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SOLUTIONS TO THE BARGAINING PROBLEM

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I. Motivation and Definition of the Problem

The bargaining problem has received considerable attention over the last several years. New axiomatic solutions have emerged and new conditions, testing the performance of these solutions, have been suggested and studied. The problem is one of a choice of a unique feasible alternative by a group of people with possibly conflicting preferences in a cooperative environment. It may be viewed as a theory of concensus, since when it is applied it is often assumed that a final choice can be made if and only if every member of the group supports this choice.

Because this theory deals with the aggregation of peoples' preferences over a set of feasible alternatives it bears close similarities to theories of social choice and the design of social welfare functions. However, there are two fundamental differences that enable us to reach a rich variety of positive results.

An important assumption made in bargaining theory which distinguishes it from other social choice theories is that there is a threat, or a disagreement, outcome. This is the outcome that would result if the bargainers fail to reach agreement. The existence of such an outcome makes the analysis of such situations easier since it enables us to start with a reference point from which comparisons of utility gains may be considered.

In bargaining theory, as in much of cooperative game theory, the physical outcomes involved in the bargaining process are ignored and only the resulting cardinal utility combinations of the players are considered. In others words, the theory assumes that any two bargaining situations are the same if they yield the same set of feasible utility combinations.

A large portion of the literature dealing with this theory concentrates on the case in which the group of players consists of only two members. For groups larger than two, this theory is irrelevant in many instances, since partial consensus, reached by intermediary coalitions (proper subsets of the entire group), may substantially effect the outcome chosen by the group as a whole. Bargaining theory deals with environments in which these partial gains are irrelevant or do not exist. However, the case of two individuals and the cases involving no profitable intermediary coalitions are important cases to solve and to gain intuition from before addressing the very general problem of solving general cooperative games. These games do allow and consider intermediary coalitions.

In this chapter we survey some of the axiomatic solutions that have been suggested in this theory and some of their properties. No attempt is made here for this survey to be complete, and some major contributions have been left out. We also made no attempt to give the strongest mathematical versions of the theorems that are presented. Our goal is to keep the presentation simple and unified, emphasizing more the later contributions. For more comprehensive coverage of this and related literature, the reader is refered to Luce-Raiffa [1957], Owen [1968], Schelling [1960], Harsanyi [1977], Binmore [1980], and Schmitz [1977]. A very comprehensive coverage of the literature can be found in Roth [1979]. For some references and results about experimental studies in bargaining we refer the reader to Rapoport-Perner [1974], Haggatt, et al. [1978], Nydegger-Owen [1975], O'Neill [1976], and Heckathorn [1978]. Comprehensive coverage of the experimental literature may be found in Roth-Malouf [1979] and Roth [1983].

Formally we describe a 2-person bargaining game by a pair (d,S) where

$d \in \mathbb{R}^2$ (the 2-dimensional Euclidean space) and $S \subseteq \mathbb{R}^2$. We assume that the pair (d,S) satisfies the following conditions.

1. $d \in S$.
2. S is compact and convex.
3. There is at least one $u \in S$ with $u > d$ ($u_i > d_i$ for i=1,2.)

We let B be the set of all bargaining games satisfying these three conditions.

The intuitive interpretation of such a pair (d,S) is the following. The elements of S, the <u>feasible set</u>, are the utility pairs that the players can receive under cooperation if they reach a unanimous agreement. The <u>disagreement point</u> d (sometimes refered to as threat point or status quo point) is the utility pair that the players have for the state of "negotiations failed, proceed without attempting to reach unanimity."

A second interpretation is that S consists of all the compromises that an arbitrator deciding the case may choose where d stands for the utilities outcome of the situation if the arbitrator was not involved.

More precise interpretations of S and of d depend on the particular situation that is being modeled. For example, when the two bargainers represent a seller and a buyer of a certain item we may let d stand for the utilities of no exchange. S then represents all the feasible utilities that arise from all the possible exchanges between them.

Often the bargaining game is used to convert a noncooperative two-person strategic form game into a cooperative one. We then let S be the convex hull of all the utility pairs that may be obtained by correlating strategies in the noncooperative game where each of the two players commits to play his specific part of the correlated strategy. The disagreement point then is the pair of utilities resulting from noncooperative (without commitment) private play, for example, a prespecified Nash equilibrium of the game.

A third type of application is for making social choices. Here we consider an organization which has to choose one state out of many feasible states. For example, S may represent the utilities arising from different agendas for running the organization, or S may represent utilities of different choices of public goods, or S may represent the individual utilities from different choices of a president, etc. A certain state is the current one, and we let d represent the utilities of the participants for being in this state. In this case we may think of d as being a status quo point. S then represents the convex hull of the utility combinations resulting from all feasible choices.

The assumption that S is convex is reasonable in many applications and certainly when the players' utilities are of the von Neumann Morgenstern (V-M) type. Convexity follows if we assume that randomizing among feasible alternatives is also a feasible choice since the V-M utilities are linear in probabilities.

There are two underlying questions motivating the study of solutions to the bargaining problem. The first type of a solution is a predictive one and attempts to answer the question of which feasible outcome would rational players arrive at on their own if commitments and signing contracts were possible. A second type of a solution is one which attempts to answer the question of which outcome should an arbitrator arbitrating the situation choose.

Our proposed solutions will be arrived at through axioms stating properties that a solution should satisfy. Thus, the relevancy of the various solutions to the question of predicting outcomes and to the question of arbitration may be tested by the reader through testing the underlying axioms against his intuition.

This axiomatic approach proves to be very useful, since it succeeds in choosing a unique solution through a small number of simple conditions. It saves us the need to get involved in the complicated process of bargaining that the players may be going through. Whatever this process is, the players will end up at our solution if our axioms are correct for their behavior. In the case of arbitration the proposed axioms give the arbitrator a rationale on which he is basing his decision.

Given a bargaining pair (d,S) and a point $u \in \mathbb{R}^2$ we say that u is <u>individually rational</u> if $u \geq d$ ($u_i \geq d_i$ for $i = 1,2$). u is <u>strongly individually rational</u> if $u > d$. We say that u is <u>Pareto optimal</u> if $u \in S$ and for every $w \in S$ if $w \geq u$ then $w = u$. We say that u is <u>weakly Pareto optimal</u> if for every $w \in S$ if $w > u$ than $w = u$.

A <u>solution</u> is a function $f: B \to \mathbb{R}^2$ such that for every $(d,S) \in B$, $f(d,S) \in S$.

II. <u>Scale-Independent Solutions</u>

The first type of solutions we discuss are ones which do not depend on the scales of the utility functions that the players use to represent their preferences. The motivation for studying scale-independent solutions stems from an implicit assumption that the utilities under consideration are of the von Neumann-Morgenstern type. Since V-M utilities are determined only up to a choice of an affine scale if our solutions were scale dependent, then they may vary arbitrarily by the arbitrary choices of scales made to represent the problem by the individuals. In later sections we will bring forth criticism of this condition.

An <u>affine transformation of player 1 utility</u> is a function $T_1: \mathbb{R}^2 \to \mathbb{R}^2$

such that for some $a > o$ and $b \in \mathbb{R}$ $T_1(u_1,u_2) = (au_1 + b, u_2)$. We similarly define affine transformations of player 2 utility.

Given a bargaining pair $(d,S) \in B$ and an affine transformation of player 1's utility T_1 we define $T_1(d,S) = (T_1(d), T_1(S))$ where $T_1(S) = \{T_1(u_1,u_2)$: for some $(u_1,u_2) \in S\}$. Similarly we define $T_2(d,S)$ for an affine transformation of 2's utility. Thus $T_1(d,S)$ is a new bargaining pair which may be viewed as the old bargaining pair (d,S) but represented by a different utility scale of player i.

We say that a solution to the bargaining problem is <u>invariant under affine transformations of utility scale</u> if for every player i, for every bargaining pair (d,S) and for every affine transformation of utility scale T_i we have

$$T_i(f(d,S)) = f(T_i(d,S)).$$

If in the definition above we consider only T_i's for which $a = 1$ then we say that the solution <u>is invariant under additive transformations in utility scales</u>. If we consider only T_i's for which $b = 0$ then we say that the solution <u>is invariant under multiplicative transformations in utility scales</u>. Clearly a solution is invariant under affine transformations of utility scales if and only if it is invariant under both additive and multiplicative transformations.

A second condition that we may want to impose on a solution is that it always chooses a Pareto optimal outcome. We say that a solution f is <u>(weakly) Pareto optimal</u> if for every $(d,S) \in B$, $f(d,S)$ is (respectively weakly) Pareto optimal. A solution f is (strongly) <u>individually rational</u> if for every $(d,S) \in B$, $f(d,S)$ is (respectively strongly) individually rational in S relative to d.

The next condition is one of symmetry. This condition guarantees that the outcome does not depend on the labeling of the players. Consider a bargaining pair (d,S) and its solution f(d,S). Suppose we now change our modeling of the bargaining situation by calling player 1 player 2 and calling player 2 player 1. Since it is basically the same problem, we expect that the players with their new labeling receive the same utility as they did with their old labeling.

Formally let $\pi: \mathbb{R}^2 \to \mathbb{R}^2$ $\pi(x,y) = (y,x)$. A solution f is called symmetric[1] if for every $(d,S) \in B$ $\pi(f(d,S)) = f(\pi(d),\pi(S))$ where $\pi(S) = \{\pi(u): u \in S\}$.

Another implication of the symmetry condition which we will discuss later is that all the relevant characteristics of the two players are described by d and S and that only information which is described by this pair (d,S) may cause us to discriminate between them. In particular if (d,S) is a symmetric problem, i.e., $d_1 = d_2$ and for every $(u_1,u_2) \in S$ we have $(u_2,u_1) \in S$, then $f_1(d,S)$ must equal to $f_2(d,S)$.

II.1 The Nash Solution

The Nash solution is the function $\eta: B \to \mathbb{R}$ which selects the individually rational utility pair with a maximal Nash product, $(u_1-d_1)(u_2-d_2)$. Formally for every bargaining pair (d,S), $\eta(d,S)$ is the individually rational utility pair with the property that for every individually rational feasible utility pair $(w_1,w_2) \in S$

$$[\eta_1(d,S) - d_1][\eta_2(d,S) - d_2] \geq (w_1 - d_1)(w_2 - d_2).$$

Thus the objective of the Nash solution is to maximize the product of the utility gains of the players. This maximization takes place over the individually rational outcomes. Since the Nash product and its square root attain their maximum at the same point we can also view the objective of the Nash solution as maximizing the geometric average of the utility gains of the bargainers.

It is easy to check that the maximum of the Nash product is attained at a unique point since the feasible set is convex. Thus $\eta(d,S)$ is a unique feasible point for every bargaining pair $(d,S)\varepsilon B$.

Our next objective is to bring forth a rationale underlying the Nash solution. We say that a solution is <u>independent of irrelevant alternatives</u> (IIA) if for every two bargaining pairs (d,S) and (d,T) with $S \subseteq T$ if $f(d,T)\varepsilon S$ then $f(d,S) = f(d,T)$.

There are two ways to view the IIA condition. Starting with the pair (d,T) and its solution $f(d,T)$, imagine that the feasible set was reduced in size to S yet the solution $f(d,T)$ is still feasible. Then we require that it remains the solution in the smaller set. Thus if $f(d,T)$ was the "best choice" among the alternatives in T, then it is still the best choice among any subset of alternatives containing it.

A mathematically analagous way of viewing the IIA condition is the following. Starting with a feasible set of alternatives S and its solutions $f(d,S)$ and assuming that some new additional alternatives become available, we require that the choice in the new set be either $f(d,S)$, the old choice, or one of the new alternatives. In other words, we do not choose a different alternative among the old ones because of the availability of additional alternatives.

We can now state Nash [1950] Theorem.

<u>Theorem 1</u>. A solution is Pareto optimal, symmetric, independent of irrelevant

alternatives, and independent of affine transformations in utility scales if and only if it is the Nash solution.

It is easy to see that the Nash solution satisfies these four properties. It is suprising that it is the only solution which satisfies them. Thus if we accept that a solution should satisfy these conditions we must adopt the Nash solution and only it as our choice.

The axiomatization presented above is the main rationalization of the Nash solution. We now present a second rational underlying the Nash solution. Here we consider only one bargaining problem at a time and we do not apply any considerations relating the solution of one bargaining pair to another as is done by the IIA condition in Nash's axiomatization. Thus with this approach we could have defined the solution to one problem without considering a solution as a function of all bargaining pairs.

When the players attempt to compromise on an alternative (u_1,u_2) as the final outcome, two considerations may arise. They may want to maximize the total combined utility gains due to their cooperation and thus maximize the sum of $(u_1 - d_1) + (u_2 - d_2)$. They may also argue for equality and desire to have $u_1 - d_1 = u_2 - d_2$. Immediately two problems come to mind. The first difficulty is that these two different objectives may not lead to the same choice of (u_1,u_2). The second difficulty is that as we have written these objectives they depend on the scale of the utility functions used to represent the players' preferences. The Nash solution turns out to be the unique way to resolve these difficulties. Let (d,S) be any bargaining pair. Can we normalize the players utilities in such a way that in the normalized utilities we are both maximizing the sum of the utility gains and preserving equality of gains? Formally consider the following problem.

Find $(u_1,u_2) \varepsilon S$ such that for some positive real numbers λ_1 and λ_2 we have

1. $\lambda_1(u_1 - d_1) = \lambda_2(u_2 - d_2)$, and
2. $\lambda_1(u_1 - d_1) + \lambda_2(u_2 - d_2) \geq \lambda_1(w_1 - d_1) + \lambda_2(w_2 - d_2)$ for every $(w_1, w_2) \epsilon S$.

The following theorem of Shapley[2] [1969] answers our question.

Theorem 2. (u_1, u_2) solves the above problem if and only if (u_1, u_2) is the Nash solution of (d,S).

Because of the IIA condition the Nash solution depends on the feasible set through a neighborhood of the solution only. Formally we say that two problems (d,S) and (d,T) agree in a neighborhood of a point $x \epsilon \mathbb{R}^2$ if there is a neighborhood 0 of x such that for every $u \epsilon 0$, $u \epsilon S$ if and only if $u \epsilon T$. We say that a solution f is local if for every $(d,S) \epsilon B$ and for every $(d,T) \epsilon B$ which agrees with (d,S) in a neighborhood of f(d,S), f(d,T) = f(d,S). It is easy to check that the Nash solution is local. Conversely, it can be shown that Theorem 1 is true with the condition that a solution be continuous and local replacing the IIA condition.

The next two solutions successively weaken the dependency of the solution on the neighborhood of only one point. The Kalai-Smorodinsky solution depends crucially on three points in the feasible set and the Maschler-Perles solution depends on the entire Pareto frontier of the individually rational portion of the feasible set.

II.2. The Kalai-Smorodinsky Solution

For every bargaining pair $(d,S) \epsilon B$ we define the ideal point I of the pair by

$$I_1 = \text{Max}\left\{u_1\text{: for some } u_2 \epsilon \mathbb{R}\ (u_1,u_2) \text{ is an individually rational feasible point in } (d,S)\right\}.$$

We define I_2 similarly. The ideal utility levels have the interpretation that they are the most that the players can hope for assuming feasibility and individual rationality of their opponents.

The Kalai-Smorodinsky (KS) solution[3] is the function μ that chooses for every bargaining pair (d,S) the unique Pareto optimal point (u_1, u_2) with $(u_1 - d_1)/(I_1 - d_1) = (u_2 - d_2)/(I_2 - d_2)$. Thus the players choose the best outcome subject to getting the same proportions of their ideal gains.

We can supply an axiomatic rationale for the KS solution as we did for the Nash solution. Here we would not accept the independence of irrelevant alternatives condition. We adopt instead a condition of individual monotonicity. This condition requires that if the feasible set is changed in favor of one of the players then this player should not end up loosing because of it. For every bargaining pair (d,T) we say that u_2 _is a rational demand for player_ 2 if there is a pair (u_1, u_2) which is feasible and individually rational in (d,S).

We say that the bargaining pair (d,W) _is better for player_ 1 _than_ the bargaining pair (d,S) if the rational demands of player 2 are the same in both pairs, and for every such rational demand u_2 we have

$$\sup\{u_1 : (u_1,u_2)\varepsilon W\} \geqslant \sup\{u_1 : (u_1,u_2)\varepsilon S\}$$

In other words with every rational demand of his opponent, player 1 can get more in W than in S.

We say that a solution f is _individually monotonic_ for player 1 if whenever (d,W) is better for him than (d,S), then $f_1(d,W) \geqslant f_1(d,S)$. f is _individually monotonic_ if the same property holds for both players.

A rationale behind the individual monotonicity condition is the following. Imagine the players facing a bargaining situation (d,S). Suppose that some additional resources are made available to player 1 as a function of his agreements with 2. Thus a new bargaining pair (d,W) is obtained in which player 1's feasible utility levels are increased. Player 1's outcome should not be made worse off than it was in the old situation.

The Nash solution does not satisfy this individual monotonicity condition. This is an immediate consequence of (see Kalai-Smorodinsky [1975]):

Theorem 3. A solution is symmetric, Pareto optimal, invarient under affine transformations of utility scale, and individually monotonic if and only if it is the Kalai-Smorodinsky solution.

II.3 The Maschler-Perles Solution

In this section we restrict our attention to a subset $B_o \subseteq B$ of bargaining pairs $(d,S) \in B$ that satisfy the following additional properties.

1. For every $x \in S$, $x \geqq d$, i.e., S consists only of individually rational outcomes.
2. Free disposal of utility, if $x \in S$ and $d \leqq y \leqq x$ then $y \in S$.
3. Existence of small utility transfers, if $d < (u_1,u_2) \in S$ then there is a pair $(v_1, v_2) \in S$ with $v_1 > u_1$ and there is a pair $(w_1, w_2) \in S$ with $w_2 > u_2$.

Conditions 1 and 2 are self-explanatory. Condition 3 requires that for every feasible utility allocation that assigns both players positive gains, each one of the players can be made better off by some small amount (possibly at the expense of his opponent). This could be accomplished for example by any small transfer of money from one player to the other. This condition is also equivalent to the strong Pareto boundary being the same as the weak

Pareto boundary of S. We denote this boundary by ∂S.

Consider a bargaining pair $(d,S)\varepsilon B_o$ and let $p = p(d,S) = (d_1, I_2(d,S))$ and $q = q(d,S) = (I_1(d,S), d_2)$, where I is the ideal point defined in the previous section.

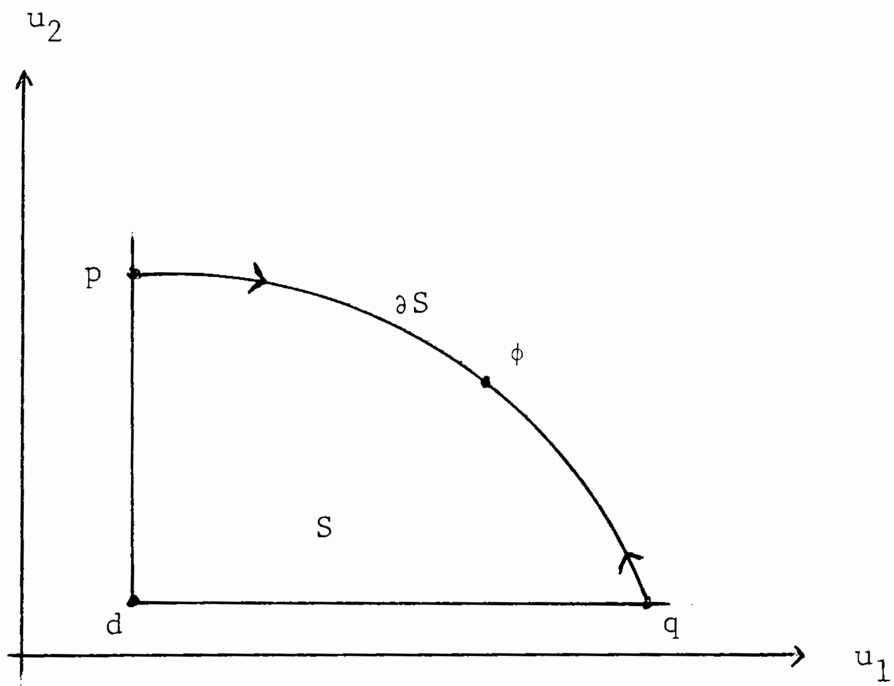

Figure 1

The Maschler-Perles (MP) solution (see Perles-Maschler [1980]) is the unique point ϕ that satisfies

$$\int_p^{\phi} \sqrt{-du_1 du_2} = \int_{\phi}^{q} \sqrt{-du_1 du_2}$$

Where these are the line integrals taken along the corresponding arcs of ∂S.

Perles and Maschler propose two intuitive procedures that yield their solution.

Procedure 1. We imagine two points moving towards each other on ∂S starting from p and q respectively. Each point moves in such a way that the products of its velocities in the u_1 and u_2 directions is a constant, say -1. ϕ is the point on the boundary where the two points meet. The above integrals are the traveling time until the points meet.

Procedure 2. The players start from d on a continuous path that would lead them to ϕ. Each point on the path may be thought of as an intermediary agreement. These intermediary agreements preserve the balance of power in the following sense. If τ is on the path and we consider the new bargaining problem (τ, S), its outcome would also be ϕ.

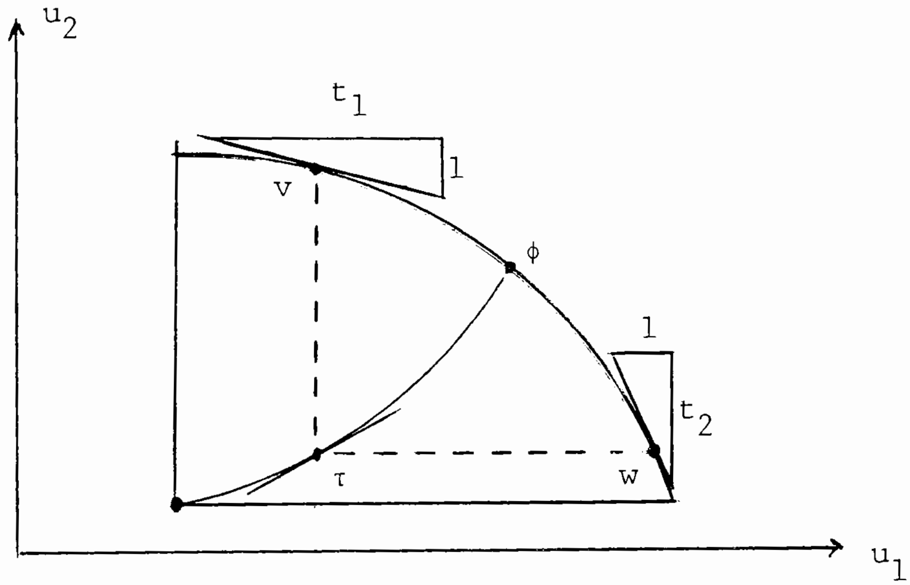

Figure 2

Consider a point τ on this status quo path and the two points v and w described by Figure 2. We let

$$t_1(v) = \sup \left\{\frac{u_1 - v_1}{v_2 - u_2} : u \varepsilon S \text{ and } u_1 > v_1\right\}$$

$$t_2(w) = \sup \left\{\frac{u_2 - w_2}{w_1 - u_1} : u \varepsilon S \text{ and } u_2 > w_2\right\}.$$

Notice that by the convexity of S it follows that the sup in the definition of

t_1 is obtained as $u \to v$. Thus t_1 is the local rate of utility gain of player 1 per unit of utility loss of 2 at v. In other words t_1 may be thought of as the local ratio of utility transfers at v. t_2 has the symmetric interpretation. The status quo path turns out to have the property that at every such τ its slope is $\sqrt{t_2/t_1}$. Thus along the status quo path the players' instantaneous ratio of utility gains equals the square root of the ratio of their per unit instantaneous exchanges at w and v.

The main rationale behind the MP solution comes from its axiomatic characterization. We say that a solution f is <u>super additive</u> if for every two bargaining pairs (d,S), $(d,T) \varepsilon B_0$ and for every λ, $0 \leq \lambda \leq 1$,

$$f(d,\lambda S + (1-\lambda)T) \geq \lambda f(d,S) + (1-\lambda)\, f(d,T),$$

where

$$\lambda S + (1-\lambda)T = \{ \lambda u + (1-\lambda)v \ : \ u \varepsilon S,\ v \varepsilon T \}.$$

The necessity of the superadditivity stems from the following argument. Suppose there is uncertainty about the bargaining problem to be played. For example it may be the case that a lottery is about to be performed in a way that with probability λ (d,S) will be played, and with probability $1-\lambda$ (d,T) will be played. Before the lottery is performed the players can make conditional agreement on what to choose in S and T under each of the two outcomes. If the players are risk neutral in the utilities (as is the case of Von Neuman-Morgenstern) then their set of feasible utilities obtained by conditional agreements is exactly $\lambda S + (1-\lambda)T$, and the bargaining problem that they face is $(d,\lambda S+(1-\lambda)T)$.

On the other hand if they do not agree on conditional outcomes before the lottery is performed, then with probability λ, the outcome would be $f(d,S)$ and with probability $(1-\lambda)$, it would be $f(d,T)$. Thus their expected outcome is $\lambda f(d,S)+(1-\lambda)f(d,T)$.

It is a reasonable game theoretic assumption that if binding agreements are possible then binding conditional agreements should also be possible in a cooperative environment. Also agreeing not to agree until after the lottery is performed is a feasible agreement. This may be thought of as the disagreement state at the stage prior to the performance of the lottery. Thus individual rationality would require that the outcome of the bargaining process at this stage should assign each player a utility not smaller than his expected utility outcome without an agreement at this stage. This is precisely the requirment of superadditivity.

A solution is <u>continuous in feasible sets</u> if for every $(d,S)\varepsilon B_o$ and every sequence $\{(d,S^i)\}_{i=1}^{\infty}$ of bargaining problems in B_o if $S^i \to S$ in the Hansdorff metric then $f(d,S^i) \to f(d,S)$.

<u>Theorem 4</u>. A solution defined on B_o is symmetric, Pareto optimal, invariant under affine transformations of utility scales, superadditive, and continuous if and only if it is the Maschler-Perles solution.

III. <u>Scale-Dependent Solutions</u>.

The scale independent condition used in the previous part is very appealing. The argument there is that it overcomes the difficulty presented by the indeterminacy of the scale in V-M utility. However while it accomplishes this task it brings about some other difficulties that make us question its validity as an undisputed axiom.

Consider for example the two bargainers 1 and 2 facing the following four possible allocations of money: (\$0,\$0) (\$10,\$0), (\$0,\$10), and (\$0,\$1000). We assume that both bargainers are selfish (their utility for these allocations depend only on their own component) and that they have a monotonically increasing utility for money. We also assume for simplicity that their utility functions have been normalized so that for $i = 1,2$ $u_i(\$0) = 0$ and $u_i(\$10) = 1$.

Now we will consider two bargaining pairs A and B. In both pairs the disagreement outcome is the (\$0,\$0) allocation resulting in the utility combination (0,0). In A the feasible set consists of all the lotteries among the three outcomes (\$0,\$0), (\$10,\$0) and (\$0,\$10). In B the feasible set consists of all the lotteries between the three outcomes (\$0,\$0), (\$10,\$0) and (\$0,\$1000). Thus the only difference between the two bargaining situations is that in B the alternative which is most attractive for player 2 was made much more attractive. Formally

$$A = \big((0,0),\ \text{Convex Hull}\ \big(\{(0,0),\ (1,0),\ (0,1)\}\big).$$

$$B = \big((0,0),\ \text{Convex Hull}\ (\{(0,0),\ (1,0),\ (0,u_2(\$1000))\})).$$

We observe that the scale of player 2 utility can be changed by representing his utility with the function $w_2 = u_2/u_2(\$1000)$ and then B is described by the bargaining pair

$$((0,0),\ \text{Convex Hull}\ (\{(0,0),\ (1,0),\ (0,1)\}))$$

Thus, with the new scale of player 2's utility, B becomes identical to A. It follows from the axiom of invariance of utility scale that whatever lottery is the solution for A it should also be the solution for B. So for example if in A the players agree on an equal probability lottery between the two outcomes (0,$10) and ($10,0) then by the scale invariance axiom they must also agree on an equal probability lottery between the two outcomes (0, $1000) and ($10,0) in B.

It is not "axiomatically" obvious to us that these two situations are identical and would or should yield the same outcome. While an equal probability lottery is probably reasonable in A (especially if the players are "similar") it seems that in B player 1 could extract better than .50 probability for his good outcome. Imagine starting the negotiations in B with the equal probability proposal. Player 2 stands to lose significantly more than player 1 if this proposal fails, and negotiations break off. Both players are aware of this fact and it seems like a threat of player 1 to break the negotiation would have significant credibility behind it. Player 2 would have to compromise the equal probability position and suggest a new probability division, one which would make player 1 happier.

The argument just presented suggests that in bargaining situations interpersonal comparisons of utility may take place. For example, this can be detected in the statement "player 2 stands to lose more than player 1." Many game theorists like to disregard theories which involve these type of interpersonal comparisons on the ground that they cannot be done using Von Neumann-Morgenstern utilities with their arbitrary scales. However if one believes that interpersonal comparisons do take place in bargaining situations then it would be a mistake to ignore them because they inconvenience us when put together with individual utility theory.

In this part of the paper, we will not rule out interpersonal comparisons by assuming invariance with respect to utility scale. We will discuss two solution concepts of this type. We will show that conflicting axioms described in the previous part stop contradicting each other (in the sense of leading to different solution concepts) when this scale invariance axiom is removed. And also new appealing axioms can be satisfied.

While interpersonal comparison of utilities is not assumed in the rationales leading to these solution concepts it does follow as a consequence of more primitive axioms. We will discuss how these solution concepts can be made useful despite their dependencies on the utility scales of the individuals.

In this part of this chapter we implicity assume that every one of our players is using the same utility function with the same scale as we vary the bargaining pairs under consideration. More precisely consider two bargaining pairs (d,S) and (d,W) in B. For $u \varepsilon S$ and $w \varepsilon W$ if $u_1 > w_1$ then our implicit assumption implies that player 1 prefers the prize that gave rise to u over the prize that gave rise to w. Notice that in our previous part this did not have to be the case since the scales of the utilities that described (d,S) might have been different from the scales describing (d,W). We feel that the implications of axioms comparing different games (such that IIA and monotonicity) are clearer with this assumption since they compare only what they intended to compare and do not involve comparisons of things that are not really comparable. Since we assume that the players' utility scales are fixed and since their choice may be arbitrary it would make no sense to assume symmetry of the solution. The issue of symmetry will be discussed later in this part.

III.1 The Utilitarian Solution

The utilitarian solution has been discussed extensively in the social welfare literature. It was axiomatized and argued for extensively by Harsanyi [1975]. Harsanyi's arguments can be easily used in the bargaining context as well. We therefore give only a very short description of the solution here.

A solution will be called <u>utilitarian</u> if there are weights $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2_+$ such that for every bargaining pair (d,S)

$$f(d,S) = \arg\max\ [\lambda_1(u_1-d_1) + \lambda_2(u_2-d_2)] = \arg\max\ [\lambda_1 u_1+\lambda_2 u_2]$$

where the maximization takes place over the pairs (u_1,u_2) which are feasible individually rational elements of S.

Notice that the arg max in the above definition may not be unique. However we can restrict our attention to bargaining pairs (d,S) in which the Pareto surface of S is strictly concave and then uniqueness is guaranteed.

The interpretation of a given utilitarian solution (for a fixed (λ_1,λ_2)) is obvious. According to it, the right solution is the one that maximizes the sum of the utility gains. However weights λ_1,λ_2 should be assigned to the utility scales of the two players, and then for every bargaining pair we would be maximizing the weighted sum of the utilities with the same (λ_1,λ_2) as weights. The question of how to determine these weights will be addressed later.

Notice that for a given utilitarian solution, with weights (λ_1,λ_2), if the scales that the players use were changed by defining new utility functions $\bar{u}_i = \lambda_i u_1$ then the objective of the utilitarian solution is to always choose the outcome that maximizes the symmetric sum of the gains in units of $\bar{u}_i$.

III.2 <u>The Egalitarian Solution</u>

In this section we restrict our discussion again to bargaining pairs belonging to B_o as described in section II.3.

A solution f is called egalitarian[4] if there are weights $\lambda_1, \lambda_2 > 0$ such that for every $(d,S) \varepsilon B$, $f(d,S)$ is Pareto optimal in S and satisfies

$$\lambda_1(f_1(d,S) - d_1) = \lambda_2(f_2(d,S) - d_2).$$

Thus an egalitarian solution is characterized by interpersonal weights λ and for every bargaining pair it chooses the highest level of utilities for the players subject to the constraint that their λ normalized gains are equal. Every choice of λ will uniquely determine an egalitarian solution and vice versa, every egalitarian solution defines uniquely the weights that it uses.

Let f be an egalitarian solution with some fixed weights λ_1 and λ_2. The reader can easily check that it satisfies the IIA condition of Nash, the individual monotonicity condition of Kalai-Smorodinsky, and the superadditivity condition of Maschler-Perles. Thus while these conditions where contradictory in the presence of the scale invariance condition they are not contradictory when this condition is removed.

We next define two weak versions of the scale invariance condition that the egalitarian solutions do satisfy. A solution f is invariant under translations if for every $a \varepsilon \mathbb{R}^2$ and every $(d,S) \varepsilon B_0$

$$f(a+d,\ a+S) = a + f(d,S).$$

A solution f is homogeneous if for every $\alpha > 0$ and every $(0,S) \varepsilon B_0$

$$f(0,\alpha S) = \alpha f(0,S).$$

The invariance under translations guarantees that if each of the players receives a fixed prize regardless of reaching agreement and independently of the bargaining process, the prize will not effect the outcome of the bargaining.

The homogeneity condition guarantees that if the feasible set S is available but only with probability[5] α (and disagreement payoff with probability $1-\alpha$) then the players would still agree on the same outcome in S as they would in the case when S was available with certainty.

A major justification of the egalitarian solution follows from an axiom of monotonicity that it satisfies. f is called monotonic if for every two bargaining pairs, (d,S) and (d,T) , if $T \supseteq S$ then $f(d,S) \leqq f(d, T)$. This condition is very appealing on normative grounds. If the opportunities of the players become greater then none of them should be made worse off. However aside from the normative issues this condition must be satisfied in many bargaining situations by strategic considerations.

Let us recall our original interpretation of a bargaining pair (d,T). The elements of T are the utility combinations that may arise from prizes provided that both players agree to support these prizes. If $S \subsetneq T$ and for player i $f_i(d,S) > f_i(d,T)$ then player i can in effect block the alternatives giving rise to the utilities in T-S and improve his outcome. Thus, viewing f as the outcome selected by an arbitrator, the monotonicity condition guarantees that none of the players will have an incentive to misrepresent his resources or to destroy some of them before coming to the arbitrator. Viewing f as a solution arrived at by the players, on their own, monotonicity follows from a general principle of individual rationality (Nash equilibrium) in the underlying noncooperative game that the players play in the process of negotiation.

Owen [1968] pointed out the incompatibility of the monotonicity condition with the utility scale invariance. However it was shown in Kalai [1977] that when scale invariance is not assumed we obtain:

Theorem 5. A solution satisfies Pareto optimality, strong individual rationality, translation invariance, homogeneity, and monotonicity if and only if it is egalitarian.

A second rationale for the egalitarian solution was proposed by Kalai [1977]. This is a condition that requires that the bargaining can be done in stages without effecting the final outcome. Mathematically, it is closely related to the monotonicity condition and it is often observed in actual processes of negotiations.

Suppose (d,S) and (d,T) are bargaining pairs with $S \subseteq T$. The players bargaining could break the process into two stages. In the first stage they would agree on an outcome in (d,S) and then use this outcome as a disagreement point for a second stage of negotiations where they may agree on a new alternative in T-S. We would like the final outcome of this two-stage process to be the same as the outcome resulting when the entire bargaining process is done in one step. A solution satisfying this property would have the appealing feature that the players would be willing to do the negotiations in stages addressing one small problem at the time. (Clearly if a solution can be decomposed into two stages as described here it can decompose into any finite number of stages by induction.)

A second appealing consequence of such a negotiation-by-stages condition is that it resolves many issues of uncertainties about future negotiations. If the players are uncertain at one point of time about which bargaining problems they may face in the future, they may hesitate in reaching agreement presently, thinking that their long-run outcome may be effected negatively by their present agreement. The condition discussed above will eliminate this type of consideration.

Formally for every (d,S), (d,T)εB_o with $S \subsetneq T$ and for a solution f we define $R = (0, (T-f(d,S)) \cap \mathbb{R}^2_+)$.

Thus R is a bargaining pair whose threat point is 0 and whose feasible points are the individually rational net gains remaining after agreeing upon f(d,S) in the first stage.

We say that a solution <u>decomposes into stages</u> if whenever (d,S), (d,T) are as described above and $R \varepsilon B_o$ then

$$f(d,T) = f(d,S) + f(R).$$

<u>Theorem 6.</u> A solution satisfies Pareto optimality, strong individual rationality, translation invariance, homogeneity, and decomposes into stages if and only if it is egalitarian.

III.3 <u>The Uses of Scale-Dependent Solutions.</u>

Scale dependent solutions may not be as useful as scale independent solutions. If we consider them as solutions to the arbitrator's problem then they do not resolve his problem in the sense that he still has to choose the appropriate λ_i's. As predictive solutions the same problem arises. How can we predict the outcome of the bargaining if we do not know the appropriate λ_i's. However they still provide us with a tremendous simplification of these two problems.

The arbitrator, in order to choose the appropriate λ_i's, can consider a simple hypothetical bargaining problem and think of the "right" outcome for it. For example he may try to think of two monetary prizes A = (\$a,\$0) and B = (\$0,\$b)) for which an equal probability lottery would be "right." He can then set $\lambda_1 = 1/u_1(A)$ and $\lambda_2 = 1/u_2(B)$ as the appropriate weights for the scale-dependent solution and proceed to solve the original problem.

A similar simplification is possible when we try to use a scale-dependent solution for prediction. If these two players have bargained before we can compute their λ_i's from their past games and use them to predict the outcome of the new game. If past games are not available we can again try to predict the outcome of a simpler hypothetical bargaining game and use the λ_i's obtained from it to predict the outcome of the game under consideration.

III.4 An Ordinal Egalitarian Solution

In this section we depart from one of the underlying assumptions of this chapter. That is, we do not assume that the utilities of the players giving rise to bargaining pairs are cardinal. We make the weaker assumption that the players' utility functions are ordinal. This means that if A is a set of possible outcomes and $u_i: A \to \mathbb{R}$ then u_i is a utility function for player i provided that $u_i(a) > u_i(b)$ if and only if player i prefers a to b. No assumption about preferences over lotteries on A, or any other assumptions giving rise to cardinal measurements, are made.

In such a situation, any other utility function $w_i: A \to \mathbb{R}$ is as meaningful as u_i provided that for every two outcomes a, b ε A

$$u_i(a) > u_i(b) \text{ if and only if } w_i(a) > w_i(b).$$

It can easily be shown that two utility functions of player i, u_i, and w_i, are equivalent in this sense if and only if there exists an order-preserving transformation $g_i: \mathbb{R} \to \mathbb{R}$ such that $w_i = g_i \circ u_i$ (for every outcome $a \in A$ $w_i(a) = g_i(u_i(a))$. What we mean by an order-preserving transformation is a strictly increasing function. Order-preserving transformations in ordinal utility theories play the same role as affine transformations (changes of scale) in the case of cardinal utilities. The idea being that if we start with a given utility function and apply an order-preserving transformation to it, we obtain a new utility function which expresses the exact same preferences as did the old one.

To ensure that we stay within the family of bargaining sets we restrict our attention to order-preserving transformations of utilities which are continuous. Thus we define an <u>order-preserving transformation of player</u> i's <u>utility</u> to be a strictly increasing continuous function $g_i: \mathbb{R} \to \mathbb{R}$.

With the above interpretations in mind, Myerson [1977] defines a notion of ordinal egalitarian solution which operates on an appropriately defined set of bargaining pairs OB. A function $f: OB \to \mathbb{R}^2$ is an <u>ordinal egalitarian solution</u> if there are two order-preserving transformations $g_i: \mathbb{R} \to \mathbb{R}$ $(i=1,2)$ such that for every bargaining pair $(d,S) \in OB$ $f(d,S)$ is Pareto optimal and

$$g_1(f_1(d,S)) - g_1(d_1) = g_2(f_2(d,S)) - g_2(d_2).$$

Thus the ordinal egalitarian solutions capture in an ordinal setup the same idea as the egalitarian solutions do in a cardinal setup. It states that there is some appropriate individual normalization of the players' utilities under which for every bargaining pair the equal division would arise. This normalization is inter- and intrapersonal. Myerson then presents an ordinal version of Theorem 6. That is, if an ordinal solution decomposes into stages in the presence of a few other natural conditions, it must be an ordinal egalitarian solution.

IV. Risk Sensitivity of Solutions

In this part we study the behavior of some of the solutions discussed earlier as one of the bargainers becomes more risk averse. We ask ourselves what happens to a player, say player one, as his opponent, player two, becomes more risk averse. One plausible expectation is that his final outcome would improve.

We break our analysis into two cases. In one case we restrict our attention to bargaining problems in which the individually rational Pareto optimal payoffs all result from lotteries among individually rational pure outcomes. In this case it turns out that indeed player one should prefer a more risk-averse opponent. This is true if we assume the Nash solution, the Kalai-Smorodinsky solution, or the Maschler-Perles solution.

The second case is where some Pareto optimal individually rational outcomes can only arise as a result of a lottery between pure outcomes which themselves are not individually rational (even though the lottery is). In this case, there is no definitive answer. Our analysis follows results of Kannai [1977], Kihlstrom-Roth-Schmeidler [1981], and Roth-Rothblum [1982].

We first motivate our notion of risk aversion comparisons based on the works of Arrow [1965], Pratt [1964], Kihlstrom and Mirman [1974] and Yaari [1969]. We consider a convex subset C of $\mathbb{R}^n$ and a von Neumann-Morgenstern utility function u defined on it (and on all the lotteries over it). For every pure outcome $c \in C$ we let

$$A_u(c) = \{m\colon m \text{ is a lottery with } u(m) \geq u(c)\}.$$

Thus $A_u(c)$ is the set of lotteries which are weakly preferred to the sure outcome c.

Now consider two utility functions u and w defined on C and which coincide on C (i.e., for every c, d ε C $u(c) > u(d)$ if and only if $w(c) > w(d)$). We say that w is more risk averse than u if for every $c \in C$ $A_w(c) \subsetneq A_u(c)$. In other words, a player with a utility function u is willing to take more lotteries instead of c than a player with a utility function w. It was shown in Khilstrom and Mirman [1974] that w is more risk averse than u if and only if there is an increasing concave function k such that $w = k \circ u$.

We consider first the set B_o of bargaining problems defined in section II.3. Motivated by the discussion above we define the following. For two bargaining problems (d,S) and $(\hat{d},\hat{S})$ in B_1 we say that player two <u>is more risk averse in</u> $(\hat{d},\hat{S})$ <u>than in</u> (d,S) if for some increasing concave function $k: \mathbb{R} \to \mathbb{R}$ $(\hat{d}_1,\hat{d}_2) = (d_1,k(d_2))$ and

$$\hat{S} = \{(u_1,k(u_2)): (u_1,u_2) \in S\}.$$

Our interpretation is that the utility function of player two that gave rise to $(\hat{d},\hat{S})$ is one which was obtained from his utility function in (d,S) but after he became more risk averse.

Given a solution to the bargaining problem f, defined on B_o, we say that <u>under</u> f, <u>player one prefers more risk averse opponents</u> if

$$f_1(d,S) \geq f_1(\hat{d},\hat{S})$$

whenever player two is more risk averse in $(\hat{d},\hat{S})$ than in (d,S).

From Kihlstrom-Roth-Schmeidler [1981] we obtain

Theorem 7. Under the Nash, Kalai-Smorodinsky, and Maschler-Perles solutions player one (and two) prefers more risk-averse opponents.

From de Koster et al. [1983], it follows that preference to bargain against risk-averse opponents follows from a property of a solution called twisting (see Thomson-Myerson [1980]).

For two bargaining problems (d,S) and (d,T) in B_o and for a solution f defined on B_o we say that T is a twist of S in favor of player one if

1. for every $(u_1,u_2) \varepsilon$ T-S $u_1 \geqslant f_1(d,S)$, and
2. for every $(u_1,u_2) \varepsilon$ S-T $u_1 \leqslant f_1(d,S)$.

An intuitive way of viewing twists is the following. Starting with the problem (d,S) and its solution f(d,S) we compare it with (d,T). Every new allocation which is feasible in T but was not feasible in S is better for 1 than the solution to S, and every allocation that was lost in the transition from S to T was worse for 1 than the solution to S.

We say that f is monotonic in favorable twists for player one if whenever (d,T) is a player one favorable twist of (d,S) then $f_1(d,T) \geqslant f_1(d,S)$.

Lemma: If f is monotonic in favorable twists for player 1 then under f player 1 prefers risk-averse opponents.

It is easy to see that both the Nash solution and the Kalai-Smorodinsky solution are monotonic in favorable twists (see Thomson-Myerson [1980]), and hence one can prove the first two thirds of Theorem 7 from the above lemma.

Once we leave the class B_o of bargaining problems then there are no longer definitive answers (as in Theorem 7) to the question of how a solution performs with regard to risk aversion. An intuitive explanation for this is that now the risk involves also the disagreement point. More specifically a certain disagreement outcome may appear more favorable to a risk-averse opponent than a lottery that involves outcomes which are worse for him than the disagreement one. Thus in a relative sense his disagreement payoff is pushed up when he becomes more risk-averse. As a result, in some instances he would have to be compensated in the final outcome for having a higher utility for disagreement. This issue is discussed in Roth-Rothblum [1982] and we illustrate it by one example.

Consider two bargaining pairs over a set of three outcomes: D, A, and B. D is the disagreement outcome. Suppose $u_1(D) = u_1(B) = 0$ and $u_1(A) = 2$. Now consider two possible utility functions, u_2 and $\hat{u}_2$ for player two. $u_2(B) = \hat{u}_2(B) = 1$. $u_2(D) = \hat{u}_2(D) = 0$, $u_2(A) = -1$ and $\hat{u}_2(A) = -4$. It is clear that under $\hat{u}_2$ player two is more risk averse than under u_2. For example, under u_2 player two is indifferent between D and the lottery L which chooses between A and B with equal probability. On the other hand, under $\hat{u}_2$ we have $\hat{u}_2(D) = 0$ and $\hat{u}_2(L) = -1.5$. If we let (d,S) and $(\hat{d},\hat{S})$ be the bargaining pairs induced by the utility functions (u_1,u_2) and $(u_1,\hat{u}_2)$ respectively, we observe the following. In (d,S) both the Nash solution and the Kalai-Smorodinsky solution assign player one an outcome with utility 1/2. In $(\hat{d},\hat{S})$ both solutions assign him a utility of 1/5. Thus, bargaining against the more risk averse $\hat{u}_2$ ended up in a loss of .3 in his final outcome.

V. Generalizations to More than Two Players

Generalizations of solution concepts and various conditions discussed earlier to the cases of more than two players have been studied extensively. The Nash solution, the utilitarian solution and the egalitarian solution generalize uniquely in a natural way. The Kalai-Smorodinsky solution can be generalized in several ways over different sets of bargaining problems. The axiomatization of the Maschler-Perles solution using the superadditivity condition cannot be generalized to n-person bargaining problems even when we make some restrictions on the class of problems[6] (see Perles [1983]).

There are two types of generalizations that have been suggested. The first type are generalizations to a fixed number n of bargainers where $n \geq 2$. The second type are ones that construct solutions that solve the bargaining problems simultaneously for any finite group of players. The second approach allows for the possibility of making comparisons and imposing consistency conditions as we vary the set of players. We will refer to the first approach as n-person generalizations and to the second one as multiperson generalizations.

V.1. n-Person Generalizations

We fix n to be a positive integer greater than 1, and we define the set of n-person bargaining pairs, B^n, in the same way as we did earlier but with $\mathbb{R}^2$ being replaced by $\mathbb{R}^n$. The definitions of individual rationality, Pareto optimality, solution, invariance under affine transformations of utility scale, symmetry, IIA, and monotonicity are all modified in the natural way by replacing $\mathbb{R}^2$ with $\mathbb{R}^n$. We also let B_1^n be the subset of B^n consisting of pairs (d,S) in which every element of S is individually rational and (d,S) allows for free disposal of utility (see sec.II.3.). We let B_o^n be the subset of B_1^n in which small transfers of utilities are possible (again, see sec.II.3).

The characterization of the Nash solution given by Theorem 1 carries over easily to the n-person case.

In general Theorem 1 and its (n-person generalization) can be strengthened by omitting the symmetry condition. An n-person nonsymmetric Nash solution is the function $f: B^n \to \mathbb{R}$ such that for some $\alpha \in \mathbb{R}^n$ with $\alpha > 0$ we have: for every $(d,S) \in B^n$ $f(d,S)$ is the unique point in the individually rational part of S which maximizes the nonsymmetric Nash product $\prod_{i=1}^{n} (u_i - d_i)\alpha_i$ among all the individually rational utility allocations in S. Clearly the (symmetric) Nash solution is the special case of the nonsymmetric ones with $\alpha = (1,1,\ldots,1)$.

Theorem 8. An n-person solution is Pareto optimal, strongly individually rational, independent of irrelevant alternative, and independent of affine transformations of utility scale if and only if it is a nonsymmetric Nash solution.

An explanation of why nonsymmetric Nash solutions may arise was given in Kalai [1977a]. Consider a 2-person bargaining problem $(d,S) \in B^2$. Now we would replicate player two to create a 3-person bargaining problem $(\bar{d},\bar{S})$ in which player three has identical interest to player two. We define

$$\bar{d} = (d_1,d_2,d_2) \quad \text{and} \quad \bar{S} = \{(u_1,u_2,u_2) : (u_1,u_2) \in S\}$$

When the (symmetric) Nash solution is applied to $(\bar{d},\bar{S})$ the solution is the argmax $(u_1 - d_1)(u_2 - d_2)(u_3 - d_3)$. But $(u_3 - d_3) = (u_2 - d_2)$ in such a replicate game. Thus, in effect $(u_1 - d_1)(u_2 - d_2)^2$ is being maximized. Now consider the following scenario. Since player two and player three have identical interest in $(\bar{d},\bar{S})$, player three gives player two a power of attorney to negotiate on his behalf. Now players one and two are playing the two-person game (d,S). If they use the symmetric 2-person Nash solution then $(u_1 - d_1)(u_2 - d_2)$ would be maximized and they would be worse off (argmax $(u_1 - d_1)(u_2 - d_2)^2$ must give them a higher payoff) than if they played the 3-

person game.

The conclusion that we can draw from this example is that under the Nash philosophy, when a negotiator is representing m players with identical interest then his weight in the Nash product must be raised to the m^{th} power. Thus all the nonsymmetric Nash solutions with α's that have integer and even rational coordinates may be explained by situations where the bargainers represent unequal size constituencies to some underlying problems. In particular, it follows that if we have one player bargaining against another one, who represents the interest of m players, the first player will have to concede more and more as m gets larger. It is not clear whether this is a reasonable phenomenon. For example, in union negotiations it would suggest that as the number of workers gets larger they may individually get better conditions for themselves. What should we predict would happen if the number of stock holders gets larger? The reader can verify that the n-person Kalai-Smorodinsky solution, defined later, is invariant to replications of this type.

The utilitarian and the egalitarian solutions also generalize to the n-person case is one natural way. The rationales justifying these solutions as demonstrated by Harsanyi [1955] and by Kalai [1977] (the analogues of Theorems 5 and 6) carry over easily to the n-person case.

Generalizations of the Kalai-Smordoinsky solution have been suggested by several authors with a variety of results. The reason for it is that the notion of individual monotonicity can be generalized in several ways. When applying one particular generalization over the bargaining problems in B^n Roth [1979a] obtains an impossibility result. However, his notion of individual monotonicity is very strong and the impossibility result relies strongly on the use of the full domain of problems in B^n. If one does not use as strong a notion of individual monotonicity and one is willing to be satisfied by solving the problems in B_o^n (accepting the availability of small utility transfers) then this difficulty disappears. Examples of such generalizations can be found in Imai [1983], Heckathorn and Carlson [1980], Segal [1980], Peters and Tijs [1982], as well as others.

Given an n-person bargaining pair $(d,S) \in B_1^n$ we define the ideal point $I \in \mathbb{R}^n$ by

$$I_i = \max\{u_i : u \text{ is an individually rational element of } S\}.$$

The n-person Kalai-Smorodinsky solution is then defined to be the unique weakly Pareto optimal point in S, μ, with the property that for every two bargainers, i and j, $(\mu_i - d_i)/(I_i - d_i) = (\mu_j - d_j)/(I_j - d_j)$. We observe that on B_1^n, μ always exists (because of the free disposal property) and that μ may be weakly and not strongly Pareto optimal for some pairs in B_1^n. Clearly this second difficulty would disappear if we restrict our attention to B_o^n.

V.2 Multiperson Generalizations

We now take a more powerful approach to the multiperson problem suggested by Thomson [1981]. We assume an infinite countable population of potential bargainers that we denote by $N = \{1,2,3,...\}$. We let F be the set of groups of players which are finite nonempty subsets of N. The idea is that the solutions we will consider should assign an outcome for every potential group which may be involved in bargaining. This approach enables us to impose consistency and rationality conditions on a solution as we vary the group of participants. We will denote groups of players from F by a capital letter with the corresponding lower case letter denoting the number of members of the group.

For every group $P \varepsilon F$ we let $\mathbb{R}^P$ denote the p-dimensional Euclidean space indexed by the members of P. For every $P \varepsilon F$ we define B^P, B_1^P and B_o^P as in the previous section to denote the set of bargaining problems in $\mathbb{R}^P$. We let B, B_1 and B_o denote the collections of the B_i^P over the P's. A solution to the multiperson problem is a collection $f = \{f^P : P \varepsilon F\}$, where each f^P is a solution to the p-person problem. We say that f is a solution on B, B_1 and B_o correspondingly to denote that each f^P is a solution on the corresponding B_i^P. We say that a solution f satisfies a given property if each of its f^P's satisfy the given property. We also define the Nash and the Kalai-Smorodinsky solution of the multiperson problem in the obvius way to be the collections of these solutions over the P's in F.

We first describe a multiperson characterization of the Nash solution suggested by Lensberg [1981]. The key condition used here is one which requires that the solution is stable with respect to negotiations by subgroups of the bargainers. More specifically let f be a multiperson solution and suppose that P and Q are groups of potential bargainers with $Q \subsetneq P$. Given a bargaining pair $(d,S) \varepsilon B^P$ we define the set

$$(S: Q,f) = \{u \varepsilon \mathbb{R}^Q : \text{for some } w \varepsilon S \text{ } w \text{ coincides with } f(d,S) \text{ on } P - Q \text{ and with } u \text{ on } Q.$$

In other words, (S: Q,f) are all the feasible allocations to the members of Q after meeting the demands of the members of P − Q according to f. For $d \varepsilon \mathbb{R}^P$ we let d^Q be the restriction of d to $\mathbb{R}^Q$. We say that f is stable for partial groups if for every two groups P and Q in F with $Q \subsetneq P$ and for every $(d,S) \varepsilon B_1^P$ if $(d^Q,(S: Q,f)) \varepsilon B_1^Q$ then $f^Q(d^Q,(S: Q,f))$ coincides with the restriction of $f^P(d,S)$ to the members of Q. The motivation for this condition is that in every subgroup of P that may be formed to renegotiate their part of the final agreement, no member has an incentive to deviate from the original agreement.

Theorem 9. A solution defined on B_1 satisfies Pareto optimality, invariance with respect to affine transformations of scales of utility, symmetry, continuity, and stability for partial groups if and only if it is the multiperson Nash solution.

A characterization of the multiperson Kalai-Smorodinsky solution was established by Thomson [1981]. Thomson motivates his bargaining problems by considering problems of fair divisions of bundles of goods among n agents. Based on the results of Billera-Bixby [1973] we know that every bargaining pair $(d,S) \varepsilon B^n$ may be viewed as a bargaining over how to divide a fixed and finite bundle of goods among n agents. The point d is the Von Neumann-Morgenstern utility image that the players have for the zero allocation which we view as the disagreement, or status quo outcome. The feasible set S represents the utilities induced by all the possible divisions of the given bundle among the n agents. Assuming monotonicity in every good, free disposal of goods, and concave utility functions, we generate bargaining problems in the class B_1^n. If we assume strict monotonicity in every good then we generate bargaining problems in B_o^n.

With this interpretation of the pairs $(d,S) \varepsilon B_o^n$ in mind we may wish to impose the following condition on a solution f. Starting with a fixed bundle, a fixed set P of agents and their utility allocation $f^P(d,S)$, assume that another agent is added to the group to share the same original bundle. Now there is one more "mouth to feed" and we should expect that none of the old agents in P should become better off because of this extra partner to the division problem.

Given two bargaining pairs $(a,S) \varepsilon B_o^Q$ and $(b,T) \varepsilon B_o^P$ with $Q \subsetneq P$ we say that (b,T) is derived from (d,S) by introducing new agents if

1. $a_i = b_i$ for every $i \varepsilon Q$, and
2. $u \varepsilon S$ if and only if there is a $w \varepsilon T$ with $u_i = w_i$ for every $i \varepsilon Q$.

We say that a solution f is monotonic in agents if for every $(a,S) \varepsilon B_o^Q$ and every $(b,T) \varepsilon B_o^P$ which is derived from (a,S) by introducing new agents we have $f_i^Q(d,S) \leqslant f_i^P(b,T)$ for every $i \varepsilon Q$.

Theorem 10. A solution defined on B_o is Pareto optimal, symmetric, invariant with respect to affine transformations of scales of utility and monotonic in agents if and only if it is the Kalai-Smorodinsky solution.

A multiperson solution f is egalitarian if for some sequence of positive numbers $\{\alpha_i\}_{i \varepsilon N}$, for every group of players $P \varepsilon F$ f^P is the p-person egalitarian solution with weights $\{\alpha_i\}_{i \varepsilon P}$. If all the α_i's are 1 then f is called the symmetric egalitarian solution. We can now present another characterization of the symmetric egalitarian solution due to Thomson [1982].

Theorem 11. A multiperson solution defined on B_1 is weakly pareto optimal, independent of irrelevant alternatives, monotonic in agents, continuous, and symmetric if and only if it is the symmetric egalitarian solution.

NOTES

[1]Some authors refer to this condition as anonomity. They use the term symmetry for solutions which choose a symmetric outcome on symmetric problems.

[2]There have been several generalizations of the Nash solution to general n-person characteristic function games in which this characterization plays a crucial role. An axiomatic generalization of this type was just developed by Aumann [1983].

[3]There has been some confusion in the literature between this solution and a procedure suggested by Raiffa as an "ad hoc method" to do interpersonal comparisons of utilities for a given game (d,S). One of Raiffa's suggestions in Luce and Raiffa [1957] is to let $b = (b_1,b_2)$ be the highest levels of utilities in S that may be attained individually by the players, and $w = (w_1,w_2)$ be the lowest individual utilities that may occur in S. The players then choose the Pareto optimal outcome $\bar{u}$ such that the line segment $(d,\bar{u})$ parallels the line segment (w,b).

[4]These solutions were originally introduced by Kalai [1977] under the name proportional solutions.

[5]Allowing for $\alpha > 1$ will follow because if $\alpha > 1$ let $W = \alpha S$ then $S = (1/\alpha)W$ and the same conclusion follows.

[6]It is quite possible however that an n-person generalization of the second procedure in the description of the Maschler-Perles solution is possible.

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