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THE NASH BARGAINING SOLUTION IS OPTIMAL

by

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Abstract

We consider the problem faced by players in a two-person bargaining game who have different opinions about what is the appropriate solution concept to use. A procedure is proposed to resolve such a conflict and it is shown that the Nash bargaining solution constitutes the unique equilibrium of the game induced by the procedure. As a by-product we obtain that in the axiomatic characterization of the Nash solution, the controversial "independence of irrelevant alternatives" axiom can be replaced by a much weaker recursivity axiom, which amounts to requiring independence of alternatives that cannot be obtained as an outcome of a risk sensitive solution.

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1. Introduction

This paper considers two-person bargaining problems with fixed threats of the type originally studied in Nash [1950b]. For such games, besides Nash's solution, many other solutions have been proposed, e.g., Kalai [1977b], Kalai and Smorodinsky [1975], Myerson [1977], Perles and Maschler [1981], Rosenthal [1976], and Yu [1973]. For a survey see Kalai [1983], Roth [1979] or Schmitz [1977]. Most of these solutions can be characterized by a set of axioms. The Nash solution, for example, is the unique bargaining solution with the following properties:

(1.1) feasibility,

(1.2) individual rationality,

(1.3) Pareto optimality,

(1.4) scale invariance

(1.5) symmetry, and

(1.6) independence of irrelevant alternatives

(For formal definitions of these properties, see section 2.) The properties (1.1)-(1.5) are not controversial,[1] but a rather large group of people does not consider (1.6) as being reasonable.[2] The fact that people can differ in their opinions about what should be considered as being reasonable, can explain the variety of solution concepts that exist. For example, the Kalai/ Smorodinsky solution is the unique one satisfying (1.1)-(1.5) together with some monotonicity property, while in the Perles/Maschler solution (1.6) is replaced by a continuity and a superadditivity axiom.

However, the state of the art is unsatisfactory. In particular, the possibility is an uncomfortable one that players supporting different bargaining solutions may fail to reach an agreement because each of them insists that his favored solution concept is the appropriate one to use in the situation at hand. This paper, therefore, proposes and analyzes a (dynamic) procedure which can be used to resolve the conflict in such a situation.[3]

The procedure is based upon two principles.

Postulate 1. If a player advocates one particular solution concept in some bargaining situation, then he should adhere to this solution concept in any other bargaining situation.

Postulate 2. If the demands of the players are not compatible, the players should continue bargaining over the set of payoffs not exceeding their previous demands.

Postulate 1, is most easily motivated by taking a normative point of view, i.e., by thinking of bargaining solutions as "fair" division schemes. In this case, the postulate expresses that a player's notion of fairness should be objectively given and should not depend upon the actual situation at hand. Actually, the results are still valid if this postualte is not imposed and players are allowed to switch from one bargaining solution to another during the game. Hence, it is not necessary to impose Postulate 1, but doing so makes the notation and presentation simpler.

The second postulate is more important and is in fact crucial for obtaining the results. It is based on the idea that payoffs larger than a player's demand should be considered as being irrelevant as a consequence of the fact that a rational player will always demand as much as he considers to be attainable (if some payoff larger than his demand could be attainable, he should ask for this amount). Hence, when a player asks for a certain amount, then he is acknowledging that he should not get more than this, so it is natural to view payoffs exceeding his demand as being irrelevant. (For another justification of Postulate 2, see the discussion on the recursivity axiom in section 5.)

Suppose a bargaining game is given and suppose the players have agreed to use Postulates 1 and 2 to resolve conflicts. Then the questions arise of what is the optimal solution concept to propose and what is the payoff the players will finally agree upon. The main result of this paper is that, if players are restricted to propose risk sensitive solutions, then all optimal choices lead to an eventual agreement on the payoffs as proposed by the Nash solution. Furthermore, in case the players do not know yet which bargaining game(s) they will have to play, there is only one bargaining solution which is optimal to propose viz. the Nash solution. Hence, like Zeuthen's process (Zeuthen [1930], Harsanyi [1956, 1977], the procedure here can be viewed as a dynamic model of negotiation which justifies the Nash solution. Actually, these results provide a partial justification for the assumption underlying Zeuthen's process, in the sense one obtains that (in equilibrium) the player proposing the outcome with the smallest Nash product always has to yield. This work is also related to recent contributions of Binmore [1980] and Rubinstein [1982] in which a different noncooperative implementation of the Nash solution is derived.

The present work also leads to a characterization of the Nash solution which avoids Nash's independence of irrelevant alternatives axiom. This characterization involves a recursivity axiom which requires that the solution of a game should not depend on alternatives that are considered to be not fair by both players. To be more precise, alternatives that cannot be obtained as outcomes of any risk sensitive solution.

The paper is organized as follows. Section 2 introduces the notation and contains some preliminary material on bargaining solutions. The procedure which is proposed is formally introduced in section 3 and its properties are studied in section 4. In section 5 some possible extensions are discussed and some open problems are mentioned.

2. Bargaining Games and Bargaining Solutions

A two-person bargaining game[4] is a nonempty, compact, convex and comprehensive subset S of $\mathbb{R}^2_+$. We think of S as representing the set of payoff vectors which the players can obtain by cooperating. In case the players do not cooperate both receive payoff zero. Assuming that S is convex amounts to allowing randomization between different payoff vectors. Comprehensiveness follows from an assumption of free disposal of utility.

The set of all bargaining games will be denoted by Σ. For $S \in \Sigma$, we write $P(S)$ for the (strong) Pareto optimal boundary of S:

$$P(S) = \{x \in S \text{ if } y \in S \text{ and } y \geqslant x, \text{ then } y = x\}.$$

There exists, $u(S)$, the utopia point associated with S, and non-increasing concave functions P^1_S and P^2_S such that

$$u_1(S) = \max\{x_1;\ \exists_{x_2} \text{ such that } (x_1,x_2) \in S\}$$

$$u_2(S) = \max\{x_2;\ \exists_{x_1} \text{ such that } (x_1,x_2) \in S\}$$

$$S = \{x \in \mathbb{R}^2_+;\ x_1 \leqslant u_1(S),\ x_2 \leqslant P^2_S(x_1)\}$$

$$= \{x \in \mathbb{R}^2_+;\ x_2 \leqslant u_2(S),\ x_1 \leqslant P^1_S(x_2)\}.$$

In a bargaining game S, the problem is which outcome in S should be chosen. To resolve this problem, the players can invoke a bargaining solution, i.e., a mapping $f\colon \Sigma \to \mathbb{R}^2$ such that for every $S \in \Sigma$ the following are satisfied:

Axiom PO (Pareto Optimality). $f(S) \in P(S)$.

Axiom SI (Scale Invariance). if $a_1, a_2 > 0$ and if A is the transformation of $\mathbb{R}^2$ given by $A(x_1,x_2) = (a_1x_1, a_2x_2)$, the $f(AS) = Af(S)$.

Axiom SY (Symmetry). If π is the transformation given by $\pi(x_1,x_2) = (x_2,x_1)$, then $f(\pi S) = \pi f(S)$.

These axioms have been discussed extensively in the literature (e.g., Roth [1979]) and they are accepted by the majority of workers in the field. However, there exist infinitely many solutions satisfying these axioms (cf. Kalai and Smordinsky [1975]) and, even worse, there exist many counter-intuitive solutions satisfying them (see, e.g., Thomson and Myerson [1980]). By a counterintuitive solution we mean one which favors a player when the bargaining situation is changed to this player's disadvantage. To exclude such perverse solutions, one needs an additional (monotonicity) axiom. The monotonicity axiom that we will consider in this paper is the so-called "risk sensitivity" property (cf. Roth [1979], Kihlstrom, Roth and Schmeidler [1981]). A bargaining solution is said to be risk sensitive if the payoff assigned to a player does not decrease when his opponent becomes more risk averse. Formally:[5]

Axiom RS (Risk Sensitivity). For any bargaining game S and any nondecreasing concave function $k\colon \mathbb{R} \to \mathbb{R}$ with $k(0) = 0$ we have $f_i(k_j(S)) \geqslant f_i(S)$ where $i \neq j \in \{1,2\}$ and k_1 and k_2 are given by

$$(2.1) \qquad k_1(x_1,x_2) = (k(x_1),x_2),\ k_2(x_1,x_2) = (x_1,k(x_2))$$

This paper will be restricted to the set F of all bargaining solutions that satisfy PO, SI, SY and RS. Hence, we assume that the players have agreed that any reasonable bargaining solution should satisfy at least these axioms. One particular bargaining solution that is reasonable in this sense is the Nash bargaining solution (Nash [1950b, 1953]). The Nash solution f^N prescribes

$$(2.2) \qquad f^N(S) = \operatorname*{argmax}_{x \in P(S)} x_1 x_2$$

as the solution of the game S. Nash's solution is characterized by the basic axioms PO, SI and SY together with the independence of irrelevant alternatives axiom:

Axiom IIA (Independence of Irrelevant Alternatives). If $S,T \in \Sigma$, $S \subset T$ and $f(T) \in S$, then $f(S) = f(T)$.

It follows from Kihlstrom et al. [1981] that, besides the Nash solution, also the Kalai/Smorodinsky solution and the Perles/Maschler solution belong to F. From this result together with the following proposition it follows that F has in fact infinitely many elements.

Proposition 1. For any bargaining game S the set $\{f(S); f \in F\}$ is closed and connected.

Proof. First, it is shown that the set is closed. Let $\{f^n\}_n$ be a sequence of elements in F. For $S \in \Sigma$, let $L(S)$ be the set of all those points in $P(S)$ that are limit points of $\{f^n(S)\}_n$. Let $\ell(S) = u(L(S))$ be the utopia point of $L(S)$ and let $f(S)$ be the intersection of $P(S)$ with the line through 0 and $\ell(S)$. Note that the intersection indeed exists, so that $f(S)$ is well defined. It will be proved that f satisfies PO, SI, SY and RS. This will establish closedness since $f(S) = \lim_{n\to\infty} f^n(S)$ for every S for which the limit exists.

It is easily seen that f satisfies the basic axioms PO, SI and SY, so the concentration will be on RS. Let k be a nondecreasing, concave function with $k(0) = 0$, let $S \in \Sigma$ and write $T = k_2(S)$. First of all, notice that for all n

$$f_1^n(S) \leq f_1^n(T) \quad \text{and} \quad k(f_2^n(S)) \geq f_2^n(T)$$

since every f^n satisfies PO and RS. Consequently,

$$\ell_1(S) \leq \ell_1(T) \quad \text{and} \quad k(\ell_2(S)) \geq \ell_2(T) \tag{2.3}$$

Let us write $s = f_1(S)$ and $t = f_1(T)$. It has to be shown that $s \leq t$. By definition of f, one has

$$\frac{P_S^2(s)}{s} = \frac{\ell_2(S)}{\ell_1(S)}, \quad \frac{P_T^2(t)}{t} = \frac{\ell_2(T)}{\ell_1(T)}. \tag{2.4}$$

(If any of the denominators is zero, the result is immediate.) Since $P_T^2 = kP_S^2$ this is equivalent to

$$s = \frac{\ell_1(S)}{\ell_2(S)} P_S^2(s), \quad t = \frac{\ell_1(T)}{\ell_2(T)} kP_S^2(t). \tag{2.5}$$

To prove $s \leqslant t$, it is sufficient to show that

$$s \leqslant \frac{\ell_1(T)}{\ell_2(T)} kP_S^2(s). \tag{2.6}$$

Substituting the expression for $P_S^2(s)$ found in (2.4) into (2.6), shows that it suffices to show

$$s \leqslant \frac{\ell_1(T)}{\ell_2(T)} k(\frac{\ell_2(S)}{\ell_1(S)} s).$$

Now, $s \leqslant \ell_1(S)$ and since k is concave with $k(0) = 0$, it suffices to show that

$$s \leqslant \frac{\ell_1(T)}{\ell_2(T)} \frac{s}{\ell_1(S)} k(\ell_2(S)),$$

but this follows immediately from (2.3). Hence, f satisfies RS which proves the first assertion of the proposition.

The second assertion can be proved by the same methods. Let $f^1, f^2 \in F$ and, for $S \in \Sigma$, define $L(S) = \{f^1(S), f^2(S)\}$. If we construct f(S) from L(S) as above, then similarly as above it is seen that $f \in F$. If $f^1(S) \neq f^1(S)$, then f(S) will be between $f^1(S)$ and $f^2(S)$, hence, using that $\{f(S); f \in F\}$ is closed, one sees that the complete Pareto boundary between $f^1(S)$ and $f^2(S)$ can be obtained by repeating this procedure. This shows that the set $\{f(S), f \in F\}$ is connected. [].

3. A Meta Bargaining Game

In this section we formally define the procedure by means of which we propose to resolve conflicts arising from different players supporting different bargaining solutions.

Let S be a bargaining game and let $\phi = (f^1, f^2)$ be a pair of bargaining solutions. For $t \in \mathbb{N}$, define $S^t(\phi)$, the t^{th} stage bargaining game by

(3.1) $$S^1(\phi) = S$$

(3.2) $$S^{t+1}(\phi) = \{x \in S^t(\phi);\ x_1 \leq f_1^1(S^t(\phi)),\ x_2 \leq f_2^2(S^t(\phi))\}.$$

$S^{t+1}(\phi)$ consists of all those payoff vectors for which the i^{th} coordinate does not exceed the demand of player i in stage t.

The procedure requires that in case of conflict (i.e., incompatible demands) the players continue bargaining over the set of payoffs not exceeding their previous demands. The underlying idea is that, when agent i is proposing $f^i(S)$, then he is acknowledging that he should not get more than $f_i^i(S)$. Hence, if the players propose $\phi = (f^1,f^2)$, then they should be willing to replace $S^1(\phi)$ with $S^2(\phi)$, etc.

Figure 1 illustrates the sequence of games arising when player 1 proposes the Kalai/Smordinsky solution f^K (i.e., that point on the Pareto boundary that is on the line through 0 and the utopia point), player 2 proposes the Nash solution f^N and S is the convex comprehensive hull of (1,1) and (2,0). Note that in this case $f^N(S^t(\phi)) = f^N(S) = (1,1)$ for all t, because of IIA and that ultimately the players reach an agreement on (1,1).

[INSERT FIGURE 1 HERE]

Assume that the underlying bargaining game is S and that the players have agreed to use the algorithm (3.1)-(3.2) to resolve possible conflicts. Then, if $\phi = (f^1,f^2)$ is the proposed pair of solutions, the conflict will be resolved only if $f^1(S^t(\phi))$ and $f^2(S^t(\phi))$ have the same limit. If the limits do not coincide we have an everlasting conflict, in which case a player cannot expect more than the disagreement outcome zero. Hence, if we define

$$U(\phi;S) = \begin{cases} \lim\limits_{t\to\infty} f^1(S^t(\phi)) & \text{if } \lim\limits_{t\to\infty} f^1(S^t(\phi)) = \lim\limits_{t\to\infty} f^2(S^t(\phi)), \\ 0 & \text{otherwise.} \end{cases}$$

Then $U(\phi;S)$ is the payoff vector which will result from applying the procedure.

The first question to be answered is whether everlasting conflict can indeed occur. In the next section we will show that such everlasting conflict cannot occur if the players restrict themselves to bargaining solutions in F (i.e., if they agree that any reasonable solution should satisfy at least PO, SI, SY and RS). Furthermore, it will be seen that if both players choose their bargaining solution optimally, then they will eventually agree on the payoffs as proposed by the Nash solution. To be more precise, once the players have agreed to resolve possible conflicts in S by using the algorithm (3.1)-(3.2), the situation can be described by the noncooperative game $\Gamma(S) = (F,F,U_1(\cdot;S), U_2(\cdot;S))$ and the question of which bargaining solution to propose amounts to asking what are the Nash equilibria of $\Gamma(S)$ (Nash [1950a]), i.e., which pairs $(\bar{f}^1,\bar{f}^2)$ satisfy

$$U_1(\bar{f}^1,\bar{f}^2;S) \geqslant U_1(f^1,\bar{f}^2;S) \quad \text{for all } f^1 \in F,$$

$$U_2(\bar{f}^1,\bar{f}^2;S) \geqslant U_2(\bar{f}^1,f^2;S) \quad \text{for all } f^2 \in F$$

It will be shown that both players proposing the Nash solution is an equilibrium in $\Gamma(S)$ and that every equilibrium of $\Gamma(S)$ results in the payoff

$f^N(S)$. Hence, $\Gamma(S)$ has the structure of a zero-sum game and it can be said that $f^N(S)$ is the value of $\Gamma(S)$. For some bargaining games there may exist many equilibria (e.g., if S is symmetric, then every pair of bargaining solutions is an equilibrium). But, as will be seen, only (F^N, f^N) is an equilibrium for every bargaining game. Hence, if it is not known yet which bargaining games one must play in the future, and if it has already been agreed to resolve possible conflicts by this procedure (3.1)-(3.2), then only proposing the Nash solution is optimal. Thus, the title of this paper is indeed justified.

To conclude this section, it will be shown that conflicts will not always be resolved by our procedure if "perverse" solutions are allowed that do not satisfy RS (see, however, Proposition 6 in section 5).

Let Σ^1 be the set of all bargaining games with utopia point (1,1). Notice that every nondegenerate bargaining game (one in which both players can possibly profit) is equivalent to exactly one game in Σ^1 via Axiom SI. Define the bargaining solution f on Σ^1 by

$$f(S) = \begin{cases} (1, P_S^2(1)) & \text{if } F_1^N(S) > f_2^N(S), \\ f^N(S) & \text{if } f_1^N(S) = f_2^N(S), \\ (P_S^1(1), 1) & \text{otherwise.} \end{cases}$$

and extend f to Σ by SI (if S is degenerate, f(S) is determined by PO). Then f satisfies PO, SI and SY but not RS as one can see by taking S to be the triangle with corners (0,0), (1,0) and (0,1) and $T = k_2(S)$ where k is given by

$$k(\xi) = \begin{cases} 2\xi & \text{if } \xi \leq 0, \\ 2\xi - \xi^2 & \text{if } 0 \leq \xi \leq 1, \\ 1 & \text{if } \xi \geq 1. \end{cases}$$

Then player 2 is more risk averse in T, but player 1 would rather play S if f is the solution, since $f_1(T) = 0 < 1/2 = f_1(S)$.

If player 1 proposes f and player 2 proposes the Nash solution, then a conflict will arise for every bargaining game in Σ^1 for which the Nash solution is not symmetric. Consequently, our basic Proposition 2 is not correct and also our other results break down. For example, if RS is not imposed, then everlasting conflict can occur even in equilibrium. Namely, for $S \in \Sigma^1$ let g(S) be given by

$$g(S) = \begin{cases} (P_S^1(1),1) & \text{if } f_1^N(S) > F_s^N(S), \\ f^N(S) & \text{if } F_1^N(S) = f_1^N(S), \\ (1,P_S^2(1)) & \text{otherwise.} \end{cases}$$

and extend g to Σ by SI. Then for every $S \in \Sigma^1$ with $(1,0)$, $(0,1) \in P(S)$ and $f_1^N(S) \neq f_2^N(S)$, the pair (f,g) is an equilibrium resulting in everlasting conflict.

4. Analysis of the Meta Bargaining Game

It will be shown first that in case of conflict a player has to yield considerably until he offers the other player at the least the Nash payoff or at least as much as this player asks for in case this is less than the Nash payoff. Our main results then follow easily from this one together with the fact that a player proposing the Nash solution never has to yield.

Proposition 2. Let $S \in \Sigma$ and assume $f^1, f^2 \in F$ are such that $f_1^1(S) > f_1^2(S)$. Define $T \in \Sigma$ by

(4.1) $$T = \{x \in S,\ x_1 \leqslant f_1^1(S),\ x_2 \leqslant f_2^2(S)\}$$

If $f_1^N(S) > f_1^2(S)$, then

$$f_1^2(T) \geqslant 1/2 \min \{f_1^N(S), f_1^1(S)\} + 1/2\ f_1^2(S).$$

<u>Proof.</u> Write $f^2(S) = (\alpha,\beta)$ and $f^1(S) = (\gamma,\delta)$. We have $\gamma > \alpha \geqslant 0$, hence $\beta > \delta \geqslant 0$. Therefore, (Axiom 5), we may assume $\beta = \gamma = 1$. We will first consider the case in which $f_1^1(S) \leqslant f_1^N(S)$.

From (2.2) it follows that

$$P_S^2(\xi) + \xi(P_S^2)'(\xi) \geqslant 0 \quad \text{for} \quad 0 \leqslant \xi \leqslant f_1^N(S).$$

Hence, $\xi P_S^2(\xi)$ is increasing on $[0,F_1^N(S)]$ and consequently

(4.2) $$\alpha < \delta \quad \text{and} \quad -(P_S^2)'(1) \leqslant \delta.$$

Let R be the triangle with corner points $(0,0)$, $(1,\alpha)$ and $(\alpha,1)$. Since $\alpha < \delta$, we have $(1,\alpha) \in T$ and $R \subset T$. Consider the function $k\colon [\alpha,1] \to [\delta,1]$ defined by

$$k(\xi) = P_S^2(-\xi + 1 + \alpha)$$

which is chosen such that $k_2(P(R)) = P(T)$, where k_2 is as in (2.1). The situation is illustrated in figure 2 in which k_2 is indicated by the arrows.

[INSERT FIGURE 2 HERE]

Note that k is concave, being the composition of an affine and a concave mapping and that k is nondecreasing. It suffices to show that k can be extended to a concave nondecreasing function on $\mathbb{R}$ with $k(0) = 0$, for in this case PO, SY and RS yield

$$f_1(T) \geqslant f_1(R) = 1/2\ (1 + \alpha) = 1/2\ f_1^1(S) + 1/2\ f_1^2(S).$$

It will be shown next that k can be extended as desired. First, define $k(\xi) = 1$ for all $\xi \geqslant 1$. Next, note that from (4.2) it follows that

$$k'(\alpha) = -(P_S^2)'(1) \leqslant \delta < \frac{\delta}{\alpha}$$

and so, if one defines $k(\xi) = \frac{\delta}{\alpha}\xi$ for $\xi \leqslant \alpha$, then k satisfies all our requirements. This completes the proof in the case $f_1^1(S) \leqslant f_1^N(S)$.

Next, consider the case in which $f_1^1(S) > F_1^N(S)$. Let T^* be the set

(4.3) $$T^* = \{x \in S;\ x_1 \leqslant f_1^N(S),\ x_2 \leqslant f_2^2(S)\}.$$

Then T^* results from T by making player 1 more risk averse, namely, $T^* = k_1(T)$, where k is given by

$$k(\xi) = \begin{cases} \xi & \text{for } \xi \leqslant f_1^N(S), \\ f_1^N(S) & \text{otherwise.} \end{cases}$$

Because of the special structure of k, one can conclude from f_2 satisfying RS and PO that

$$(4.4) \qquad f_1^2(T) \geqslant f_1^2(T^*);$$

but from the first part of the proof we know that

$$f_1^2(T^*) \geqslant 1/2\ f_1^N(S) + 1/2\ f_1^2(S)$$

which completes the proof. []

Notice that proposition 2 shows that the player proposing the outcome with the smallest Nash product has to make a concession in the next round. In the case in which the Nash solution is inbetween the outcomes proposed, both players have to yield. Hence, Proposition 2 provides a partial justification for the behavioral assumption underlying Zeuthen's process (Zeuthen [1930], Harsanyi [1956, 1977]).

Our first main result is that our procedure resolves every possible conflict, i.e., in every bargaining game the players will eventually reach an agreement no matter which bargaining solutions they propose.

Proposition 3. For any bargaining game S and for any pair of bargaining solutions f^1 and f^2 satisfying PO, SI, SY and RS

$$(4.5) \qquad \lim_{t\to\infty} f^1(S^t(\phi)) = \lim_{t\to\infty} f^2(S^t(\phi)).$$

Proof. In view of PO it suffices to consider the case in which

$$(4.6) \qquad f_1^1(S^t(\phi)) > f_1^2(S^t(\phi)) \quad \text{for all } t.$$

If it is the case that

$$f_1^1(S^t(\phi)) \geqslant f_1^N(S) \geqslant f_1^2(S^t(\phi)) \quad \text{for all } t$$

then it follows from Proposition 2 (and its analogue with the players interchanged) that for $i \neq j \in \{1,2\}$ and $t \in \mathbb{N}$

$$f_i^N(S) \geqslant f_i^j(S^{t+1}(\phi)) \geqslant (1 - (1/2)^t)f_i^N(S)$$

so that both limits in (4.5) are equal to $f^N(S)$. In the same way it is seen that the limits of (4.5) coincide in case a player sometimes asks for less than his Nash payoff. []

Let S be a bargaining game, let $f \in F$ and write $\phi = (f^N, f)$. Since the Nash solution satisfies IIA, we have

$$f_1^N(S^t(\phi)) = f_1^N(S) \quad \text{for all } t.$$

Hence, from Proposition 3 we see that

$$U_1(f^N, f; S) = f_1^N(S) \quad \text{for all } S \in \Sigma \text{ and } f \in F. \tag{4.7}$$

Similarly we have

$$U_2(f, f^N; S) = f_2^N(S) \quad \text{for all } S \in \Sigma \text{ and } f \in F. \tag{4.8}$$

Hence, each player can guarantee the Nash payoff against any other bargaining solution in F. From this observation and the fact that the Nash solution is

Pareto optimal, we immediately deduce

Corollary 1. Let S be a bargaining game and let $\Gamma(S)$ be as in section 3. Then

(i) (f^N, f^N) is an equilibrium of $\Gamma(S)$,

(ii) if (f^1, f^2) is an equilibrium of $\Gamma(S)$, then $U(f^1, f^2; S) = f^N(S)$.

Because of the second property in this corollary $f^N(S)$ can be called the value of the game $\Gamma(S)$. Proposition 3 in fact shows that f^N is not only an equilibrium strategy in $\Gamma(S)$ but is also a maximin strategy, i.e., choosing f^N guarantees the value of the game no matter which bargaining solution the other player proposes. As for zero-sum games, we have that every maximin strategy is an equilibrium strategy, but the author does not know whether the converse is also true. Also, the related question of whether all equilibria are interchangeable is still open. Nevertheless, we have

Corollary 2. For every bargaining game S

(i) Every maximin strategy of $\Gamma(S)$ is an equilibrium strategy of $\Gamma(S)$;

(ii) f^N is a maximin strategy in $\Gamma(S)$.

In has been already remarked in section 3 that for some bargaining games there may exist more equilibria than just (f^N, f^N). For example, in the game of figure 1, also (f^K, f^N) is an equilibrium; in fact, f^K (the Kalai/ Smorodinsky solution) is even a maximin strategy in this game. Note, however, that it is not at all difficult to construct a bargaining game in which f^K is not an equilibrium strategy (just take the symmetric image of the game in figure 1). Hence, if a player has to announce his strategy (bargaining solution) before he actually knows which bargaining game will be played, then it is not optimal to propose the Kalai/Smorodinsky solution. A model of this situation is the game Γ consisting of the following stages:

Stage 1: The players choose bargaining solutions f^1 and f^2 in F.

Stage 2: Chance chooses a bargaining game $S \in \Sigma$.

Stage 3: Player i receives the payoff $U_i(f^1,f^2;S)$.

The appropriate solution concept in such a dynamic game is the subgame perfect equilibrium[6] (Selten [1965, 1975]). A pair (f^1,f^2) is a <u>subgame perfect equilibrium</u> of Γ if it is an equilibrium in $\Gamma(S)$ for every $S \in \Sigma$. We have

<u>Proposition 4.</u> (f^N,f^N) is the unique subgame perfect equilibrium of Γ.

<u>Proof.</u> Corollary 1 shows that (f^N,f^N) is a subgame perfect equilibrium in Γ. Assume (f^1,f^2) is a subgame perfect equilibrium in Γ. First, it is shown that $f^1 = f^2$. Obviously,

$$f_1^1(S) \geqslant f_1^2(S) \quad \text{for all } S \in \Sigma,$$

since otherwise player 1 could improve by proposing f^2 in some game. On the other hand, if the inequality would be strict for $S^* \in \Sigma$, then

$$f_1^1(\pi S^*) < f_1^2(\pi S^*)$$

where πS^* is the symmmetric image of S^* and this is impossible. Hence, $f^1 = f^2$ and so

$$U(f^1,f^2;S) = f^1(S) = f^2(S) \text{ for all } S \in \Sigma$$

but this implies $f^1 = f^2 = f^N$ in view of Corollary 1. $\square$

To conclude this section it will now be shown that the results so far lead to a characterization of the Nash solution that does not involve Nash's independence of irrelevant alternatives axiom, but rather a recursivity principle[7] which amounts to asking for independence of alternatives that cannot be obtained as outcomes of risk sensitive solutions.

Assume a two-person bargaining game S is given and assume that the players have agreed that any reasonable bargaining solution should satisfy PO, SI, SY and RS. Then the players agree that the outcome should be an element of the set

$$S_F = \{f(S);\ f \in F\}$$

of all those points that can be obtained as the result of some bargaining solution in F (i.e., as the outcome of some "fair division scheme"). Hence, disagreement is limited to S_F, bargaining will continue over this set (or rather, over its comprehensive hull $S_F^{\leq}$ which is in Σ by Proposition 1) and it is natural to require that points outside S_F should have no influence on the outcome. This leads to

<u>Axiom R (Recursivity).</u> $f(S) = f(S_F^{\leq})$ for all $S \in \Sigma$.

It is clear that the Nash solution satisfies this recursivity axiom, since it satsifies IIA. On the other hand, if $f \in F$ satisfies R, then one must have $f(S^t) = f(S)$ for all t, where S^t is defined by $S^0 = S$, $S^{t+1} = (S^t)_F^{\leq}$. However, from Proposition 2 it follows that for i = 1,2

$$|x_i - f_i^N(S)| \leq (1/2)^t f_i^N(S) \quad \text{for all } x \in P(S^t)$$

which shows that only f^N satisfies the recursivity axiom. We have proved

Proposition 5. A bargaining solution satisfies PO, SI, SY, RS and R if and only if it is the Nash solution.

Hence, in the axiomatic characterization of the Nash solution, Nash's independence of irrelevant alternatives axiom can be replaced by a weaker axiom which requires that the solution should not depend on outcomes which cannot be obtained as the result of any risk sensitive solution.

Obviously, if the players agree that a bargaining solution should satsify even more stringent requirements than PO, SI, Sy and RS--i.e., if they agree on a subset G of F, then, as long as f^N belongs to G, it is the unique element of g which satisfies recursivity with respect to G.

5. Extensions and a Related Result

It is interesting to see whether (and how far) the present assumptions can be relaxed without changing the results. The assumptions fall into three clasess: (i) assumptions about the feasible bargaining games, (ii) restrictions on the bargaining solutions that are allowed, and (iii) assumptions about the procedure. These topics will be discussed separately.

(i) The Bargaining Domain. The assumptions of S being nonempty, compact and convex are innocuous and are made in any theory of bargaining. The comprehensiveness assumption has been made only to facilitate the notation--all the results remain correct if this assumption is dropped and only some minor modifications are needed in the proofs. We can drop the requirement of S containing only individually rational outcomes by imposing that every bargaining solution should be individually rational. In fact, in this case we can extend our results to the class of bargaining games in which the individually rational Pareto optimal outcomes all result from lotteries among individually rational pure outcomes (see Roth and Rothblum [1982]).

The restriction to two-person games is obviously a significant one. The problem of whether the results can be generalized to n-person games is still outstanding, but it seems safe to conjecture that with a suitable generalization of Axiom RS (or Axiom SL (see below)) the results will still be true.

(ii) The Axioms. Our basic axioms PO, SI, and SY are standard in the theory of bargaining and, in fact, all can be weakened to some extent without affecting the results.

For instance, if instead of PO the requirement would be only for weak Pareto optimality (i.e., that the solution not be strongly dominated), then Proposition 4 and Corollaries 1 and 2 would still be correct. In fact, to establish these results we only need that the solution is (weakly) Pareto optimal for every symmetric set. To prove the other results, we have used (4.4) and the author does not know whether this formula remains correct in case of weak Pareto optimality. Note, however, that if Axiom RS is strengthened a little and $f_i(k_i(S)) \leq k(f_i(S))$ is required for every $S \in \Sigma$ and any nondecreasing concave function k with $k(0) = 0$, then weak Pareto optimality would also be sufficient to establish these results.

If Pareto optimality is dropped completely, then the results are no longer valid. For example, consider Corollary 1: if Pareto optimality is not required, then both players proposing the disagreement outcome is an equilibrium.

The symmetry axiom SY used thus far is called anonomity by some authors. They call a solution symmetric if it prescribes a symmetric solution in every symmetric game. Under this weaker symmetry requirement all the results thus far, except Proposition 4, remain valid. To see that this proposition is not correct in this case, consider the bargaining solution f defined by

$$f(S) = \begin{cases} f^K(S) & \text{if } f_1^K(S) > f_1^N(S), \\ f^N(S) & \text{otherwise.} \end{cases}$$

Then f satisfies PO, SI, RS and the weak version of the symmetry axiom, while (f,f^N) is an equilibrium of Γ.

If the symmetry axiom is dropped completely, then for every $S \in \Sigma$ every element of P(S) can be obtained as an equilibrium payoff in $\Gamma(S)$. Namely, for $\lambda \in [0,1]$, let the nonsymmetric Nash solution f^λ be given by

$$f^\lambda(S) = \underset{x \in P(S)}{\operatorname{argmax}}\; x_1^\lambda\, x_2^{1-\lambda}$$

(see Harsanyi and Selten [1972], Kalai [1977a]). Then f^λ satisfies PO, SI and IIA and therefore (see DeKoster et al. [1982]) also satisfies RS. Since f^λ satisfies IIA, it is clear that (f^λ,f^λ) is an equilibrium of $\Gamma(S)$ if nonsymmetric solutions are allowed.

Of the basic set of axioms, the scale invariance property SI is the most controversial one. The axiom arises naturally, since von Neumann-Morgenstern utility scales are determined only up to a positive affine transformation. However, the axiom precludes interpersonal utility comparisons. In fact (and this may surprise some readers), this axiom can be dispensed with completely. The reason is that PO and RS together imply SI (see Kihlstrom et al. [1981]). This shows that the risk sensitivity axiom is stronger than one a priori might expect, which leads to the question of whether this axiom can be weakened without affecting the results. It will now be indicated that this is indeed the case. However, rather than considering a weaker version of RS, it will be proven that for a related axiom (not implying SI) the main results are still true although the rate of convergence is somewhat different. The axiom to be considered is the so-called slice property axiom of Tijs and Peters [1983], which is a weaker version of the cutting axiom of Thomson and Myerson [1980]. A solution f is said to have the slice property when it favors player j in case a piece of the bargaining set S which is favored by player i to $f_i(S)$ is cut off, the utopia point remaining the same. Formally:

<u>Axiom SL (Slice Property).</u> For all $i,j \in \{1,2\}$ and for all $S,T \in \Sigma$ with $T \subset S$ and $u(T) = u(S)$

$$\text{if } S\backslash T \subset \{x;\ x_i > f_i(S)\},\ \text{then } f_j(T) \geqslant f_j(S).$$

The axioms RS and SL are closely related, but neither implies the other. Namely, the Perles/Maschler solution satisfies RS but not SL (See Tijs and Peters [1983]) and the solution f defined by[8]

$$f(S) = \underset{x \in P(S)}{\operatorname{argmax}}\ (x_1 + u_1(S))(x_2 + u_2(S))$$

satisfies SL, but does not satisfy RS. That f satisfies SL is most easily seen if one realizes that f prescribes the Nash solution of S corresponding to the threat point $-(u_1(S),u_2(S))$. That f does not satisfy RS can be seen by taking S to be the triangle with corners (0,0), (1,0) and (0,1) and $T = \{x \in S;\ x_1 \leqslant 3/4\}$. Then player 1 is more risk averse in T, but player 2 would rather play S ($f_2(S) = 1/2$, $f_2(T) = 3/8$).

We have the following analogue to Proposition 2, showing that the player proposing the outcome with the smallest Nash product has to yield

Proposition 6. Let $S \in \Sigma$ and assume f^1, f^2 satisfying PO, SI, SY and SL are such that $f_1^2(S) < f_1^1(S)$. Let T be as in (4.1). If

$$f_1^1(S)f_2^1(S) \geqslant f_1^2(S)f_2^2(S)$$

then

$$f_1^2(T) \geqslant \frac{f_1^1(S)f_2^1(S)}{f_2^2(S)} \geqslant f_1^2(S) \tag{5.1}$$

Proof. We only have to prove the first inequality. Notice that the denominator is positive by PO. As in the proof of Proposition 2 we may assume $f^2(S) = (\alpha,1)$ and $f^1(S) = (1,\delta)$ with $\alpha \leqslant \delta < 1$. Notice that

$$P_S^2(\delta) > \delta, \text{ since } \delta < 1 \text{ and } (\delta, P_S^2(\delta)) \in P(S) \tag{5.2}$$

(if the inequality would not be satisfied, $(\delta, P_S^2(\delta))$ would be Pareto dominated by $(1,\delta)$). Furthermore, we have

$$P_S^1(\xi) \leqslant P_S^2(\xi) \quad \text{for} \quad P_S^2(\delta) \leqslant \xi \leqslant 1$$

since P_S^2 and P_S^1 are nonincreasing and since

$$P_S^1(P_S^2(\delta)) = \delta = P_S^2(1).$$

Define the map $\rho\colon [0,1] \to [0,1]$ by

$$\rho(x) = \begin{cases} P_S^2(x) & \text{for } 0 \leqslant x \leqslant \delta, \\ -x + \delta + P_S^2(\delta) & \text{for } \delta \leqslant x \leqslant P_S^2(\delta), \\ P_S^1(x) & \text{for } P_S^2(\delta) \leqslant x \leqslant 1. \end{cases}$$

Then ρ is continuous and nonincreasing with $\rho\rho(x) = x$ for all x and $\rho(x) \leqslant P_S^2(x)$ for all x. Since P_S^2 is concave we have

$$P_S^2(1) \leqslant P_S^2(\delta) + (P_S^2)'(\delta)(1 - \delta)$$

hence

$$(P_S^2)'(\delta) \geqslant \frac{\delta - P_S^2(\delta)}{1 - \delta} \geqslant -1$$

from which it follows that ρ is concave. Therefore, if $R \in \Sigma$ is the game with Pareto boundary defined by ρ (i.e., $P_R^2 = \rho$), then R is a symmetric bargaining game contained in T. Graphically, the situation is illustrated in figure 3

[INSERT FIGURE 3 HERE]

Now assume (5.1) is not true, i.e., $f_1^2(T) \leqslant \delta$. Then SL would apply to the pair R,T and we would have

$$f_2^2(R) \geqslant f_2^2(T) \geqslant P_S^2(\delta),$$

but since R is symmetric we have

$$f_2^2(R) = 1/2\ (\delta + P_S^2(\delta)),$$

and therefore (5.2) yields a contradiction. []

Let $S \in \Sigma$ and let f^1, f^2 satisfying PO, SI, SY and SL be given and consider the procedure (3.1)-(3.2). Proposition 6, together with its analogue with the players interchanged, shows that a player proposing the outcome with the smallest Nash product has to yield in the next round and that both players have to yield in case they propose outcomes with the same Nash products. Hence, again the behaviorial assumption underlying Zeuthen's process is obtained. If in some stage t player 1 asks for not more than player 2 offers $(f_1^1(S^t(\phi)) \leqslant f_1^2(S^t(\phi))$, the players will reach an agreement at time t + 1 by PO. So it suffices to analyze the situation in which there is conflict in any stage—i.e., $f_1^1(S^t(\phi)) > f_1^2(S^t(\phi))$ for all t. In this case, Proposition 6 implies that in the limit the players will propose outcomes with the same Nash product, i.e.,

$$(5.3) \qquad \prod_{i=1}^{2} \lim_{t\to\infty} f_i^1(S^t(\phi)) = \prod_{i=1}^{2} \lim_{t\to\infty} (S^t(\phi)).$$

Now, if a player proposes the Nash solution, then he never has to yield and (5.3) shows that in this case the players will finally reach an agreement on the Nash payoffs. Hence, the formulas (4.7) and (4.8) and, consequently, the Corollaries 1 and 2 and Proposition 4 remain correct if the players are allowed to propose solutions satisfying the slice property SL. Hence, in equilibrium, everlasting conflict cannot occur: the players will rather reach an agreement on the Nash payoffs.

It is unknown whether the procedure actually resolves the conflict between anytwo solutions satisfying SL. However, this will be the case if we restrict ourselves to solutions satisfying some weak continuity requirement (upper semi-continutiy as in van Damme [1983] will do). If f^1 and f^2 satisfy this continuity condition then for the limit set S^∞

$$S^{\infty} = \{x \in S;\ x_i \leqslant \lim_{t\to\infty} f_i^i(S^t(\phi)) \text{ for } i = 1,2\}$$

we will have

$$f^i(S^{\infty}) = \lim_{t\to\infty} f^i(S^t(\phi)) \text{ for } i = 1,2$$

Hence,

$$f_i^i(S^{\infty}) = u_i(S^{\infty}) \text{ for } i = 1,2$$

while in view of (5.3)

$$\prod_{i=1}^{2} f_i^1(S^{\infty}) = \prod_{i=1}^{2} f_i^2(S^{\infty})$$

but this implies that $f^1(S^{\infty}) = f^2(S^{\infty})$ by Proposition 6. Hence, if players are restricted to proposing upper semi-continuous solutions satisfying SL, then every conflict will be resolved by this procedure. Consequently, also, the analogue of Proposition 5 would remain valid in this case.

(iii) The Procedure. It can be easily seen that all the results remain correct if the stationarity assumption of Postulate 1 is not imposed, i.e., if players are allowed to switch from one bargaining solution to another during the process. Another requirement of the procedure of minor importance is that the players should stop bargaining as soon as their demands are feasible, even though the outcome may be Pareto inferior. One could modify the procedure to allow players to continue bargaining in such a case until a Pareto optimal outcome is achieved and it is easily seen that this does not affect the results. In fact, the results are very robust with respect to what is prescribed in case of anticonflict, since for any reasonable rule anticonflict will never arise in equilibrium.

The results, however, do depend crucially on Postulate 2, i.e., on how to proceed in case of conflict. The reader might have the opinion that by imposing this postulate we implicitly put axiom IIA into the model. Indeed, in motivating this postulate we argued that payoffs exceeding a player's demand should be considered as being irrelevant and the closely related recursivity axiom was motivated by saying that non-solution alternatives are irrelevant. Nevertheless, a priori it is not clear at all that Corollary 1 and Proposition 5 should hold, since one could imagine that some bargaining solution when confronted with the Nash solution would give rise to an everlasting conflict. Anyhow, even readers disagreeing with the paper's title will probably agree that the results obtained elucidate Nash's independence of the irrelevant alternatives axiom.

Finally, let us briefly consider different procedures that could be used for resolving the type of conflict as analyzed in this paper. The basic assumption underlying its procedures is that when player i proposes $f^i(S)$ then he is acknowledging that he should not get more than $f^i_i(S)$. However, one could also interpret the proposal $f^i(S)$ as a signal of player i that in his opinion player j should not get more than $f^i_j(S)$. In this case, the sequence of games arising from S and $\phi = (f^1,f^2)$ is

$$S^1(\phi) = S \tag{5.4}$$

$$S^{t+1}(\phi) = \{x \in S^t(\phi);\ x_1 \leqslant f^2_1(\phi)),\ x_2 \leqslant f^1_2(S^t(\phi))\}. \tag{5.5}$$

In this case, a player cannot guarantee the Nash payoff by proposing the Nash solution, but rather he can guarantee that the other cannot obtain more than the Nash payoff. Consequently, both players proposing the Nash solution is an equilibrium, but there are other equilibria and other equilibrium payoffs as well. For instance, in the game S of figure 1, the pair (f^K,f^N) is an equilibrium with payoffs $(f_1^N(S),f_2^K(S))$.

Alternatively, one might interpret the proposal $f^i(S)$ as a signal that agent i is willing to give at least $f_j^i(S)$ to player j. In this case, the $(t + 1)$-st stage bargaining game is given by (3.2), but the threat point is

$$d^{t+1}(\phi) = (f_1^2(S^t(\phi)),f_2^1(S^t(\phi)))$$

rather than zero. The reader can easily verify that, in general, both players proposing the Nash solution will not be an equilibrium of this game. It is also easily seen that any of the other well-known bargaining solutions, such as the ones of Kalai/Smorodinsky and Perles/Maschler, in general, are not equilibrium strategies and the problem of what are the equilibria of this game (if any) is still open.

Of course, one could think of many more procedures that could be used and this leads naturally to a "bargaining on procedures" stage. This stage of the game has not yet been analyzed and so it is not known whether there are reasons to prefer our procedure (3.1)-(3.2) above the others; hence, it is not known whether this procedure is an equilibrium outcome of the larger game.

Notes

[1]However, there is some discussion about (1.4) and (1.5). The motivation for (1.4) is the implicit assumption that players have von Neumann-Morgenstern utility functions, which are determined only up to a positive affine transformation. This property rules out interpersonal utility comparison and there are some solutions in which such comparisons are allowed (e.g., Kalai [1977b], Myerson [1977]; also see Shapley [1969]).

Property (1.5) expresses that the solution should depend only on information contained in the model. Loosely speaking, this axiom (which is dropped in Kalai [1977b]) assumes equal bargaining ability.

[2]Property (1.6) is extensively discussed in Luce and Raiffa [1957] and Roth [1979]. For an informal discussion, see Aumann [1983].

[3]For an alternative approach in the special case of a conflict between the Nash and Kalai/Smorodinsky solutions, see Richter [1981].

[4]A subset S of $\mathbb{R}^2_+$ is comprehensive if $y \in S$ for all $y \in \mathbb{R}^2_+$ with $y \leqslant x$ for some $x \in S$. We confine ourselves to bargaining games with threat point zero which is no restriction as long as we consider only translation invariant solutions. By defining S to be a subset of $\mathbb{R}^2_+$, we implicitly impose that a bargaining solution should be individually rational. By imposing individual rationality as an axiom (i.e., every player should get at least his threat payoff), we can extend our results to the class of all bargaining games in which all individually rational Pareto optimal outcomes result from lotteries among individually rational pure outcomes (see Roth and Rothblum [1982]).

[5]The requirement that $k(0) = 0$ arises from the fact that we consider bargaining games with threat point zero. This definition is taken from Tijs and Peters [1983] and it is stronger than the definition in Kihlstrom et al., [1982]. In the latter paper, k is required to be increasing, but all results from that paper remain valid for this broader definition. It can also be shown that for upper semi-continuous solutions (see van Damme [1983]), the two definitions of risk sensitivity are equivalent.

The notion of risk aversion comparisons is based on the works of Arrow [1965], Pratt [1964], Yaari [1969], and Kihlstrom and Mirman [1974].

[6]Ordinary Nash equilibria are not appropriate, since they may prescribe irrational behavior at bargaining games that are reached with zero

probability. In this case, the subgame perfectness concept is only slightly stronger than the Nash concept, since a Nash equilibrium can only be imperfect at a set that occurs with probability zero.

[7]A similar recursivity principle is advocated by Green [1983] in a slightly different context.

[8]This example was provided by William Thomson, who pointed out the possibility of generalizing the results by allowing solutions satisfying the slice property. He also gave the first proof that Corollary 1 remains correct if the players are allowed to propose upper semi-continuous solutions satisfying the slice property. Note that Proposition 6 implies that the corollary remains valid even without this continuity requirement.

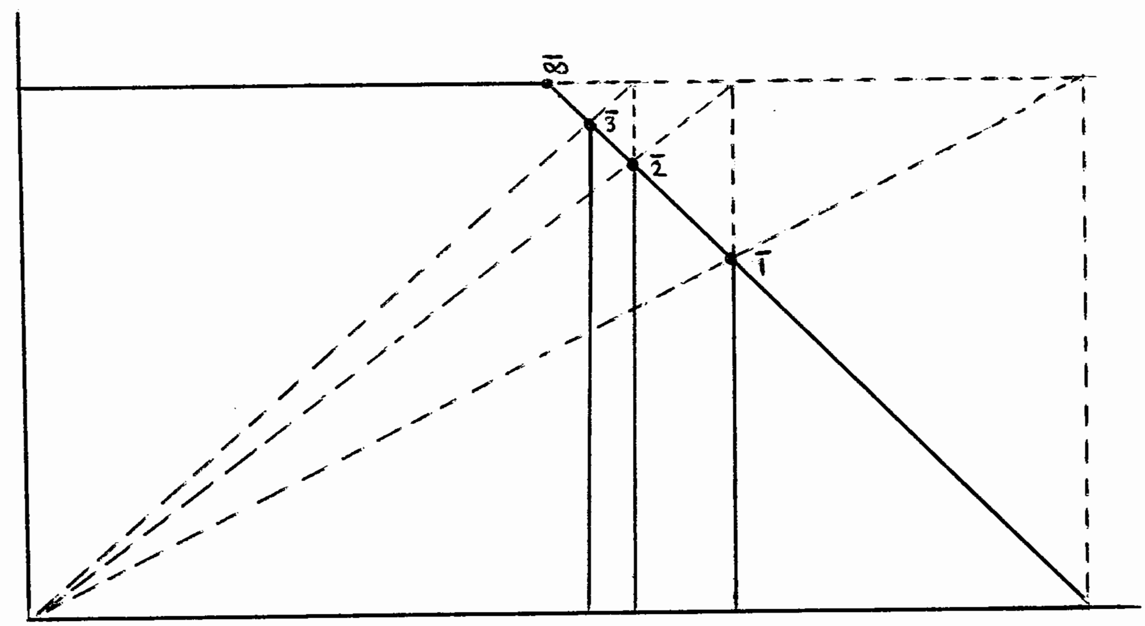

Figure 1. If S is the convex comprehensive hull of (1,1) and (2,0) and $\phi = (f^K, f^N)$, then $S^t(\phi)$ is the convex comprehensive hull of (1,1) and $(\lambda_t, 2 - \lambda_t)$ where λ_t is given by $\lambda_{t+1} = 2\lambda_t(1 + \lambda_t)$ and $\lambda_1 = 4/3$.

Key: $\bar{t} = f^K(S^t(\phi)) = (\lambda_t, 2 - \lambda_t)$, $\bar{\infty} = f^N(S) = (1,1)$.

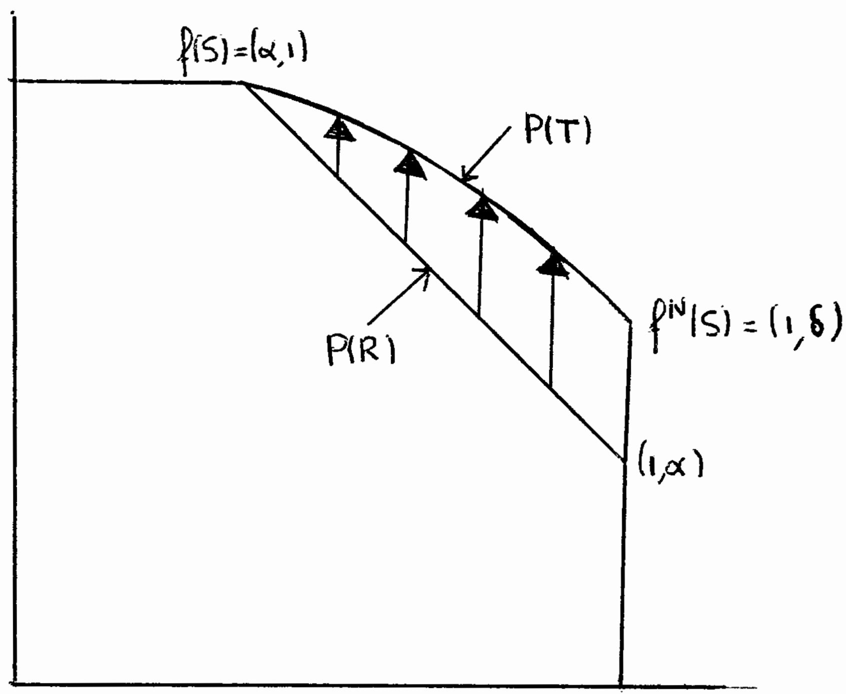

Figure 2: The sets R and T of the proof of Proposition 2.

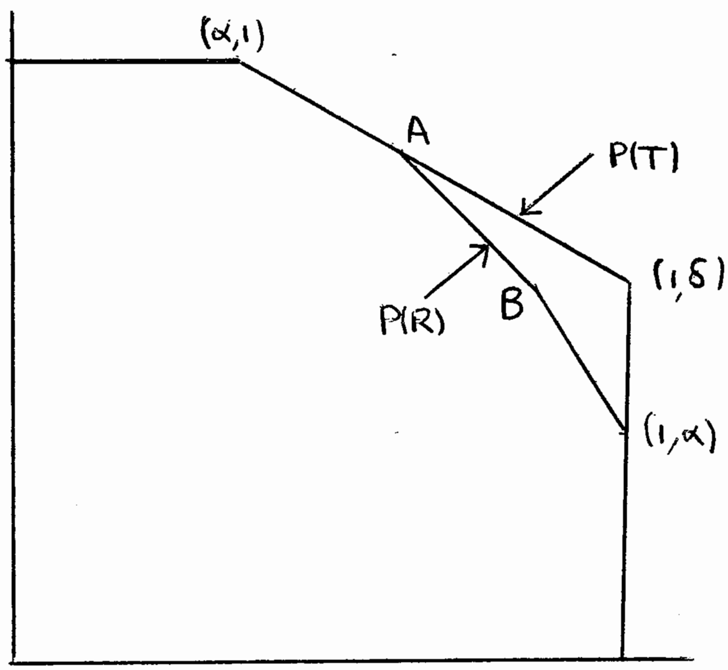

Figure 3. The sets R and T of the proof of Proposition 6. $A = (\delta, P_S^2(\delta))$, $B = (P_S^1(\delta), \delta)$.

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