly were drastically curtailed by the Coase Conjecture, one would probably also suspect the same of durable goods oligopoly.[3] After all, how could the presence of a rival make the monopolist better off? In order to understand why this intuitive argument is misleading, it is instructive to reflect on the reason why a monopolist may be forced to act competitively. The problem is essentially this: once an initial quantity of the good has been sold, the monopolist will find it tempting to sell some additional output as long as her accumulated output sold remains below the competitive level. If there is virtually no restraint to the speed at which the monopolist can sell additional units (if the time interval between successive offerings is very small), the market will almost immediately be saturated with the competitive output. Rational consumers will foresee this, and purchase the good at no more than the competitive price. It is this inability of the monopolist to control the speed at which she sells, or put somewhat differently, her inability to punish her future types for deviant behavior[4], that impairs her monopoly power. The monopolist may thus be better off having a competitor around to punish her (or her future types) for selling off the good too quickly.

We consider a market for a good which is infinitely durable, and which is demanded only in quantity zero or one. Consumers, who are infinitely lived, have reservation prices for the good which are distributed over the interval $V = [0,1]$ according to the distribution function $F(v) = v^{\alpha}$ $(0 < \alpha < \infty)$. Because they discount future utility at the interest rate r, individuals who have a reservation value of v, and who obtain the good at time t for the price p_t, derive a net surplus given by

$$e^{-rt}[v - p_t]$$

Two producers serve this market which is open at discrete times spaced z apart $(t = 0,z,2z,\ldots,nz,\ldots)$. The timing of moves within the period is as follows: firms first name their prices. Consumers (who did not buy in previous periods) then decide whether or not to purchase the good in the current period. We will sometimes refer to the "period" n rather than to the time t (=nz). There is no constraint on the amount of output any producer can supply at a given date, and production occurs at a common marginal (and average) cost of c, which we assume, for convenience, to be equal to zero. Firms are interested in maximizing the net present value of profits, discounted at the interest rate r.

A _strategy_ for producer i specifies the price he will charge at each moment in time as a function of the history of prices charged by both competitors, and the history of purchases by consumers. A strategy for a consumer specifies, given the current price charged and the history of past prices and purchases, whether or not to buy the good in the current period.

The next two paragraphs are technical in nature and may be skipped on a first reading. Let G(z,r) denote the above game when the time interval between successive sales is z and payoffs are discounted at the rate r. Let

σ_i be a <u>pure</u> strategy for producer i (i = 1,2). σ_i is a sequence of functions $\{\sigma_i(n)\}_{n=0}^{\infty}$. The function $\sigma_i(n)$ at date nz determines i's price in period n as a function of the prices charged by both competitors in previous periods, and the actions chosen by consumers in the past. The latter history is conveniently summarized by the set $V_n = \{v$: consumer v did not buy at any time $t < nz\}$.[5] We impose measurability restrictions on joint consumer strategies below which imply that the set V_n will be a measurable set, i.e., $V_n \in \Omega$, where Ω is the Borel σ-algebra on V. Then $\sigma_i(n)$: $S^{n-1} \times \Omega \to S_i$, with $S_i = [0,1]$ for i = 1,2 and $S = S_1 \times S_2$. A strategy combination for consumers is a sequence of functions $\{\tau^n\}_{n=0}^{\infty}$ where τ^n: $S^n \times \Omega \times V \to \{0,1,2\}$ is such that for each $s^n \in S^n$ and each $B \in \Omega$, $\tau^n(s^n,B,\ )$ is measurable. Decision "0" here is to be interpreted as "do not buy from either supplier." The decisions "i" (i = 1,2) mean "buy from supplier i." Without loss of generality, we may assume that τ^n is zero in any coordinate v such that $\tau^j > 0$ in the coordinate v for some period prior to n.

Let Σ_i be the strategy space for player i (i = 1,2) and T the set of strategy combinations for consumers. Furthermore let $\sigma = (\sigma_1,\sigma_2)$ and $\tau = \{\tau^n\}_{n=0}^{\infty}$. The strategy profile $\{\sigma,\tau\}$ generates a path of prices and sales which can be computed recursively. The pattern of prices and sales over time in turn determines the payoff to the players. Let $\pi^i(\sigma,\tau)$ be the discounted present value of the profits of firm i, generated by the strategy profiles (σ,τ), and let $u^v(\sigma,\tau)$ be the discounted net surplus derived by consumer v. The profile (σ,τ) is a Nash equilibrium if:

$$\pi^i(\sigma_i,\sigma_{-i},\tau) \geq \pi^i(\sigma_i',\sigma_{-i},\tau), \quad \forall\ \sigma_i' \in \Sigma_i\ , \forall\ i.$$

and

$$u^v(\sigma,\tau_v,\tau_{-v}) \geq u^v(\sigma,\tau_v',\tau_{-v}), \quad \forall\ \tau_v' \in T_v\ , \forall\ v \in V.$$

where τ_v is the projection of τ onto the v-th coordinate (and similarly for T_v). An n-period history of the game is a sequence of prices for each firm in periods $0,\ldots,(n-1)$ and a specification of the set of consumers who did not buy in any period prior to n. We denote such a history by the symbol H_n. The symbol H_n' refers to H_n followed by prices announced by both firms in period n. The strategy profile (σ,τ) induces strategy profiles $(\sigma|_{H_n},\tau|_{H_n})$ and $(\sigma|_{H_n'},\tau|_{H_n'})$ after the histories H_n and H_n', respectively. (σ,τ) is a <u>subgame perfect</u> equilibrium if and only if (σ,τ) is a Nash equilibrium and $(\sigma|_{H_n},\tau|_{H_n})$ is a Nash equilibrium in the game remaining after the history H_n, for all H_n and for all n, and similarly after any history H_n'.

Denote the price charged by firm i in period n along the outcome path generated by (σ,τ) as $p_n^i(\sigma,\tau)$, and let $p_n(\sigma,\tau) = \min_i p_n^i(\sigma,\tau)$. In our first theorem, we prove that a necessary condition for subgame perfection is that prices be set to eventually induce <u>all</u> customers (with valuations exceeding marginal cost) to purchase. Otherwise, the profits from following the equilibrium approach zero, while the profits from deviating remain positive.[6]

<u>Theorem 1</u>: In any subgame perfect equilibrium of $G(z,r)$, $\inf p_n(\sigma,\tau) = 0$.

<u>Proof</u>: Suppose to the contrary that $\hat{p} = \inf_n p_n(\sigma,\tau) > 0$. We will show that there exists an n sufficiently large such that each firm has an incentive to deviate from its strategy σ_i.

First, we calculate a <u>lower bound</u> on the net present value of profits generated by a deviation in period n consisting of charging a price $p < p_n$ (for some firm $i = 1,2$). Such a deviation will be the least profitable when it leads to maximally competitive behavior in future periods (i.e., both firms charge a zero price in the next period). In that case, a deviation engenders zero profits in subsequent periods and makes consumers extremely averse to buying in period n. The profits from deviating in period n can in turn be bounded below by assuming that all customers with valuations greater than $\hat{p}$ have already been served. By charging a price $p \leq \hat{p}$, our deviant will attract all customers with valuation exceeding $\tilde{v}$ such that $\tilde{v} - p \geq e^{-rz}\tilde{v}$. Letting $\delta = e^{-rz}$, we have $\tilde{v} \geq p/(1 - \delta)$. Indeed, those customers will find it attractive to buy in period n at the price p rather than waiting until period (n + 1) for a zero price. Recall that the number of customers with valuation less than v is given by v^{α}. By setting a price p then, profits in period n are at least

$$[\hat{p}^{\alpha} - (\frac{p}{1 - \delta})^{\alpha}]p$$

This expression is maximized at $p = \rho(1 - \delta)\hat{p}$, where $\rho = (1 + \alpha)^{-1/\alpha}$, with a corresponding profit of $\mu(1 - \delta)\hat{p}^{(1+\alpha)}$, where $\mu = \alpha\rho/(1 + \alpha)$. Thus, a firm will certainly choose to deviate if for some n:

$$\mu(1 - \delta)\hat{p}^{(1+\alpha)} > \pi_n(\sigma,\tau)$$

where $\pi_n(\sigma,\tau)$ is the net present value (evaluated in period n) to a single firm in the subgame starting after the equilibrium history H_n. It is clear that $\pi_n(\sigma,\tau) \to 0$ as $n \to \infty$ (for large enough n, either $p_n = \hat{p}$ or p_n gets arbitrarily close to $\hat{p}$, so that the set of customers who remain to be served becomes arbitrarily small). Thus, the above inequality will be satisfied for large enough n unless $\hat{p} = 0$. $\square$

Observe that ρ (= $[1 + \alpha]^{-1/\alpha}$) in the above proof has been defined to equal the static monopoly price when $F(v) = v^{\alpha}$, and μ (= $\rho[1 - \rho^{\alpha}]$) equals the static monopoly profits.

The worst subgame perfect equilibrium payoff (from the sellers' viewpoint) is easily seen to be equal to zero (it is implemented by the strategies $\sigma_i(n) = 0$ and $\tau_v^0 = 1$ or 2). In fact, this payoff coincides with the sellers' minmax payoff. This equilibrium will be useful as a "punishment" regime following deviations from more collusive price paths (as in Friedman [1971], Aumann and Shapley [1976], and Abreu [1984]). Next, we derive necessary and sufficient conditions which any subgame perfect price path $\{p_n\}$ must satisfy (we suppress the dependence of p_n and π_n on the equilibrium strategies (σ,τ) whenever no confusion is likely to arise). Let $v_n = \sup\{v: \tau_v^j = 0 \text{ for all } j \leq n-1\}$, i.e., v_n is the highest consumer valuation remaining as we enter period n. Observe that the single number v_n summarizes along the equilibrium path the actions chosen by consumers in periods prior to n. Indeed, consumer optimization implies that whenever some consumer v has bought prior to period n, any consumer with valuation $v' \geq v$ should also have bought prior to n. Theorem 2 below states that in every period, the net present value of profits along the equilibrium path, for each firm, must exceed the profits from optimally deviating.

Theorem 2: Any subgame perfect price path $\{p_n\}$ of G(z,r) must satisfy:

(1) $$\pi_n^i \geq \mu(1-\delta)v_n^{1+\alpha} \text{ for all n such that } p_n \geq \rho(1-\delta)v_n$$

(2) $$\pi_n^i \geq p_n[v_n^\alpha - (\frac{p_n}{1-\delta})^\alpha] \text{ for all n such that } p_n < \rho(1-\delta)v_n$$

Conversely, any price path $\{p_n\}$ that satisfies (1) and (2) is an equilibrium price path.

Proof: Suppose that the inequality of (1) is not satisfied for some n where $p_n > \rho(1-\delta)v_n$. By setting the price $p = \rho(1-\delta)v_n$ at date n, and thereby undercutting p_n, a firm will earn at least $\mu(1-\delta)v_n^{1+\alpha}$ in immediate profits in period n. Since this exceeds π_n, any firm has an incentive to deviate from $\{p_n\}$. Similarly, suppose that inequality (2) is not satisfied for some n where $p_n < \rho(1-\delta)v_n$. Then by charging a price slightly below p_n, any firm can earn, at least, profits arbitrarily close to $[v_n^\alpha - (\frac{p_n}{1-\delta})^\alpha]p_n$, again contradicting the hypothesis that $\{p_n\}$ is a subgame perfect price path.

Conversely, any price path $\{p_n\}$ satisfying (1) and (2) can be supported by the threat (and corresponding consumer expectations) of reversion to the subgame perfect equilibrium that involves pricing at marginal cost as soon as any firm deviates.[7] □

3. Optimally Collusive Subgame Perfect Equilibria

It is evident that, in general, a multiplicity of SPE's satisfy the necessary and sufficient conditions of Theorem 2. In this section, we find the optimally collusive equilibrium, i.e., the SPE which maximizes joint profits. We also find the minimal (nonzero)[8] joint profits associated with any SPE. In order to keep this characterization elegant, we will assume--for reasons extraneous to the model--an "equal division rule." In any period in which both firms charge identical prices, an equal fraction of consumers chooses to patronize each firm.[9]

The potentially formidable task of characterizing optimally (and minimally) collusive SPE's is greatly simplified by Theorems A.1 and A.2 of Appendix A. These theorems demonstrate that there is no loss of generality in restricting attention to simple strategy profiles. Such profiles are completely described by a pair of numbers (p_0,ε). Here p_0 is the price to be charged in period 0; ε denotes the fraction by which firms lower prices in each subsequent period. The equilibrium price sequence corresponding to the simple strategy profile (p_0,ε) is thus $p_n = p_0\varepsilon^n$. This price sequence implies a pattern of historic variables v_n of:

$$v_n = \frac{p_{n-1} - \delta p_n}{(1-\delta)} = \frac{(1-\delta\varepsilon)}{(1-\delta)} p_{n-1} \equiv \gamma p_{n-1}, \quad n \geq 1$$

Since v_n is strictly decreasing in n as long as $\varepsilon < 1$, a simple strategy profile implies sales in all periods if $v_1 = \gamma p_0 < 1$. Sales in period n can be calculated to be:

$$v_n^\alpha - v_{n+1}^\alpha = (1-\varepsilon^\alpha)\gamma^\alpha p_{n-1}^\alpha$$

for $n \geq 1$, provided $p_0 < \gamma^{-1}$. We can now compute π_n along the sequence $p_n = p_0\varepsilon^n$ (with $p_0 < \gamma^{-1}$):

$$\pi_n = \frac{1}{2}\sum_{k=0}^{\infty} \delta^k p_{n+k}(v_{n+k}^\alpha - v_{n+k+1}^\alpha) = \frac{1}{2}\frac{\gamma^\alpha\varepsilon(1-\varepsilon^\alpha)}{1-\delta\varepsilon^{1+\alpha}} p_{n-1}^{1+\alpha}$$

for $n \geq 1$, and for $n = 0$,

$$\pi_0 = \frac{1}{2}[1 - \gamma^\alpha p_0^\alpha]p_0 + \frac{\delta}{2}\frac{\gamma^\alpha\varepsilon(1-\varepsilon^\alpha)}{1-\delta\varepsilon^{1+\alpha}} p_0^{1+\alpha}$$

To complete the description of simple strategy profiles, we still have to describe the off-equilibrium-path behavior of firms. We let deviations from the path $p_n = p_0\varepsilon^n$ be punished by both firms reverting to the <u>worst</u> subgame perfect equilibrium, i.e., pricing at marginal cost in every future period.

We can immediately rule out any simple profile (p_0,ε) for which $p_0 \geq \gamma^{-1}$ as potentially optimally collusive; it is dominated by the simple profile

$(\varepsilon p_0, \varepsilon)$.[10] In our quest for optimal simple strategy profiles, the following lemma will be useful:

Lemma 1: A simple strategy profile (p_0, ε) with $\rho(1 - \delta) \leq p_0 < \gamma^{-1}$ is a subgame perfect equilibrium if and only if $\varepsilon \geq \rho/(1 + \rho\delta)$,

$$\text{(i)} \quad \frac{\varepsilon(1 - \varepsilon^{\alpha})}{1 - \delta\varepsilon^{1+\alpha}} \geq 2\mu(1 - \delta\varepsilon)$$

and

$$\text{(ii)} \quad p_0 - \gamma^{\alpha} p_0^{1+\alpha}\left[\frac{1 - \delta\varepsilon}{1 - \delta\varepsilon^{1+\alpha}}\right] \geq 2\mu(1 - \delta)$$

If $0 < p_0 < \rho(1 - \delta)$, the simple strategy profile (p_0, ε) is an SPE if and only if $\varepsilon \geq \rho/(1 + \rho\delta)$, (i) holds and

$$\text{(iii)} \qquad p_0 - \gamma^{\alpha} p_0^{1+\alpha}\left[\frac{1 - \delta\varepsilon}{1 - \delta\varepsilon^{1+\alpha}}\right] \geq 2p_0\left[1 - \left(\frac{p_0}{1 - \delta}\right)^{\alpha}\right].$$

Proof: The condition $\varepsilon \geq \rho/(1 + \rho\delta)$ implies that $p_n \geq \rho(1 - \delta)v_n$ for $n \geq 1$. If $p_0 \geq \rho(1 - \delta)$, Theorem 2 implies (p_0, ε) is an SPE iff:

$$\pi_n = \frac{\gamma^{\alpha}\varepsilon(1 - \varepsilon^{\alpha})}{2(1 - \delta\varepsilon^{1+\alpha})} p_{n-1}^{1+\alpha} \geq \mu(1 - \delta)v_n^{1+\alpha} = \mu(1 - \delta)\gamma^{1+\alpha} p_{n-1}^{1+\alpha}$$

$$\pi_0 = \frac{1}{2}\left\{p_0 - \gamma^{\alpha} p_0^{1+\alpha}\left(1 - \frac{\delta\varepsilon(1 - \varepsilon^{\alpha})}{1 - \delta\varepsilon^{1+\alpha}}\right)\right\} \geq \mu(1 - \delta)$$

Simple algebraic manipulations then yield (i) and (ii). If $p_0 < \rho(1 - \delta)$, inequality (2) of Theorem 2 is applicable for $n = 0$, yielding (iii) in place of (ii).

If $\varepsilon < \rho/(1 + \rho\delta)$, $p_n < \rho(1 - \delta)v_n$ and Theorem 2 now requires

$$\pi_n = \frac{\gamma^{\alpha}\varepsilon(1 - \varepsilon^{\alpha})}{2(1 - \delta\varepsilon^{1+\alpha})} p_{n-1}^{1+\alpha} \geq \left[\gamma^{\alpha} - \left(\frac{\varepsilon}{1 - \delta}\right)^{\alpha}\right]\varepsilon p_{n-1}^{1+\alpha}$$

or, $(1 - \varepsilon^{\alpha}) \geq 2(1 - \delta\varepsilon^{1+\alpha})(1 - (\varepsilon/(1 - \delta\varepsilon))^{\alpha})$. Some straightforward, but tedious calculations show that no $\varepsilon < \rho/(1 + \rho\delta)$ satisfies this inequality. □

Lemma 2: For every $\alpha > 0$, there exists a $\bar{\delta}(\alpha)$, such that for all δ satisfying $\bar{\delta}(\alpha) \leq \delta < 1$, the optimally collusive simple strategy profile uses a rate of descent $\varepsilon_{+}(\delta)$ defined by:

$$\varepsilon_{+} \equiv \sup E, \text{ where } E = \{\varepsilon: \varepsilon(1 - \varepsilon^{\alpha}) \geq 2\mu(1 - \delta\varepsilon)(1 - \delta\varepsilon^{1+\alpha})\}$$

and sets an initial price:

$$p_0 = \frac{\rho(1 - \delta)(1 - \delta\varepsilon_{+}^{1+\alpha})}{(1 - \delta\varepsilon_{+})^{1+1/\alpha}}$$

Proof: We have already argued that the optimal simple strategy profile must satisfy $p_0 < \gamma^{-1}$. For any ε, such simple strategy profiles yield joint profits of:

$$\Pi_0 = p_0 - \frac{(1 - \delta\varepsilon)^{1+\alpha}}{(1 - \delta)^{\alpha}(1 - \delta\varepsilon^{1+\alpha})} p_0^{1+\alpha}$$

The unconstrained maximum of this function occurs at:

$$p_0 = \frac{\rho(1 - \delta)(1 - \delta\varepsilon^{1+\alpha})}{(1 - \delta\varepsilon)^{1+1/\alpha}}$$

which exceeds $\rho(1 - \delta)$ for all (ε,δ). Since the choice of p_0 does not affect subsequent incentive constraints (i.e., (i) in Lemma 1), the above equation defines the optimal p_0 as a function of ε. Substituting the optimal value for

p_0 into the objective function yields joint profits of

$$\hat{\Pi}(\varepsilon) = \mu\left(\frac{1-\delta}{1-\delta\varepsilon}\right)\left(\frac{1-\delta\varepsilon^{1+\alpha}}{1-\delta\varepsilon}\right)^{1/\alpha}$$

The <u>optimal</u> ε maximizes $\hat{\Pi}(\varepsilon)$ subject to the incentive constraints (i). Since $\partial\hat{\Pi}/\partial\varepsilon > 0$, this amounts to choosing the largest value of ε which satisfies the incentive compatibility constraints. It is proven in Appendix A that the set E is nonempty if and only if $\delta \geq \bar{\delta}(\alpha)$, where $\bar{\delta}(\alpha) \leq .59$ for all $\alpha > 0$. $\square$

Minimally collusive simple strategy profiles (with $\Pi_0 > 0$) are established in Theorem A.2 of Appendix A.

Next, we wish to show that the optimally collusive joint profits, denoted $\Pi_+(\delta)$ ($\equiv \hat{\Pi}(\varepsilon_+(\delta))$), converge to the profits which a monopolist with commitment power could make. (Recall, from Stokey [1979], that this monopolist maximizes by charging the static monopoly price ρ forever.) Consider any fixed, positive <u>real-time</u> rate of descent in price. If price follows the rule ρe^{-st}, where $s > 0$, and the interval between successive periods is z, we have $p_n = \rho e^{-nsz}$. Now let z approach zero. The equilibrium joint profit stream (given by (A.2)) is negligibly affected by the choice of z; in particular, Π_n converges to a positive constant times $v_n^{1+\alpha}$ as $z \to 0$, for all $n \geq 1$. However, the profits Π_n^{dev} from optimally deviating equal $\mu(1-\delta)v_n^{1+\alpha}$, and $\mu(1-\delta) \to 0$ as $z \to 0$. Hence, for small but positive s, there exists a largest z such that subgame perfection is satisfied in all but (possibly) the initial period whenever the time interval between periods does not exceed z. Let $\delta = e^{-rz}$ and $\varepsilon = e^{-sz}$. Observe that ε equals $\varepsilon_+(\delta)$ of Lemma 2.

By setting s near zero, Stokey's (constant) intertemporal pricing scheme can be arbitrarily closely approximated. Incentive compatibility is preserved (in all periods $n \geq 1$), provided z is sufficiently small. Then, virtually all consumers with valuations exceeding ρ will purchase at price ρ in period zero, yielding joint profits $\Pi_0 \approx \mu$ (and guaranteeing incentive compatibility in period zero). We conclude that the optimally collusive joint payoff $\Pi_+(\delta)$ converges to static monopoly profits μ, as $\delta \to 1$ (i.e., $z \to 0$). Furthermore, the minimally collusive joint payoff converges to zero.

Lemma 3: $\varepsilon_+(\delta) \to 1$ as $\delta \to 1$. Furthermore, $p_0(\varepsilon_+) \to \rho$, $\Pi_+(\delta) \to \mu$ and $\Pi_-(\delta) \to 0$.

Proof: $\varepsilon_+(\delta)$ satisfies the equation:

$$\psi(\varepsilon) = 2\mu\delta^2\varepsilon^{2+\alpha} + (1 - 2\mu\delta)\varepsilon^{1+\alpha} - (1 + 2\mu\delta)\varepsilon + 2\mu = 0$$

Appendix A establishes that $\varepsilon_+(\delta) \to 1$ as $\delta \to 1$, and that $\lim_{\delta \to 1} \partial\varepsilon_+/\partial\delta = 0$. Using l'Hopital's rule, it is then easily verified that $p_0(\varepsilon_+(\delta)) \to \rho$ and $\hat{\Pi}(\varepsilon_+(\delta)) \to \mu$. Also observe that the price path corresponding to $v_n = \varepsilon_+^n$ is feasible and yields joint profits of $2\mu(1 - \delta)$. Thus, $\Pi_-(\delta) \leq 2\mu(1 - \delta)$ and $\lim_{\delta \to 1} \Pi_-(\delta) = 0$. □

4. The Set of Equilibrium Joint Profits

For every distribution function F, real interest rate $r > 0$, and time interval between periods $z > 0$, let SPE (F,r,z) denote the set of equilibrium joint profits (i.e., the set of all $\Pi_0 = \pi_0^1 + \pi_0^2$ arising from subgame perfect equilibria). We have the following theorem.

Theorem 3: For every $\alpha > 0$, consider the durable goods oligopoly model where consumers' valuations v are distributed by $F(v) = v^\alpha$ and where consumers are allocated by an equal division rule when firms charge identical prices. Then there exist positive valued functions $\bar{\delta}(\alpha)$, $\Pi_+(\alpha,\delta)$ and $\Pi_-(\alpha,\delta)$ such that:

$$\mathrm{SPE}(v^{\alpha},r,z) = \begin{cases} \{0\} \cup [\Pi_{-}(\alpha,\delta),\Pi_{+}(\alpha,\delta)], & \text{if } \delta \equiv e^{-rz} \geq \bar{\delta}(\alpha) \\ \{0\}, & \text{if } \delta \equiv e^{-rz} < \bar{\delta}(\alpha) \end{cases}$$

$\Pi_{+}(\alpha,\delta)$ was explicitly developed for duopoly in section 3, and $\Pi_{-}(\alpha,\delta)$ and $\bar{\delta}(\alpha)$ are developed in Appendix A. Furthermore:

$$\lim_{\delta \to 1} \Pi_{+}(\delta) = \mu \text{ and } \lim_{\delta \to 1} \Pi_{-}(\delta) = 0.$$

Proof: Suppose $\delta \geq \bar{\delta}(\alpha)$. By Lemma 2, Theorem A.1 and Theorem A.2, optimally collusive and minimally collusive joint payoffs are supported by simple strategy profiles (p_0,ε), both setting $\varepsilon = \varepsilon_{+}$. Now observe that Π_0 is a continuous function in p_0 for fixed ε. Furthermore, the set of all p_0 satisfying the conditions of Lemma 1, for fixed ε, is connected. By continuously varying the initial price, we can obtain all intermediate payoffs while preserving subgame perfection. The Bertrand equilibrium of $p_n = 0$ for all n is also an equilibrium.

If $\delta < \bar{\delta}(\alpha)$, subgame perfection requires $p_0 = 0$. The limiting result follows from Lemma 3. □

(Insert Figure 1 About Here)

In Figure 1, we graphically depict the set of equilibrium joint profits when consumer valuations are uniformly distributed over [0,1] (i.e., $\alpha = 1$). When the discount factor δ is less than $\bar{\delta}(1) \approx .585$, the inequality $\varepsilon(1 - \varepsilon) \geq .5(1 - \delta\varepsilon)(1 - \delta\varepsilon^2)$ has no solutions for ε in [0,1], and so the Bertrand equilibrium is the unique SPE. At $\delta = \bar{\delta}$, the unique rate of descent $\varepsilon = .7235$ is viable for positive p_0. However, p_0 may range anywhere from .345

to .522, giving joint profits anywhere from .207 to .217. Then, as δ increases to one, the set of equilibrium joint profits spreads to the entire interval from zero to static monopoly profits.

The behavior away from the limit implied by Theorem 3 (and Figure 1) is striking. First, observe that for $\bar{\delta}(\alpha) \leq \delta < 1$, the equilibrium set is not connected. Marginal cost pricing is always an equilibrium, but to support tacit collusion, profits must be bounded away from zero. The intuition for this result is that future profits are the inducement to prevent present deviations. If future profits, compared to the highest remaining consumer valuation, are excessively low, it necessarily pays for one firm to undercut its rival. Second, observe that effective collusion is quite possible far away from the limit. Even when $\delta = .585$, the duopolists, if they collude at all, must earn at least 83 percent of the static monopoly surplus, and possibly as much as 87 percent.

The effect of varying the time interval z between successive offers on optimal collusion by oligopolists is exactly opposite of that on a durable goods monopolist confined by the Coase Conjecture. When z approaches infinity (i.e., $\delta \to 0$), the durable goods monopolist becomes, for all practical purposes, a static monopolist, and hence can approach static monopoly profits. Meanwhile, the oligopolists find themselves playing, for all practical purposes, a one-shot Bertrand game. On the other hand, as z approaches zero (i.e., $\delta \to 1$), the monopolist in a Coase Conjecture[11] SPE drops to marginal cost "in the twinkling of an eye" (Coase, 1972). However, when there are at least two firms in the market, the ability to collude increases as z goes to zero.

The intuition for the oligopoly result is clear: the incentive to cheat on any collusive agreement (enforced by the type of trigger strategies discussed above) goes to zero as z approaches zero, whereas the loss from future retaliation stays roughly constant. The reason why the incentive to cheat vanishes is the anticipatory behavior of consumers. Indeed, when z becomes smaller, fewer and fewer consumers will be willing to buy from the price-cutter, as they expect a better deal (a price equal to marginal cost) in the next round.

This intuition makes it clear that the limiting result of Theorem 3 does not depend on our assumption that $F(v) = v^{\alpha}$. Indeed, the main theorem in Gul [1987] is stated in terms of arbitrary demand curves. Furthermore, it is easy to argue that the behavior away from the limit (depicted in Figure 1) holds quite generally. For arbitrary F( ), define $\underline{v}$ and $\bar{v}$ by $F(\underline{v}) = 0$, $F(\bar{v}) = 1$, and $0 < F(v) < 1$ whenever $\underline{v} < v < \bar{v}$. Now suppose there exist $\alpha > 0$ and $L \geq M > 0$ such that:

$$(3) \qquad M(v - \underline{v})^{\alpha} \leq F(v) \leq L(v - \underline{v})^{\alpha}, \text{ for all } v \text{ such that } \underline{v} \leq v \leq \bar{v},$$

i.e., F(v) is "enveloped" by $L(v - \underline{v})^{\alpha}$ and $M(v - \underline{v})^{\alpha}$). In Appendix B, we demonstrate that when (3) is satisfied,[12] tight bounds showing rapid convergence may also be established. Broadly speaking, large profits are possible even when the time interval between periods is very long.

5. Potential Entry and the Coase Conjecture

The "Contestable Markets" literature attempts to modify the theory of static monopoly by arguing that the existence of a potential entrant may force a monopolist to price at marginal cost. The "Coase Conjecture" literature further amends our understanding of monopoly by observing that a monopolist in a durable good may be forced, due to lack of commitment power and anticipatory behavior on the part of consumers, to price at close to marginal cost. Below we combine these forces, to demonstrate that the existence of a potential entrant may enable a durable goods monopolist to price close to the static monopoly price. In other words, one (and one potential entrant) is enough for monopoly.

Let us make precise the sequencing of moves. At the beginning of every period, the entrant has the choice of whether or not to enter if he has not entered previously. If the entrant decides to stay out, the incumbent names a price (after observing the entrant's decision), and consumers make their purchasing decisions. The game then repeats, with the same description, in the next period. If the potential entrant decides to join the market, both firms simultaneously and independently call out prices (after observing that decision), which the consumers can then either accept or reject.[13] The play proceeds in subsequent periods with the two firms naming prices followed by the consumers making purchases.

In order to make the description of the equilibrium we have in mind compact, it is convenient to consider three types of outcome paths. The equilibrium will incorporate these three paths, both in specifying an initial outcome path and in determining punishments for any deviation from the initial outcome path, or from ongoing punishments.

Path 1: The incumbent charges close to the static monopoly price; the potential entrant does not enter.

This is the equilibrium path. Let $p_0(\varepsilon)$ denote the initial price of the optimally collusive duopoly solution, and s the real-time rate of descent of prices. The incumbent charges a price $p_n = \varepsilon^n p_0(\varepsilon)$ in each period n, where $\varepsilon = e^{-sz}$. Meanwhile, the potential entrant stays out. Thus, along path 1, the incumbent earns the joint duopoly profits associated with the simple strategy profile (p_0, ε).

Path 2: The potential entrant enters; the two firms then follow a subgame perfect equilibrium price path.

This is the punishment path if the incumbent deviates from path 1. The punishment path is characterized by two numbers: (ω,ε). The interpretation of ω and ε is the following: if x is the price at which a deviation from path 1 occurred, and if an offer of x induces all customers with valuations $\geq$ v to buy, both entrant and incumbent charge a price of ωv in the next period. Afterwards, the price is discounted by the fraction $\varepsilon = e^{-rs}$ every period. The punishment path after such a deviation is thus $(\omega v, \omega\varepsilon v, \omega\varepsilon^2 v, \omega\varepsilon^3 v, \ldots)$.

Path 3: The incumbent and the entrant revert to marginal-cost pricing.

This is the punishment path if the entrant deviates from path 1, or if either firm deviates from path 2.

The equilibrium goes as follows: The incumbent follows path 1. If the incumbent ever deviates (singly) from path 1, she triggers a reversion to path 2. If either the incumbent or the entrant deviate from path 2, or if the entrant ever deviates from path 1 (singly or jointly with the incumbent), both players revert to path 3. Deviations from path 3 are punished by starting that path over again. Observe that under the above strategies, the potential entrant has a strict incentive to enter whenever he is supposed to, but may as well stay out as long as the monopolist sticks to path 1.

Before showing that the above strategies form an equilibrium strategy pair for large enough δ, we would like to offer one possible interpretation of the equilibrium. As long as the monopolist sticks to path 1, the potential entrant stays out of the market, because he interprets this behavior as a sign of determination on the part of the monopolist. Mean and nasty monopolists not only hurt consumers but also trash new entrants. Deviations from path 1 are considered to be a sign of weakness on the part of the monopolist, and lead to the entrant's inference that the monopolist will not spoil the market once entry occurs (because weak monopolists are soft on entrants as well as consumers). Such behavior invites entry.

In order to establish that the above strategies induce an equilibrium for large enough δ, we need to show the existence of a value of ω such that path 2 is an SPE starting from any deviation x, and such that the monopolist has no incentive to trigger a reversion to that path.[14] These two results are proven in Appendix B. Observe that as the real-time rate of descent s approaches zero, the monopolist's initial price converges to ρ, with corresponding profits of μ. Hence we obtain:

<u>Theorem 4</u>: For every $r > 0$ and $\theta > 0$, there exists a $\bar{z} > 0$ such that for every z satisfying $0 < z < \bar{z}$, the incumbent firm in the potential entry model earns profits greater than $\mu - \theta$ in the equilibrium described above.

6. Conclusion

In this paper, we have proved that a durable goods monopolist may benefit from entry or potential entry. Whereas the monopolist lacked the means to force her future self to follow a strategy which her present self would like, the oligopolist finds commitment power in the actions of her rivals. To put it bluntly: 'If you cannot punish yourself, find someone else to punish you'. We shall conclude the paper by comparing our model with related literature in which punishment enables improved payoffs, and by relating our results obtained here to the literature on durable goods monopoly.

Traditional supergame analysis of the oligopoly problem (for example, Abreu [1985]) also makes extensive use of punishment strategies to support collusive outcomes. It is instructive to compare the modeling technique of that literature with the present paper's. As a supergame is typically an infinitely-repeated version of a one-shot game, the supergame analysis of oligopoly assumes that firms face the same static demand curve in every period. In contrast, our model contains a convenient (albeit extreme) version of intertemporal substitutability in demand. Our paper indicates that the presence of intertemporal substitutability in demand facilitates collusion, via rational expectations on the part of consumers. Suppose consumers witness a deviation from cooperative behavior by one firm. Given the subgame perfect equilibrium (supported by punishment strategies), consumers anticipate an all-out price war in subsequent periods and postpone their purchases. *Ceteris paribus*, a firm's one-period gain associated with deviation will be less than in the standard supergame treatment of oligopoly, enabling more collusive outcomes to be supported as subgame perfect equilibria. Moreover, the effect we are describing appears to be a real phenomenon: for example, a consumer who sees one airline cut its fare can often profit by deferring purchase of his ticket, as price-matching by other airlines and further cuts may reasonably be expected.

Closely related to the Coase Conjecture is the literature on the time-consistency problem of macroeconomics (for example, Kydland and Prescott [1977]). In a variety of contexts, the government may seek to choose a sequence of policy actions over time which are not "time-consistent" in the sense that if the government were able to re-optimize in subsequent periods, it would not choose the policy actions called for under the original maximization. Hence, unless the government possesses commitment power, it cannot follow such a strategy. The approach of our paper suggests a way out of this conundrum: separation of power may permit "mutually-assured commitment." For example, if we give Congress the authority to run deficits but only give an independent Fed the power to monetize the debt, we may permit a subgame perfect equilibrium where the debt is never monetized.

Finally, an illuminating comparison may be made between the models of oligopoly and potential entry in this paper and the model of pure durable goods monopoly[15] in a sequel (Ausubel and Deneckere, 1986). There are actually two cases which need to be distinguished. First, consider the case of a "gap" between seller's cost and buyers' valuations. When $\underline{v} > 0$, it has been shown (Fudenberg, Levine and Tirole [1985]; Gul, Sonnenschein and Wilson [1986]) that there generically exists a unique SPE in the pure monopoly model. Moreover, as the time interval z approaches zero, the initial price p_0 necessarily approaches $\underline{v}$, which may be much lower than the static monopoly price. In contrast, for oligopoly and monopoly with potential entry, a Folk Theorem holds. The explanation for the oligopolists' advantage lies in the fact that, when $\underline{v} > 0$ and $\delta < 1$, all sales occur in finite time in any monopoly SPE. Backward induction from the last period of positive sales drives the initial price near $\underline{v}$. But with entry or potential entry, a pricing rule similar to $p_n - \underline{v} = \varepsilon^n(p_0 - \underline{v})$ becomes incentive compatible. Sales are then extended over an infinite time, and backward induction fails.

Second, consider the case where there is "no gap" between the seller's marginal cost and the lowest buyer valuation. We demonstrate in Ausubel and Deneckere [1986] that when $\underline{v} = 0$, and under very general distributional assumptions, there exist subgame perfect equilibria in which the monopolist initially charges the static monopoly price and then follows a very slow rate of price descent. This main equilibrium path is supported by a reversion to a Coase Conjecture SPE, if the monopolist ever deviates. Hence, in the case of no gap, a durable good monopolist may earn monopoly profits. Nevertheless, observe (somewhat counterintuitively) that for a wide range of discount factors, the duopolists may do strictly better than the monopolist. Reversion to the Bertrand equilibrium is more severe than reversion to a Coase Conjecture SPE, so the optimally collusive duopoly equilibrium yields higher joint profits than the maximally profitable monopoly SPE.

Combining the results of the two papers, we may conclude that in the case of no gap, one firm is enough for a durable goods industry to earn monopoly profits (though more than one may do better). In the case of a gap, one is not enough for monopoly, but one is almost enough.

Appendix A

In this appendix we derive conditions on the subgame perfect equilibria of a durable goods duopoly with an "equal division" rule: optimally collusive payoffs and minimally collusive payoffs are proved to be supported by "simple strategy profiles" for all $\alpha > 0$. Analogous results can be derived for an oligopoly with N firms.

We first write equations which express $\{p_n\}_{n=0}^{\infty}$ and $\{\Pi_n\}_{n=0}^{\infty}$ in terms of $\{v_n\}_{n=0}^{\infty}$. If the consumer with valuation v_{n+1} is indifferent between purchasing in periods n and n + 1, $v_{n+1} - p_n = \delta(v_{n+1} - p_{n+1})$, giving the difference equation $p_n = (1 - \delta)v_{n+1} + \delta p_{n+1}$. Telescoping the right side yields:

$$\text{(A.1)} \qquad p_n = (1 - \delta) \sum_{i=0}^{\infty} \delta^i v_{n+i+1}, \text{ for all } n \geq 0$$

Equation (A.1) was developed in Stokey [1981]. Meanwhile, the net present value of joint profits, Π_n, is defined by:

$$\text{(A.2)} \qquad \Pi_n = \sum_{i=0}^{\infty} \delta^i p_{n+i}(v^{\alpha}_{n+i} - v^{\alpha}_{n+i+1}), \text{ for all } n \geq 0.$$

Conditions (1) and (2) of Theorem 2 hold for all SPE's. The first lemma of this appendix shows that (2) is vacuous when $\alpha \geq 1$.

<u>Lemma A.1</u>: Consider any SPE such that $p_n > 0$ for all $n \geq 0$, and let $\alpha \geq 1$. Then $p_n \geq \rho(1 - \delta)v_n$, for all $n \geq 0$.

<u>Proof</u>: Suppose not. Then there exists n such that $0 < p_n < \rho(1 - \delta)v_n$. Using inequality (2) of Theorem 2, observe that $\Pi_n = \pi^1_n + \pi^2_n \geq 2(\frac{\alpha}{1 + \alpha})p_n v^{\alpha}_n$.

However, note that joint profits along the equilibrium path are bounded above by the current price times the number of remaining customers. Hence, $\Pi_n < p_n v_n^{\alpha}$, giving a contradiction when $\alpha/(1 + \alpha) \geq 1/2$. □

Lemma A.1 is not true for $\alpha < 1$. However, it still can be shown that (2) is irrelevant for optimally-collusive SPE's.

<u>Lemma A.2</u>: Suppose $\{v_n\}_{n=0}^{\infty}$ maximizes Π_0 subject to:

(a) $\quad \Pi_n \geq 2\mu(1 - \delta)v_n^{1+\alpha}$, for all $n \geq 1$ s.t. $p_n \geq \rho(1 - \delta)v_n$

(b) $\quad \Pi_n \geq 2p_n(v_n^{\alpha} - (\frac{p_n}{1 - \delta})^{\alpha})$, for all $n \geq 1$ s.t. $p_n < \rho(1 - \delta)v_n$

and

(c) $\quad v_0 = 1$, and $v_n \geq v_{n+1} \geq 0$, for all $n \geq 0$

where $\{p_n\}_{n=0}^{\infty}$ is defined by (A.1) and $\{\Pi_n\}_{n=0}^{\infty}$ is defined by (A.2).

Then for any $\alpha > 0$, if $p_n > 0$ (for all $n \geq 0$), we have:

$$\Pi_n \geq 2\mu(1 - \delta)v_n^{1+\alpha}, \text{ for all } n \geq 1.$$

<u>Outline of Proof</u>: Suppose not. Let $k = \inf\{n \geq 1: \Pi_n < 2\mu(1 - \delta)v_n^{1+\alpha}\}$. Observe that $v_k > v_{k+1}$. For β satisfying $0 < \beta < v_k/v_{k+1}$, define $\{v_n'\}_{n=0}^{\infty}$ by equation (A.4) (in proof of next lemma). $\{p_n'\}_{n=0}^{\infty}$ and $\{\Pi_n'\}_{n=0}^{\infty}$ are defined by (A.1) and (A.2). Define $a_n(\beta)$ to be the slack in the n^{th} constraint (a), evaluated at $\{v_n'\}_{n=0}^{\infty}$ given by β, and define $b_n(\beta)$ to be the slack in the n^{th} constraint (b). The following statements can be established by direct calculation:

1. $\left.\frac{\partial \Pi_0'}{\partial \beta}\right|_{\beta=1} > 0$

2. $\left.\frac{\partial a_n}{\partial \beta}\right|_{\beta=1} > 0$, and $\left.\frac{\partial b_n}{\partial \beta}\right|_{\beta=1} > 0$, for all $n \leq k - 1$

3. $\left.\frac{\partial b_k}{\partial \beta}\right|_{\beta=1} > 0$

4. $a_n(\beta) = \beta^{1+\alpha} a_n(1)$ and $b_n(\beta) = \beta^{1+\alpha} b_n(1)$ for all $n \geq k + 1$

Hence, we can conclude there exists β $(1 < \beta < v_k/v_{k+1})$ such that $\{v_n'\}_{n=0}^{\infty}$ is in the feasible region of the above maximization problem, but $\Pi_0' > \Pi_0$, contradicting our hypothesis that $\{v_n\}_{n=0}^{\infty}$ maximizes Π_0. □

Using Theorem 2, Lemma A.1 and Lemma A.2, the optimally collusive SPE must solve the following optimization problem (denoted MAX), for all $\alpha > 0$:

(MAX) Maximize Π_0 subject to:
$\{v_n\}_{n=0}^{\infty}$

(*) $\Pi_n \geq 2\mu(1 - \delta)v_n^{1+\alpha}$, for all $n \geq 1$

and

(**) $v_0 = 1$ and $v_n \geq v_{n+1} \geq 0$, for all $n \geq 0$

Lemma A.3: For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves MAX and $\Pi_0 > 0$. Then constraint (*) holds with equality, i.e.,

(A.3) $$\Pi_n = 2\mu(1 - \delta)v_n^{1+\alpha}, \text{ for all } n \geq 1.$$

Proof: Suppose not. We will demonstrate an alternative sales path $\{v_n'\}_{n=0}^{\infty}$ which improves upon Π_0, while still satisfying constraint (*).

Observe that $v_n > 0$ for all $n \geq 0$, by induction: $\Pi_n > 0$ implies $p_n > 0$, and so $v_{n+1} > 0$, but by (*), $\Pi_{n+1} > 0$, etc.

Define k to be the first period in which constraint (*) has slack and in which sales are positive: $k = \inf\{n \geq 1: \Pi_n > 2\mu(1-\delta)v_n^{1+\alpha}$ and $v_n > v_{n+1}\}$. Note that $k < \infty$ if and only if the hypothesis of the lemma is not satisfied. For suppose there exists n such that $\Pi_n > 2\mu(1-\delta)v_n^{1+\alpha}$ but $v_n = v_{n+1}$. Since there are no sales in period n, $\Pi_n = \delta\Pi_{n+1}$, implying $\Pi_{n+1} > 2\mu(1-\delta)v_{n+1}^{1+\alpha}$. Eventually, we must reach $m > n$ such that $v_m > v_{m+1}$, since v_n is positive but $\lim_{i\to\infty} v_i = 0$ (by Theorem 1).

Let β satisfy $0 < \beta < v_k/v_{k+1}$. Define $\{v_n'\}_{n=0}^{\infty}$ by

$$(A.4) \qquad v_n' = \begin{cases} v_n & , \text{ if } n \leq k-1 \\ [p_{k-1} - \beta\delta p_k]/[1-\delta], & \text{if } n = k \\ \beta v_n & , \text{ if } n \geq k+1 \end{cases}$$

Observe that $\{v_n'\}_{n=0}^{\infty}$ has been conveniently chosen so that, by equation (A.1), it determines $\{p_n'\}_{n=0}^{\infty}$ given by

$$(A.5) \qquad p_n' = \begin{cases} p_n , & \text{if } n \leq k-1 \\ \beta p_n, & \text{if } n \geq k \end{cases}$$

It is also straightforward to verify, from equation (A.2), that:

$$(A.6) \qquad \Pi_n' = \beta^{1+\alpha}\Pi_n, \text{ for all } n \geq k+1$$

Furthermore, for $n \leq k-2$:

$$\Pi_n' = [\sum_{i=0}^{k-2-n} \delta^i (v_{n+i}'^{\alpha} - v_{n+i+1}'^{\alpha}) p_{n+i}'] + \delta^{k-1-n}\Pi_{k-1}'$$

and

$$\Pi_n = [\sum_{i=0}^{k-2-n} \delta^i (v^\alpha_{n+i} - v^\alpha_{n+i+1}) p_{n+i}] + \delta^{k-1-n} \Pi_{k-1}$$

implying:

(A.7) $$\Pi'_n = \Pi_n + \delta^{k-1-n}[\Pi'_{k-1} - \Pi_{k-1}], \text{ for all } n \leq k-2$$

Claim: $\partial\Pi'_{k-1}/\partial\beta$, evaluated at $\beta = 1$, is strictly positive.

Proof of Claim: Joint profits, Π'_{k-1}, may be written:

$$\Pi'_{k-1} = (v'^{\alpha}_{k-1} - v'^{\alpha}_k) p'_{k-1} + \delta(v'^{\alpha}_k - v'^{\alpha}_{k+1}) p'_k + \delta^2 \Pi'_{k+1}$$

Its derivative with respect to β is calculated using (A.4), (A.5) and (A.6):

$$\frac{\partial\Pi'_{k-1}}{\partial\beta} = [\frac{\partial v'^{\alpha}_{k-1}}{\partial\beta} - \frac{\partial v'^{\alpha}_k}{\partial\beta}] p'_{k-1} + \delta[\frac{\partial v'^{\alpha}_k}{\partial\beta} - \frac{\partial v'^{\alpha}_{k+1}}{\partial\beta}] p'_k$$

$$+ \delta[v'^{\alpha}_k - v'^{\alpha}_{k+1}] \frac{\partial p'_k}{\partial\beta} + \delta^2 \frac{\partial\Pi'_{k+1}}{\partial\beta}$$

$$= \alpha\delta p_k v'^{\alpha-1}_k \{\frac{p'_{k-1} - \delta p'_k}{1-\delta}\} - (1+\alpha)\delta p_k v'^{\alpha}_{k+1}$$

$$+ \delta p_k v'^{\alpha}_k + (1+\alpha)\delta^2 \beta^\alpha \Pi_{k+1}$$

The above expression in braces equals v'_k. Moreover, at $\beta = 1$, we have $v'_k = v_k$ and $v'_{k+1} = v_{k+1}$. Hence:

$$\left.\frac{\partial\Pi'_{k-1}}{\partial\beta}\right|_{\beta=1} = (1+\alpha)[\delta p_k (v^\alpha_k - v^\alpha_{k+1}) + \delta^2 \Pi_{k+1}] > 0$$

proving the claim.

Remainder of the Proof of Lemma A.3: By hypothesis, constraint (*) has slack for $n = k$ and sales are positive in period k. Using the claim and (A.4), there exists β satisfying $1 < \beta < v_k/v_{k+1}$ such that:

$$\Pi_k' > 2\mu(1 - \delta)v_k'^{1+\alpha} \text{ and } \Pi_{k-1}' > \Pi_{k-1} \tag{A.8}$$

Recall that $\{v_n\}_{n=0}^{\infty}$ satisfies constraint (*). By equations (A.4) and (A.6):

$$\Pi_n' = \beta^{1+\alpha}\Pi_n \geq 2\mu(1 - \delta)\beta^{1+\alpha}v_n^{1+\alpha} = 2\mu(1 - \delta)v_n'^{1+\alpha}, \text{ for all } n \geq k + 1$$

and by equations (A.4), (A.7) and (A.8):

$$\Pi_n' > \Pi_n \geq 2\mu(1 - \delta)v_n^{1+\alpha} = 2\mu(1 - \delta)v_n'^{1+\alpha}, \text{ for all } n \leq k - 1$$

Thus, $\{v_n'\}_{n=0}^{\infty}$ satisfies constraint (*) and $\Pi_0' > \Pi_0$. We conclude that $\{v_n\}_{n=0}^{\infty}$ is not a solution to MAX, contradicting the hypothesis. □

Next, we use Lemma A.3 to derive a (nonlinear) difference equation which the solution to MAX must satisfy. Observe that:

$$\Pi_n = p_n(v_n^{\alpha} - v_{n+1}^{\alpha}) + \delta\Pi_{n+1}, \text{ for all } n \geq 0 \tag{A.9}$$

$$p_n - \delta p_{n+1} = (1 - \delta)v_{n+1}, \text{ for all } n \geq 0 \tag{A.10}$$

The solution to MAX satisfies equation (A.3). Substituting (A.3) into (A.9) gives:

(A.11) $$p_n = \frac{\Pi_n - \delta\Pi_{n+1}}{v_n^{\alpha} - v_{n+1}^{\alpha}} = 2\mu(1-\delta)\left(\frac{v_n^{1+\alpha} - \delta v_{n+1}^{1+\alpha}}{v_n^{\alpha} - v_{n+1}^{\alpha}}\right)$$

Then, substituting (A.11) into (A.10) yields:

$$\frac{v_n^{1+\alpha} - \delta v_{n+1}^{1+\alpha}}{v_n^{\alpha} - v_{n+1}^{\alpha}} - \delta\left(\frac{v_{n+1}^{1+\alpha} - \delta v_{n+2}^{1+\alpha}}{v_{n+1}^{\alpha} - v_{n+2}^{\alpha}}\right) = \frac{v_{n+1}}{2\mu}$$

Let $\varepsilon_n = v_{n+1}/v_n$. Then this equation may be rewritten as:

$$\varepsilon_n^{-1}\left(\frac{1 - \delta\varepsilon_n^{1+\alpha}}{1 - \varepsilon_n^{\alpha}}\right) - \delta\left(\frac{1 - \delta\varepsilon_{n+1}^{1+\alpha}}{1 - \varepsilon_{n+1}^{\alpha}}\right) = \frac{1}{2\mu}$$

Setting $h_\alpha(z) = (1 - \delta z^{1+\alpha})/(1 - z^{\alpha})$, we obtain our difference equation:

(A.12) $$\varepsilon_n^{-1} h_\alpha(\varepsilon_n) - \delta h_\alpha(\varepsilon_{n+1}) = \frac{1}{2\mu}, \text{ for all } n \geq 1.$$

We may establish the following properties of $h_\alpha(z)$:

$\lim_{z\to 0} h_\alpha(z) = h_\alpha(0) = 1$; $\lim_{z\to 1} h_\alpha(z) = +\infty$; and $h_\alpha(\cdot)$ is monotone <u>increasing</u> on $[0,1]$. Let us rewrite (A.12) as:

(A.13) $$h_\alpha(\varepsilon_{n+1}) = \frac{1}{\delta}\left[\varepsilon_n^{-1} h_\alpha(\varepsilon_n) - \frac{1}{2\mu}\right]$$

The above properties show that a solution $\varepsilon_{n+1} = g(\varepsilon_n)$ to (A.13) exists for ε_n close to 0 and 1, and that $\lim_{\varepsilon_n\to 0} g(\varepsilon_n) = \lim_{\varepsilon_n\to 1} g(\varepsilon_n) = 1$.

Furthermore, one easily establishes that the set of ε_n for which no solution $0 \leq \varepsilon_{n+1} \leq 1$ to (A.13) exists, i.e., the set of all ε_n s.t. $\varepsilon_n^{-1} h_\alpha(\varepsilon_n) - \frac{1}{2\mu} < \delta$, is an open interval. When no solution $\varepsilon_{n+1} \leq 1$ to (A.13) exists, we extend the definition of g to obtain a continuous function by setting $\varepsilon_{n+1} = 0$. Some additional algebra shows that $g(\cdot)$ is first decreasing and then increasing, so we obtain the picture in figure 2.

(Insert Figure 2 About Here)

As drawn in figure 2, $g(\cdot)$ has a flat section and exactly two fixed points in $[0,1)$. But it is also possible that $g(\varepsilon)$ contains no flat sections ($g > 0$ on $[0,1]$) or that g has no fixed points (the graph of g lies completely above the 45° line in $[0,1)$). We now establish the existence of a $\bar{\delta}(\alpha)$ such that g has exactly two fixed points in $[0,1)$ for $\delta > \bar{\delta}(\alpha)$ and no fixed points if $\delta < \bar{\delta}(\alpha)$. Fixed points of $g(\cdot)$ are zeros of $\psi(\varepsilon;\ \alpha,\delta)$, where

$$\psi(\varepsilon;\ \alpha,\delta) = 2\mu\delta^2\varepsilon^{2+\alpha} + (1 - 2\mu\delta)\varepsilon^{1+\alpha} - (1 + 2\mu\delta)\varepsilon + 2\mu$$

ψ is a quasiconcave function, decreasing on $[0,\rho]$ and convex on $[\rho,1]$, with $\psi(0) = 2\mu > 0$ and $\psi(1) = 2\mu(1 - \delta)^2 > 0$. Thus, either ψ has exactly two roots or no roots at all, expect at some critical $\bar{\delta}(\alpha)$. Now, for each α, $\psi(1;\ \alpha,1) = 0$, $\frac{\partial\psi}{\partial\delta} = 2\mu\varepsilon[\varepsilon^{\alpha}(2\delta\varepsilon - 1) - 1)] < 0$ and $\frac{\partial\psi}{\partial\varepsilon}(1;\alpha,1) = \alpha > 0$. An application of the implicit function theorem thus yields that there exists a root in $[0,1)$ to $\psi(\varepsilon;\alpha,\delta) = 0$ for δ close to 1. Since $\frac{\partial\psi}{\partial\delta} < 0$, we see there exists $\bar{\delta}(\alpha)$ such that ψ has roots for $\delta > \bar{\delta}(\alpha)$. Some numerical calculations yield the following graph of $\bar{\delta}(\alpha)$ versus $\alpha/(1 + \alpha)$.

(Insert Figure 3 About Here)

For $\bar{\delta}(\alpha) < \delta < 1$, let ε_- and ε_+ denote the smallest and largest fixed point of g, respectively, in $[0,1)$. Construct a square with vertices $(\varepsilon_+,\varepsilon_+)$

and $(\hat{\varepsilon},\hat{\varepsilon})$, where $\hat{\varepsilon}$ is the unique $\varepsilon < \varepsilon_+$ s.t. $g(\hat{\varepsilon}) = \varepsilon_+$.

Consider $g^n(\varepsilon)$. If $\varepsilon \in [0,\hat{\varepsilon})$ or $\varepsilon \in (\varepsilon_+,1]$, then $g_n(\varepsilon) \uparrow 1$. If $\varepsilon \in [\hat{\varepsilon},\varepsilon_+]$, then one of three possibilities hold:

(A.14) $\quad g^n(\varepsilon) > 0$ for all n but $g^k(\varepsilon) \notin [\hat{\varepsilon},\varepsilon_+]$ for some k.

(A.15) $\quad g^n(\varepsilon) = 0$ for some n.

(A.16) $\quad \hat{\varepsilon} \leqslant g^n(\varepsilon) \leqslant \varepsilon_+$ for all n.

<u>Lemma A.4</u>: For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves MAX and $\Pi_0 > 0$. Then:

$$\text{(A.17)} \qquad \hat{\varepsilon} \leqslant \frac{v_{n+1}}{v_n} \leqslant \varepsilon_+, \text{ for all } n \geqslant 1$$

<u>Proof</u>: Suppose $v_2/v_1 < \hat{\varepsilon}$ or $v_2/v_1 > \varepsilon_+$. We have just argued that $\lim_{n\to\infty} (v_{n+1}/v_n) = 1$. For any $c < 1$, there exists n such that for every $i \geqslant n$, $v_{i+1} > cv_i$. Then

$$v_i^{\alpha} - v_{i+1}^{\alpha} < (1 - c^{\alpha})v_i^{\alpha} \leqslant (1 - c^{\alpha})v_n^{\alpha}, \text{ for all } i \geqslant n$$

Meanwhile, $p_i \leqslant v_n$ for all $i \geqslant n$, so by (A.2):

$$\text{(A.18)} \qquad \Pi_n < \sum_{i=1}^{\infty} \delta^i(1 - c^{\alpha})v_n^{1+\alpha} = (\frac{1 - c^{\alpha}}{1 - \delta})v_n^{1+\alpha}$$

But c may be chosen arbitrarily close to 1, and then (A.18) contradicts (A.3) (or simply (*)).

Suppose $\hat{\varepsilon} \leqslant v_2/v_1 \leqslant \varepsilon_+$. We have just argued that one of (A.14), (A.15), or (A.16) holds. If (A.14) holds, then $v_{k+2}/v_{k+1} < \hat{\varepsilon}$ or $v_{k+2}/v_{k+1} > \varepsilon_+$, and we generate the contradiction of the previous paragraph. If (A.15) holds, let $k = \inf\{n\colon g^n(\varepsilon) = 0\}$. Then $v_{k+2}/v_{k+1} \leqslant 0$, implying $v_{k+2} \leqslant 0$ but $v_{k+1} > 0$. Note that $v_{k+2} \leqslant 0$ implies $p_{k+1} \leqslant 0$. But then $\Pi_{k+1} \leqslant 0$, contradicting (A.3) (or simply (*)). The only remaining possibility is (A.16). □

<u>Theorem A.1</u>: For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves MAX and $\Pi_0 > 0$. Then $\{v_n\}_{n=0}^{\infty}$ corresponds to a simple strategy profile, i.e., $v_{n+1}/v_n = \varepsilon_+$, for all $n \geqslant 1$.

<u>Proof</u>: By Lemma A.4, $\{v_n\}_{n=0}^{\infty}$ satisfies (A.17). Define $\{v_n'\}_{n=0}^{\infty}$ by $v_0' = v_0$ and $v_n' = \varepsilon_+^{n-1}v_1$, for all $n \geqslant 1$. Note by (A.9) and (A.3) that:

$$\Pi_0' = (v_0'^{\alpha} - v_1'^{\alpha})p_0' + 2\mu\delta(1 - \delta)v_1'^{1+\alpha}$$

and

$$\Pi_0 = (v_0^{\alpha} - v_1^{\alpha})p_0 + 2\mu\delta(1 - \delta)v_1^{1+\alpha}$$

Since $v_0' = v_0$ and $v_1' = v_1$, these imply: $\Pi_0' - \Pi_0 = (v_0^{\alpha} - v_1^{\alpha})(p_0' - p_0)$. Note that $v_0 - v_1 > 0$. Furthermore, $p_0' - p_0 = (1 - \delta) \sum_{i=0}^{\infty} \delta^i(v_{i+1}' - v_{i+1})$, by (A.1). Observe that $v_n' \geqslant v_n$ for all n, since $\{v_n\}_{n=0}^{\infty}$ satisfies (A.17). If $v_{n+1}/v_n < \varepsilon_+$ for some $n \geqslant 1$, then $v_{n+1}' > v_{n+1}$, implying $\Pi_0' > \Pi_0$ and contradicting our hypothesis that $\{v_n\}_{n=0}^{\infty}$ solves MAX. □

The minimally collusive SPE (with $\Pi_0 > 0$) solves a second optimization problem, analogous to MAX, which we denote MIN: minimize Π_0 subject to (1), (2), (**), and $p_n > 0$ (for all $n \geqslant 0$). We have:

<u>Lemma A.5</u>: Suppose MIN is feasible and $\{v_n\}_{n=0}^{\infty}$ solves MIN. If $\alpha < 1$, then constraint (2) is applicable to period zero, and (2) is satisfied with equality.

Proof: Observe that if MIN is feasible, there exist feasible $\{v_n\}_{n=0}^{\infty}$ such that $\Pi_0 < 2\mu(1 - \delta)v_0^{1+\alpha}$. Hence, (2) is applicable in period zero. Suppose that $\{v_n\}_{n=0}^{\infty}$ yields slack in the period zero constraint: define $\{v_n'\}_{n=0}^{\infty}$ by $v_0' = v_0$ and $v_n' = \beta v_n$, for all $n \geq 1$. Then there exists β such that $\{v_n'\}_{n=0}^{\infty}$ is feasible and $\Pi_0' < \Pi_0$, so $\{v_n\}_{n=0}^{\infty}$ does not solve MIN. []

Now consider a variant on optimization problem MAX, which we denote MAX'(p_0): maximize Π_0, given p_0, and subject to (1), (2) and (**). We have:

Lemma A.6: If MAX'(p_0) has a solution, it is uniquely given by the simple strategy profile (p_0,ε_+).

Proof: We follow literally the same proof we used in Lemmas A.1 through A.4 and Theorem A.1. The previous argument hinged on defining alternative sales path $\{v_n'\}_{n=0}^{\infty}$, which improved upon $\{v_n\}_{n=0}^{\infty}$, using (A.4) for $k \geq 1$. But by (A.5), $p_0' = p_0$, so $\{v_n'\}_{n=0}^{\infty}$ is feasible for the same MAX'(p_0) as $\{v_n\}_{n=0}^{\infty}$. []

Theorem A.2: Suppose δ satisfies $\bar{\delta}(\alpha) \leq \delta < 1$. If $\alpha \geq 1$, then $v_n \equiv v_0\varepsilon_+^n$ is one solution to MIN (there also exist others), and $\Pi_0 = 2\mu(1 - \delta)v_0^{1+\alpha}$.

If $\alpha < 1$, MIN has a unique solution. It corresponds to the simple strategy profile (p_0,ε_+), where $p_0 = \inf\{p > 0: \Pi(p,\varepsilon_+) \geq 2p\{v_0^{\alpha} - [p/(1 - \delta)]^{\alpha}\}\}$.

Proof: If $\alpha \geq 1$, Lemma A.1 implies that $\Pi_n \geq 2\mu(1 - \delta)v_n^{1+\alpha}$, for all $n \geq 0$. Observe that $v_n \equiv v_0\varepsilon_+^n$ satisfies $\Pi_n = 2\mu(1 - \delta)v_n^{1+\alpha}$, for all $n \geq 0$, necessarily solving MIN.

If $\alpha < 1$, suppose to the contrary that $\{v_n'\}_{n=0}^{\infty}$ solves MIN and $v_n' \neq \varepsilon_+^{n-1}v_1'$ for some $n \geq 2$. Define p_0' and Π_0' by (A.1) and (A.2). By Lemma A.6, the simple strategy profile (p_0',ε_+) yields joint profits strictly greater than Π_0'. Observe that $\partial\Pi_0(p,\varepsilon_+)/\partial p > 0$ when $p < \rho(1 - \delta)v_0$. Hence, there exists

$p_0'' < p_0'$ such that $\Pi_0(p_0'', \varepsilon_+) = \Pi_0'$. Furthermore, $2p\{v_0^\alpha - [p/(1 - \delta)]^\alpha\}$ is monotone increasing in p when $p < \rho(1 - \delta)v_0$. Hence, (p_0'', ε_+) satisfies constraint (2) with slack in period zero. We conclude there exists $p_0''' < p_0''$ such that (p_0''', ε_+) is in the feasible region for MIN and $\Pi_0(p_0''', \varepsilon_+) < \Pi_0'$, a contradiction.

Finally, suppose $v_n' = \varepsilon_+^{n-1} v_1'$ for all $n \geq 2$, but p_0' does not equal p_0 of the theorem. If $p_0' > p_0$, observe using Lemma A.5 that $p_0' < \rho(1 - \delta)v_0$, so $\Pi_0(p_0, \varepsilon_+) < \Pi_0(p_0', \varepsilon_+)$. If $p_0' < p_0$, observe by the definition of p_0 that (p_0', ε_+) is infeasible. □

Appendix B

In the first part of this appendix, we show that for quite general distribution functions, analysis similar to that of the main text is possible. In particular, for any distribution function F(v) satisfying inequality (3) (for some $\alpha > 0$ and $L \geq M > 0$), we develop a function $\varepsilon_+(\delta)$ such that $\varepsilon_+(\delta)$ implies a subgame perfect equilibrium whenever the discount factor exceeds δ. $\varepsilon_+(\delta)$ is defined similarly to the ε_+ of Lemma 2, so it is again the case that maximally collusive joint profits converge rapidly to static monopoly profits.

Suppose that F(v) satisfies (3) and that $F^{-1}(\cdot)$ is well defined. (Even if F^{-1} is not defined, one can proceed as below, but the notation becomes more cumbersome.) For any $p_0' \geq \underline{v}$ and ε $(0 < \varepsilon < 1)$, define $\{p_n'\}_{n=1}^{\infty}$ by

$$(B.1) \qquad p_n' - \underline{v} = \varepsilon^n(p_0' - \underline{v}).$$

Let $\{v_n'\}_{n=0}^{\infty}$ denote the sequence of cutoff valuations induced by $\{p_n'\}_{n=0}^{\infty}$. Define $\{v_n\}_{n=0}^{\infty}$ to be the sequence of cutoff valuations which induces the <u>same sales</u> on F(v) as $\{v_n'\}_{n=0}^{\infty}$ induced on $L(v - \underline{v})^{\alpha}$:

$$(B.2) \qquad v_n = F^{-1}[L(v_n' - \underline{v})^{\alpha}], \text{ for all } n \geq 0$$

Now define $\{p_n\}_{n=0}^{\infty}$ to be a price sequence which induces $\{v_n\}_{n=0}^{\infty}$, using equation (A.1) in Appendix A. We will establish conditions on δ and ε such that $\{p_n\}_{n=0}^{\infty}$ is the equilibrium price path of an SPE.

We need to establish that each firm's share of joint profits along the equilibrium starting from period n $(\frac{1}{2}\Pi_n)$ exceeds any firm's optimal deviation in period n (denoted Π_n^{dev}), for all $n \geq 1$. (Clearly, $\frac{1}{2}\Pi_0 \geq \Pi_0^{dev}$ will also be satisfied, unless v_1 is too close to $\bar{v}$ or zero.) Observe that $p_n \geq p_n'$ for all $n \geq 0$ (using (A.1) and (3)). Hence:

$$\Pi_n = \sum_{i=0}^{\infty} \delta^i p_{n+i}[F(v_{n+i}) - F(v_{n+i+1})]$$

$$\geq \sum_{i=0}^{\infty} \delta^i p'_{n+i}[L(v'_{n+i} - \underline{v})^\alpha - L(v'_{n+i+1} - \underline{v})^\alpha]$$

by (B.2). Substituting (B.1) and using $v'_{n+1} - p'_n = \delta(v'_{n+1} - p'_{n+1})$ eventually yield:

$$\text{(B.3)} \qquad \Pi_n \geq \frac{L(\varepsilon^{-1} - \delta)^\alpha(1 - \varepsilon^\alpha)(p_n - \underline{v})^\alpha \underline{v}}{(1 - \delta)^\alpha(1 - \delta\varepsilon^\alpha)} + \frac{L(\varepsilon^{-1} - \delta)^\alpha(1 - \varepsilon^\alpha)(p_n - \underline{v})^{1+\alpha}}{(1 - \delta)^\alpha(1 - \delta\varepsilon^{1+\alpha})}$$

We also derive an upper bound on Π_n^{dev}. Observe that $\Pi_n^{dev} \leq \max_v \{(1 - \delta)v[F(v_n) - F(v)]\}$. The expression in braces may be further bounded[16] by

$$\text{(B.4)} \qquad \Pi_n^{dev} \leq (1 - \delta)\underline{v}F(v_n) + \max_v \{(1 - \delta)(v - \underline{v})[F(v_n) - F(v)]\}.$$

Using $F(v) \geq M(v - \underline{v})^\alpha$, and performing extensive algebra yields:

$$\text{(B.5)} \qquad \Pi_n^{dev} \leq \frac{L(\varepsilon^{-1} - \delta)^\alpha(p_n - \underline{v})^\alpha \underline{v}}{(1 - \delta)^{\alpha-1}} + \left(\frac{L}{M}\right)^{1/\alpha} \frac{L\alpha(\varepsilon^{-1} - \delta)^{1+\alpha}(p_n - \underline{v})^{1+\alpha}}{(1 + \alpha)^{1+1/\alpha}(1 - \delta)^\alpha}$$

If the first term of (B.3) exceeds twice the first term of (B.5) <u>and</u> the second term of (B.3) exceeds twice the second term of (B.5), we have $(\frac{1}{2})\Pi_n \geq \Pi_n^{dev}$. Observe that $\mu \leq 1$ and $(\frac{L}{M})^{1/\alpha} \geq 1$. Hence, a sufficient condition for this is:

(B.6) $$\psi(\varepsilon) \equiv \varepsilon(1 - \varepsilon^{\alpha}) - 2(\frac{L}{M})^{1/\alpha}(1 - \delta\varepsilon)(1 - \delta\varepsilon^{1+\alpha}) \geq 0$$

Observe that, as in Appendix A, there exists $\bar{\delta}(\alpha,L,M) < 1$ such that $\psi(\varepsilon)$ has a root in [0,1] whenever $\delta \geq \bar{\delta}(\alpha,L,M)$. Define $\varepsilon_{+}(\delta) = \sup\{\varepsilon \leq 1: \psi(\varepsilon) \geq 0\}$. As before, $\varepsilon_{+}(\delta)$ converges to 1 as $\delta \to 1$, and the convergence is rapid. By choosing v_1 very close to the static monopoly price ρ, we obtain nearly static monopoly profits; by choosing v_1 very close to $\bar{v}$, we obtain nearly zero profits. By continuously varying v_1 from $\bar{v}$ to ρ, we generate all payoffs between.

In the second part of this appendix, we prove two lemmas which immediately imply Theorem 4.

<u>Lemma B.1</u>: For every $0 < s < \infty$ and every $0 < \omega < r/(r + s)$, there exists $\bar{z}_1 > 0$, such that for all $0 < z \leq \bar{z}_1$ the path (ω,ε) with $\varepsilon = \varepsilon^{-sz}$ is an SPE path after any deviation from path 1.

<u>Proof</u>: After any deviation x in a particular period (which we identify, without loss of generality, as period -1), all customers with valuations $\leq v_0$ remain to be served. Assuming $\omega \leq \gamma^{-1} = (1 - \delta)/(1 - \varepsilon\delta) + r/(r + s)$, one readily calculates that $v_0 = x/(1 - \delta(1 - \omega))$ if $x \leq (1 - \delta(1 - \omega))v_{-1}$, and $v_0 = v_{-1}$ otherwise. Since ω represents a <u>fraction</u> of the highest remaining consumer valuation v_0, there is, as far as incentive compatibility is concerned, no loss of generality in assuming $v_0 = 1$. Conditions for (ω,ε) to induce an SPE thus coincide with conditions for the simple strategy (ω,ε) to give an SPE. The result then immediately follows from the same reasoning we use for Theorem 3. □

Lemma B.2: For every $0 < s < \infty$, there exist $\omega > 0$ and $\bar{z}_2 > 0$ such that for all $0 < z \leq \bar{z}_2$, the monopolist has no incentive to deviate from path 1.

Proof: Confine the choice of ω to $(0,\gamma^{-1})$ so that, in the period after the deviation x, there are positive sales. If $x \leq [1 - \delta(1 - \omega)]v_{-1}$, profits from deviating will be bounded above by: $\Pi^{dev} \leq xv_{-1}^{\alpha} \leq [1 - \delta(1 - \omega)]v_{-1}^{1+\alpha}$. Observe that $\lim_{z\to 0} \Pi^{dev} \leq \omega v_{-1}^{1+\alpha}$. If $x \geq [1 - \delta(1 - \omega)]v_{-1}$, profits from deviating will satisfy $2\Pi^{dev} = \omega v_{-1}^{1+\alpha}[1 - \gamma^{\alpha}\omega^{\alpha}(1 - \delta\varepsilon)/(1 - \delta\varepsilon^{1+\alpha})]$. Again, observe that $\lim_{z\to 0} \Pi^{dev} \leq \omega v_{-1}^{1+\alpha}$.

Meanwhile, profits Π_{-1} from continuing on the main path satisfy $\Pi_{-1} \geq \frac{(1 - \delta)\varepsilon(1 - \varepsilon^{\alpha})}{(1 - \delta\varepsilon)(1 - \delta\varepsilon^{1+\alpha})} v_{-1}^{1+\alpha}$ (with equality unless we are in the initial period). The coefficient of $v_{-1}^{1+\alpha}$ in this inequality approaches $\frac{\alpha rs}{(r + s)(r + (1 + \alpha)s)}$ as $z \to 0$. Hence, for any ω such that $\omega < \min\{\frac{r}{r + s}, \frac{\alpha rs}{(r + s)(r + (1 + \alpha)s)}\}$, there exists $\bar{z}_2 > 0$ such that the monopolist has no incentive to deviate from path 1 when $0 < z \leq \bar{z}_2$. []

Notes

[1]Kahn [1986] considered the durable goods monopolist with increasing marginal cost; Bond and Samuelson [1984] examined a durable good subject to depreciation. Both found the Coase Conjecture modified.

[2]There may also exist subgame perfect equilibria which are qualitatively different. See Ausubel and Deneckere [1986].

[3]Coase did, in 1972. He wrote (p.144), "With complete durability, the price becomes independent of the number of suppliers and is thus always equal to the competitive price" (emphasis added).

[4]Once a time period has lapsed, the original monopolist transforms into a greedy type who wants to maximize profits from then on rather than to implement the sales plan her previous type would have liked to commit herself to.

[5]To capture the idea that customers are anonymous and nonatomic, we will assume that plays which result in the same sequence of prices and the same measure of consumer acceptances yield identical strategy choices for both the monopolist and the consumers in subsequent periods.

[6]If instead, consumer valuations are bounded away from zero (see last paragraph of Section 4), Theorem 1 becomes: in any SPE, either $\inf_n p_n = \underline{v}$ or $p_n = 0$ (for all $n \geq 0$). The result $\inf_n p_n \leq \underline{v}$ follows analogously to Theorem 1. If $p_m < \underline{v}$ for some $m \geq 0$, it is easy to show that a consumer with

valuation $\underline{v}$ purchases in finite time, implying the Bertrand equilibrium.

[7]Off-equilibrium behavior on the part of consumers can be handled in the following way: if a set of measure zero of consumers deviates, those deviations are ignored. If a set of positive measure deviates, firms revert to pricing at marginal cost in all future periods (and the remaining consumers' strategies are optimal subject to this prediction).

[8]Obviously, the lowest joint profits in an SPE are zero. We seek the next higher profits attainable.

[9]As shown in Appendix A, the equal division rule implies that there is no loss of generality in further focusing attention on *symmetric* equilibria, i.e., SPE's with the property that $\sigma_1 = \sigma_2$.

[10]If $p_0 \geqslant \gamma^{-1}$, identical sales at identical prices occur under $(\varepsilon p_0, \varepsilon)$ as under (p_0, ε). However, they occur one period sooner.

[11]But also see the Conclusion (and Ausubel and Deneckere, 1986), which discusses the class of *all* SPE's for monopoly, in the case $\underline{v} = 0$.

[12]Suppose, for example, that F is differentiable in a neighborhood of $\underline{v}$, and there exist $\ell \geqslant m > 0$ such that $m \leqslant F'(v) \leqslant \ell$ in that neighborhood. Then condition (3) may be shown to be satisfied, using $\alpha = 1$. Even if $F'(\underline{v}) = 0$ or $F'(\underline{v}) = \infty$, it is still often possible to satisfy (3), using some $\alpha \neq 1$.

[13]More true to the spirit of the contestable markets literature, one

could allow the entrant a first-mover advantage by not permitting the incumbent to react until the period following entry. Uninvited entry will yield the entrant only limited profits, since consumers expect an all-out price war in subsequent periods. Small sunk entry costs proportional to the size of the market remaining (representing, e.g., introductory advertising costs) still make the entrant prefer to stay out when uninvited, but wanting to enter when invited. The monopolist's profits, however, will be bounded away from static monopoly profits by an amount equal to sunk entry costs.

We thank Vijay Krishna for raising this issue.

[14]Consumer expectations are fixed (in a fashion similar to the previous section) to implement this equilibrium. See footnote 7.

[15]It has been observed by a number of authors that the durable goods monopoly model is formally equivalent to a sequential bargaining model of one-sided offers and one-sided uncertainty. Thus, the present paper has implications for the multilateral bargaining problem with two sellers (of known valuation) and one buyer (of unknown valuation), etc.

[16]The intuition for the bound in (B.4) is: the right side of equation (B.4) amounts to selecting a two-part price discriminatory schedule, where the firm charges $(1 - \delta)v$ to all customers with valuation exceeding v and charges $(1 - \delta)\underline{v}$ to all other customers. Profits without this price discrimination are necessarily lower.

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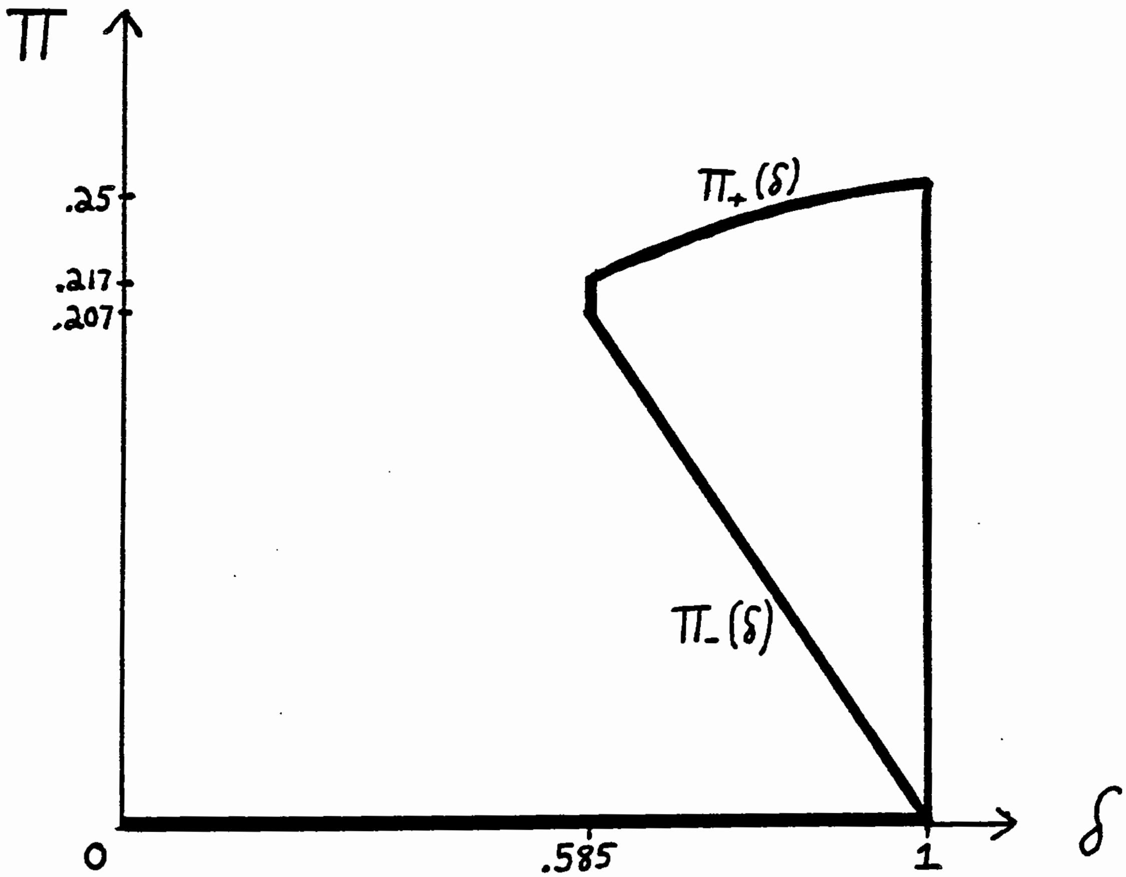

Figure 1: The set of equilibrium joint profits, with F(v) = v, for all discount factors δ.

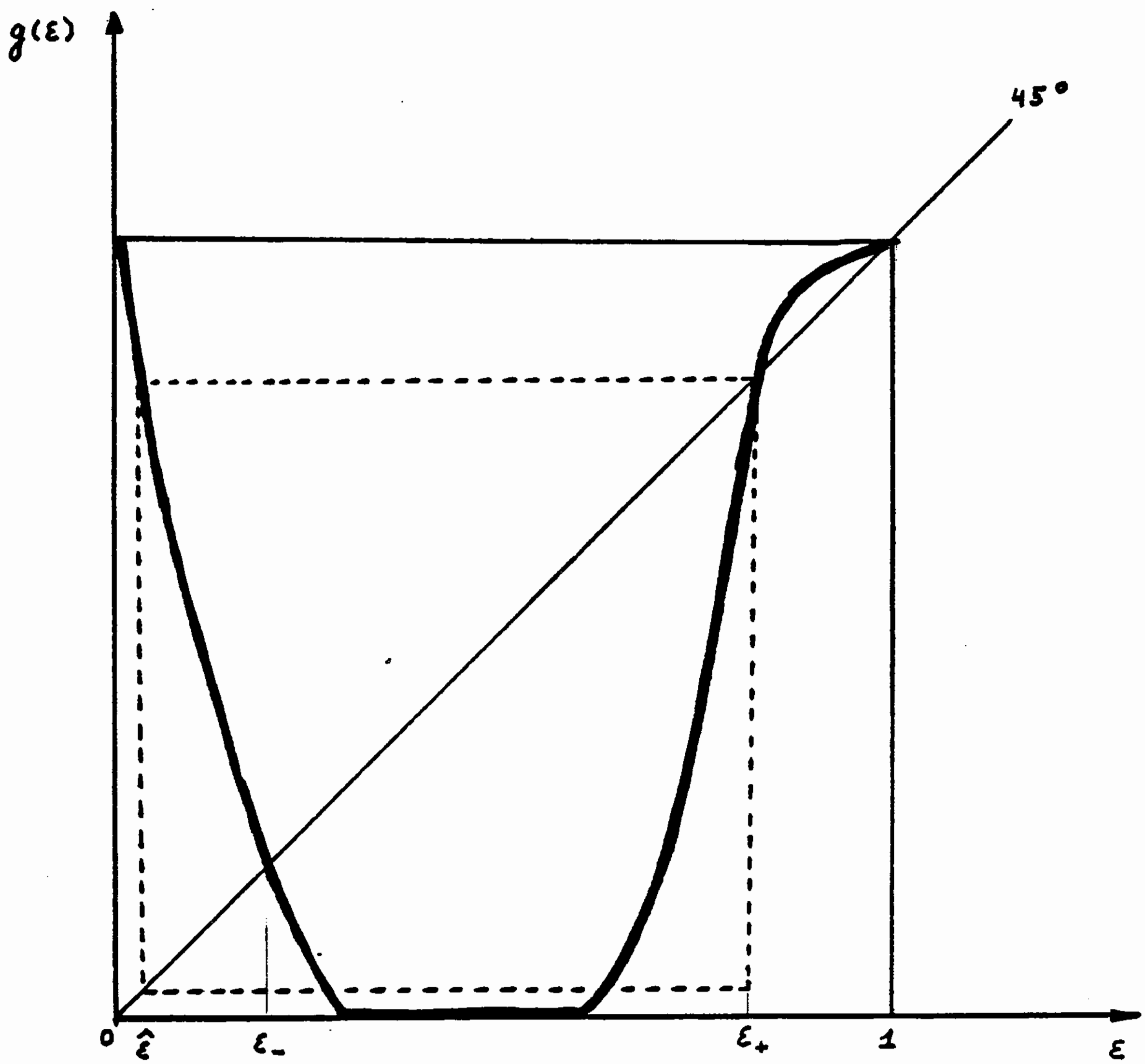

Figure 2: Graph of g(•), the solution to the difference equation (A.13)

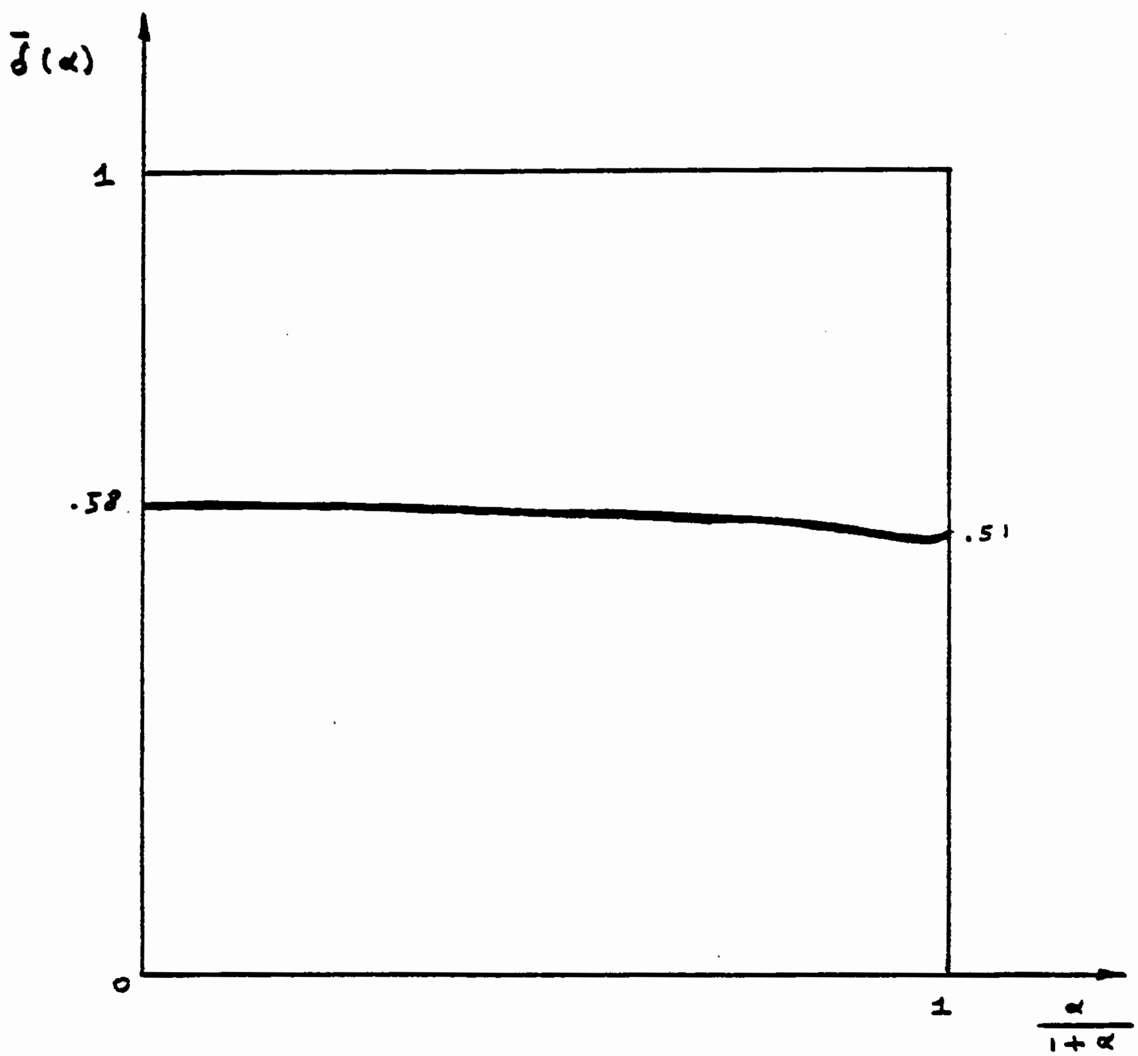

Figure 3: Graph of $\bar{\delta}(\alpha)$, the critical value at which roots of ψ appear.