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Random Behavior In Numerical Analysis, Decision Theory, and Macrosystems : Some Impossibility Theorems

by

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RANDOM BEHAVIOR IN NUMERICAL ANALYSIS, DECISION THEORY, AND MACROSYSTEMS: SOME IMPOSSIBILITY THEOREMS[1]

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Abstract: For many topics, including decision analysis, policy making, and the normative study of certain macrosystems, tools of analysis are applied to determine the essence or the state of a problem. The one commonality among these tools is that we want them to be "reliable". For certain standard tools, it is shown that this goal of reliability may be impossible to attain. For some of these impossibility statements, alternative approaches are suggested.

1. Introduction

Certain basic tools are commonly used both with decision analysis and with macrosystems. Some of these tools are devices designed to be incorporated within the system in order to assist and to influence the subsequent dynamics. For instance, this includes any method used to facilitate the decision making within an organization. Here an obvious example would be voting methods used to aggregate individual differing rankings over several alternatives into one common group ranking. Other types of tools are the techniques used in systems analysis. On a theoretical level, this may be an algorithm designed to seek a zero of a smooth function - such a zero may correspond to an equilibrium or an optimal point for a system. Or, it may be an integer programming problem used to determine an efficient policy. It may be the statistical and probabilistic tools developed to understand and to interpret data - perhaps to aid in a decision analysis or a policy decision.

Central to the selection of any tool is the requirement that it is reliable. Here there are at least two criteria. First of all, the tool should apply for all of the situations within a class of interest; that is, we seek <u>universal mechanisms</u>. For instance, when we search for a zero or an equilibria of a function g, we prefer to use an algorithm which always will work as long as g is sufficiently smooth. Indeed, this is part of the historical attraction of the tatonnement story from economics; it has been viewed as being a universal mechanism where the market forces of supply and demand iteratively converge to a market equilbrium price.

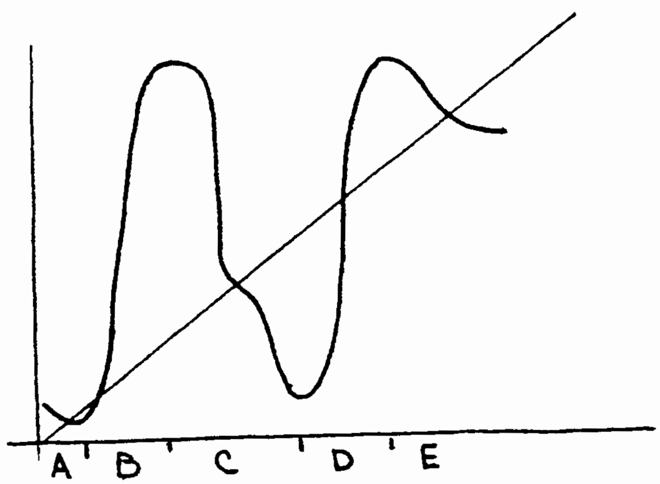
A mechanism or tool is selected to achieve a specified goal. Consequently, a second crucial condition is that the tool doesn't lead to unexpected surprises, conclusions, or consequences which may violate or vitiate our objectives; we want <u>the outcomes to consistently reflect these objectives</u>. For instance, in the choice of a voting method, we want the final result to accurately reflect the preferences of the electorate. As a hypothetical example, consider the problem of selecting a common beverage for lunch where a voting method is used to guide us in the decision process. Suppose a vote leads to the ordering wine > water > beer. Should wine be unavailable, we would expect to be able to replace it with the second ranked choice of water free of fear that a majority of the people really would have preferred beer.

An impossibility theorem arises when certain basic objectives are frustrated; when there doesn't exist a devise or a mechanism which satisfies the specified criteria. Therefore, the theme of this paper, which is that impossibility theorems play an important role in the system sciences, is somewhat disturbing. Often such theorems arise because mechanisms violate conditions which are "intuitively obvious"; in this setting, an impossibility statement is called a "paradox".

In this paper, I'll consider several paradoxes and impossibility theorems with three goals in mind. The first is to introduce several new impossibility theorems related to the topics mentioned above. The second is to take these seemingly disparate results and to unify them by showing that they have a common explanation. (Although I will not develop the theme here, this approach relates these new results to several important paradoxes such as Arrow's Theorem, the Alabama paradox of apportionment, etc.) Finally, I'll briefly note some research, still in its infancy, which has the goal either to circumvent, or to handle the disturbing consequences of these impossibility results.

2. The source of the problem

All of the results to be discussed here are caused by <u>the inverse image of certain functions being multivalued in a particular manner</u>. To understand the basic idea, consider the function f represented in Figure 1.



The inverse function, f^{-1}, clearly is multi-valued; indeed, in this figure, the intervals A, B, C, D, E designate those regions over which f^{-1} is single valued. To see what mischief this multivalued property can create, consider the trajectories of the deterministic system

$$2.1) \qquad x_{N+1} = f(x_N).$$

This system admits five equilibria given by the intersection of the graph of

$y=f(x)$ with the diagonal $y=x$; and the three asymptotically stable points are those equilibria in the regions A, C, E. (At these points $|f'|<1$.) But, there is much more interesting dynamics going on in the intervals B, C, and D than can be captured by any stability analysis. For instance, I contend that there exist trajectories of this deterministic system which are as "random" as you desire! By this I mean it is possible to specify in advance in which of these intervals the N^{TH} iterate will land <u>for all values of</u> $N=0,1,...$ This selection can be made by any means desired, say a random process, and there will exist initial iterates with trajectories which will follow the specified pattern. For instance, choose the interval in which you wish the initial point, x_0, to be, say D. Then choose the interval in which you wish $x_1=f(x_0)$ to be, say B. Continuing in this fashion, specify the interval in which the i^{TH} iterate should land. This defines a sequence of labelled intervals, say

2.2) $S = \{D, B, C, ...\}$

where the K^{TH} symbol designates the interval in which x_{K-1} should be.

To establish the existence of a trajectory which will follow this designated future, a judicious choice of the multiple inverse images will be made. To do this, let S_N denote the finite sequence consisting of the first N entries of S. Then, in an iterative fashion, we will determine all initial iterates for which at least the first (N-1) iterates do what they are supposed to do; they follow the pattern given by S_N. Let $K(S_N)$ be this set. For instance, for the above choice of S, $K(S_2)$ is the set of points in D which are mapped to B′, the closure of B; i.e., $K(S_2) = f^{-1}(B') \cap D$. Because the image of f restricted to D, f_D, covers the set B′, this is the closed subset $f^{-1}_D(B')$.

To determine $K(S_3)$, we first determine, as above, the set $K(\{B,C\}) = f^{-1}_B(C')$. This is a closed subset of the interval B. As a result, $K(S_3) = f^{-1}_D(K(\{B,C\})) = f^{-1}_D(f^{-1}_B(C'))$ is a closed subset of $K(S_2)$. Continuing in this fashion, it follows that $K(S_N)$ is a nonempty, closed subset of

$K(S_{N-1})$. (It is nonempty because the image of f_K, K=B,C,D, covers the union of all three intervals.)

The sought after set is given by

2.3) $$K(S) = \bigcap K(S_N).$$

This set is nonempty because it is given by the intersection of a nested, decreasing sequence of nonempty, compact sets. This establishes the existence of orbits with the behavior specified by S.

Some properties can be extracted immediately from this derivation. While most of them will not be explictly exploited in what follows, it should be clear that they provide additional information about the types of behavior of systems which we will be discussing.

1. Because there are an uncountable number of possible sequences S, there are an uncountable number of sets K(S). For an uncountable number of these sets, convergence to equilibria is impossible.

2. The system can exhibit sensitivitiy with respect to initial conditions. By using the figure to determine the set K((D,B,C)), it is clear that these sets decrease in size quite rapidly. However, the initial points for any two trajectories which define the same S_N, but which differ quite radically after the N^{TH} iterate, are in the same set $K(S_N)$. This means that near-by points may have radically different futures.

3. For any S_N, $K(S_N)$ contains a nonempty open set of points which converge to one of the equilibiria. This is because the image of f restricted to any of these three intervals meets A and E. Consequently, there are open sets of points where the first N-1 iterates obey the specified pattern of S_N, and then the N^{TH} iterates are either in A or in E. In either case, all subsequent iterates asymptotically tend to an equilibria. This has several implications:

a) The basin of attraction for an equilibrium can be extremely complex.
b) There has been about 10^{27} seconds of time since the "Big Bang". The fastest computers on the drawing boards are projected to do 10^{12} operations

per second. Hence, for computational purposes, any stable point in $K(S_N)$, where $N \geq 10^{33}$ and where the last two entries of S_N differ, are unstable for any practical purposes.

The above nested set construction, which depends upon the properties of the multivalued aspects of f^{-1}, and the accompanying properties listed above are the essential ideas behind the following impossilbity theorems. (For a comprehensive discussion of iterative dynamics, chaos, and random motion, I highly recommend [1,2].)

3. Applications

Based upon the discussion in Section 2, it is clear that iterative dynamics of deterministic systems can lead to random, unexpected behavior. This is particularly so should f^{-1} be multivalued in a sense indicated above. As such, examples exhibiting this behavior are plentiful and easy to find. For instance, a macrosystem where all sorts of examples and applications of this type arise is the general area of biology; this type of random motion occurs in discrete preditor-prey models, in the Volterra equations adjusted for seasonal effects, in population genetics, etc. Other areas include the motion of a projectile entering the atmosphere (depending upon its angle of entry), and on and on.

It isn't as obvious that the same behavior is manifested in common tools of analysis. These are the topics I'll consider here.

Numerical Analysis [3]

Consider the problem of determining a zero of a smooth function g. To be more specific, let

3.1) $G^K = \{ g \in C^{K+1}[0,1] \mid g(0)g(1) \leq 0\}$.

(The product condition is imposed only to ensure that a function from G^K has a zero. It can be replaced with a more restrictive condition such as $g(0)>0$, $g(1)<0$, or with a weaker condition that g has a zero in [0.1].) The goal is to find a universal algorithm; an algorithm which will determine a zero for any function g in G^K

Perhaps the best known algorithm is the Newtonian iterative scheme

$x_{N+1} = x_N - (g(x_N)/g'(x_N))$.

However, from the work of Barna [4] (also see [5,6,7]), we know that such a scheme isn't universal; there exist polynomials and initial points so that this scheme never converges. So, the issue becomes to determine what information we need in order to design an universal algorithm. A standard approach, which is in the spirit of Newton's method, is to seek this information in terms of the values of g, its first K derivatives, and the location of the initial iterates. (Actually, the goal is to find the class of <u>all</u> such universal algorithms so that an optimal choice, say in terms of computer costs or complexity, could be made.)

Definition 1. A mechanism is given by (M,D) where M is a piecewise, smooth function from R^{K+1} to R, and D is a subset of [0,1]. The mechanism defines the iterative scheme

3.2) $x_{N+1} = x_N + M(g(x_N),..,g^{(K)}(x_N))$

where x_0 is in the set D. The mechanism (M,D) is a <u>universal</u> <u>mechanism</u> if for any g in G^K, the sequence defined by Eq 3.2 converges to some zero of g.

For the Newton algorithm, $M(u,v,...) = -u/v$. The problem is to determine all choices of (M,D) which are universal mechanisms.

Theorem 1. There does not exist a universal algorithm for G^K which is of the type specified in Eq. 3.2.

The basic idea for the proof of this theorem is that no matter how you choose M and D, there still exists an open set of functions in G^K such that $(x+M)^{-1}$ is multivalued in a sense similar to that described in the previous section. As such, convergence will not occur.

A natural question is whether one could possibly design a more creative algorithm - based upon additional information - to overcome this negative statement. For instance, perhaps by incorporating a memory of the last "A" iterates, an algorithm could be designed to recognize earlier mistakes and to make the appropriate adjustments. In this case, the algorithm would assume the form

3.3) $x_{N+1}=x_N + M(g(x_N),.,g^{(K)}(x_N),.,g(x_{N-A}),.,g^{(K)}(x_{N-A}))$.

Again, the goal is to characterize the set of all (M,D)'s which are universal mechanisms; again an impossibility theorem results. In fact,

Theorem 2. For any (M,D) where M is of the type given in Eq 3.3 and for any integer $s>A$, there is an open set, B, of functions in G^K so that if g is in B then the trajectories of Eq. 3.3 tend toward an attractive periodic orbit of period s.

In other words, for any such g, there is an open set of initial conditions where the trajectories oscillate with a periodicity just outside the limits of memeory of the algorithm; these iterates will never approach a small neighborhood of any of the zeros of g. These results suggest that to define an effective, universal algorithm, other techniques and approaches are required.

Recently, questions concerning the "complexity of algorithms" have been studied to determine whether one is better than another. Again, some of these concepts are stated from the viewpoint of "universality". Namely, these definitions are in terms of the "worse case" situations; in the worse case, how many iterates are needed to determine a zero of a function. A combination of the ideas in the proof of the above theorem and Comment 3 in the last section can be used to show that for any M of the type given in Eq. 3.3, (which includes Eq 3.2), there exists an open set of functions B in G^K with the property that if g is in B, then there is an open set of "convergent" points for which convergence could never be discovered on any computer; the bounds on the number of required iterates to reach a small neighborhood of a zero can be made arbitrarily large.

Incidently, these theorems extend to functions from R^N to R^N.

Price Dynamics [3].

The standard tatonnement story from economics describes how the market forces of supply and demand adjust the prices so that the iterates converge to a price equilibrium where supply equals demand. One attraction of this story is that it describes a self-regulating universal mechanism which determines an equilibrium. But, is this story correct?

It is known from the work of H. Sonnenschein, G. Debreu, and others [8] that any function in the set $\{g \in C^k | g(0)>0, g(1)<0\}$ serves as the excess demand function for some standard, neo-classical economy. Thus, it follows from the above that the standard tatonnement story will not always work even for the highly restrictive setting of only 2 commodities! (The tatonnement story corresponds to the mechanism $M(g(x))=g(x)$.) Moreover, it also follows from the above that there doesn't exist a mechanism using the past history of market forces as captured by the excess demand function (which may be used to model speculation, anticipation, etc.) of the nature given in Eq 3.3 which will serve as a universal price mechanism. Consequently, the existence of such an universal mechanism, if one even exists, must depend upon a different form of information, and at this stage it isn't clear what it should be.

4. The tools of decision analysis

The iteration $x_{N+1}=f(x_N)$ can be expressed as $x_N=f(x_{N-1})=f(f(x_{N-2}))=\ldots=f^N(x_0)$; or $x_N=f^N(x_0)$. This means we are examining a specific sequence of functions $\{f^N\}$ (which happen to be obtained by composition), and analyzing the images as governed by a common domain point, x_0. A natural extension is to eliminate the restriction that these functions $\{f^N\}$ are derived in this special manner. Instead, let $\{f_N\}$ be any given sequence of functions where $f_N:B_0 \dashrightarrow B_N$.

The question is the same; for a specified sequence of behavior described in the different spaces B_N, does there exist a common point p in B_0 such that $f_N(p)$ will have the desired designated "random" future? This is the type of model I"ll discuss in this section. (Traditionally, examples of the type I will be describing are analyzed in a "static" setting. An advantage of the "dynamic" approach advocated here is that it suggests the natural extensions of well-known paradoxes, and it suggests the approach to determine whether they exist.)

Voting [9,10]

Consider the earlier hypothetical example concerning the choice of a luncheon beverage among wine (wi), water (wa), and beer (be). Assume there are 9 voters where 4 have the ranking wi>be>wa, 3 have the ranking wa>be>wi, and 2 have the ranking be>wi>wa. Using the customary plurality voting scheme where you vote for your first place alternative, the group ranking is wi>wa>be with the tally of 4,3,2.

For this voting model, B_0 is the space of all of the ways in which the 9 people could linearly rank the three alternatives in a linear, ordinal fashion. So, a point p in B_0 represents a specific choice of the individual rankings for the voters, $f_3(p)$ is the resulting ranking for the group, and B_3 is the set of all linear, ordinal rankings of the three alternatives. In general, if there are N alternatives, then B_N corresponds to the N! ways in which these N alternatives can be linearly ordered. Note that in this setting, the spaces B_N change with the value of N=0,1,...

This example illustrates that plurality voting doesn't provide us with a desired consistency property. For instance, if wine isn't available, then the above ranking suggests that water would be the group's next choice. But is it? In fact, 2/3 of these people prefer beer to water. Indeed, a majority of them even prefer beer to wine! Thus, the outcome obtained by using this tool for decision making is inconsistent with majority sentiment over any of the possible pairwise comparisons!

In this example, the voting method is characterized by the vector $\underline{W}=(1,0,0)$. One source of the difficulty is the fact that lower level preferences aren't accounted for. A possible remedy would be to use a vector, such as $\underline{W}=(2,1,0)$. (Here, a first place alternative is assigned 2 points, a second place alternative is assigned 1 point and a third place alternative is assigned 0 points.) With this system, the group's ranking becomes be>wi>wa with a tally of 11, 10, 6. Notice that for this particular example, the resulting ranking <u>is</u> consistent with how a majority view each of the three possible pairwise comparisons.

In general, the problem becomes one of choosing a voting method $\underline{W}^N=(w_1,\ldots,w_N)$, $w_{K+1}\leq w_K$, $w_1>w_N$, where w_K points are tallied for a voter's K^{TH} place alternative. The objective is to choose a $\underline{W}^N$ to "avoid" surprises; to find a method which will preserve consistency in the group's rankings as alternatives are eliminated. However, a classical result due to K. Arrow [11] asserts the impossibility of choosing a $\underline{W}^N$, $N\geq 3$, without running into a phenomena of the type exhibited above -- for any voting method it is possible to find examples where the group's ordering ranks some one alternative over another even though a majority of the voters would have the opposite ranking for this particular pair.

The goal for social choice must be modified; the new goal isn't to find absolute consistency, but rather to find a set of vectors $\{\underline{W}^j\}$, j=2,3,..,N, which will preserve as <u>much</u> consistency in the different rankings of alternatives as possible. But, the following theorem shows that even should alternatives be eliminated (or added) in a simple monotonic fashion, there are serious obstacles in achieving this goal - for any choice of a voting method, no relationship whatsoever need be retained among the rankings of the different subsets of alternatives! Notice what this conclusion implies about "run-off elections" and other procedures such as the "Hare method".

Theorem. Let $N>2$ alternatives, $\{a_K\}$, $K=1,2,..,N$, be given. For $j=2,3,..,N$, let $S_j=\{a_1,..,a_j\}$. Let Al_j be an arbitrary, linear, ordinal ranking of S_j; that is, Al_j is some element of B_j. Let $\underline{W}_j$ be any voting vector used to rank S_j. Then, there exist examples of voters, each with a fixed, linear, ordinal ranking of the N alternatives, such that for each $j=2,..,N$, when these same voters rank the alternatives in S_j by use of $\underline{W}^j$, the outcome is Al_j.

This result is an impossibility theorem asserting the inability ever to design voting (ballot) methods even with liberal allowances for inconsistencies suggested by this theorem; leave alone the stricter requirements imposed by Arrow. This result means that all sorts of counter-intuitive examples can be created; e.g., we can find examples where the outcome changes periodically with the number of alternatives. For instance, for N alternatives, there exist examples of voters' preferences (i.e., a point p in B_0) so that the outcome is $a_1>a_2>...>a_j$ when j is even, but just the opposite whenever j is odd. Consequently, even though the voters vote in a consistent fashion, the group's outcomes oscillate as $a_1>a_2$ for S_2, $a_3>a_2>a_1$ for S_3, $a_1>a_2>a_3>a_4$ for S_4, ... In other words, Arrow's theorem, asserting the inconsistency of an outcome at the binary level was only the tip of this iceberg of possible inconsistencies!

Compare the statement of this theorem with that of the iterative example given in Section 2. In each setting, the image of f_N is selected in a random fashion. In each case, f^{-1}_N is multi-valued. (For instance, for $\underline{W}^3=(2,1,0)$, there are many choices of the rankings for the individuals which lead to the same ranking be>wi>wa.) Although the technical details differ significantly, the proof of this theorem can be viewed as being based on an intersection argument similar to that given in Section 2. (Incidently, a related argument will provide an alternative proof and extensions for the classical Arrow theorem.)

However, this goal of finding a proper choice of a voting system - a $\underline{W}^N$

which minimizes the number and the types of paradoxes and inconsistencies - still holds, and it can be answered. By increasing the number of subsets of the N alternatives which need to be ranked (e.g., by requiring that not just those subsets stated in the theorem must be ranked, but all possible subsets of the alternatives must be ranked), it turns out that the Borda method, $B^N=(N-1,N-2,..,0)$, is the unique "best choice" method to reduce the inconsistencies of voting. This will be described in a subsequent paper. (Also, see [9].)

Probability and Statistics [9]

The tools of probability and statistics are not only indispensible for the analysis of data, but their concepts have become crucial in the development of several other areas. For instance, such basic ideas are fundamental for decision making (e.g., the Nash Baysian decision analysis which currently is popular in management science), in theoretical constructs (such as in the evolutional stable strategies in Biology), and in numereous other areas. But, are these tools reliable? Can they cause surprises and unexpected, undesired behavior?

That they can should be expected from the fact that the inverse image of standard probability constructs generally are multi-valued. A simple illustration of these unexpected inconsistencies can be obtained by identifying an important ranking problem from statistics with the voting discussion given above. Namely, consider the problem of determining which one of N firms produces the highest quality product, say a certain type of steel. An obvious approach to solve this problem would be to collect samples from the N firms; after the samples are compared, they are rank ordered. In this way, each sample describes a rank ordering of the firms, so a sample point can be identified with a voter who has a particular ordering of the N alternatives.

Before a decision is made, the problem is to aggregate the information embedded in several samples. But this problem is equivalent to the voting problem of aggregating voters' preferences. For instance, a natural approach would be to select the firm which has the top rating over most samples. This is equivalent to using the plurality voting method $\underline{W}^N=(1,0,..,0)$. Thus, this problem inherits all of the difficulties, inconsistencies, impossibility theorems, and complexities described above in the section on voting. (Moreover, the Borda Count is the unique set of weights to reduce the number of inconsistencies.)

Even more can occur; it turns out that pairwise comparisons of the firms can lead to any desired paradox. To see this, consider the process where from the samples we compare the quality of firm k with that of firm j, where firm j is better than firm k iff for a majority of the samples, firm j had a better product than that of firm k. Now, for each of the N(N-1)/2 different choices of firms, designate (in a completely random fashion if you like) which firm is to be the better one. It turns out that there exist examples of data which will satisfy all of these (possibly inconsistent) rankings simultaneously! [9]

In both of these examples, the rankings are determined by "inequalities", thus the inverse image of the defining functions are, in general, multivalued. From this and an independence condition, the above conclusions follow, and they should be expected.

This difficulty for probability and statistics extends to other constructs for much the same reason. For instance, the concept of "conditional probability" has been used in all sorts of models of decision analysis ("do this if that occurs"), economic and political science models ("because we are in such a situation, we can expect ..."), etc. But, is there a consistency which is preserved as the conditions change - even if they change in a simple, monotonic fashion? The answer is no, not necessarily!

The easiest way to illustrate this is to consider the following game which involves a decision. There are two urns marked I and II; both contain a mixture of red and blue marbles. The game is for you to select one of the two urns, and then, at random, select a marble from this urn. Success is if you selected a red marble. The decision problem is, of course, to select the urn which maximizes your chance for success.

Now suppose there are two sets of urns labelled (I^1_j, II^1_j), j=1,2, where it is known that from either set of urns it is more likely to select a red marble from the urn I^1_j than from the urn II^1_j; i.e., $P(R|I^1_j) > P(R|II^1_j)$, j=1,2, where R is the random variable indicating that a red marble has been selected. For this setting, the decision analysis is trivial to resolve - select urn I^1_j.

Suppose that the marbles from urns I^1_1 and I^1_2 are poured into an urn I^0, while those from II^1_1 and II^1_2 are poured into an urn II^0. You have the same decision problem, but now with these new urns. Which urn should you select? Presumably by now the reader has developed enough caution to avoid the "intuitively obvious" answer of urn I^0. This is fortunate because the same random behavior exists; for either choice of the sign of $P(R|I^0) - P(R|II^0)$, many examples illustrating this behavior can be found. (For the skeptic, consider the following allocation: I^1_1 has 9 red out of 24 marbles, II^1_1 has 2 red out of 6 marbles, I^1_2 has 3 red out of 6 marbles, and II^1_2 has 11 red out of 24 marbles. A simple computation shows that $P(R|I^1_j) > P(R|II^1_j)$, j=1,2, but $P(R|I^0) < P(R|II^0)$.)

This behavior is known as Simpson's paradox [12,13], and it can be extended in many directions [9]. For instance, the number of urns in each set can be more than two, the number can vary, etc. It can be extended to N levels, where initially there are 2^N sets of urns. Then, the marbles from urns I^N_{2j-1} and I^N_{2j} are poured into an urn labelled I^{N-1}_j, while those from II^N_{2j-1} and II^N_{2j} are poured into an urn II^{N-1}_j. This defines the contents of the 2^{N-1} sets of urns at the $(N-1)^{TH}$ level. This process is continued through the different levels, $k=0,..,N$. The problem concerns whether there needs to be any consistency in the signs of $P(R|I^K_s)-P(R|II^K_s)$, $k=0,1,2,..,N$, $s=1,2,..,2^K$. It turns out that there need not be; for each choice of the indices, you can choose the sign in an arbitrary fashion, and there exist initial apportionments of marbles so that all conclusions will be simultaneously satisfied! Again, the proof can be viewed as being based upon an "intersection argument", where B_0 is the space of initial allocations of marbles to the urns.

The implications of this result for decision making should be obvious. It points out that the component parts of a decision analysis can differ sharply from its aggregate. For instance, suppose the problem is comparing success rates of two different methods; say a comparison of the proposed cure of a disease with the standard method. In this setting, the urns I^K_J correspond to pool of people being subjected to the new treatment, while II^K_J corresponds to those receiving the standard treatments; the indices identify the locations where the experiments are being conducted and the level of aggregation of these figures. (If $K=N$, then the figures are the raw figures at the experimental locations. If $K<N$, then the figures correspond to a partial aggregration of the result at different locations.) If R corresponds to "regaining health", then the sign of $P(R|I^K_s)-P(R|II^K_s)$ indicates which treatment was more successful at that particular site and level of aggregation. The above indicates that the conclusions from such a study can be random and highly unexpected; local conclusions may differ from a global, or from a partially aggregated analysis.

This result impacts on decision analysis on the comparison of two (or more) strategies I and II, say in a military context or in an economic plan for a society. Here we see the existence of the apparent anomaly that, on the global level, strategy I is better than strategy II, $P(R|I^0)>P(R|II^0)$, even though this same strategy is <u>weaker</u> in each of the local situations, $P(R|I^K_J)<P(R|II^K_J)$ for all $K>0$ and j. (A phenomena of this type occurred at

Berkely [14]. The objective was for the university to improve the percentage of women it hired in one year (I) over the preceeding year (II). The stategy to accomplish this was for each of the academic units to increase its hiring percentage of women ($P(R|I_j) > P(R|II_j)$). But, the aggregated results for the total university showed a reversal, ($P(R|I) < P(R|II)$)!)

5. Summary

From this brief description, it should be clear that random, unexpected behavior can occur not only in deterministic dynamics, say the dynamics of decision theory or of macrosystems, but also in the basic tools employed to analyze them, and in the devises designed to be implanted within systems to facilitate certain processes (as price mechanisms in economics or voting in political science). In this survey, I've selectively described only standard tools - tools coming from numerical methods, voting, and probability - which are common and familiar to most readers. However, it is easy to demonstrate that the same phenomena extends to optimization problems [15] (such as optimal growth or overlapping generations problems), to integer programming problems (such as an apportionment problem of the type coming from the assignment of legislative seats or draft quotas to regions, or in the economics of decision making), and on and on. Indeed, <u>it probably is safe to speculate that such behavior is prevalent in the tools of decision making and in macrosystems</u>; such a speculative comment is based upon the fact that for most tools of analysis, inverse images of the key defining functions generally are multivalued!

But, if random, unexpected behavior is an inherient part of these tools -- a fact which reduces their reliability -- then we are faced with a serious problem. Decisions must be made, systems must be analyzed, certain tools must be used! This means that these systems must be analyzed to determine the root causes for this behavior; then this information must be used either to develop indicators which will warn when such counterintuitive behavior is occurring, or to design different approaches which will eliminate these problems. The first approach is needed where a system is already specified, as in a gambling problem where the dice are given, as in the dynamics of evolution, as in a strategic situation where the laws of probability are already defined (an analysis of population data, strategic planning, gambling, etc.), or as in a voting situation where the type of ballot tally is designated by law. Here the issue becomes to develop the appropriate tools of information to ascertain, in advance, whether or not the random, unexpected behavior applies to the existing, current situation. This will warn us when an accompanying decision analysis need not be "monotone" - when the conclusions need not mean what we are assuming they mean - and we need to know this.

A different avenue is open for the "design" of tools; the design of algorithms to achieve a specified goal, the design of probabilistic techniques which avoid certain pitfalls, or the design of mechanisms which achieve a desired, self-enforcing status within a macrosystem such as in economics and in political science. Here the objective or the goal of the desired mechanism is stated; the design problem is to determine what type of information structures, communication rules, and decision approaches can be applied to implement these goals. On one hand, this may involve finding or designing a procedure which does minimal damage to these goals (as in the choice of a Borda count for voting). On the other hand, it may involve finding structures which <u>eliminate</u> the existence of these random behaviors. Here, following the lead of L. Hurwicz, some work at an initial stage has been done. [16].

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References

1. Collet, P. and J. P. Eckmann, Iterated Maps of the Interval as Dynamical Systems, Birkhauser, Boston, 1980.

2. Preston, C., Iterates of Maps on an Interval, Lecture Notes in Mathematics Series, no. 999, Springer-Verlag, New York, 1983.

3. Saari, D. G., Iterative price mechanisms, to appear in Econometrica.

4. Barna, B., Uber die divergenzpunkte des Newtonschen verfahrens zur bestimmung von wurzeln algebraischer gleichungen II, Publicationes Mathematicae Debrecen 4,(1956) 384-397; and III, same journal, 8 (1961), 193-207.

5. Saari, D. G., and J. B. Urenko, Newton's method, circle maps, and chaotic motion, American Mathematical Monthly, 91 (1984), 3-17.

6. Martin, C., and R Hurley, Newton's algorithm and chaotic dynamical systems, SIAM Journal on Math Anal. 15 (1984), 238-252.

7. Smale, S., The fundamental theorem of algebra and complexity theory, Bull Amer. Math Soc., 4 (1981), 1-37.

8. Shafer, W., and H. Sonnenschein, Market demand and excess demand functions, in Handbook of Mathematical Economics, vol 2, pp 671-693, ed by K Arrow and M Intriligator, North Holland, 1982.

9. Saari, D. G., The source of some paradoxes from social choice and probabiliity, Center for Mathematical Studies in Economics Discussion paper no. 609, May, 1984.

10. Saari, D. G., The ultimate of chaos resulting from weighted voting systems, Advances in Applied Math, 5 (1984), 286-308.

11. Arrow, K., Social Choice and Individual Values, Crowles Foundation for Research in Economics, Monograph 12, 1963.

12. Blyth, C., On Simpson's paradox and the sure-thing principle, Jour. of Amer. Stat. Assoc., 67 (1972), 364-366.

13. Wagner, C., Simpson's paradox in real life, The American Statistician 36 (1982), 46-48.

14. Bickel, P. J., Hammel, E. A., and J. W. O'Connel, Sex bias in graduate admissions: data from Berkeley, Science, 187 (1975), 398-404.

15. Saari, D. G., Mathematical Economics and Dynamical Systems, to be published in a vol. ed by H. Weinberger and H Sonnenschein.

16. Saari, D. G., A method for constructing message systems for smooth

performance functions, Jour. of Econ. Theory, 33 (1984), 249-274.