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UNANIMITY GAMES AND PARETO OPTIMALITY*

by

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1. Introduction and Summary of Results:

A group of players make one choice from a set of feasible states. This situation occurs frequently in real life and is addressed by economic theorists, game theorists and political scientists.

When the participants try to deal with the situation individually and selfishly, without the aid of some social mechanism or an arbitrator, inefficient outcomes often result. One observes this type of phenomena in strikes, wars, excessive competition between individuals and free rider type of situations. The classical game theoretic example is the prisoners' dilemma game where the only equilibrium that may be chosen noncooperatively by the group is inefficient.

When the situation is repetitive and is analyzed as an infinitely repeated game, the problem becomes less severe (see Aumann [1981]). One observes new equilibrium outcomes which are group efficient, but there are still many inefficient ones.

Purely cooperative game theory assumes that a group efficient outcome would, or should, result in such situations. This assumption is motivated by the observation that in real life binding agreements and contracts can be signed by players. However, the process describing how the group of players reaches this final agreement has not been explicitly studied. Clearly, any such process must give rise to a noncooperative game in which every player wishes to maximize his utility from the final outcome. Given that such a game is played, it is not clear that a group efficient outcome would necessarily result.

In this paper we study a noncooperative process called a unanimity game. The outcomes of this game are binding contracts that the players commit to. We show that under certain individual behavioral asumptions, group efficient outcomes would necessarily result.

The environment we deal with is described as follows. There is a finite group of n (n>0) players denoted by $N = \{1,2,...,n\}$. There is a finite nonempty set of states, C. There is one distinct state, denoted by d, in C which is referred to as the status quo state. Our motivating intuition is that the group attempts to agree (sign a binding contract) on transforming into one of the states in C. If no such agreement is reached, then they would continue in the state d. Thus, d may be thought of as the state "procedure failed".

Examples of such environments are numerous. In an economic context, one may view each state as a full description of the variables of the economy, such as trades and production allocations. The status quo state is the current state of the economy. The efficient outcomes that our procedure yield are the Pareto optimal states.

In a social choice context, our procedure may be viewed as one implementing the unanimity social welfare function (see Sen [1970]). Here, the states may describe possible candidates to occupy an office or possible social preferences. The status quo is the prevailing state. Our procedure will end up with a state which is maximal with respect to the unanimity social welfare function.

In using our procedure to overcome lack of efficiency exhibited by noncooperative games, we assume that each state is a choice of a joint correlated strategy of the n-players. The status quo state stands for "play the noncooperative game without the use of the procedure".

We assume that every player $i \varepsilon N$ has a Von-Neumann Morgenstern utility function u_i: $C \to R$ and we denote by u the vector of the utility functions. Without loss of generality, we assume that every u_i has been normalized so that $u(d) = 0$. We also assume that there exists at least one uniformly positive state, i.e., there is a state $c \varepsilon C$ such that for every player $i \varepsilon N$, $u_i(c) > 0$. We say that a utility allocation vector $w = (w_1,w_2,\ldots,w_n)$ is strongly individually rational if $w > 0$ (i.e., if $w_i > 0$ for $i=1,2,\ldots,n$). We say that w is weakly Pareto optimal if there is no state $c \varepsilon C$ with $u(c) = (u_1(c), u_2(c),\ldots,u_n(c)) > w$.

Our unanimity game is described as follows. There is a nonnegative integer T which describes the number of attempts the group makes in order to reach unanimity. This number is exogenously given and is publicly known to the players prior to the beginning of the game. We define the extensive form of the unanimity game inductively on T as follows.

If $T = 0$, the game has no moves by any of the players and the outcome is the status quo state d. For $T > 0$, the following game is played. In the first attempt, simultaniously, every player $i \varepsilon N$ chooses a state $c_i \varepsilon C$. These choices are publicly announced. If all the c_i's coincide, i.e., $c_i = c_j$ for every two players i and j, then we say that an agreement was reached. This agreed upon state is then the outcome of the game. However, if no agreement was reached in this first attempt, then the players proceed to play the T-1 attempts unanimity game.

The one attempt unanimity game (T=1) has been studied before in the game theory literature (see Harsanyi [1981]). These games have the nice mathematical structure that in the normal matrix description of the game all the "of the diagonal" entries of the matrix are zero. It turns out that every uniformly positive "on the diagonal" entry in this game is a Nash equilibrium. However, it turns out that there are also (mixed and not mixed) equilibrium strategies resulting in off the diagonal zero payoffs.

When studying the behavior according to more sophisticated equilibrium notions (Selten's [1975] perfect equilibrium, Myerson's [1978] proper equilibrium and hence also Kreps-Wilson's [1980] sequential equilibria) this phenomenon still occurs. That is, while there may be uniformly positive diagonal entries in the game, according to these equilibrium notions, the players may end up with a zero gain.

The notion of persistent equilibrium (see Kalai-Samet [1983]) is different. It was shown to pick only the uniformly positive diagonal entries as persistent equilibria. In this paper we study the use of the persistent equilibrium notion in the case of T-attempts unanimity games where $T \geq 1$.

We show that if the number of attempts is sufficiently large (for example if $T \geq |C|$) then the outcome resulting from every persistent equilibrium is weakly Pareto optimal (provided that it satisfies some symmetry condition). We show also the converse, that every state which is weakly Pareto optimal and strongly individually rational (relative to the status quo state) is a possible outcome of some persistent equilibrium. Thus, the unanimity games proposed here may serve as a mechanism to reach Pareto optimality without distinguishing among the Pareto optimal outcomes.

In the next two sections, we give a formal description of the results discussed above and their proofs. This includes a formal definition of persistent equilibrium for extensive form games and some of its properties.

In the last section, we give a modified version of the unanimity game which we refer to as a nonterminating unanimity game. The basic difference is that in the nonterminating version, if an agreement is reached at any stage of the game, the game does not terminate, but continues with the agreement state, serving as a new status quo state for the next iterations. It can be shown that the nonterminating unanimity game possesses the same properties regarding group efficiency and individual rationality as the regular unanimity game.

2. Formal Description of the Unanimity Game

For a given N, C, and T, as described in the previous section, we let U(T) denote the T attempts unanimity game described there. We first proceed to develop the formal notations for the description of the strategies in T.

We call an n-tuple of states $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ a proposal (combination). If a proposal γ is of the form $\gamma = (c, c, \ldots, c)$ then we call it an agreement on c. We denote such an agreement by c^n; otherwise we call it a disagreement proposal. For a positive integer ℓ a history of length ℓ is defined to be an ℓ-tuple $h = (\gamma^1, \gamma^2, \ldots, \gamma^\ell)$ where every γ^t is a disagreement proposal. We denote the set of all histories of length ℓ by H^ℓ. H^0 is defined to be $\{0\}$ where 0 is defined to be the only history of length zero. The set of all histories in U(T) is defined by $H = H(T) = \bigcup_{\ell=0}^{T-1} H^\ell$. For every history $h \in H$ we define $\ell(h)$ to be its length.

We let $\Delta(C)$ be the set of probability distributions over C. Thus $\sigma \in \Delta(C)$ means that $0 \leq \sigma(c) \leq 1$ for every $c \in C$, and $\sum_{c \in C} \sigma(c) = 1$. With some abuse of notations we sometimes let c stand for the probability distribution on C which puts its entire mass on c. In the next two paragraphs we describe the strategies and payoff structure of the game.

Strategies: An (individual) strategy of a player i at h in U(T) is a point in $\Delta(C)$. A strategy of i is a function $s_i : H \to \Delta(C)$. Thus at every history the player has to decide how to randomize over the states that he may submit for the next proposal. This is what would be considered a behavioral strategy in the usual theory of extensive form games. A strategy combination is a vector $s = (s_1, s_2, \ldots, s_n)$ consisting of n individual strategies.

Utilities: It was assumed in the previous section that every player i has a utility function u_i: $C \to R$. The utility of a player for a lottery in $\Delta(C)$ is computed in the usual way. To extend the utility definitions to the entire game we first define the probability p(h,s) induced on history h by a given strategy combination s. If h is the history of length 0 then p(0,s) = 1, that is, the game will start with probability 1. Now we assume for induction on the length of histories that for all histories of length t, p(h,s) has been defined. Let $h' = (h,\gamma^{t+1})$ (the first t components of h' are those of h and the last one is γ^{t+1}) be a history of length t+1. We define

$$p(h',s) = p(h) \prod_{i=1}^{n} s_i(h)(\gamma_i^{t+1}).$$

Thus, $p(h',s)$ is the probability of arriving at h times the product of the probabilities assigned by the players to their corresponding parts in the proposal γ^{t+1} at the history h.

Now we define p(c,s), the probability that a state c be the outcome of the unanimity game for a given strategy combination s:

$$p(c,s) = \sum_{h \in H} p(h,s) \prod_{i=1}^{n} s_i(h)(c).$$

It is now easy to extend the utility functions u_i from $\Delta(C)$ to the space of strategy combinations in U(T):

$$u_i(s) = \sum_{c \in C} p(c,s) u_i(c).$$

3. Equilibrium Analysis

We say that a strategy combination $s = (s_1, s_2, \ldots, s_n)$ is a Nash equilibrium if for every player i and for every individual strategy of player i, $\bar{s}_i$, we have

$$u_i(s) \geq u_i(s_1, \ldots, s_{i-1}, \bar{s}_i, s_{i+1}, \ldots, s_n).$$

When this inequality holds we also say that s_i is a best reply of player i to s.

In what follows we would also need a notion of local best reply, that is, a best reply after a history h has been observed and assuming that the game will proceed according to a specified strategy combination s. For a player i, an individual strategy s_i, history h, and $\sigma \in \Delta(C)$ we define $(s_i|h,\sigma)$ by

$$(s_i|h,\sigma)(h) = \sigma \text{ and for every } \bar{h} \neq h \text{ and } (s_i|h,\sigma)(\bar{h}) = s_i(\bar{h}).$$

Thus the strategy $(s_i|h,\sigma)$ is the same as s_i except that after the history h player i plays σ.

The best reply set of player i to s at h is defined now by

$$BR_i(h,s) = \underset{\sigma}{\operatorname{argmax}}\ u_i(s_1, \ldots s_{i-1}, (s_i|h,\sigma), s_{i+1}, \ldots, s_n).$$

We proceed now to define the notion of a persistent strategy in the T-attempts unanimity game. We use for this purpose the definition given by Kalai and Samet of persistency for games in normal form and we apply it to the agent normal form of the game. (This development is analogous to the definition of perfect equilibrium (Selten [1975]) of a tree game as the pefect equilibrium of the normal agent form of the tree.)

An (individual) retract of player i's strategies is a correspondence R_i: $H \twoheadrightarrow \Delta(C)$ with nonempty convex and closed values. We say that $s_i \in R_i$ if $s_i(h) \in R_i(h)$ for every $h \in H$. $R = (R_1, R_2, \ldots, R_n)$ is a retract combination if it consists of n individual retracts. A strategy combination s belongs to the retract combination R if for every player i, $s_i \in R_i$. A neighborhood 0 of a retract combination R is an open valued correspondence 0: $H \twoheadrightarrow \Delta(C)$ with $R(h) \subseteq 0(h)$ for every $h \in H$.

A retract R is absorbing if it has a neighborhood 0 with the following property. For every strategy $s \in 0$ (i.e., $s(h) \in 0(h)$ for every $h \in H$) every history h and every player i there is a $\sigma \in R_i(h)$ such that $\sigma \in BR_i(h,s)$. That is, the agent of player i playing at history h can find a best reply to s within $R_i(h)$.

For two individual retracts R_i and T_i we say that $R_i \subseteq T_i$ if $R_i(H) \subseteq T_i(h)$ for every history h. given two retract combinations R and T we say that $R \subseteq T$ if $R_i \subseteq T_i$ for every player i.

A retract is called persistent if it is a minimal absorbing retract, i.e., it does not properly contain another absorbing retract. A strategy combination s is persistent if it belongs to some persistent retract.

In the unanimity game U(T) after every history h the remaining game is a new unamity game $U(T-\ell(h))$. The proof of our main theorem involves considerations of what are the equilibria of every such subgame. Given a strategy s in U(T) and a history $h \in H(T)$ we define s^h, the strategy combination induced by s on the subgame starting at h, $U(T-\ell(h))$, as follows. For every $h' \in H(T-\ell(h))$

$$s^h(h') = s(h,h').$$

((h,h') is the vector whose first ℓ componenets are those of h and others are those of h').

We say that a strategy combination s is subgame symmetric if for any two histories, h and $\bar{h}$ of the same length ℓ, $s^h(h') = s^{\bar{h}}(h')$ for every history $h' \in H(T-\ell)$. Thus a subgame symmetric strategy induces the same strategies on every two identical subgames.

We also define the restriction of retracts to subgames. Given a retract combination R and a history h we define the <u>retract combination induced by</u> R <u>on the subgame starting at</u> h, $U(T-\ell(h))$ as follows.

For every $h' \in H(T-\ell(h))$

$$R^h(h') = R(h,h').$$

Our first theorem relates persistent retracts to the retracts they induce on the subgames of $U(T)$.

<u>Theorem 1: Subgame Persistency:</u>

Let R be a persistent retract in $U(T)$ and let h be a history in $H(T)$. Then R^h is a peristent retract in $U(T-\ell(h))$.

<u>Proof:</u>

First we show that R^h is absorbing. Let O be a neighborhood absorbed by R. We will show that $O^h(h')$ $(= O(h,h'))$ is absorbed by R^h. Let s' be a strategy combination in O^h. Choose a strategy s in O such that $s^h = s'$ and $p(h,s) > 0$. Since R is absorbing, for every $h' \varepsilon H(T-\ell(h))$, each player i has a best response s_i to s at (h,h'), which belongs to $R_i(h,h') = R_i^h(h')$. Since $p(h,s) > 0$, s_i^h is a best response to $s'(=s^h)$ at h'.

Next we show that R^h is minimal. If not then there exists a retract $R' \subseteq R^h$ which is absorbing in $U(T-\ell(h))$. Define the retract $\hat{R}$ in $U(T)$ as follows. For each $h' = (h,h'')$, $\hat{R}(h') = R'(h')$ and $\hat{R}(h') = R(h')$ for all other h'. It is easy to see that $\hat{R}$ is absorbing in $U(T)$ which contradicts the minimality of R. Q.E.D.

The next lemma is needed in order to study the structure of certain persistent retracts in $U(T)$. We say that two states, c_1 and c_2 are <u>equivalent for player i at the history h</u> if for each strategy combination $s = (s_1,\ldots,s_n)$

$$u_i(s_1,\ldots,(s_i|h,c_1),\ldots s_n) = u_i(s_1,\ldots,(s_i|h,c_2),\ldots,s_n)$$

<u>Lemma 1</u>

For each history h in $H(T)$ with $\ell(h) < T-1$, no player i has two distinct equivalent states at h.

<u>Proof</u>

Let c_1 and c_2 be two distince states and let c be a state such that $u_i(c) \neq u_i(c_1)$. (Either $u_i(c_1) \neq 0$ and then choose $c = d$ or $u_i(c_1) = 0$, then choose c to be a uniformly positive state.) Let $s = (s_1,\ldots,s_n)$ be a strategy combination such that $p(h,s) = 1$ and such that for each player j, $s_j(h) = c_1$ and $s_j(h,\gamma) = c$ where γ is the proposal in which every player $j \neq i$ propose c, and i proposes c_2. Obviously $u_i(s) = u_i(c_1)$ but $u_i(s_1,\ldots,(s_i|h,c_2),\ldots,s_n) = u_i(c)$. Q.E.D.

From Lemma 1 we conclude

Lemma 2

If $R = (R_1,\ldots,R_n)$ is a persistent retract in U(T) then for each player i and each history $h \in H(T)$, $R_i(h) = \Delta(\hat{C})$ for a subset $\hat{C}$ of C, i.e., $R_i(h)$ is the set of all probability distributions over a certain subset of states $\hat{C}$.

Proof

Observe that $R_i(h)$ is the retract of the agent of player i at h (in the agent normal form of U(T)). Notice also that for each history h of length T and each player i, there are no two distinct equivalent states c_1 and c_2 for player i among his undominated strategies. The lemma follows then by Lemma 1 and Corollary 2 in Kalai and Samet [1983]. Q.E.D.

The structure of persistent retracts for the game U(1) is very simple as was shown in Kalai and Samet [1983]. A retract in this game is persistent iff it is a singleton which contains an agreement on a uniformly positive state. A similar simplicity can be found in a certain type of persistant retracts in U(T).

Lemma 3

Let $R = (R_1,\ldots,R_n)$ be a persistent retract which contains a subgame symmetric strategy. If T is sufficiently large, then there exists $0 \leq t_0 < T$ and states $c_{t_0},\ldots,c_{T-1}$ such that

(i) for each player i and history h of length t ($t_0 \leq t < T$), $R_i(h) = \{c_t\}$.

(ii) $u(c_{t_0}) > u(c_{t_0+1}) > \ldots > u(c_{T-1}) > 0$ (greater than in every coordinate).

(iii) c_{t_0} is not strongly dominated by any state, i.e., there is no state c with $u(c) > u(c_{t_0})$.

Proof

We prove the lemma by showing first that t_0 = T-1 satisfies (i) and (ii) and by showing next that if a certain t_0' satisfies (i) and (ii) and not (iii) then $t_0 = t_0'-1$ also satisfies (i) and (ii). Observe first that for each history h of length T-1, $R^h(0)(=R(h))$ is a persistent retract in U(1). It follows then by Theorem 6 of Kalai and Samet [1983] that for each such history R(h) is an agreement on a uniformly positive state. Since R contains a subgame symmetric strategy if follows that there exists a positive state c_{T-1} such that $R(h) = \{c_{T-1}\}$ for each history of length T-1. Now assume that (i) and (ii) hold for $t_0 = t'$ and that there exists a state c with $u(c) > u(c_{t'})$. Let us fix a history $h = (\gamma_1,\ldots,\gamma_{t'-1})$, and consider the retract R^h (for the game $U(T-t'+1)$) which by Theorem 1 is persistent. We first show that $R^h(0)(=R(h))$ contains an agreement on a state which strongly dominates $c_{t'}$. Assume the contrary that $R^h(0)$ does not include such an agreement and let c be a state with $u(c) > u(c_{t'})$. We contradict the assumption by showing $c^n \varepsilon R^h(0)$ $(c^n = (c,c,\ldots,c))$. Now by Lemma 2, for each player i, $R_i^h(0)$ contains a point from C (i.e., an extreme point of $\Delta(C)$). We examine two cases.

(i) <u>First Case</u>: $R^h(0) = \{(\hat{c})^n\}$ for some $\hat{c}\ \varepsilon$ C (i.e., there is no disagreement at 0). By the negation assumption there exists a player i for which $u_i(\hat{c}) \leq u_i(c_{t'})$. Consider a strategy s close to R^h in which at history 0 each player proposes $\hat{c}$ with probability $1-\varepsilon$ and c with probability ε. Also at every history of length 1, $s_i(h) = c_{t'}$. By responding to s at 0 with c, player i gets $\varepsilon^{n-1}u_i(c) + (1 - \varepsilon^{n-1})u_i(c_{t'})$, which is more than $u_i(c_{t'})$. By responding with any other proposal i gets no more than $u_i(c_{t'})$. Therefore $R_i^h(0)$ should contain also c which shows that the first case is impossible.

(ii) Second Case: A disagreement $\gamma = (c^1,\dots,c^n)$ belongs to $R^h(0)$. We consider the case in which there are more than two players $(n > 2)$. In γ each player, except perhaps one player, i_0, faces a group of n-1 players which do not agree among themselves on one state. Consider now a strategy s close to $R^h(0)$ in which at history 0 each player i proposes c in probability ε and c^i in probability $1-\varepsilon$. For each $i \neq i_0$ the best response to s at 0 is c and therefore $c \in R_i^h(0)$. Clearly also $c \in R_{i_0}^h(0)$ since c is the best response for i_0 at h when the rest of the players propose c at 0. Therefore $c^n \in R^h(0)$. The case of n=2 is similarly proved.

We have shown that for the persistant retract R^h, $c^n \in R^h(0)$ for some state c which strongly dominates c_t. We now show that by deleting from $R^h(0)$ everything but c^n we are left with an absorbing retract which, by the persistency of R^h proves that $R(h) = \{c^n\}$.

Define a retract $\hat{R}$ by $\hat{R}(h') = R^h(h')$ for each $h' \neq 0$ and $\hat{R}(0) = \{c^n\}$. We show that $\hat{R}$ is an absorbing retract. Since by Theorem 1, R^h is absorbing it is sufficient to show that for a certain neighborhood 0 of $\hat{R}$, c is the best response at 0 to each strategy in 0 for each player. The expected payoff of i is:

$$P(c \text{ is agreed upon at } 0) \cdot u_i(c) + \varepsilon$$

where ε is the expected payoff if there was no agreement on c at 0. When a strategy $\hat{s}$ is close to $\hat{R}$ $\hat{s}(0)$ is close to c^n and therefore the probability of reaching an agreement on c at 0 is high while ε is small. By responding to $\hat{s}$ at 0 with c, player i gets payoffs close to $u_i(c)$. If i is responding to $\hat{s}$ at 0 by $\hat{c} \neq c$ then his expected payoff is

$$p(\hat{c} \text{ is agreed upon at } 0)\cdot u_i(\hat{c}) + \sum_{\{h':\ell(h')=1\}} p(h',\hat{s})u_i(\hat{s}^{h'}).$$

Here the summation ranges over all histories h' of length 1. Agreement on $\hat{c}$ is not very likely at 0 when $\hat{s}$ is played. On the other hand, $\hat{s}(h')$ is close to $\hat{R}(h') = \{c^n_{t'}\}$ for each $h' = (\gamma)$ and thus $u_i(\hat{s}^{h'})$ is close to $u_i(c_{t'})$. It follows that by responding to $\hat{s}$ at 0 with $\hat{c}$, i's payoff is close to $u_i(c_{t'})$. Since $u_i(c) > u_i(c_{t'})$ it follows that i's best response to $\hat{s}$ at 0 is c.

We have proved now that for each history h of length $t'-1$, R(h) is a singleton which contains an agreement on a state c which strongly dominates $c_{t'}$. Because R contains a subame symmetric strategy this state c should be the same for all such histories which finishes the proof. Q.E.D.

We are now ready to prove the main theorem.

Theorem 2

If T is sufficiently large (the cardinality of C suffices) then for every subgame symmetric Nash equilibrium of U(T) which is persistant, u(s) is strongly individual rational and is weakly pareto optimal.

Proof:

By Lemma 3 there exists t_0 and weakly pareto optimal state c_0 such that $u(s^h) = u(c_0)$ for each history of length t_0. It suffices therefore to prove that $u(s) \geqslant u(s^h)$ for each history h. Indeed, if for some history h, and player i, $u_i(s) < u_i(s^h)$, then there exists a state c for which $u_i(c) > u_i(s)$. Define now the strategy $\hat{s} = (s_1,\ldots,s_n)$ such that $\hat{s}_j = s_j$ for each player $j \neq i$ and $\hat{s}_i(h') = c$ when $\ell(h') < \ell(h)$ and $\hat{s}_i(h') = s_i(h')$ otherwise. If the game with $\hat{s}$ is over in no more than $\ell(h)$ steps then c is agreed with $u_i(c) = u_i(\hat{s}) > u_i(s)$. Otherwise the game is over after more than

$\ell(h)$ steps and $u_i(\hat{s}) = \sum P(\hat{s},h')\cdot u_i(s^{h'})$ where the summation is over all h' with $\ell(h') = \ell(h)$. But since s in symmetric $u_i(\hat{s}) = u_i(s^h) > u_i(s)$. It follows that by playing $\hat{s}_i$ player i improves upon his payoff in contradiction to s being a Nash equilibrium. Q.E.D.

Theorem 3: Let T be a positive integer. If c is strongly individually rational and weakly Pareto optimal then there is a persistent subgame symmetric Nash equilibrium s in U(T) whose outcome is c (i.e., $p(c,s) = 1$.)

Proof: Let s be the constant strategy c^n, $s_i(h) = c$ for every player i at every history h. It is obvious that s is subgame symmetric and since c is strongly individually rational, s is a Nash equilibrium. We have to show that s belongs to some persistent retract.

Consider the following retract R. For every history h with $\ell(h) = T - 1$, $R_i(h) = \{c\}$ for every player i. For every history h with $\ell(h) < T - 1$ and every player i $R_i(H) = \Delta(C)$.

To show that R is a persistent retract we first observe that it is absorbing. Since for histories with length less than T-1, $R_i(h)$ is the entire set of strategies, every strategy is absorbed by R at these histories. For histories of length T-1, we have an induced one stage unanimity game and hence, by Kalai and Samet [1983], R(h) is absorbing.

To show that R is a minimal absorbing retract it suffices to show that for every history h with $\ell(h) < T - 1$, for every player i, and for every state $\bar{c}$, there is a strategy $\bar{s}$, arbitrarily close to R, with $\bar{c}$ being a unique best reply of player i to $\bar{s}$ at h. For such h, i and $\bar{c}$ consider a strategy combination $\bar{s}$ which satisfies the following conditons.

(i) $p(h,\bar{s}) = 1$.

(ii) For each player $j \neq i$ and history h' with $\ell(h) \leq \ell(h') < T-1$, $\bar{s}_j(h') = d$, and $\bar{s}_i(h') = c$.

(iii) For histories h' with $\ell(h') = T-1$, we distinguish two cases

a. If player i's part of the proposal made at period number $\ell(h)$ in h' was $\bar{c}$ then $\bar{s}_j(h') = c$ for each player j (including i).

b. Otherwise $s_j(h') = (1-\varepsilon)c + \varepsilon d$ for small $\varepsilon > 0$ for all players j.

It is clear that the only best reply of player i at h is $\bar{c}$. By backwards induction on t, this illustrates that R is a minimal absorbing retract, since $\bar{s}$ is arbitrarily close to R. Thus, the constant strategy s is persistent.

4. Nonterminating Unanimity Games

A second natural generalization of the one stage unanimity game to a multistage one is in a game which does not terminate when an agreement is reached. We define inductively G(d,T), the <u>nonterminating</u> T ($T \geq 1$) <u>attempts unanimity game with the status quo state</u> d as follows

If $T = 0$, then G(d,0) is the game with no strategies whose outcome is d. For $T > 0$, consider the following game. At the first iteration, each player i proposes a state $\gamma_i \in C$. The joint proposal $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is then publicly announced. If $\gamma_1 = \gamma_2 = \gamma_3 \ldots = \gamma_n = c$ then the players proceed to play G(c,T-1). Otherwise they proceed to play G(d,T-1). (The T-1 attempts games are assumed to have been defined inductively.)

We believe that the exact analyses of Theorems 2 and 3 would hold for nonterminating games also.

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