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LEAST SQUARES AND STOCHASTIC DIFFERENCE EQUATIONS

by

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I. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we study asymptotic properties of least squares estimates of parameters in a stochastic difference equation. The results we obtain represent an extension of results previously obtained by H. B. Mann and A. Wald [2], H. Rubin [5], T. W. Anderson [1], M. M. Rao [4], and T. J. Muench [3].

The basic result obtained before can be stated as follows:

<u>Theorem 1:</u> Let $\{x(t);\ t = -n+1, -n+2, \ldots\}$ be a family of real-valued random variables which satisfy the conditions

(i) $x(t) \equiv \bar{x}_t$ w.pr. 1 ($\equiv$ with probability 1),

$t = -n+1, \ldots, 0;$

(ii) $Ex(t)^2 < \infty,\ t = 1,2,\ldots;$ [2/] and

(iii) $\sum_{k=0}^{n} a_k x(t-k) = \eta(t), t = 1,2,\ldots,$

where the $\bar{x}_t$ and the a_k are real constants with $a_0 = 1$, and where $\{\eta(t);\ t = 1,2,\ldots\}$ is a family of non-degenerate, independently and identically distributed real random variables with mean zero. Next, let $a \equiv (a_1, \ldots, a_n)$ and let $\hat{a}(N) \equiv (\hat{a}_1(N), \ldots, \hat{a}_n(N))$, $N > n$, be a sequence of random vectors which for each N and "amost all" realizations of the $x(t)$ satisfy

$$(1.1) \quad \sum_{t=1}^{N}(x(t) - \sum_{k=1}^{n} \hat{a}_k(N)x(t-k))^2 = \min_{\alpha_1,\dots,\alpha_n} \sum_{t=1}^{N}(x(t) - \sum_{k=1}^{n} \alpha_k x(t-k))^2.$$

Then $\hat{a}(N)$ converges in probability to $-a$, i.e.

$$(1.2) \quad \operatorname{plim}_{N\to\infty} \hat{a}(N) = -a.$$

This theorem was originally established by Mann and Wald under the additional assumptions that $E\,\eta(1)^4 < \infty$,and that the moduli of the roots of the characteristic polynomial

$$(1.3) \quad A(z) \equiv \sum_{k=1}^{n} a_k z^{n-k}$$

are all less than 1. Anderson established the theorem for the case when the roots of $A(z)$ all have moduli greater than 1, and Rao proved the theorem for the case when $A(z)$ has two roots, one with modulus less than 1 and one with modulus greater than 1. Finally, Rubin proved the theorem for $n = 1$, and Muench proved it for an arbitrary n.

In this paper we will establish the following theorem:

<u>Theorem 2</u>: Let $\{x(t);\ t = -n+1, -n+2, \dots\}$ be a family of real random variables which satisfy conditions (i) - (iii) of Theorem 1. Moreover, let $\hat{a}(N)$ and $A(z)$ be as defined in (1.1) and (1.3) respectively, and assume that the moduli of the roots of $A(z)$ are all different from 1. Then

$$(1.4) \quad \lim_{N\to\infty} \hat{a}(N) = -a \quad \text{w.pr. } 1$$

The proof of the theorem is given in Section two of the paper. It is based on the validity of five auxiliary lemmas which we state and prove in Section three. [3/]

II. Proof of Theorem 2: It is easy to show that, if

$\tilde{x}(t) \equiv (x(t),\ldots,x(t-n+1))'$, $t = 0,1,\ldots,$ then

$$(2.1)\quad \hat{a}(N)' = \{\sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)'\}^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)x(t)$$

$$= -a' + \{\sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)'\}^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)\eta(t).$$

To prove that $\hat{a}(N)$ converges to $-a$ we must consider three different cases separately.

Suppose first that the roots of $A(z)$ all have moduli less than 1. Then

$$(2.2)\quad (\hat{a}(N) + a)' = \{N^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)'\}^{-1} N^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)\eta(t).$$

Moreover, Lemma 1 in Section 3 shows (cf. equation (3.1)) that the matrices

$$N^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)'$$

converge w.pr. 1 to a matrix that is well known to be invertible. Finally, Lemma 2 in Section 3 shows (cf. equation (3.19)) that the vectors

$$N^{-1} \sum_{t=1}^{N} \tilde{x}(t-1)\eta(t)$$

converges to the zero vector w.pr. 1. Consequently, from (2.2), and from

Lemmas 1 and 2 it follows that, when all the roots of $A(z)$ have moduli less than 1,

$$(2.3) \quad \lim_{N\to\infty} (\hat{a}(N) + a)' = 0 \quad \text{w.pr. } 1.$$

Suppose next that all the roots of $A(z)$ have moduli greater than 1, and let

$$A = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{pmatrix}.$$

Then

$$(2.4) \quad (\hat{a}(N) + a)' = A'^{-N}\{A^{-N} \sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)'A'^{-N}\}^{-1} A^{-N} \sum_{t=1}^{N} \tilde{x}(t-1)\eta(t).$$

Moreover, Lemma 3 in Section 3 shows (cf. equation (3.24)) that the matrices

$$A^{-N} \sum_{t=1}^{N} \tilde{x}(t-1)\tilde{x}(t-1)' A'^{-N}$$

converge w.pr. 1 to a random matrix that according to a result of T. Muench (cf. [3, pp. 11-14]) is non-singular w.pr. 1. Finally, Lemma 4 shows (cf. equation (3.44)) that the vectors

$$N^{-\frac{1}{2}} A^{-N} \sum_{t=1}^{N} \tilde{x}(t-1)\eta(t)$$

converge to the zero vector w.pr. 1. Consequently, since $N^{\frac{1}{2}}A^{-N}$ converges to the zero matrix, it follows from (2.4), and from Lemmas 3 and 4 that, when all the roots of $A(z)$ have moduli greater than 1, (2.3) must be valid. Actually, in this case the convergence in (2.3) occurs at an exponential rate. [4/]

Lastly, suppose that $A(z)$ has at least one root with modulus less than 1 and one with modulus greater than 1. To establish the validity of (2.3) for this case we proceed as follows: Let $A(z)$ be factored as in

$$(2.5)\qquad A(z) = \prod_{j=1}^{\ell} (z-z_j)^{\ell_j} ,$$

where $\ell_j \geq 0$ and $\sum_{j=1}^{\ell} \ell_j = n$, and suppose that the z_j have been numbered so that

$$(2.6)\qquad |z_j| < 1,\ j = 1,\ldots,h, \text{ and}$$

$$(2.7)\qquad |z_j| > 1,\ j = h+1,\ldots,\ell .$$

Moreover, let $m \equiv \sum_{j=1}^{h} \ell_j$, $p \equiv \sum_{j=h+1}^{\ell} \ell_j$, and let $b \equiv (b_1,\ldots,b_m)$, and $c \equiv (c_1,\ldots,c_p)$ be defined by

$$(2.8)\qquad \sum_{k=0}^{m} b_k z^{m-k} \equiv \prod_{j=1}^{h} (z-z_j)^{\ell_j}, \text{ and}$$

$$(2.9)\qquad \sum_{k=0}^{p} c_k z^{p-k} \equiv \prod_{j=h+1}^{\ell} (z-z_j)^{\ell_j} .$$

Finally, let

$$(2.10)\quad y(t) \equiv \sum_{k=0}^{p} c_k x(t-k),\ t = -m+1,\ -m+2,\ldots,$$

$$(2.11)\quad u(t) \equiv \sum_{k=0}^{m} b_k x(t-k),\ t = -p+1,\ -p+2,\ldots,\ \text{and}$$

$$(2.12)\qquad R \equiv \begin{pmatrix} 1 & 0 & & 0 & 1 & 0 & \cdots & 0 \\ b_1 & 1 & & 0 & c_1 & 1 & & \cdot \\ \vdots & b_1 & & & \vdots & c_1 & & \\ b_m & & & & c_p & & & 1 \\ & b_m & & 1 & \cdot & c_p & & c_1 \\ & & & b_1 & \cdot & & & \vdots \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & & \\ 0 & 0 & \cdots & b_m & 0 & 0 & \cdots & c_p \end{pmatrix}$$

Then: $\{y(t);\ t = -m+1,\ -m+2,\ldots\}$ satisfies

$$(2.13)\quad \sum_{k=0}^{m} b_k y(t-k) = \eta(t),\ t = 1,2,\ldots,$$

and also Conditions (i) and (ii) of Lemma 1. Moreover, $\{u(t);\ t = -p+1,\ -p+2,\ldots\}$ satisfies the equations

$$(2.14)\quad \sum_{k=0}^{p} c_k u(t-k) = \eta(t),\ t = 1,2,\ldots,$$

and also Conditions (i) and (ii) of Lemma 3. Finally,

$$(2.15)\quad R'\sum_{t=1}^{N}\tilde{x}(t-1)\tilde{x}(t-1)'\cdot R=\begin{pmatrix}\sum_{t=1}^{N}\tilde{u}(t-1)\tilde{u}(t-1)' & \sum_{t=1}^{N}\tilde{u}(t-1)\tilde{y}(t-1)'\\ \sum_{t=1}^{N}\tilde{y}(t-1)\tilde{u}(t-1)' & \sum_{t=1}^{N}\tilde{y}(t-1)\tilde{y}(t-1)'\end{pmatrix}.$$

It is easy to verify that R is non-singular (for a proof see [7, p. 30]). Consequently, if we let

$$D^N \equiv \begin{pmatrix} C^{-N} & O \\ O & N^{-\frac{1}{2}}I\end{pmatrix}, \quad N = n+1,\ldots,$$

where I is the $m \times m$ identity matrix, and where

$$\underset{\sim}{C} \equiv \begin{pmatrix} -c_1 & -c_2 & \cdots & -c_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \cdot \\ \vdots & \vdots & & \cdot \\ 0 & 0 & \cdots & 1\,0 \end{pmatrix},$$

then (2.15) and (2.1) imply that [5/]

$$(2.16)\quad (\hat{a}(N)+a) = D'^N R\left\{D^N \begin{pmatrix}\sum_{t=1}^{N}\tilde{u}(t-1)\tilde{u}(t-1)' & \sum_{t=1}^{N}\tilde{u}(t-1)\tilde{y}(t-1)'\\ \sum_{t=1}^{N}\tilde{y}(t-1)\tilde{u}(t-1)' & \sum_{t=1}^{N}\tilde{y}(t-1)\tilde{y}(t-1)'\end{pmatrix} D'^N\right\}^{-1}\cdot D^N \begin{pmatrix}\sum_{t=1}^{N}\tilde{u}(t-1)\eta(t)\\ \sum_{t=1}^{N}\tilde{y}(t-1)\eta(t)\end{pmatrix}.$$

Now Lemma 5 in Section 3 shows (cf. equation (3.50)) that the matrices

$$N^{-\frac{1}{2}} C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{y}(t-1)'$$

converge to the zero matrix w.pr. 1. From this fact, from Lemmas 1 and 3, and from Muench's result referred to above it follows that the matrices

$$D^{N} \begin{pmatrix} \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{\ }(t-1)' & \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{y}(t-1)' \\ \sum_{t=1}^{N} \tilde{y}(t-1)\tilde{u}(t-1)' & \sum_{t=1}^{N} \tilde{y}(t-1)\tilde{y}(t-1)' \end{pmatrix} D'^{N}$$

converge w.pr. 1 to a random block-diagonal matrix that is non-singular w.pr. 1.

Next observe that Lemmas 2 and 4 imply that the vectors

$$N^{-\frac{1}{2}} D^{N} \begin{pmatrix} \sum_{t=1}^{N} \tilde{u}(t-1)\eta(t) \\ \sum_{t=1}^{N} \tilde{y}(t-1)\eta(t) \end{pmatrix}$$

converge to the zero vector w. pr. 1. Since $N^{\frac{1}{2}} D^{N}$ is uniformly bounded in N, it follows from this fact, from the result obtained in the preceding paragraph, and from (2.16) that (2.3) is valid when $A(z)$ has at least one root of modulus less than 1 and one of modulus greater than 1.

Since there are no other cases to consider, the proof of Theorem 2 is complete.

Q.E.D.

III. Auxiliary Lemmas. In the proof of Theorem 2 we made use of five auxiliary Lemmas. These are stated and proved in this section.

The first two lemmas were needed to show that Theorem 2 is valid when the roots of $A(z)$ all have moduli less than 1.

Lemma 1: Let $\{y(t); t = -m+1, -m+2,\ldots\}$ be a sequence of real random variables which satisfy the conditions:

(i) $y(t) \equiv \bar{y}_t$ w.pr. 1, $t = -m+1, -m+2,\ldots,0$;

(ii) $Ey(t)^2 < \infty$, $t = 1,2,\ldots$; and

(iii) $\sum_{k=0}^{m} b_k y(t-k) = \eta(t)$, $t = 1,2,\ldots,$

where the $\bar{y}_t$ and b_k are real constants with $b_0 = 1$, and where the $\eta(t)$ are as specified in Theorem 1. Moreover, let $B(z) \equiv \sum_{k=0}^{m} b_k z^{m-k}$, and assume that the roots of $B(z)$ all have moduli less than 1. Finally, let $\sigma_\eta^2 \equiv E\eta(1)^2$, let $\tilde{y}(t) \equiv (y(t),\ldots,y(t-m+1))'$, $t = 1,2,\ldots,$ and let

$$B \equiv \begin{pmatrix} -b_1 & -b_2 & \cdots & & -b_m \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \cdot \\ \vdots & & & & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Then

$$\text{(3.1)}\quad \lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} \tilde{y}(t-1)\tilde{y}(t-1)' = \sum_{s=0}^{\infty} B^s \begin{pmatrix} \sigma_\eta^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix} B'^s \quad \text{w.pr. 1.}$$

Proof: It follows easily from Condition (iii) of Lemma 1 that

$$(3.2)\qquad \tilde{y}(t) = B\tilde{y}(t-1) + \hat{\eta}(t) = \dots$$

$$= B^t\tilde{y}(0) + \sum_{s=0}^{t-1} B^s \hat{\eta}(t-s),\ t = 1,2,\dots,$$

where

$$\hat{\eta}(t) \equiv (\eta(t), 0_{m-1})',\ t = 1,2,\dots,$$

and 0_{m-1} denotes the (m-1) - dimensional zero vector. Thus

$$(3.3)\qquad N^{-1}\sum_{t=1}^{N} \tilde{y}(t-1)\tilde{y}(t-1)' =$$

$$= N^{-1}\{\tilde{y}(0)\tilde{y}(0)' + \sum_{t=2}^{N} [B^{t-1}\tilde{y}(0) + \sum_{s=0}^{t-2} B^s\hat{\eta}(t-1-s)][B^{t-1}\tilde{y}(0) + \sum_{s=0}^{t-2} B^s\hat{\eta}(t-1-s)]'\}$$

$$= N^{-1}\sum_{t=1}^{N} B^{t-1}\tilde{y}(0)\tilde{y}(0)'B'^{t-1} + N^{-1}\sum_{t=2}^{N}\sum_{s=0}^{t-2} B^{t-1}\tilde{y}(0)\hat{\eta}(t-1-s)'B'^{s}$$

$$+ N^{-1}\sum_{t=2}^{N}\sum_{s=0}^{t-2} B^s\hat{\eta}(t-1-s)\tilde{y}(0)'B'^{t-1}$$

$$+ N^{-1}\sum_{t=2}^{N}\sum_{s,r=0}^{t-2} B^s\hat{\eta}(t-1-s)\hat{\eta}(t-1-r)'B'^{r}$$

$$\equiv \alpha(N) + \beta(N) + \gamma(N) + \varphi(N).$$

It is quite obvious that

$$(3.4)\qquad \lim_{N\to\infty} \alpha(N) = 0 \ \text{ w.pr. 1.}$$

Next we will show that

$$(3.5)\qquad \lim_{N\to\infty} \beta(N) = 0 \ \text{ w.pr. 1.}$$

To do that we first note that by Lemma 2 in [6] there exists a $\lambda \in (0,1)$ and a finite constant K_1 such that

(3.6) $|B^s| \leq K_1\lambda^s, \; s = 0,1,2,\ldots,$

where $|B^s|$ denotes the matrix whose components are the absolute values of the components of B^s. Then we let

$$e \equiv (1, 0_{m-1}),$$

and compute

(3.7) $\text{tr. } E\beta(N)\beta(N)' =$

$$= N^{-2}\text{tr.}E \sum_{t,r=2}^{N} \sum_{s=0}^{t-2} \sum_{v=0}^{r-2} B^{t-1}\tilde{y}(0)\,\hat{\eta}(t-1-s)'B'^{s}B^{v}\hat{\eta}(r-1-v)\tilde{y}(0)'B'^{r-1}$$

$$= \sigma_\eta^2 N^{-2}\text{tr.} \sum_{t=2}^{N} \sum_{r=2}^{t} \sum_{s=t-r}^{t-2} [B^{t-1}\tilde{y}(0)eB'^{s}B^{s+r-t}e'\tilde{y}(0)B'^{r-1}]$$

$$+ N^{-2}\sigma_\eta^2\text{tr.} \sum_{t=2}^{N-1} \sum_{r=t+1}^{N} \sum_{s=0}^{t-2} B^{t-1}\tilde{y}(0)eB'^{s}B^{s+r-t}e'\tilde{y}(0)'B'^{r-1}$$

$$\leq K_2N^{-2}\{\sum_{t=2}^{N} \sum_{r=2}^{t} \sum_{s=t-r}^{t-2} \lambda^{2(s+r-1)} + \sum_{t=2}^{N-1} \sum_{r=t+1}^{N} \sum_{s=0}^{t-2} \lambda^{2(s+r-1)}\} = O(N^{-2}),$$

where K_2 is a suitably large constant. From (3.7), and from the Borel-Cantelli Lemma it follows that (3.5) is true as stated. A similar argument suffices to show that

(3.8) $\lim_{N\to\infty} \gamma(N) = 0$ w.pr. 1

as well.

We can also show that, if

$$\varphi_1(N) \equiv N^{-1} \sum_{t=2}^{N} \sum_{\substack{s,r=0 \\ s\neq r}}^{t-2} B^s \hat{\eta}(t-1-s)\hat{\eta}(t-1-r)'B'^r,$$

then

(3.9) $\lim_{N\to\infty} \varphi_1(N) = 0$ w.pr. 1.

To do that we compute

$$(3.10)\quad N^{-2}\mathrm{tr.}\ E\,\varphi_1(N)\varphi_1(N)'$$

$$= N^{-2}\mathrm{tr.} \sum_{t,q=2}^{N} \sum_{\substack{s,r=0 \\ s\neq r}}^{t-2} \sum_{\substack{v,\ell=0 \\ v\neq \ell}}^{q-2} B^s e'eB'^r B^v e'eB'^\ell E[\eta(t-1-s)$$

$$\eta(t-1-r)\eta(q-1-v)\eta(q-1-\ell)]$$

$$= 2\sigma_\eta^4 N^{-2}\mathrm{tr.}\{ \sum_{t=2}^{N} \sum_{q=2}^{t} \sum_{\substack{v,\ell=0 \\ v\neq \ell}}^{q-2} B^{v+(t-q)} e'eB'^{\ell+(t-q)} B^v e'eB'^\ell$$

$$+ \sum_{t=2}^{N-1} \sum_{q=t+1}^{N} \sum_{\substack{v,\ell=q-t \\ v\neq \ell}}^{q-2} B^{v+(t-q)} e'eB'^{\ell+(t-q)} B^v e'eB'^\ell\}$$

$$\leq K_3 N^{-2}\{ \sum_{t=2}^{N} \sum_{q=2}^{t} \sum_{\substack{v,\ell=0 \\ v\neq 0}}^{q-2} \lambda^{2(v+\ell+(t-q))} \sum_{t=2}^{N-1} \sum_{q=t+1}^{N} \sum_{\substack{v,\ell=q-t \\ v\neq \ell}} \lambda^{2(v+\ell+(t-q))}\} = O(N^{-1}),$$

where K_3 is a suitably large constant. If we now let $N_m \equiv m^2$, then it follows from (3.10), and from the Borel-Cantelli Lemma that

$$(3.11)\quad \lim_{m\to\infty} \varphi_1(N_m) = 0 \quad \text{w.pr. 1.}$$

On the other hand, for each and every m

$$(3.12)\quad E\{\max_{N_m \le N < N_{m+1}} \operatorname{tr}.[\varphi_1(N) - (N_m/N)\varphi_1(N_m)][\varphi_1(N)-(N_m/N)\varphi(N_m)]'\}$$

$$\le N_m^{-2}\operatorname{tr}.E \sum_{t,q=N_m+1}^{N_{m+1}} \sum_{\substack{s,r=0\\ s\ne r}}^{t-2} \sum_{\substack{v,\ell=0\\ v\ne\ell}}^{q-2} [B^s \hat{\eta}(t-1-s)\hat{\eta}(t-1-r)'B'^r B^v \hat{\eta}(q-1-v)$$

$$\hat{\eta}(q-1-\ell)'B'^\ell\ ]$$

$$\le K_3 N_m^{-2}\{\sum_{t=N_m+1}^{N_{m+1}} [\sum_{q=N_m+1}^{t-1} \sum_{\substack{v,\ell=0\\ v\ne\ell}}^{q-2} \lambda^{2(v+\ell+(t-q))} + \sum_{q=t}^{N_{m+1}} \sum_{\substack{v,\ell=q-t\\ v\ne\ell}}^{q-2} \lambda^{2(v+\ell+(t-q))}]\}$$

$$= O(N_m^{-3/2}) = O(m^{-3}).$$

From (3.12), and from the Borel-Cantelli Lemma it follows that

$$(3.13)\quad \lim_{N\to\infty} |\varphi_1(N) - (N_m/N)\varphi(N_m)| = 0 \quad \text{w.pr. 1 for} \quad N_m \le N < N_{m+1}.$$

Since, for all $N \in [N_m, N_{m+1})$ $\lim_{m\to\infty} (N/N_{m+1}) = 1$, the second factor in (3.13) can be replaced by $\varphi(N_m)$ so that

$$(3.14)\quad \lim_{N\to\infty} |\varphi_1(N) - \varphi(N_m)| = 0 \quad \text{w.pr. 1} \quad \text{for} \quad N_m \le N < N_{m+1}$$

From (3.11) and (3.14) we conclude that (3.9) is true as stated.

To complete the proof of the lemma we must show that, if

$$\varphi_2(N) \equiv N^{-1} \sum_{t=2}^{N} \sum_{s=0}^{t-2} B^s \hat{\eta}(t-1-s) \hat{\eta}(t-1-s)' B'^s,$$

then

$$(3.15) \quad \lim_{N\to\infty} \varphi_2(N) = \sum_{s=0}^{\infty} B^s \begin{pmatrix} \sigma_\eta^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix} B'^s \quad \text{w.pr. 1.}$$

To do that we note first that

$$(3.16) \quad \varphi_2(N) = N^{-1} \sum_{s=0}^{N-2} \sum_{t=s+2}^{N} B^s \hat{\eta}(t-1-s) \hat{\eta}(t-1-s)' B'^s$$

$$= N^{-1} \sum_{s=0}^{N-2} \sum_{v=1}^{N-s-1} B^s \hat{\eta}(v) \hat{\eta}(v)' B'^s$$

$$= N^{-1} \sum_{s=0}^{N_0} \sum_{v=1}^{N-s-1} B^s \hat{\eta}(v) \hat{\eta}(v)' B'^s + N^{-1} \sum_{s=N_0+1}^{N-2} \sum_{v=1}^{N-s-1} B^s \hat{\eta}(v) \hat{\eta}(v)' B'^s$$

$$\equiv \varphi_{2N_0}(N) + \varphi_{3N_0}(N).$$

Next we observe that, by the law of large numbers

$$(3.17) \quad \lim_{N\to\infty} \varphi_{2N_0}(N) = \sum_{s=0}^{N_0} B^s \begin{pmatrix} \sigma_\eta^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} B'^s \quad \text{w.pr. 1,}$$

and that

$$(3.18)\quad \lim_{N\to\infty} |\varphi_{3N_0}(N)| = \lim N^{-1} \Big| \sum_{s=N_0+1}^{N} \sum_{v=1}^{N-s-1} B^s \hat{\eta}(v)\hat{\eta}(v)'B'^s \Big|$$

$$= \lim_{N\to\infty} N^{-1} \Big| \sum_{s=N_0+1}^{N} \sum_{v=1}^{N-s-1} B^s e'eB'^s \eta(v)^2 \Big|$$

$$\leq \lim_{N\to\infty} N^{-1} \sum_{s=N_0+1}^{N} \sum_{v=1}^{N-s-1} |B^s e'eB'^s| \eta(v)^2$$

$$\leq \lim_{N\to\infty} K_4 \sum_{s=N_0+1}^{N} \lambda^s N^{-1} \sum_{v=1}^{N-s-1} \eta(v)^2$$

$$\leq \lim_{N\to\infty} K_4 \left\{ \frac{1-\lambda^{N-N_0+1}}{1-\lambda} \right\} \lambda^{N_0+1} N^{-1} \sum_{v=1}^{N} \eta(v)^2 .$$

$$= (K_4\sigma_\eta^2/1-\lambda)\lambda^{N_0+1} \quad \text{w.pr. 1.}$$

From (3.17) and (3.18) it follows that (3.15) is true as stated. Q.E.D.

<u>Lemma 2:</u> If $\{y(t);\ t = -m+1, -m+2, \ldots\}$ is as specified in Lemma 1, then

$$(3.19)\quad \lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} \tilde{y}(t-1)\eta(t) = 0 \quad \text{w.pr. 1.}$$

Proof: It follows from (3.2) that

$$(3.20)\quad N^{-1} \sum_{t=1}^{N} \tilde{y}(t-1)\eta(t) = N^{-1} \sum_{t=1}^{N} B^{t-1}\tilde{y}(0)\eta(t) + N^{-1} \sum_{t=2}^{N} \sum_{s=0}^{t-2} B^s \hat{\eta}(t-1-s)\eta(t)$$

$$\equiv \Psi_1(N) + \Psi_2(N).$$

Moreover, it follows easily from (3.6) and from the fact that $\eta(1)$ has finite variance that $\sum_{t=1}^{N} B^{t-1}\eta(t)$ converges with probability one to a random vector with finite covariance matrix. Consequently,

$$(3.21)\quad \lim_{N\to\infty} \Psi_1(N) = 0 \quad \text{w.pr. } 1.$$

To show that

$$(3.22)\quad \lim_{N\to\infty} \Psi_2(N) = 0 \text{ w.pr. } 1$$

as well, we proceed in the following way. We first note that

$$(3.23)\quad \text{tr. } E\,\Psi_2(N)\Psi_2(N)' = N^{-2}\text{tr.}E \sum_{t,u=2}^{N}\sum_{s=0}^{t-2}\sum_{r=0}^{u-2} B^s\hat{\eta}(t-1-s)\hat{\eta}(u-1-r)'B'^r\eta(t)\eta(u)$$

$$= N^{-2}\text{tr.} \sum_{t,u=2}^{N}\sum_{s=0}^{t-2}\sum_{r=0}^{u-2} B^s e'eB'^r E\eta(t-1-s)\eta(u-1-r)\eta(t)\eta(u)$$

$$= N^{-2}\sigma_\eta^4 \text{tr.} \sum_{t=2}^{N}\sum_{s=0}^{t-2} B^s e'eB'^s$$

$$= O(N^{-1}).$$

But if that is so, then arguments similar to those used to establish (3.9) can be used to verify the validity of (3.22). For brevity's sake we will omit the necessary details here and consider the lemma proved. Q.E.D.

The next two lemmas were used to show that Theorem 2 is valid when all the roots of $A(z)$ have moduli greater than 1.

Lemma 3: Let $\{u(t); t = -p+1, -p+2, \ldots\}$ be a sequence of real random variables which satisfy the conditions

(i) $u(t) \equiv \bar{u}_t$ w.pr. 1, $t = -p+1, -p+2, \ldots, 0$;

(ii) $Eu(t)^2 < \infty$, $t = 1,2,\ldots$;

(iii) $\sum_{k=0}^{p} c_k u(t-k) = \eta(t)$, $t = 1,2,\ldots$,

where the $\bar{u}_t$ and the c_k are real constants with $c_0 = 1$, and where the $\eta(t)$ are as specified in Theorem 1. Moreover, let $C(z) \equiv \sum_{k=0}^{p} c_k z^{p-k}$, and assume that the roots of $C(z)$ all have moduli greater than 1. Finally, let $\tilde{u}(t) \equiv (u(t),\ldots,u(t-p+1))'$, $t = 1,2,\ldots$, and let

$$C \equiv \begin{pmatrix} -c_1 & -c_2 & \cdots & -c_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \cdot \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\ 1 & 0 \end{pmatrix}$$

Then

$$(3.24)\quad \lim_{N\to\infty} C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{u}(t-1)' C'^{-N}$$

$$= \sum_{s=1}^{\infty} C^{-s}\{\tilde{u}(0) + \sum_{r=1}^{\infty} C^{-r}\eta^*(r)\}\{\tilde{u}(0) + \sum_{r=1}^{\infty} C^{-r}\eta^*(r)\}' C'^{-s} \text{ w.pr. } 1,$$

where

$$\eta^*(t) \equiv (\eta(t), 0_{p-1})' \quad t = 1,2,\ldots$$

Proof: It follows easily from Condition (iii) of Lemma 3 that

$$(3.25)\quad \tilde{u}(t) = C\tilde{u}(t-1) + \eta^*(t) = \cdots =$$

$$= C^t\tilde{u}(0) + \sum_{s=0}^{t-1} C^s \eta^*(t-s),\ t = 1,2,\ldots \quad .$$

Thus

$$(3.26)\quad C^{-N}\sum_{t=1}^{N}\tilde{u}(t-1)\tilde{u}(t-1)'C'^{-N}$$

$$= C^{-N}\{\tilde{u}(0)\tilde{u}(0)' + \sum_{t=2}^{N}[C^{t-1}\tilde{u}(0) + \sum_{s=0}^{t-2} C^{s}\eta^{*}(t-1-s)][C^{t-1}\tilde{u}(0) + \sum_{s=0}^{t-2} C^{s}\eta^{*}(t-1-s)]'\}C'^{-N}$$

$$= C^{-N}\sum_{t=1}^{N} C^{t-1}\tilde{u}(0)\tilde{u}(0)'C'^{t-1}\cdot C'^{-N}$$

$$+ C^{-N}\sum_{t=2}^{N}\sum_{s=0}^{t-2} C^{t-1}\tilde{u}(0)\eta^{*}(t-1-s)'C'^{s}\cdot C'^{-N}$$

$$+ C^{-N}\sum_{t=2}^{N}\sum_{s=0}^{t-2} C^{s}\eta^{*}(t-1-s)\tilde{u}(0)'C'^{t-1}\cdot C'^{-N}$$

$$+ C^{-N}\sum_{t=2}^{N}\sum_{s,r=0}^{t-2} C^{s}\eta^{*}(t-1-s)\eta^{*}(t-1-r)'C'^{r}\cdot C'^{-N}$$

$$\equiv \bar{\alpha}(N) + \bar{\beta}(N) + \bar{\gamma}(N) + \bar{\varphi}(N).$$

It is easy to see by a change of variable that

$$(3.27)\quad \lim_{N\to\infty}\bar{\alpha}(N) = \lim_{N\to\infty}\sum_{t=1}^{N} C^{-t}\tilde{u}(0)\tilde{u}(0)'C'^{-t}\ .$$

By Lemma 2 in [6] there exists a $\mu \in (0,1)$ and a finite constant K_6 such that

$$(3.28)\quad |C^{-t}| \le K_6\mu^{t},\quad t = 0,1,2,\ldots\ .$$

Consequently,

$$(3.29)\quad \lim_{N\to\infty} \bar{\alpha}(N) = \sum_{t=1}^{\infty} C^{-t}\tilde{u}(0)\tilde{u}(0)'C'^{-t} < \infty.$$

Next we observe that

$$(3.30)\quad \bar{\beta}(N) = \sum_{v=1}^{N-1}\sum_{q=1}^{v} C^{-(N-v)}\tilde{u}(0)\eta^*(q)'C'^{-(N-v)}\cdot C'^{-q}$$

$$= \sum_{q=1}^{N-1}\sum_{s=1}^{N-q} C^{-s}\tilde{u}(0)\eta^*(q)'C'^{-s}\cdot C'^{-q}$$

$$= \sum_{s,q=1}^{N-1} C^{-s}\tilde{u}(0)\eta^*(q)'C'^{-s}C'^{-q} - \sum_{q=2}^{N-1}\sum_{s=N-q+1}^{N-1} C^{-s}\tilde{u}(0)\eta^*(q)'C'^{-s}C'^{-q}$$

$$\equiv \bar{\beta}_1(N) - \bar{\beta}_2(N).$$

We will show that

$$(3.31)\quad \lim_{N\to\infty} \bar{\beta}_2(N) = 0 \quad \text{w.pr. } 1.$$

To do that we note that, if

$$e_1 \equiv (1, 0_{p-1}),$$

$$(3.32)\quad \text{tr. } E\bar{\beta}_2(N)\bar{\beta}_2(N)' =$$

$$= \text{tr. } E\sum_{\ell,q=2}^{N-1}\sum_{s=N-\ell+1}^{N-1}\sum_{r=N-q+1}^{N-1} C^{-s}\tilde{u}(0)e_1C'^{-(s+\ell)}C^{-(r+q)}e_1'\tilde{u}(0)'C'^{-r}\eta(\ell)\eta(q)$$

$$= \sigma_\eta^2 \text{tr. } \sum_{\ell=2}^{N-1}\sum_{s,r=N-\ell+1}^{N-1} C^{-s}\tilde{u}(0)e_1C'^{-(s+\ell)}C^{-(r+\ell)}e_1'\tilde{u}(0)'C'^{-r}$$

$$\leq K_7 \sum_{\ell=2}^{N-1}\sum_{s,r=N-\ell+1}^{N-1} \mu^{2(s+\ell+r)} = O(\mu^{2N}).$$

for a suitably large constant K_7. From this, and from the Borel-Cantelli lemma it follows that (3.31) is true as stated. But if that is so, then (3.30), (3.31), and (3.28) can easily be seen to imply that

$$(3.33)\quad \lim \bar{\beta}(N) = \lim_{N\to\infty} \bar{\beta}_2(N) = \sum_{s=1}^{\infty} \sum_{q=1}^{\infty} C^{-s}\tilde{u}(0)\eta^*(q)'C'^{-q} \cdot C'^{-s} \quad \text{w.pr. 1.}$$

A similar argument suffices to show that

$$(3.34)\quad \lim_{N\to\infty} \bar{\gamma}(N) = \sum_{s=1}^{\infty} \sum_{q=1}^{\infty} C^{-s}C^{-q}\eta^*(q)\tilde{u}(0)'C'^{-s} \quad \text{w.pr. 1} \quad .$$

To conclude the proof of Lemma 3 we now observe that

$$(3.35)\quad \bar{\varphi}(N) = C^{-N} \sum_{v=1}^{N-1} \sum_{s,r=0}^{v-1} C^{s}\eta^*(v-s)\eta^*(v-r)'C'^{r}C'^{-N}$$

$$= C^{-N} \sum_{v=1}^{N-1} \sum_{\ell,q=1}^{v} C^{v-\ell}\eta^*(\ell)\eta^*(q)'C'^{v-q} \cdot C'^{-N}$$

$$= C^{-N} \sum_{v,\ell,q=1}^{N-1} C^{v-\ell}\eta^*(\ell)\eta^*(q)'C'^{v-q} \cdot C'^{-N}$$

$$- C^{-N} \sum_{v,\ell=1}^{N-2} \sum_{q=v+1}^{N-1} C^{v-\ell}\eta^*(\ell)\eta^*(q)'C'^{v-q} \cdot C'^{-N}$$

$$- C^{-N} \sum_{v=1}^{N-2} \sum_{\ell=v+1}^{N-1} \sum_{q=1}^{v} C^{v-\ell}\eta^*(\ell)\eta^*(q)'C'^{v-q} \cdot C'^{-N}$$

$$\equiv \bar{\varphi}_1(N) - \bar{\varphi}_2(N) - \bar{\varphi}_3(N).$$

It is easy to see that

(3.36) $\quad \mathrm{tr.}\ E\ \bar{\varphi}_3(N)\ \bar{\varphi}_3(N)' =$

$$= \sum_{v,r=1}^{N-2} \sum_{\ell=v+1}^{N-1} \sum_{q=1}^{v} \sum_{s=r+1}^{N-1} \sum_{t=1}^{r} [C^{-(N-v+\ell)} e_1' e_1 C'^{-(N-v+q)} C^{-(N-r+t)} \cdot$$

$$e_1' e_1 C'^{-(N-r+s)} E \eta(\ell) \eta(q) \eta(s) \eta(t)]$$

$$= \sigma_\eta^4 \sum_{v,r=1}^{N-2} \sum_{\ell=\max(v+1,r+1)}^{N-1} \sum_{q=1}^{\min(v,r)} [C^{-(N-v+\ell)} e_1' e_1 C'^{-(N-v+q)} \cdot$$

$$C^{-(N-r+q)} e_1' e_1 C'^{(N-r+\ell)}]$$

$$\leq K_8 \sum_{s,t=2}^{N-1} \sum_{\ell=\max(N-s+1,N-t+1)}^{N-1} \sum_{q=1}^{\min(N-s,N-t)} \mu^{2(s+\ell+q+t)}$$

$$= O(\mu^{2N}),$$

for some suitably large constant K_8, and hence by the Borel-Cantelli lemma that

(3.37) $\quad \lim_{N\to\infty} \bar{\varphi}_3(N) = 0 \quad$ w.pr. 1.

It is also easy to see that

(3.38) $$\bar{\varphi}_2(N) = \sum_{s=2}^{N-1} C^{-s} \sum_{\ell=1}^{N-2} \sum_{q=N-s+1}^{N-1} C^{-\ell} \eta^*(\ell) \eta^*(q)' C'^{-q} \cdot C'^{-s}$$

$$= \sum_{s=2}^{N-1} C^{-s} \sum_{\ell=N-s+1}^{N-2} C^{-\ell} \eta^*(\ell) \eta^*(\ell)' C'^{-\ell} \cdot C'^{-s}$$

$$+ \sum_{s=2}^{N-1} C^{-s} \sum_{\ell=1}^{N-2} \sum_{\substack{q=N-s+1 \\ q\neq\ell}}^{N-1} C^{-\ell} \eta^*(\ell) \eta^*(q)' C'^{-q} \cdot C'^{-s}$$

$$\equiv \bar{\varphi}_{21}(N) + \bar{\varphi}_{22}(N).$$

Since

$$(3.39) \quad E|\varphi_{21}(N)| \leq \sigma_\eta^2 \sum_{s=2}^{N-1} \sum_{\ell=N-s+1}^{N-2} |C^{-(s+\ell)} e_1' e_1 C'^{-(s+\ell)}|$$

$$\leq K_9 \sum_{s=2}^{N-1} \sum_{\ell=N-s+1}^{N-2} \mu^{2(s+\ell)} = O(N\mu^{2N}),$$

for a suitably large constant K_9, it follows from the Borel-Cantelli lemma that

$$(3.40) \quad \lim_{N\to\infty} \bar{\varphi}_{21}(N) = 0 \quad \text{w.pr. } 1.$$

To see that

$$(3.41) \quad \lim_{N\to\infty} \bar{\varphi}_{22}(N) = 0 \quad \text{w.pr. } 1$$

as well, we compute

$$(3.42) \quad \text{tr. } E\bar{\varphi}_{22}(N)\bar{\varphi}_{22}(N)'$$

$$= \text{tr. } \sum_{s,t=2}^{N-1} \sum_{r,\ell=1}^{N-2} \sum_{\substack{q=N-s+1 \\ q\neq r}}^{N-1} \sum_{\substack{k=N-t+1 \\ k\neq \ell}}^{N-1} [C^{-(s+r)} e_1' e_1 C'^{-(s+q)} C^{-(t+k)} e_1' e_1 C'^{-(t+\ell)} \cdot$$

$$\cdot \; E\eta(r)\eta(q)\eta(\ell)\eta(k)]$$

$$\leq K_{10} \left\{ \sum_{s,t=2}^{N-1} \sum_{r=1}^{N-1} \sum_{q=\max(N-s+1,N-t+1)}^{N-1} \mu^{2(s+r+t+q)} \right.$$

$$\left. + \sum_{s,t=1}^{N-1} \sum_{r=N-t+1}^{N-1} \sum_{q=N-s+1}^{N-1} \mu^{2(s+r+t+q)} \right\}$$

$= O(\mu^{2N})$,

for a suitably large constant K_{10}, which together with the Borel-Cantelli lemma implies the validity of (3.41). Now (3.35), (3.37), (3.40), and (3.41) can easily be seen to imply that

$$(3.43)\quad \lim_{N\to\infty} \bar{\varphi}(N) = \lim_{N\to\infty} \bar{\varphi}_1(N) = \sum_{s=1}^{\infty} C^{-s} \sum_{\ell,q=1}^{\infty} C^{-\ell}\eta^*(\ell)\eta^*(q)'C'^{-q}C'^{-s} \text{ w.pr. } 1.$$

The validity of (3.24) now follows from (3.43), (3.34), (3.33), and (3.29). So the proof of Lemma 3 is complete. Q.E.D.

<u>Lemma 4:</u> Let $\{u(t);\ t = -p+1, -p+2,\ldots\}$ be as in Lemma 3. Then

$$(3.44)\quad \lim_{N\to\infty} N^{-\frac{1}{2}}C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\eta(t) = 0 \text{ w.pr. } 1$$

Proof: It is clear that

$$(3.45)\quad N^{-\frac{1}{2}}C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\eta(t) = N^{-\frac{1}{2}}C^{-N}\{\tilde{u}(0)\eta(1) + \sum_{t=2}^{N} [C^{t-1}\tilde{u}(0) +$$

$$+ \sum_{s=0}^{t-2} C^s\eta^*(t-1-s)]\eta(t)\}$$

$$= N^{-\frac{1}{2}}C^{-N} \sum_{t=1}^{N} C^{t-1}\tilde{u}(0)\eta(t) + N^{-\frac{1}{2}}C^{-N} \sum_{t=2}^{N} \sum_{s=0}^{t-2} C^s\eta^*(t-1-s)\eta(t)$$

$$\equiv \hat{\alpha}(N) + \hat{\beta}(N).$$

Now

$$(3.46)\quad \text{tr. } E\hat{\alpha}(N)\hat{\alpha}(N)' = \sigma_\eta^2 N^{-1}\text{tr. } C^{-N} \sum_{t=1}^{N} C^{t-1}\tilde{u}(0)\tilde{u}(0)'C'^{t-1}\cdot C'^{-N}$$

$$= \sigma^2 N^{-1}\text{tr. } \sum_{v=1}^{N} C^{-v}\tilde{u}(0)\tilde{u}(0)'C'^{-v} = O(N^{-1}).$$

Consequently, by using an argument similar to that used to establish (3.9) we can show that

$$(3.47)\quad \lim_{N\to\infty} \hat{\alpha}(N) = 0 \text{ w.pr. } 1.$$

For brevity's sake we omit the detailed arguments here.

Next we observe that

$$(3.48)\quad \text{tr. } E\hat{\beta}(N)\hat{\beta}(N)' = N^{-1}\text{tr.} \sum_{t,r=2}^{N} \sum_{s=0}^{t-2} \sum_{q=0}^{r-2} [C^{-(N-s)} e_1' e_1 C'^{-(N-q)} \cdot$$

$$\cdot\ E\eta(t)\eta(t-1-s)\eta(r)\eta(r-1-q)]$$

$$= \sigma_\eta^4 N^{-1}\text{tr.} \sum_{t=2}^{N} \sum_{s=0}^{t-2} C^{-(N-s)} e_1' e_1 C'^{-(N-s)}$$

$$\leq K_{11} N^{-1} \sum_{t=2}^{N} \sum_{s=0}^{t-2} \mu^{2(N-s)} = O(N^{-1}).$$

Hence, again we can use arguments similar to those used to establish (3.9) to show that

$$(3.49)\quad \lim_{N\to\infty} \hat{\beta}(N) = 0 \quad \text{w.pr. } 1.$$

We omit the detailed arguments for the sake of brevity. The relations (3.47) and (3.49) establish the validity of (3.44). Q.E.D.

<u>Lemma 5</u>: Let $\{y(t);\ t = -m+1, -m+2,\ldots\}$ and $\{u(t); t = -p+1, -p+2,\ldots\}$ be as in Lemmas 1 and 3 respectively. Then

$$(3.50)\quad \lim_{N\to\infty} N^{-\frac{1}{2}} C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{y}(t-1)' = 0 \quad \text{w.pr. 1.}$$

Proof: By using (3.25) and (3.2) we find that

$$(3.51)\quad N^{-\frac{1}{2}} C^{-N} \sum_{t=1}^{N} \tilde{u}(t-1)\tilde{y}(t-1)'$$

$$= N^{-\frac{1}{2}} C^{-N}\{\tilde{u}(0)\tilde{y}(0)' + \sum_{t=2}^{N} [C^{t-1}\tilde{u}(0) + \sum_{s=0}^{t-2} C^{s}\eta^{*}(t-1-s)][B^{t-1}\tilde{y}(0) + \sum_{s=0}^{t-2} B^{s}\hat{\eta}(t-1-s)]'$$

$$= N^{-\frac{1}{2}} C^{-N}\{\sum_{t=1}^{N} C^{t-1}\tilde{u}(0)\tilde{y}(0)'B'^{t-1}\}$$

$$+ N^{-\frac{1}{2}} C^{-N} \sum_{t=2}^{N} \sum_{s=0}^{t-2} C^{t-1}\tilde{u}(0)\hat{\eta}(t-1-s)'B'^{s}$$

$$+ N^{-\frac{1}{2}} C^{-N} \sum_{t=2}^{N} \sum_{s=0}^{t-2} C^{s}\eta^{*}(t-1-s)\tilde{y}(0)'B'^{t-1}$$

$$+ N^{-\frac{1}{2}} C^{-N} \sum_{t=2}^{N} \sum_{q,s=0}^{t-2} C^{s}\eta^{*}(t-1-s)\hat{\eta}(t-1-s)'B'^{q}$$

$$\equiv \alpha^{*}(N) + \beta^{*}(N) + \gamma^{*}(N) + \varphi^{*}(N).$$

It is quite clear that

$$(3.52)\quad \lim_{N\to\infty} \alpha^{*}(N) = 0.$$

To determine the behavior of $\beta^*(N)$ we compute

$$(3.53)\quad \text{tr. } E\beta^*(N)\beta^*(N)' =$$

$$= N^{-1}\text{tr.} \sum_{t,r=2}^{N} \sum_{s=0}^{t-2} \sum_{q=0}^{r-2} [C^{-(N-t+1)}\tilde{u}(0)eB'^{s}B^{q}e'\tilde{u}(0)'C'^{-(N-r+1)} \cdot$$

$$E\eta(t-1-s)\eta(r-1-q)]$$

$$= \sigma_\eta^2 N^{-1}\text{tr.} \sum_{t=2}^{N} \sum_{r=2}^{t} \sum_{q=0}^{r-2} C^{-(N-t+1)}\tilde{u}(0)eB'^{q+(t-r)}B^{q}e'\tilde{u}(0)'C'^{-(N-r+1)}$$

$$+ \sigma_\eta^2 N^{-1}\text{tr.} \sum_{t=2}^{N-1} \sum_{r=t+1}^{N} \sum_{q=r-t}^{r-2} [C^{-(N-t+1)}\tilde{u}(0)eB'^{q+(t-r)} \cdot$$

$$B^{q}e'\tilde{u}(0)'C'^{-(N-r+1)}]$$

$$\leq K_{12}N^{-1} \sum_{t=2}^{N} \{\sum_{r=2}^{t-1} \sum_{q=0}^{r-2} \mu^{2(N+1)-t-r}\lambda^{2q+(t-r)} +$$

$$+ \sum_{r=t}^{N} \sum_{q=r-t}^{r-2} \mu^{2(N+1)-t-r}\lambda^{2q+(t-r)}\}$$

$$\leq K_{12}N^{-1} \sum_{t=2}^{N} \{\sum_{r=2}^{t-1} \sum_{q=0}^{r-2} \beta^{2(N+1-r+q)} + \sum_{r=t}^{N} \sum_{q=r-t}^{r-2} \beta^{2(N+1-r+q)}\}$$

$$\leq K_{13}N^{-1} \sum_{v=1}^{N-1} v\beta^{2v} = O(N^{-1}),$$

where K_{12} and K_{13} are suitably large constants, and where $\beta = \max(\mu,\lambda)$. From (3.53), and from an argument similar to that used to establish (3.9) it follows that

$$(3.54)\quad \lim_{N\to\infty} \beta^{*}(N) = 0 \quad \text{w.pr. 1.}$$

By analogy we also find that

$$(3.55)\quad \lim_{N\to\infty} \gamma^{*}(N) = 0 \text{ w. pr. 1.}$$

To conclude the proof of the lemma we must show that

$$(3.56)\quad \lim_{N\to\infty} \varphi^{*}(N) = 0 \quad \text{w.pr. 1.}$$

This we do in the following way. We first note that

$$(3.57)\quad \varphi^{*}(N) = N^{-\frac{1}{2}} C^{-N} \sum_{v=1}^{N-1} \sum_{\substack{q,s=0\\ q\neq s}}^{v-1} C^{s} \eta^{*}(v-s)\, \hat{\eta}(v-q)' B'^{q}$$

$$+ N^{-\frac{1}{2}} C^{-N} \sum_{v=1}^{N-1} \sum_{q=0}^{v-1} C^{q} \eta^{*}(v-q)\, \hat{\eta}(v-q)' B'^{q}$$

$$\equiv \varphi_1^{*}(N) + \varphi_2^{*}(N).$$

Next we compute

$$(3.58)\quad E|\varphi_2^{*}(N)| \leq \sigma_\eta^2 N^{-\frac{1}{2}} \sum_{v=1}^{N-1} \sum_{q=0}^{v-1} |C^{-(N-q)} e_1' e B'^{q}|$$

$$\leq K_{14} N^{-\frac{1}{2}} \sum_{v=1}^{N-1} \sum_{q=0}^{v-1} \beta^{N-q+q} = O(N^{3/2}\beta^{N}),$$

for a suitably large constant K_{14}, which by the Borel-Cantelli lemma implies that

$$(3.59)\quad \lim_{N\to\infty} \varphi_2^{*}(N) = 0 \text{ w.pr. 1.}$$

Finally we compute

$$(3.60)\quad \text{tr. } E\,\varphi_1^*(N)\varphi_1^*(N)' =$$

$$= N^{-1}\text{tr.} \sum_{t,v=1}^{N-1} \sum_{\substack{q,s=0\\ q\neq s}}^{v-1} \sum_{\substack{r,\ell=0\\ r\neq \ell}}^{t-1} [C^{-(N-s)} e_1' e B'^q B^r e' e_1 C'^{-(N-\ell)} \cdot$$

$$E\eta(v-s)\eta(v-q)\eta(t-r)\eta(t-\ell)]$$

$$= 2\sigma_\eta^4 N^{-1}\text{tr.} \sum_{t=1}^{N-1} \sum_{v=1}^{t} \sum_{\substack{q,s=0\\ q\neq s}}^{v-1} C^{-(N-s)} e_1' e B'^q B^{s+(t-v)} e' e_1 C'^{-(N-q-(t-v))}$$

$$+ 2\sigma_\eta^4 N^{-1}\text{tr.} \sum_{t=1}^{N-2} \sum_{v=t+1}^{N-1} \sum_{\substack{q,s=v-t\\ q\neq s}}^{v-1} [C^{-(N-s)} e_1' e B'^q B^{s+(t-v)} \cdot$$

$$e' e_1 C'^{-(N-q-(t-v))}]$$

$$\leq K_{15} N^{-1}\{ \sum_{t=1}^{N-1} \sum_{v=1}^{t} \sum_{q,s=0}^{v-1} \beta^{2N} + \sum_{t=1}^{N-2} \sum_{v=t+1}^{N-1} \sum_{q,s=v-t}^{v-1} \beta^{2N}\}$$

$$= O(N^3\beta^{2N}),$$

for a suitably large constant K_{15}. From (3.60) and from the Borel-Cantelli lemma it follows that

$$(3.61)\quad \lim_{N\to\infty} \varphi_1^*(N) = 0 \quad \text{w.pr. 1.}$$

Now (3.61) and (3.59) imply the validity of (3.56), and (3.56), (3.55), (3.54), (3.52), and (3.51) imply the validity of (3.50). Q.E.D.

FOOTNOTES

1/ This paper was written while the author was a Fellow of the John Simon Guggenheim Memorial Foundation.

2/ Throughout this paper E denotes the expectation operator. Thus $Ex(t)^2$ denotes the second moment of $x(t)$.

3/ In this context we should mention that the asymptotic distribution of $(\hat{a}(N) + a)$ is derived in [7].

4/ Under the additional assumption that the limiting matrix of

$$A^{-N} \sum_{t=1}^{N} \tilde{x}(t-1)\, \tilde{x}(t-1)' A'^{-N}$$

is invertible. T. W. Anderson showed in [1] that (2.3) was valid for this case.

5/ I learned of the relation (2.16) from reading T. Muench's paper [3].