

DISCUSSION PAPER NO. 55

A NON-THEOREM OF SAMUELSON IN  
THE THEORY OF REVEALED PREFERENCE

by

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October 25, 1973

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1. INTRODUCTION

In "A Note on the Pure Theory of Consumer's Behavior" [1], Samuelson showed how the theory could be founded on three postulates: (I) demand is single valued as a function of the prices and income; (II) the demand function is homogenous of order zero in the variables prices and income; and (III) the individual always behaves consistently in the sense that his behavior conforms to the (Weak) Axiom of Revealed Preference. Subsequently, (in [2],) Samuelson claimed that this postulational base is redundant in the sense that postulates I and II can be deduced as theorems from postulate III alone and may, therefore, be omitted, so that the single postulate III provides complete foundations for the pure theory of consumer's behavior (excepting some reservations concerning integrability -- cf. [3] and p. 111 in [4]). A simple "proof" of this claim was presented in [2].

Although examples of preference fields obeying both III and II but not I can readily be constructed (see 3.3 below), thus

showing that Samuelson's claim is false, it has never been explicitly shown as to where exactly resides the fallacy in Samuelson's argument. Its validity remaining unquestioned for all these (thirty-five) years, the proof continues to appear intact in "Foundations of Economic Analysis" to this day (see pp. 110-112, 1972 reprint of [4]).

In this paper, we present a simple counter example to show that Samuelson's claim that I is a theorem of III is false, and trace the fallacy in his argument to an error in his use of the formal definition of the revealed preference relation. When this error is corrected, we find that the argument fails to establish I and II as theorems of the postulate III taken singly. We are, however, able to obtain II as a theorem of I and III taken together as postulates. Thus, in view of the counter example, we conclude that complete foundations for the pure theory of consumer's behavior is provided if both I and III are taken as postulates, and not if only III is taken as a postulate.

Toward this end, Section 2 first attends to some needed preliminaries concerning notation, definition of the revealed preference relation and statement of the (Weak) Axiom of Revealed Preference, and then recapitulates the essence of Samuelson's argument (cf. pp. 111-112 in [4]). Section 3 is devoted to an exposition of the fallacy in the argument; Section 4 presents its corrected version; and Section 5 summarizes our conclusions.

## 2. REVEALED PREFERENCE

We consider the case of a consumer, to whom his income (or expenditure budget), and the prices of goods are specified in advance, and these are assumed to remain unaffected by the choice which the consumer makes.

2.1 Notation: For some positive integer  $n$ , let  $X$  be an  $n$ -dimensional Euclidean space, with  $x \in X$  being generic.

We let

$n$  = Number of goods, indexed by  $1, \dots, n$ ;

$x = (x_1, \dots, x_n)$  = A commodity bundle, where  $x_i$  is the quantity of the  $i^{\text{th}}$  good ( $i = 1, \dots, n$ );

$p = (p_1, \dots, p_n)$  = Price vector, where  $p_i$  is the price of the  $i^{\text{th}}$  good ( $i = 1, \dots, n$ );

$px = \sum_{i=1}^n p_i x_i$  = Total cost of the commodity bundle  $x$  at prices  $p$ ;

$I$  = Income (or expenditure budget) of the consumer;

$\Psi$  = Demand correspondence, where  $\Psi(p, I) \subset X$  is the set of commodity bundles "demanded" by the consumer in the (price, income) situation  $(p, I)$ , subject to  $x \in \Psi(p, I) \Rightarrow px = I$ .

2.2 Definition of Revealed Preference Relation: "If at the price and income situation A you could have bought the goods actually bought at a different point B and if you actually chose not to, then [the bundle of goods demanded at] A is defined to be "revealed to be better than" [the bundle of goods demanded at] B." (cf. p. 370, [3])

Denoting the 'revealed to be better than' relation by  $\Theta$ , and using superscripts a and b to refer to the situations A and B respectively, Samuelson transcribes the above definition into a formal statement as follows (cf. 5.11 - 12, [1]): Let  $x^a \in \Psi(p^a, I^a)$  and  $x^b \in \Psi(p^b, I^b)$ . Then

$$p^a x^a \geq p^a x^b \Rightarrow x^a \Theta x^b \quad - - - - (1)$$

2.3 The (Weak) Axiom of Revealed Preference: "The individual behaves consistently in the sense that he never "prefers" a first batch of goods to a second at the same time that he "prefers" the second to the first." [2]

Denoting the complement of the relation  $\Theta$  by  $\not\Theta$ , Samuelson transcribes the above statement into the formal axiom (cf. 6.01 - 6.02 [1]):

$$x^a \Theta x^b \Rightarrow x^b \not\Theta x^a \quad - - - - (2)$$

This, using (1) above, gives (cf. 5.11 - 5.32 [1]):

$$p^a x^a \geq p^a x^b \Rightarrow p^b x^a > p^b x^b \quad - - - - (3)$$

2.4 Samuelson's Claim: If the individual's demand behavior conforms to the (Weak) Axiom of Revealed Preference, then (i) the demand is a single valued function, and (ii) this function is homogenous of degree zero, in the variables prices and income.

2.5 Samuelson's Proof: ad (ii): Consider two price, income situations  $(p^\circ, I^\circ)$  and  $(p^1, I^1) = (mp^\circ, mI^\circ)$ , where  $m$  is any positive real number, and let  $x^\circ \in \Psi(p^\circ, I^\circ)$  and  $x^1 \in \Psi(p^1, I^1)$  be arbitrary. To establish the claim by way of contradiction, assume that  $x^1$  and  $x^\circ$  are different, i.e.,  $x^1 \neq x^\circ$ . Now,

$$p^\circ x^\circ = I^\circ, \text{ so that } mp^\circ x^\circ = p^1 x^\circ = mI^\circ \quad - - - - (4)$$

$$\text{Also, } p^1 x^1 = mp^\circ x^1 = I^1 = mI^\circ, \text{ so that } p^\circ x^1 = I^\circ \quad - - - (5)$$

$$\text{Thus, } p^\circ x^\circ = p^\circ x^1 \text{ and } p^1 x^\circ = p^1 x^1 \quad - - - - (6)$$

This is a contradiction, for using (3),

$$p^\circ x^\circ = p^\circ x^1 \Rightarrow p^1 x^\circ > p^1 x^1 \quad - - - - (7)$$

Hence,  $x^1$  and  $x^\circ$  cannot be different, and so  $x^1 = x^\circ$ .

ad (i) To see this, merely take the special case of  $m = 1$ .

# (!)

### 3. THE FALLACY IN SAMUELSON'S ARGUMENT

We are now ready to argue that Samuelson's proof 2.5 is invalid. After showing, in 3.1 and 3.2 below, exactly where the fallacious argument is made in the proof, we will give a simple counter-example to show that the claim in 2.4 (i) is false. (We will not need to establish whether 2.4 (ii) also is false.)

- 3.1 The Fallacy in 2.5 (ad (ii)): The claim is that if demand behavior conforms to 2.3, then  $\Psi$  is homogenous of degree zero in  $p$  and  $I$ , i.e., denoting  $\Psi^{\circ} = \Psi(p^{\circ}, I^{\circ})$  and  $\Psi^1 = \Psi(p^1, I^1) = \Psi(mp^{\circ}, mI^{\circ})$ , that  $\Psi^{\circ} = \Psi^1$ . To prove this claim by way of contradiction, Samuelson denies the claim by taking arbitrary  $x^{\circ} \in \Psi^{\circ}$  and  $x^1 \in \Psi^1$  and assuming that  $x^1$  is distinct from  $x^{\circ}$ , i.e.,  $x^1 \neq x^{\circ}$ . This denial is incorrect, for the correct negation of the claim is the set inequality  $\Psi^1 \neq \Psi^{\circ}$ . Thus, what Samuelson should assume (as a contradictory supposition) is the disjunction

$$\Psi^1 \not\subset \Psi^{\circ} \quad \underline{\text{or}} \quad \Psi^{\circ} \not\subset \Psi^1 \quad - - - - (8)$$

Notice that, if  $\Psi$  were known to be single valued, setting  $\Psi^{\circ} = \{x^{\circ}\}$  and  $\Psi^1 = \{x^1\}$ , would reduce (8) to the inequality  $x^1 \neq x^{\circ}$ , which is what Samuelson would have us assume when it is not known whether  $\Psi$  is single valued!

Since the contradiction constructed in 2.5 (ad (ii)) is based on an incorrectly asserted denial of the claim 2.4(ii),

it is invalid to conclude therefrom that 2.4 (ii) must, therefore, be true.

3.2 The Fallacy in 2.5 (ad (i)): The claim is that if the demand behavior conforms to 2.3, then  $\Psi$  is single valued, i.e., that  $x, x' \in \Psi^\circ$  implies  $x = x'$ . Samuelson seeks to establish this from 2.5 (ad (ii)) by setting  $m = 1$ . Explicitly, then, his argument runs as follows: Let  $x, x' \in \Psi^\circ$ , and assume  $x \neq x'$ . Now,  $p^\circ x = p^\circ x'$ , which, using (3) implies  $p^\circ x > p^\circ x'$ ; this is a contradiction, hence  $x = x'$ !

This argument is fallacious on two counts.

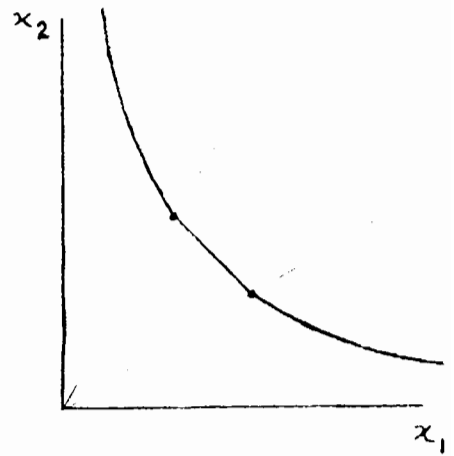
3.2.1 Firstly, the proof proceeds by setting  $m = 1$ , i.e., by setting  $(p^\circ, I^\circ) \equiv (mp^\circ, mI^\circ)$ . This violates the definitional requirement (see 2.2) that in (3) the two price, income situations A and B must be distinct. For this reason, when  $m = 1$ , there is no guarantee that (3) holds, and the argument then that  $p^\circ x = p^\circ x'$  implies  $p^\circ x > p^\circ x'$  is fallacious (cf. 4.3 below).

3.2.2 Secondly, according to Definition 2.2, we may affirm  $x \otimes x'$  if  $x$  is demanded at  $(p^\circ, I^\circ)$ , and if  $x'$  could have been demanded but is not demanded. Since, however, both  $x$  and  $x'$  are assumed to have been demanded at  $(p^\circ, I^\circ)$ , we cannot affirm  $x \otimes x'$  or  $x' \otimes x$ , so we cannot invoke (3). Thus, the argument that  $p^\circ x = p^\circ x'$  implies

$p^\circ x > p^\circ x'$  is fallacious.

The following is a counter-example to show that the (Weak) Axiom of Revealed Preference (even with the demand being homogenous of order zero in the variables prices and income) fails to guarantee that demand is single valued:

- 3.3 Consider the two commodity case for a consumer whose indifference curves are hyperbolae everywhere in the first quadrant, except between two rays from the origin, in which region they are chords connecting the points where the two rays intercept each of the hyperbolae.



4. THE CORRECTED VERSION

In order to avoid fallacious argument, the formal enunciation (1) of Definition 2.2 of the revealed preference relation needs to be stated rigorously. The definition says that  $x^a \ominus x^b$  holds if (i)  $x^b$  can be demanded at  $A$  (i.e.,  $p^a x^b \leq p^a x^a$ ); and if (ii)  $x^b$  is actually not demanded at  $A$  (i.e.,  $x^b \notin \Psi(p^a, I^a)$ ). Notice that, of these two conditions, the second one is neither incorporated implicitly in the notation, nor is it stipulated explicitly in the formal expression (1).

4.1 Using the notation of 2.2, and denoting  $\Psi^a = \Psi(p^a, I^a)$  and  $\Psi^b = \Psi(p^b, I^b)$ , the formal expression introducing  $\ominus$  should, therefore, read as follows:

$$(p^a x^a \geq p^a x^b) \ \& \ (x^b \notin \Psi^a) \Rightarrow x^a \ominus x^b \quad - - - - (9)$$

4.2 Then the (Weak) Axiom of Revealed Preference (see 2.3) yields the following implication:

$$\begin{aligned} (p^a x^a \geq p^a x^b) \ \& \ (x^b \notin \Psi^a) \Rightarrow x^a \ominus x^b \\ \Rightarrow x^b \ominus x^a \Rightarrow (p^b x^a > p^b x^b) \ \underline{\text{or}} \ (x^a \in \Psi^b) \quad - - - - (10) \end{aligned}$$

4.3 When the axiom is stated in this manner, it is unnecessary to require that  $(p^b, I^b)$  and  $(p^a, I^a)$  be distinct: For consider the case  $(p^b, I^b) = (p^a, I^a)$ , so

that  $\Psi^b = \Psi^a$ ; then, (10) reduces to the (trivially) true statement that  $p^a x^a = p^a x^b$  implies  $x^a \in \Psi^b = \Psi^a$ .

It is now easy to see that the correct version of the argument in 2.5 fails to construct a contradiction from the denial of 2.4:

4.4 Consider two price, income situations  $(p^\circ, I^\circ)$  and  $(p^1, I^1) = (mp^\circ, mI^\circ)$ , where  $m$  is any positive real number, and let  $x^\circ \in \Psi^\circ = \Psi(p^\circ, I^\circ)$  and  $x^1 \in \Psi^1 = \Psi(p^1, I^1)$  be arbitrary. Then, as in 2.5,

$$p^\circ x^\circ = p^\circ x^1 \quad \text{and} \quad p^1 x^\circ = p^1 x^1 \quad \text{--- (6)}$$

4.4.1 Concerning  $\Psi$  being homogenous: If  $m = 1$ , then  $\Psi^\circ = \Psi^1$  trivially, and there is nothing to show. So let  $m \neq 1$  and assume  $\Psi^\circ \neq \Psi^1$ . Without loss of generality, let  $x^1 \in \Psi^1$  be such that  $x^1 \notin \Psi^\circ$ . Take any  $x^\circ \in \Psi^\circ$ ; then

$$(p^\circ x^\circ = p^\circ x^1) \ \& \ (x^1 \notin \Psi^\circ) \\ \Rightarrow (p^1 x^\circ > p^1 x^1) \ \underline{\text{or}} \ (x^\circ \in \Psi^1) \quad \text{--- (11)}$$

Since, by (6),  $p^1 x^\circ = p^1 x^1$ , we have  $x^\circ \in \Psi^1$ , and this is true for any  $x^\circ \in \Psi^\circ$ . Thus,

$$\Psi^1 \not\subset \Psi^\circ \Rightarrow \Psi^\circ \subset \Psi^1 \quad \text{--- (12)}$$

Equivalently,

$$\Psi^1 \subset \Psi^0 \quad \underline{\text{or}} \quad \Psi^0 \subset \Psi^1 \quad - - - - (13)$$

4.4.2 Concerning  $\Psi$  being single valued: Let  $x, x^1 \in \Psi^0$ , and assume  $x \neq x^1$ . Then, by (10),  $p^0 x = p^0 x^1$  implies  $x \in \Psi^0$  &  $x^1 \in \Psi^0$ , which is (trivially) true.

Lastly, we state and prove the following

4.5 Theorem: If demand  $\Psi$  is a single valued function of variables  $p$  and  $I$ , and if Axiom 2.3 holds, then  $\Psi$  is homogenous of degree zero in the variables  $p$  and  $I$ .

Proof: Assume the hypothesis and let everything be as in 4.4. If  $m = 1$ ,  $\Psi^0 = \Psi^1$  trivially, and there is nothing to show. So assume  $m \neq 1$ . Then (13) holds, and since  $\Psi$  is single valued,  $\Psi^0$ , in fact, equals  $\Psi^1$ .

#

## 5. CONCLUSION

We showed that Samuelson's argument in [2] is fallacious and that, taken singly, the (Weak) Axiom of Revealed Preference fails to provide complete foundations for the pure theory of consumer's behavior. The fallacy in the said argument was traced to result from taking (1), instead of (9), as (formally) defining the revealed preference relation. The fundamental entailment of the (Weak) Axiom of Revealed Preference was shown to be (10) and not (3). Finally, we showed that complete foundations for the pure theory of consumer's behavior are provided if both, (I) single valuedness of  $\Psi$ , and (III) the (Weak) Axiom of Revealed preference are taken as the postulates of the theory.