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SOLVING THE "MARKETING MIX" PROBLEM USING GEOMETRIC PROGRAMMING

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Abstract

Solving the "Marketing Mix" Problem

Using Geometric Programming

This paper investigates the optimal allocation of the marketing budget within the marketing-mix decision variables so that sales (or profit) is maximized in a planning horizon. Since the influence of marketing mix variables upon sales are, in reality, non-linear and interactive, a geometric programming algorithm is used that solve this problem. An estimation procedure to estimate a functional of sales on the marketing mix and environmental variables utilizaing the experienced judgments of the firm's executives and the raw data is provided. The derived functional is later optimized by the Geometric Programming algorithm under a constraint set consisting of budget and strategy restrictions imposed by a firm's marketing environment, and conditions under which the optimal solution is either local or global are identified. An empirical application for a large midwestern brewery is provided which utilizes and illustrates both the estimation and optimization procedures.

I. Introduction

Many industries are currently in a situation in which relative production and financial capacities no longer provide particular firms within the industry with significant competitive advantages over their rivals. In these industries the marketing capabilities increasingly determine the relative success among the rival firms.

There are a number of marketing decision variables, to mention a few: price, quality, and advertising, that can be used to influence the customer's perceived utility. "Marketing mix" is defined as the interrelationship among the marketing decision variables [3]. One of the most challenging questions facing marketing analysts and planners is to determine the optimum marketing mix. It is the essense of marketing strategy. Should a firm increase quality, lower price, increase advertising, or increase the number of salesman? Dollars will be spent on some combination of these marketing decision variables. But, what is the optimum combination?

To answer these questions, sales must be defined as a function of predictor variables and then hopefully a method of optimizing the sales function can be applied. Most of the previous work on this problem either *a priori* ignored a number of potentially relevant marketing mix variables [7, 22] or proposes a simple linear or exponential [4, p. 45] functional forms which are tailored for easy optimization but are unrealistic representation of reality [17, p. 80]. This paper estimates a real world functional relationship between sales and predictor variables consisting of n terms, each term being a set of interacting variables, where the interaction among the variables is often nonlinear. Kotler [17, p. 68] summarizes previous work which suggests that the effects of marketing mix variables are both nonlinear and interactive. Geometric programming is a method for optimizing the above functional form subject to constraints on the coefficients associated with each term. The practical problem is in generating the above type of sales equation from historical sales data and experienced executive judgments. The requirement of all positive coefficients prevents the direct application of the known nonlinear regression techniques [8]. In fact, there is no known method of searching the response surface to find a global optimum for the type of functional form described above. Therefore, a heuristic approach is proposed which starts with the raw data and, utilizing executive judgment and a modified stepwise regression approach, estimates a functional form which can be solved by geometric programming and in estimating the parameter values of that defined functional form.

This paper is divided into four sections. First, the relevant literature is reviewed. Second, a relevant set of marketing decision variables are defined and suggestions are made as to how a firm can obtain quantitative measures of these variables. Third, since the model was developed in conjunction with a large midwestern brewery, the estimation of the functional form will be explained in the context of the brewery example. Finally, the functional form is converted into the objective function of a geometric programming model and solved.

II. Review of Literature

The authors are not aware of any solution procedures to solve the marketing mix problem if the functional is nonlinear, interactive, and nonconvex. Philip Kotler [17, pp 56-63] defines the sales functional Q as $Q = f(x_1 --- x_i --- x_n)$ where the x_i's are marketing mix variables. He then suggests using the Dorfman and Steiner [7] method of solution which essentially takes the first partial derivatives of Q with respect to each x_i, sets them to zero and solves for x_i's. Because these conditions are only necessary but not sufficient, the global optimality of the solutions cannot be guaranteed. The approach is severely limited in that it doesn't give consideration to constraints that are associated with these variables. Thus, solutions generated by this approach can call for managerial actions that are not feasible or rational, such as having negative advertising expenditures.

Nerlove and Arrow [21] extended the Dorfman-Steiner paper from a static to a dynamic formulation. Their procedure is to integrate a net revenue function over time where demand is a function of price, advertising, and an environmental variable. The environmental variable is used to represent all variables not under control of the firm such as a consumer income, population size, and competitive actions. The problem with this calculus approach is that in order to obtain optimal results the sales functional must be limited to two or, under very restrictive conditions, three variables. Thus, the basic calculus approach is not designed to optimize a functional containing more than three interacting variables.

Summarizing the literature on the marketing mix problem, there does not appear to be any work done that <u>actually used</u> empirical or real world data. The two suggested approaches to the problem are a general formulation lacking sufficiency conditions, hence allowing infeasible management solutions, and a very restrictive calculus formation that limits the number of relevant variables to two or three.

For a constrained maximization problem, the necessary and sufficient condition for any <u>local optimum</u> to be <u>globally optimal</u>, is that the objective function to be maximized is pseudoconcave and the constraints are quasi-concave (with ≥ 0 as right-hand side in the constraints) with certain added constraint) qualifications [23, p. 43]. These conditions are taken care of if the problem can be formulated in the format of geometric programming [13, chapter 3]. The discussion of the marketing mix variables in the next section indicates that for most real world marketing mix problems there are usually more than two or three relevant variables.

III. The Relevant Variables

The variables which influence demand can essentially be divided into two sets. First, "marketing mix" variables such as price, advertising, salesmen's salaries, and product quality, which may be adjusted at the discretion of the firm. Second, environmental variables such as disposable income, age composition, and population growth, which are essentially outside the control of the firm.

A number of authors have provided lists of "marketing mix" variables [3, 14, 17, 19, 20]. Some of the suggested variables are specific to a particular product, others are abstract and difficult to measure. We will now proceed to define a basic set of ten quantifiable "marketing mix" variables, indicating the data needed for quantification. Our aim is to define a set that is relatively universal and complete. Product differences will vary the degree to which the basic marketing decision variables are relevant, but the defined set contains the key decision variables with which most marketing executives seek to maximize their goals of profit, sales volume, and market share. In quantifying the variables a ratio format is used to take into account the influences of competition and time. The ratio format directly introduces the influence of competition, tends to adjust for seasonal and trend influences, and results in a dimensionless number which makes the estimation of the functional form and the interpretation of the geometric programming results straightforward. The functional forms of the ten marketing mix variables are summarized in the appendix. For notational consistency each variable is time subscripted although, as will be explained later, not all of the variables were actually lagged in analyzing the data.

Relative advertising expenditure (A_t) is defined as the ratio of dollars spent on advertising by the particular firm during a given time period to the total industry advertising effort in dollars spent There are a number of sources such as Nielson, Starch, Advertising Age, etc., that report on the amoung of advertising done by firms and industries. Furthermore, many firms set this year's advertising budget as a function of last year's sales. Thus competitive advertising budgets are often highly predictable. From some combination of the above sources most firms have the capability of measuring A_t.

Relative In-Store Promotion (I_t) is defined as the ratio of dollars spent on in-store promotion by the particular firm, during a given time period, to the total industries' in-store effort in dollars spent. In-store promotion includes such items as displays, signs, and small customer gifts. This variable is formulated in the same manner as the relative advertising effectiveness variable described above. Salesmen often provide a detailed listing of competitive in-store promotions.

Relative price (P_t) is the retail price of the firm's product divided by the average price charged by competing firms. Retail prices are highly visible and are easily obtainable for most firms. In some industries it might be advisable to weigh each competitive firm's price by quantity sold in determining the average industry price. A second aspect of price, which we have defined as a separate variable, is the firm's relative price differential from the previous time period (C_t). For many products a small change in retail price is a highly visual cue to consumers. Thus, we examine the dollar change in retail price from one time period to the next, divided by the average retail price change in the industry. For computational convenience a zero value in either the numerator or denominator was converted to .001 and the range of C_t was restricted to (.001, 10).

The deals or special discounts a firm allows its wholesalers and retailers (T_t) and the salesman's effort representing total salary and commissions (S_t) are internal values known by the firm. Both of these variables should be converted to relative terms to account for the effect of competition. In general, since it is very difficult to obtain reliable current estimates of the dollar amounts spent by each competitor on T_t and S_t, it is advisable to use the dimensionless form shown in the Appendix.

Distribution (D_t) should represent the availability of the firm's product this utility can be extremely important for less expensive convenience type goods. The retail outlets within the defined geographic market are weighted by the quantity of the product sold in each outlet.
Thus D_t represented the percentage of weighted retail outlets that carried the firm's brand.

Relative customer service (W_t) which represents the warranty and service backing for a given product, relative packaging appeal (B_t) and relative quality (Q_t), should be estimated from test market or panel data. One might argue that design engineers within the firm should provide the estimates of the relative quality, warranty value, and functional evaluation of the packaging as the engineers can more accurately measure these variables than a housewife. We reject this idea and stress that it is the perceived quality, packaging, and service that really affect the purchase rate.

The prior theoretical papers dealing with the concept of marketing mix variables [4, 7, 17] conclude by relating sales to some functional form of the marketing mix variables. We suggest that environmental variables should also be considered. Clearly, the marketing mix variables may interact with environmental variables such as G.N.P., personable disposable income, industrial production, age composition of population. Therefore, if one is attempting to explain sales in terms of marketing mix variables only, he can easily misinterpret what is really happening. By failing to include environmental variables, such as population size in the analysis, a researcher quite easily may be led to the erroneous conclusion that by immediately hiring more salesmen he will increase sales.

The Survey of Current Business provides information on a number of standard environmental variables. Since the list is quite expensive and the relevance of the variables is highly product dependent no attempt will be made to identify a standardized set of environmental variables. Rather we simply suggest that when sales are functionally related to the set of marketing mix variables a researcher should consider environmental variables as supplementary predictor variables.

IV. Deriving a Functional Form and Estimating the Parameter

The method for deriving the functional form and estimating the parameters developed in this paper is clearly a general approach applicable to many other problem areas. However, instead of first abstractly explaining the approach and then applying it, we will develop the approach in terms of the beer example in order to conserve space and provide a concrete example which is easier for the reader to follow. In our beer example, we initially had seven years of historical data on a monthly basis (84 data points per variable). Subsequently, over the course of the model development, an additional 18 data points have become available. All of our estimates as to functional form and parameter values are in terms of the original 84 data points. The 18 new data points are used for model validation.

Much of the actual data used in our case example, such as budget limits and elasticities, were felt to be proprietary information. In compliance with the wishes of the brewery supplying our data, we will not reveal the actual numbers used. We will use symbols whenever possible and in the next section, where actual numbers are provided so as to illustrate a numerical solution, these numbers have been modified from real data.

The model defined in this paper is for one geographic market segment. Obviously, through the use of additional subscripts on the variables, the model can be expanded to include m interacting market segments.

In addition to the ten standard marketing mix variables, it was felt that relevant environmental variables should also be included in the functional used to predict sales.

In order to determine relevant environmental variables, beer sales were log and linearly regressed on a number of environmental variables both singularly and in combinations. The two environmental variables that appeared to have some relationship with total beer sales are age composition of the population and disposable personal income. The age composition variable was finally defined as the percentage of the total population within the geographic market between 18 and 24 years of age. Age composition (Ac_t) was dealt with as a dimensionless number in that the percentage of the population between 18 and 24 in period t was divided by the average percentage in this age bracket over the 84 time periods used in our parameter estimation. Personal disposal income (In_t) was converted into a pure number by dividing the observed value by the de-trended estimate of (In) for period t by its average for the 84 time periods. Since the concept of warranty and service backing (W_t) for beer is not relevant, we a priori dropped this variable from our analysis.

We now have eleven variables from which we wish to construct a functional to predict sales. The following terminology used to describe sales' functionals will be used throughout the remainder of this paper: a <u>variable</u> will refer to the one of the eleven variables defined above; a <u>term</u> will be a product of one or more variables; <u>elasticity</u> will be an exponent of a variable; and, <u>coefficient</u> will be the linear weight associated with each term in the sales' functional.

A straight linear regression, where each variable represents a term of the functional, is appropriate if one assumes that the effect of each variable on sales is linear and independent of the other ten. On the other hand, if one assumes each of the variables interacts with <u>all</u> of the other independent variables, a log regression of the data is appropriate. In the log regression, the regression coefficients become the elasticities of the variables in the function form: e.g., $\log \text{sales}_t = \log k + b_2 \log A_t + b_2 \log B_t + b_3 \log C_t$ is transformed into the functional form $\text{Sales} = A_t^{b_1} B_t^{b_2} C_t^{b_3}$, where b_1, b_2, and b_3 are the regression coefficients. The constant k is removed by shifting to a zero intercept before making the transformation. For beer sales, the linear regression yields an R^2 of .42 and the log regression yields an R^2 of .36.

However, the functional form of the estimating equation provided by the two regression approaches mentioned above are not consistent with our <u>a priori</u> view of reality in that we suspect the eleven predictor variables are neither <u>completely</u> independent nor <u>completely</u> interactive. A more realistic functional form would be a polynomial consisting of terms representing higher order interaction among <u>sets</u> of the variables where the interactions are nonlinear and the total function could be nonconcave. For example, $\text{Sales} = \pm c_1(A^{b_1}Q^{b_2})_t \pm c_2 (A^{b_3}P^{b_4})_t \pm c_3 S_t^{b_5}$ implies that dollars spent on advertising interact with the relative price and preceived quality of the product but not with the dollars spent on salesmen's compensation. Also, advertising may interact quite differently with P_t than Q_t thus $b_1 \neq b_3$ is often the situation. One would expect a polynomial expressing sales in terms of a number of marketing mix and environmental variables to be rather complicated and the number of constraints associated with the variables representing budget limits and strategy considerations to be rather numerous.

Suppose, for a moment, that the true polynomial relating sales to environmental and marketing mix variables was interactive, nonlinear, and nonconcave and is known. How could the marketing manager use this information to optimize his allocation of marketing dollars? In general, this polynomial is too big and too complex to be optimized by the known nonlinear programs. Even a simple trial and error approach to increase sales through improved budget allocation would be difficult in view of interactions and nonlinearities. One would think that knowledge of the true polynomial by which the marketing mix variables are related to sales would be extremely valuable information for a marketing executive, yet because of the higher order interactions among the variables, he would, in general, be very limited in his ability to improve his budget allocation.

There is, however, one large subset of the general polynomial described above for which an optimization program is available. Geometric programming can optimize the general polynomial, subject to numerous budget and strategy constraints, if the linear coefficients (c_i) associated with each term are nonnegative in the context of minimization. Thus, one reasonable approach to dealing with this problem, without removing its reality through simplifying assumptions, would be to find a posynomial (a polynomial with positive coefficients) which is a reasonable estimator of sales and then optimize the functional using geometric programming.

This approach raises two basic philosophical research issues which the reader should be aware of. Functionals relating decision variables to management goals are seldom known with certainty; usually the function is an estimate. Because of the number of relevant variables and their rather complicated interactions, even the best estimated functions often leave substantial unexplained variance of the management goal. The first issue is: Should an optimizing program be applied to these management functionals? Clearly, if the estimated functional differs from the <u>true</u> functional, the results of the optimization program may suggest management decisions that are suboptimal or even counterproductive. On the other hand, to argue that optimizing algorithms should not be applied to estimation functionals essentially restricts math programming to engineering design type problems and precludes its use on most of the interesting marketing and management science allocation problems. Furthermore, in the real world, a marketing manager cannot avoid decision making because he lacks perfect information. He makes marketing mix decisions that implicitly or explicitly are based upon a sales functional which <u>he</u> believes to be realistic. We are suggesting that the functional be made explicit and then math programming be utilized to suggest the best decisions for the decision maker's view of reality.

The second issue concerns the selection of an estimation function which is posynomial in form. Note that the marketing mix variables as defined in the appendix are non-negative variables where the values of the variables all tend to increase as the firm increases its utilization of the variable. Intuitively one expects that sales will usually respond in the same <u>direction</u> but to different degrees to equal changes in particular marketing mix variables. Thus a sales functional where the linear weights all have the same sign is intuitively appealing. However, it is <u>possible</u> for a variable or more particularly for a term containing a number of non-linear and interactive variables to have a linear weight opposite in sign from the other terms. We will show in step four of our data analysis approach how the information contained in a term with a linear weight opposite in sign from the others can be brought into a posynomial. If a posynomial cannot be developed, or if one wishes to investigate functionals with negative terms, we suggest seeking the general polynomial with the maximum R^2 value using a modified Gauss-Newton method as suggested by H. O. Hartley [16]. In the next section we indicate how this general polynomial can be analyzed for local optimum.

In attempting to generate a reasonable posynomial, there are two basic sources of information; the experienced judgment of the executives, and the historical raw data on the variables. One obvious approach is to have the executives estimate a reasonable posynomial and then, using the historical data in sensitivity testing, try to improve on this estimate. In practice, this is asking quite a bit from an executive. In our experience we found the executives relatively good at indicating which variables interacted together but needed help in identifying the type and number of terms which would contribute relatively independent information to the estimating posynomial.

We decided to analyze the raw data prior to asking the executives to try and specify a reasonable sales functional so as not to be constrained too severely by their first estimate and in the hope we could develop some insight which would supplement and improve the executive's estimate. In order to get some feel for the raw data with respect to appropriate elasticities and which sets of variables fit together, we developed the following two-state regression procedure, consisting of four steps.

Step One - Log Regression

From the data, log sales were regressed on the logs of all singles, pairs, and triples (231 terms) of the eleven predictor variables. Also, terms were included representing combinations of variables that the marketing department felt *a priori* to be useful. The log regression coefficients gave the elasticities.

Step Two

Those terms which yielded unreasonable elasticities, like a negative elasticity for advertising or a positive elasticity for price, were modified or dropped. Similarly, those terms that the executives felt unreasonable were also dropped. The formal geometric programming problem is stated in terms of minimization of a posynomial. In our case we wish to maximize sales. Thus we wish to treat the increased use of a marketing mix variable as essentially a cost to the firm. We thus change all of the signs on the terms from positive to negative. This will change the direction of the relationship between sales and the marketing mix terms but not the form of the relationship.

Step Three

The useful terms (each term being a product of variables with associated elasticities as exponents) are to be combined into a linear model. For that, one needs independence of the selected terms. It is possible that a particular variable may be present in two terms, possibly with different elasticities which were estimated independently in the first stage. Combining these two terms into a linear model now is valid if the terms are independent of each other. Thus, at this point we look at matrix of correlation among the terms from step two. In the beer sales matrix almost all the correlations had an absolute value $< .33, (|r| < .33)$. We therefore felt we could proceed with step four. If the correlations among terms had been high we would then have proceeded to Hartley's procedure [16].

Step Four - Stepwise Multiple Regression

This stepwise regression procedure of sales on terms of step two is based on a code available as BMDO2R [6, pp. 233-247]. In this we impose two conditions: 1) when additional terms are included in the derived functional, the adjusted $\bar{R}^2$ [22, p. 311] should not decrease. Secondly, since we wish to use the derived functional in a maximization formulation, we will estimate a functional with a most one positive regression coefficient of a term (constant term excluded). This facilitates the geometric programming procedure to yield a global optimum at no sacrifice of the solution space or interpretation. The combination of terms is valid due to their independence. In a regression approach, the sign of the coefficient depends upon the direct or inverse relationship taken pairwise. The variables as defined in the appendix are non-negative dimensionless variables, which have primarily a direct relationship with sales. We therefore changed the direction of the relationship in step two prior to the stepwise regression, thus generating functionals with almost exclusively negative regression coefficients. However, there were some terms, primarily because of interaction among the variables within the term, for which the original relationship with sales was believed to be inverse by the executives. There were also some terms for which the executives were unsure if the original relationship was direct or inverse. We could accommodate one of these terms in our estimation functional in the following manner. The multiple regression program selects the regressor having the highest absolute correlation after the effect of the first regressor is eliminated from sales. At this step, the program is interrupted and we inspect for the sign of the coefficients. If both the coefficients are positive (no matter what the sign of the constant term is), we redefine the last regressor entering as its reciprocal, thus guaranteeing a negative coefficient for this regressor if the reciprocal enters the functional. This will not change the earlier coefficeints as built-in by the stepwise regression procedure. (If both are negative, or one negative and the other positive, no interruption is made and the program proceeds to include a next regressor). Thus, the procedure of redefining a regressor by its receiprocal is followed at every successive step if that variable yields the second positive regression coefficient. This scheme is continued until either one of the two following cases materialized: (i) all the regressors of Stage Two are exhaused; or, (ii) the adusted $\bar{R}^2$ starts decreasing. Only functionals with terms that are relatively independent and reasonable (if inverted terms were included) are retained.

Using the modified stepwise regression program, a number of regressions were run to get a feel for which terms tended to fit together. The maximum $\bar{R}^2$ we attained was .46, and this functional consisted of four terms: $(AC)_t^{b_1}$, $(I^{b_2}P^{b_3})_t$, $S_t^{b_4}$, and $(C^{b_5}Q^{b_6}D^{b_7})_t$. Having attempted, through the above statistical analysis of raw data, to get some feel for the raw data, we now approached the executives with these results in the hope that they could provide some insights which would improve the $\bar{R}^2$ of our sales functional.

The marketing executives of the beer firm noted that advertising was not in the maximum functional nor was it very prevalent in the list of regression equations with lower adjusted $\bar{R}^2$. They felt quite strongly that advertising was a significant marketing mix variable, particularly for the geographic market we were investigating, because of the historically high proportion of beer purchased by people in the 18-24 age range within this territory. Their theory was that people in this age range were more sensitive to advertising.

We proceeded to supplement our purely statistically generated results with the judgments and suggestions of the executives. The first thing we did was to time lag A_t, I_t, P_t, T_t, and S_t for up to four months and C_t for two months. Because of the nature of the data and the extremely large number of combinations possible, [$\binom{27}{3}$ for the 3 tuple alone] it was not possible to search this entire space optimally. We, thus, used a trial and error heuristic search procedure and eventually found some time lagged combinations that improved the regression equation to an adjusted $\bar{R}^2$ of .54. The manner in which we quantified A_t also became a source of concern. While the use of dollar amounts is the obvious method of quantification of the advertising variable, this assumes that the firm is consistently efficient and that all firms have the same efficiency per dollar spent, that all advertising messages generate the same response.

Krishnan and Gupta [18] suggest judgmental coefficients (α values) representing the firm's effectiveness be used to modify the dollar amounts. We found that the use of efficiency coefficients, while theoretically appealing, was difficult to implement practically because of a wide variance in the sets of α_i proposed by various executives

It was at this point we decided to account for differences in advertising content by separating A_t into three variables. The firm had magazine advertisements representing the various beer advertising campaigns of its rivals enabling us to classify the themes into three groups; $A1_t$ represents advertising emphasizing price, $A2_t$ represents "mood" or image advertising and $A3_t$ represents advertising emphasizing the quality of the brand. Each of these variables was ratio estimated in the same form as the original A_t variable. Segmenting the advertising variable so as to include message content in the analysis enabled us to substantially improve our regression. The following functional form transposed into linear variables gave an adjusted $\bar{R}^2 = .72$.

$$(1) \quad \text{Sales} = b_0 + b_1 \left[(A1)_{t-1} (PTC)_t\right] + b_2 \left[(A2)_{t-1} B_t\right] + b_3 (Ac\ Q)_t + b_4 \left[I_t S_{t-1}\right] + b_5 (PD)_t$$

where $b_0, \ldots, b_5$ are the regression coefficients. This functional was derived from the original 84 data points. While most of the terms in the final functional were identified by the executives, it is interesting to note that one term found in the analysis of raw data supplemented the executive judgment providing additional explaining power for the functional and a new insight for the executives.

Eighteen new observations were used in model validation. Repeating the above procedure, we found the above terms to again be significant using the new data. Running the new data through the original functional yields an $\bar{R}^2$ of .66. The dual weights associated with these terms, which under the normality condition indicate the relative contribution of each term, also were very consistent from one data set (84 points) to the other (18 points).

In summary, the regression approach used in this paper is a heuristic approach requiring subjective judgments by the researchers. Use of both log and linear regression may at first appear to be a piecemeal method of obtaining the functional form and elasticities; upon reflection, one realizes that as one treats the terms in the above equation as independent, the estimates of the elasticities associated with the variables in each term should not be influenced by the presence of the other independent terms in the linear regression. We contend that the modified stepwise regression approach is a reasonable method for obtaining a posynomial functional form and parameter estimates directly from data.

V. Marketing Mix Optimization Utilizing Geometric Programming

In this section, solution procedure for solving the marketing mix decision variables by optimizing the functional derived in the earlier section, subject to a set of constraints is provided. Two cases are identified in this context: (A) The case when the functional and the constraints are not compatible with the geometric programming requirements (e.g., the functional is a signomial [11] so that the solution may be optimal only locally; (B) The case when the functional and the constraints are, or can be converted to, a format where geometric programming [13] can be used yielding a solution that is globally optimal. This provides a general procedure for solving a marketing mix problem for any firm. Finally, optimization of the sales functional generated in our beer example (equation (1) with added constraints) is given with its interpretation as a model exercise.

A good introduction to geometric programming can be obtained from Duffin, et al. [13]. Later modifications to "Signomials," [11], "Linearizing Geometric Programs" [9], "Harmonized Geometric Programs" [10], "Geometric Programs Treated with Slack Variables" [12] are finer frills available in the literature but cannot always guarantee global optimality. We are assuming that the reader is familiar with [13]. Certain computer codes exist to solve geometric programming procedures [2].

(A) When the functional and constraints are not compatible with geometric programming requirements: This case arises when the condition of statistical independence of the terms (each term is a product of a certain subset of marketing mix variables with estimated elasticities from the first stage of log regression) is violated. Thus, the second stage of linear regression to combine the terms estimated earlier is invalid. This is because the elasticities of each of the variables in a term were computed in the first stage independent of any other terms that might be in the final sales functional. Thus, the elasticities of the variables do not reflect the possible influences of other independent terms in the sales functional. In such a situation, it is possible to estimate the functional all at one stage. H. O. Hartley [16] has provided a method by which one can estimate the elasticities and the coefficients of each term all at one stage. Here each term can be a single variable or an interaction term which is a product of certain variables. Thus, a polynomial functional could be estimated by this method. It is to be emphasized here that the regression coefficients that are estimated for each term need not be all or all but one negative. If it so happens that either all coefficients or all but one are negative, this functional satisfies the geometric programming requirement which can yield globally optimal solution (case B). If not, we are lead to the case of a functional which has signomial terms (both positive and negative coefficients). The procedure given by Duffin and Petersen [11] for signomials, or the one given by Avriel and Williams [1] utilizing "Complementary Geometric Programming" can be used to deal with signomials but lead only to a local optimum.

(B) Case when functional is, or can be converted into acceptable geometric programming format: The estimated functional Q and the constraints under which an optimization by geometric programming (G-P) must be posynomials for a minimization problem. However, since we are maximizing Q in the context of sales, this requires the functional to carry negative coefficients in each term without constraining the constant term. Generally, one requires all terms in the G-P minimization to be strictly positive. We will relax this condition to allow for one negative term. To satisfy the G-P requirements, the sales function Q to be maximized in terms of the regressors of Section 3, is to satisfy the following requirements: a) a constant term (positive or negative); b) at most, one positive regression coefficient; c) all other terms have either zero or negative regression coefficients. (Note: The exponents of the terms can be positive or negative.)

Since the basic four step procedure used in the beer sales example satisfied the requirements of case (B), we are ensured of a regression function of beer sales with, at most, one term having a positive linear coefficient. This will enable us to guarantee for global optimality as we have avoided signomials [11] or approximate solutions as given by Charnes and Cooper [5].

We now will solve the estimated functional in terms of the values given below.

$$(2)\quad \text{Max. Sales} = 16.212 + 3.937\ A2_{t-1}^{.91}\ B_t^{1.31} - .00021\ A1_{t-1}^{-.95}\ P_t^{-.68}\ T_t^{-.84}\ C_t^{-.28}$$

$$- .00305\ Ac_t^{-.18}\ Q_t^{+1.76} - .0046\ I_t^{-.9}\ S_{t-1}^{-1.1} - .0053\ P_t^{-.76}\ D_t^{-1.12}$$

Let $u(\underline{t}) = 3.937\ A2_{t-1}^{.91}\ B_t^{1.31}$ and

$$f(\underline{t}) = - [.0021\ A1_{t-1}^{-.95}\ P_t^{-.68}\ T_t^{-.84}\ C_t^{-.28} + .00305\ Ac_t^{-18}\ Q_t^{1.76}$$

$$+ .0046\ I_t^{-.9}\ S_{t-1}^{-1.1} + .0053\ P_t^{-.76}\ D_t^{-1.12}.]$$

This is equivalent to the posynomial (G-P) P: Min t_0^{-1}; S.T. $t_0/u(\underline{t}) + f(\underline{t})/u(\underline{t})$ (4) ≤ 1; $\underline{t} \geq 0$. We will now actually solve the above equation for these given values. Sales are in million dollar units. The equivalence is established below. Due to the way in which the regression model was obtained, we have at most one positive term. If the vector of variables is defined as $\underline{t}$ and the Sales function as $S(\underline{t})$ we have the objective function: Maximize $S(\underline{t}) = u(\underline{t}) - f(\underline{t})$; where $u(\underline{t})$ and $f(\underline{t})$ are posynomials with $u(\underline{t})$ having <u>only one term</u>. Since Sales cannot be negative, maximizing $S(\underline{t})$ is equivalent to minimizing $- S(\underline{t})$, if and only if, $\tau^* = [u(t^*), t_1^*, t_2^*, \ldots, t_m^*]$ maximizes the function: $g(\tau) = t_0$; subject to the constraint: $t_0 + f(\underline{t}) - u(\underline{t}) \leq 0$.

It is obvious that this maximization problem is equivalent to the primal program (P) that consists of minimizing: $h(\tau) = 1/g(\tau) = 1/t_0$. Subject to the consistent constraint: $t_0/u(\underline{t}) + [f(\underline{t})/u(\underline{t})] \leq 1$. Thus the original (regression function) objective function on Sales (3) will be the same as given as (4) in the geometric programming format.

The following are the budget constraints. These are imposed as policy constraints by the management on the marketing mix variables. The given dollar budgets of the firms are converted into the dimensionless indices given in the appendix yielding the following constraints:

I. Advertising: $A1_{t-1} \geq .10$; $A2_{t-1} \geq .10$ and $A1_{t-1} + A2_{t-1} \leq .35$

$A1_t \geq .10$; $A2_t \geq .10$ and $A1_t + A2_t \leq .35$

II. Relative Instore Promotion: $.2 \leq I_t \leq .35$

III. Relative Price: $.7 \leq P_t \leq 1.35$

IV. Relative Price Change: $.6 \leq C_t \leq 1.3$

V. Trade Allowances: $.6 \leq T_{t-1} \leq 1.3$

$.6 \leq T_t \leq 1.3$

VI. Salesmen Effort: $.8 \leq S_{t-1} \leq 1.25$

$.8 \leq S_t \leq 1.25$

VII. Distribution: $.3 \leq D_t \leq .8$

VIII. Relative Packaging Appeal: $.6 \leq B_t \leq 1.8$

IX. Relative Product Quality $.8 \leq Q_t \leq 1.8$

In addition, a constraint for age composition, in the relative market and two other strategy constraints are also laid out.

(x) Age composition $.8 \leq Ac_t \leq 1.3$

(xi) Strategies: $A_t + I_t \leq .6$; (where A_t is the current advertising index)

(xii) $S_t + T_{t-1} \leq 2.$

Representing these variables in a vector form $\underline{t}$ where

$$\underline{t} = (t_o, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17})$$

$$= (t_o, AC_t, AP_{t-1}, P_t, T_t, C_t, B_t, I_t, S_{t-1}, D_t, A_t, S_t, T_{t-1}, AM_{t-1}, AP_t, AM_t, A_{t-1})$$

equation (4) and the constraints are given below in the geometric programming format:

Minimize t_o^{-1},

Subject to the following constraints $\underline{t} > 0$ and

$$\{.25\ t_o\ t_{14}^{-.91}\ t_7^{-1.31} + .00083\ t_{14}^{-.91}\ t_7^{-1.31}\ t_3^{-.95}\ t_4^{-.68}\ t_5^{-.84}\ t_6^{-.28}$$

$$(5) \quad + .012\ t_{14}^{-.91}\ t_7^{-1.31}\ t_1^{-.18}\ t_2^{+1.76} + .0018\ t_{14}^{-.91}\ t_7^{-1.31}\ t_8^{-.9}\ t_9^{-1.1}$$

$$+ .0021\ t_{14}^{-.91}\ t_7^{-1.31}\ t_4^{-.76}\ t_{11}^{-1.12}\} \leq 1$$

We converted each of the policy constraints of the form $.2 \leq I_t \leq .35$ into the following two constraints: lower bound is $.2 \leq I_t$, that is $\frac{.2}{I_t} \leq 1$ which equals $.2\ t_8^{-1} \leq 1$; upper bound is $I_t \leq .35$, that is $\frac{I_t}{.35} \leq 1$ which equals $2.86\ t_8 \leq 1$. Similarly, the upper and lower bounds of the policy constraints are derived.

The transformed strategy constraints are:

$$2.86\ t_3 + 2.86\ t_{14} \leq 1$$

$$2.86\ t_{15} + 2.86\ t_{16} \leq 1$$

$$1.66\ t_{11} + 1.66\ t_8 \leq 1$$

$$.5\ t_{12} + .5\ t_{13} \leq 1$$

There are 18 variables, 26 budget constraints, 4 strategy constraints and a constraint due to the formulation of maximizing sales function into geometric programming format. The total number of terms are 40 which yields a <u>degree of</u> difficulty of (40-18-1) = 21 [14] which shows that dual solution is contained in a space of 21 dimensions. The dual problem can be expressed for equation 5 as shown in [14] and solved. This problem was solved by the code developed by W. Gochet and Y. Smeers [16] which yielded the following solution vector which minimized equation (5). The solution vector $\underline{t}$ is given below:

$\underline{t}$ = (1, .8, 1.8, .1, .7, .6, .6, 1.8, .2, .8, .3, .225, 1.25, .6, .125, .1, .125, .225)

From the above it is seen that the following variables: AC_t, AP_{t-1}, P_t, C_t, I_t, S_{t-1}, D_t, T_{t-1} are at their lower bounds, while the following variables: Q_t, B_t, S_t are at their upper bounds.

Note that AM_t and A_t are not attaining either of their bounds. Though A_t can take a maximum of .4 as $A_t + I_t \leq .6$ with $I_t = .2$, it has only taken a value of .225. This indicates that sales can be increased by taking the .175 of A_t that is not productive and after transforming this back into dollar budget terms, subtracting the transformed .175 from the advertising budget and adding it to either Q_t, B_t, or S_t thus raising the upper bounds of these variables. Further sensitivity testing would indicate the optimal transfer of funds from advertising to product quality, packaging, and salesmen's effort. In a similar manner the above results enable us to suggest better budget ranges than are currently used, by distributing the saved dollars from the lower bounded variables into those which reach their upper bounds.

To some degree this example is dynamic in that there are decisions for periods t and t-1. Naturally the model would be much richer in a dynamic sense if more than two periods were in the objective function, thus yielding insights into strategies fortiming the allocation of a variable's budget over successive periods. Looking at S_{t-1} and S_t we see that when salesmen's effort reaches its lower bound of .8, the model suggests that an intensive effort be made in the next period. The maximum sales under the present set of budget and strategy constraints for the values used in this example was found to be 17. 489 million dollars.

In conclusion we have suggested a reasonable method for generating a functional form and elasticities from raw data in a manner that allows a practitioner to make use of the mathematically powerful technique of geometric programming. A theoretical contribution was made in relaxing the requirement that the non-linear polynomial terms must be restricted to posynomials. The methods discussed in this paper have been empirically applied, yielding results which executives of a major brewery found insightful and useful. Actual management decisions to change the budget size of advertising appear to have been decisions influenced by this model. Through the use of additional subscripts the model described in this paper can be extended to deal with m markets and p products.

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APPENDIX

The Ten Marketing Mix Variables

1. Relative Advertising Expenditure (A_t)

$$A_t = \frac{AF_t}{AI_t}$$

where A_t = The firm's relative advertising expenditure for time period t.

AF_t = Advertising Dollars spent by the firm in time period t. (dollars)

AI_t = Advertising Dollars spent by industry in time period t. (dollars)

2. Relative In-Store Promotion (I_t)

$$I_t = \frac{IF_t}{II_t}$$

where I_t = The firm's relative in-store promotional expenditure for time period t.

IF_t = In-store promotion dollars spent by the firm in time period t. (dollars)

II_t = In-store promotion dollars spent by the industry in time period t. (dollars)

3. Relative Price (P_t)

$$P_t = \frac{PF_t}{\left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{m} P_{j,t}\right)/(n-1)}$$

where P_t = Relative price of the firm in time period t.

PF_t = Retail price charged by the firm in time period t (dollars/case.)

$P_{j,t}$ = Retail price charged by the jth firm in time period t (dollars/case.)

4. Relative Price Change (C_t)

$$C_t = \frac{PF_t - PF_{t-1}}{\sum_{\substack{j=1 \\ j \neq i}}^{n} (P_{j,t} - P_{j,t-1})}$$

$$C_t = \begin{cases} O_t & \text{if } O_t > PF_t - PF_{t-1} < 0 \\ O_t^{-1} & \text{if } O_t > 0,\ PF_t - PF_{t-1} > 0 \\ 1 & \text{if } O_t = 0 \\ .001 & \text{if } O_t < 0,\ PF_t - PF_{t-1} > 0 \\ 10 & \text{if } O_t \geq 10 \text{ or } O_t < 0, \\ & \quad PF_t - PF_{t-1} < 0 \end{cases}$$

where O_t = An Operator

PF_t = Retail price charged by the ith firm in time period t (dollars/case).

$P_{j,t-1}$ = Retail price charged by the jth firm in time period $t-1$ (dollars/case).

5. Trade Allowances (T_t)

$$T_t = \frac{R_t + W_{t-1}}{\sum_{i=0}^{n} (R_i + W_{i-1})/n+1}$$

where T_t = Trade allowance for time period t.

R_t = Trade allowances granted retailers in time period t (dollars).

W_{t-1} = Trade allowances granted wholesalers in time period $t-1$ (dollars).

R_1 = Trade allowances granted retailers in time period i (dollars) ($i = 0, \ldots\ldots, n$). (Here $n=84$, but in order to get a first data point for W_{i-1} we had to go back to the 85th observation, thus the division by $n+1$ instead of n).

W_{i-1} = Trade allowances granted wholesalers in time period $i-1$ (dollars)

6. Salesman's Effort (S_t)

$$S_t = \frac{SS_t + SC_t}{\sum_{i=1}^{n} SS_i + SC_i} \qquad (i = 1, \ldots t, \ldots n)$$

where S_t = Salesman's effort in time period t.

SS_t = Total salesman's salary paid in time period t (dollars).

SC_t = Total salesman's commissions credited to salesman for time period t (dollars).